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DIAGONALITY OF ACTIONS AND KMS WEIGHTS

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ABSTRACT. We show that in many cases a one-parameter group of automorphisms on a C^*-algebra of an étale groupoid is given by a real-valued homomorphism on the groupoid if and only if the KMS weights of the one-parameter group is given by measures on the unit space. The results are applied to graph C^*-algebras.

KEYWORDS: *KMS weights, one-parameter groups, diagonality, graph C^*-algebras.*

MSC (2010): 46L55, 46L60, 46L30.

1. INTRODUCTION

Recent years have seen an increasing interest in the investigation of KMS states for one-parameter actions on C^*-algebras. While the original motivation for the introduction of KMS states came from the interpretation of these states as equilibrium states in models from quantum statistical mechanics, the renewed interest stems also from more purely mathematical considerations, where the KMS states have been related to objects and structures from other fields, such as number theory or dynamical systems. In the present paper we investigate relations between properties of the KMS states and properties of the one-parameter action giving rise to them. As we shall now explain, we show that the existence of a "diagonal" KMS state or weight implies that the action itself must be "diagonal".

For most if not all the one-parameter actions on C^*-algebras for which we have been able to determine the structure of the KMS states or KMS weights, the underlying C^*-algebra can be presented as the C^*-algebra of a locally compact groupoid, as introduced by Renault in [9], and the action described as arising from a continuous real-valued homomorphism on the groupoid by a canonical procedure also introduced in [9]. For this reason the results of Neshveyev, [7], which extend results of Renault and give a general and abstract description of the KMS states for such actions on a groupoid C^*-algebra are of utmost importance. In the following we call these actions *diagonal*.

When the groupoid and the associated C^*-algebra is fixed, it is certainly not all one-parameter actions that are diagonal. It follows from Neshveyev's theorem, Theorem 1.3 in [7], that a diagonal action has the property that if a KMS state exists, there will also be one which factorizes through the canonical conditional expectation onto the abelian C^*-subalgebra generated by the continuous compactly supported functions on the unit space. In the following we call these states *diagonal*. The present work sprang from the realization that in many cases the property that there is a diagonal KMS state characterizes the diagonal actions. That is, for many groupoid C^*-algebras a one-parameter action is diagonal if and only if the action admits a diagonal KMS state. The simplest example of this is perhaps the following.

Consider the C^*-algebra M_n of complex n by n matrices. A continuous one-parameter group α of automorphisms on M_n is inner in the sense that there is a self-adjoint matrix $A \in M_n$ such that

$$\alpha_t(B) = e^{itA} B e^{-itA}$$

for all $t \in \mathbb{R}$ and all $B \in M_n$. For each $\beta \in \mathbb{R}$ there is a unique β-KMS state ω_β for α given by

$$\omega_\beta(B) = \frac{\mathrm{Tr}(e^{-\beta A} B)}{\mathrm{Tr}(e^{-\beta A})}.$$

It can be shown that for $\beta \neq 0$ the state ω_β factorizes through the canonical (and unique) conditional expectation from M_n onto the C^*-subalgebra of diagonal matrices if and only if A is diagonal. It is this fact we will generalize. For this note that M_n is the groupoid C^*-algebra of the groupoid $\mathcal{G} = \{1,2,3,\ldots,n\} \times \{1,2,3,\ldots,n\}$ with operations

$$(a,b)(b,c) = (a,c) \quad \text{and} \quad (a,b)^{-1} = (b,a).$$

When M_n is identified with the C^*-algebra $C^*(\mathcal{G})$ of $\mathcal{G}$, the diagonal matrices in M_n constitute the C^*-algebra $C(\mathcal{G}^{(0)})$ of (continuous) functions on $\mathcal{G}$ whose support is contained in the unit space

$$\mathcal{G}^{(0)} = \{(k,k) : k \in \{1,2,\ldots,n\}\}$$

of $\mathcal{G}$. In this picture the conditional expectation onto the diagonal matrices is the map

$$P : C^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$$

which restricts functions to $\mathcal{G}^{(0)}$. Furthermore, the matrix A will be diagonal if and only if there is a groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that

$$\alpha_t(f)(a,b) = e^{ic(a,b)t} f(a,b) \tag{1.1}$$

for all $t \in \mathbb{R}$, $(a,b) \in \mathcal{G}$ and all $f \in C^*(\mathcal{G})$. Because the whole setup is so transparent in this case, we can easily conclude that there is an equivalence between the following conditions:

(1) α is diagonal in the sense that there is a groupoid homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that (1.1) holds.

(2) There is a $\beta \neq 0$ and a β-KMS state ω_β for α which is diagonal in the sense that it factorizes through the conditional expectation $C^*(\mathcal{G}) \to C(\mathcal{G}^{(0)})$.

(3) $\alpha_t(f) = f$ for all $t \in \mathbb{R}$ and all $f \in C(\mathcal{G}^{(0)})$.

(4) All β-KMS states of α, for $\beta \neq 0$, are diagonal.

Our main result is that these equivalences hold much more generally as we shall now explain.

2. NOTATION AND MAIN RESULT

Let $\mathcal{G}$ be a second countable locally compact Hausdorff étale groupoid with unit space $\mathcal{G}^{(0)}$. Let $r : \mathcal{G} \to \mathcal{G}^{(0)}$ and $s : \mathcal{G} \to \mathcal{G}^{(0)}$ be the range and source maps, respectively. For $x \in \mathcal{G}^{(0)}$ put $\mathcal{G}^x = r^{-1}(x)$, $\mathcal{G}_x = s^{-1}(x)$ and $\mathcal{G}_x^x = s^{-1}(x) \cap r^{-1}(x)$. Note that $\mathcal{G}_x^x$ is a group, the *isotropy group* at x. The space $C_c(\mathcal{G})$ of continuous compactly supported functions is a $*$-algebra when the product is defined by

$$(f_1 * f_2)(g) = \sum_{h \in \mathcal{G}^{r(g)}} f_1(h) f_2(h^{-1}g)$$

and the involution by $f^*(g) = \overline{f(g^{-1})}$. To define the *reduced groupoid C^*-algebra* $C_r^*(\mathcal{G})$, let $x \in \mathcal{G}^{(0)}$. There is a representation π_x of $C_c(\mathcal{G})$ on the Hilbert space $l^2(\mathcal{G}_x)$ of square-summable functions on $\mathcal{G}_x$ given by

$$\pi_x(f)\psi(g) = \sum_{h \in \mathcal{G}^{r(g)}} f(h)\psi(h^{-1}g).$$

$C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ with respect to the norm

$$\|f\|_r = \sup_{x \in \mathcal{G}^{(0)}} \|\pi_x(f)\|.$$

Note that $C_r^*(\mathcal{G})$ is separable since we assume that the topology of $\mathcal{G}$ is second countable.

We shall here be concerned not only with KMS states, but more generally with KMS weights. Let A be a C^*-algebra and A_+ the convex cone of positive elements in A. A *weight* on A is a map $\psi : A_+ \to [0, \infty]$ with the properties that $\psi(a+b) = \psi(a) + \psi(b)$ and $\psi(\lambda a) = \lambda\psi(a)$ for all $a, b \in A_+$ and all $\lambda \in \mathbb{R}$, $\lambda > 0$. By definition ψ is *densely defined* when $\{a \in A_+ : \psi(a) < \infty\}$ is dense in A_+ and *lower semi-continuous* when $\{a \in A_+ : \psi(a) \leqslant \alpha\}$ is closed for all $\alpha \geqslant 0$. We will use [5], [6] as our source for information on weights, and we say that a weight is *proper* when it is non-zero, densely defined and lower semi-continuous. Let ψ be a proper weight on A. Set $\mathcal{N}_\psi = \{a \in A : \psi(a^*a) < \infty\}$ and note that

$$\mathcal{M}_\psi = \mathrm{Span}\{a^*b : a, b \in \mathcal{N}_\psi\}$$

is a dense $*$-subalgebra of A, and that there is a unique well-defined linear map $\mathcal{M}_\psi \to \mathbb{C}$ which extends $\psi : \mathcal{M}_\psi \cap A_+ \to [0, \infty)$. We denote also this densely defined linear map by ψ.

Let $\alpha : \mathbb{R} \to \operatorname{Aut} A$ be a continuous one-parameter group of automorphisms on A. Let $\beta \in \mathbb{R}$. Following [2] we say that a proper weight ψ on A is a *β-KMS weight* for α when

(i) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and

(ii) for every pair $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function F defined on the closed strip D_β in $\mathbb{C}$ consisting of the numbers $z \in \mathbb{C}$ whose imaginary part lies between 0 and β, and is holomorphic in the interior of the strip and satisfies that

$$F(t) = \psi(a\alpha_t(b)), \; F(t + \mathrm{i}\beta) = \psi(\alpha_t(b)a)$$

for all $t \in \mathbb{R}$.

Compared to [2] we have changed the orientation in order to have the same sign convention as in [1], for example. It will be important for us that there is an alternative characterization of when a proper weight is a β-KMS weight. Specifically, by Proposition 1.11 in [6] a proper weight ψ is a β-KMS weight for α if and only if it is α-invariant (as in (i) above) and

$$\psi(a^*a) = \psi(\alpha_{\mathrm{i}\beta/2}(a)\alpha_{\mathrm{i}\beta/2}(a)^*) \tag{2.1}$$

for all a in the domain $D(\alpha_{\mathrm{i}\beta/2})$ of $\alpha_{\mathrm{i}\beta/2}$; the closure of the restriction of $\alpha_{\mathrm{i}\beta/2}$ to the analytic elements for α, cf. [5]. A β-KMS weight ψ with the property that

$$\sup\{\psi(a) : 0 \leqslant a \leqslant 1\} = 1$$

will be called a *β-KMS state*.

Returning to the case $A = C_{\mathrm{r}}^*(\mathcal{G})$, note that the map $C_{\mathrm{c}}(\mathcal{G}) \to C_{\mathrm{c}}(\mathcal{G}^{(0)})$ which restricts functions to $\mathcal{G}^{(0)}$ extends to a conditional expectation $P : C_{\mathrm{r}}^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$. Via P a regular Borel measure m on $\mathcal{G}^{(0)}$ gives rise to a weight $\varphi_m : C_{\mathrm{r}}^*(\mathcal{G})_+ \to [0, \infty]$ defined by the formula

$$\varphi_m(a) = \int_{\mathcal{G}^{(0)}} P(a) \, \mathrm{d}m. \tag{2.2}$$

It follows from Fatou's lemma that φ_m is lower semi-continuous. Since $\varphi_m(faf) < \infty$ for every non-negative function f in $C_{\mathrm{c}}(\mathcal{G}^{(0)})$, it follows that φ_m is also densely defined, i.e. φ_m is a proper weight on $C_{\mathrm{r}}^*(\mathcal{G})$ if and only if m is not the zero measure. In the following we say that a proper weight ψ on $C_{\mathrm{r}}^*(\mathcal{G})$ is *diagonal* when $\psi = \varphi_m$ for some regular Borel measure m on $\mathcal{G}^{(0)}$. By the Riesz representation theorem this occurs if and only if $\psi \circ P = \psi$.

Given a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ there is a continuous one-parameter group σ^c on $C_{\mathrm{r}}^*(\mathcal{G})$ such that

$$\sigma_t^c(g)(\xi) = \mathrm{e}^{\mathrm{i}tc(\xi)} g(\xi) \tag{2.3}$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$, cf. [9]. A one-parameter action of this kind will be called *diagonal* in the following. We can then formulate our main result as follows.

THEOREM 2.1. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid such that for at least one element $x \in \mathcal{G}^{(0)}$ the isotropy group $\mathcal{G}_x^x$ is trivial, i.e. $\mathcal{G}_x^x = \{x\}$, and that $\mathcal{G}$ is minimal in the sense that $s(r^{-1}(y))$ is dense in $\mathcal{G}^{(0)}$ for all $y \in \mathcal{G}^{(0)}$. Furthermore, assume that $\mathcal{G}^{(0)}$ is totally disconnected.*

Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G})$ and assume that for some $\beta_0 \neq 0$ there is a β_0-KMS weight for α.

The following are equivalent:

(i) *There is a $\beta_1 \neq 0$ and a diagonal β_1-KMS weight for α.*

(ii) *Whenever $\beta \neq 0$ and there is a β-KMS weight for α, there is also a diagonal β-KMS weight for α.*

(iii) *$\alpha_t(f) = f$ for all $t \in \mathbb{R}$ and all $f \in C_0(\mathcal{G}^{(0)})$.*

(iv) *$\alpha_t(C_0(\mathcal{G}^{(0)})) \subseteq C_0(\mathcal{G}^{(0)})$ for all $t \in \mathbb{R}$.*

(v) *α is diagonal.*

Some of the (non-trivial) implications hold with fewer assumptions. Specifically, (i)⇒(iii) holds without the assumption that the unit space is totally disconnected by Proposition 4.1, and the implication (iii)⇒(v) holds assuming only that the points with trivial isotropy are dense in $\mathcal{G}^{(0)}$ (i.e. if $\mathcal{G}$ is topologically principal) by Proposition 4.3. The implication (v)⇒(ii) holds whenever $\mathcal{G}^{(0)}$ is totally disconnected, without any further assumptions, as it follows from Corollary 3.4. It may be that this implication is true in general and if so the theorem with (iv) removed is true also when $\mathcal{G}^{(0)}$ is not totally disconnected. However, the first two assumptions on $\mathcal{G}$ which are equivalent to topological principality and minimality of $\mathcal{G}$ are certainly necessary for the implication (iii)⇒(i) to hold, cf. Example 4.9. Finally, the gauge action on the C^*-algebra of a strongly connected (row-finite) graph with infinite Gurevich entropy does not admit any KMS weights at all, cf. [14], showing that it is necessary to assume the existence of some KMS-weight for the implication (v)⇒(i) to hold.

3. NESHVEYEV'S THEOREM FOR KMS WEIGHTS

LEMMA 3.1. *Let A be a C^*-algebra, α a continuous one-parameter group of automorphisms on A and ψ a KMS weight for α. Let $p \in A$ be a projection in the fixed point algebra of α. Then $\psi(p) < \infty$.*

Proof. Assume that $a \geqslant 0$, $\psi(a) < \infty$ and that $a^{1/2}$ is analytic for α. Then Proposition 1.11 in [6] applies to conclude that

$$\psi(pap) = \psi(\alpha_{i\beta/2}(a^{1/2})p\alpha_{i\beta/2}(a^{1/2})^*) \leqslant \psi(\alpha_{i\beta/2}(a^{1/2})\alpha_{i\beta/2}(a^{1/2})^*) = \psi(a). \tag{3.1}$$

Let $\{b_k\}$ be a sequence of positive elements in A such that $\lim_{k\to\infty} b_k = p$ and $\psi(b_k) < \infty$ for all k. For each $n \in \mathbb{N}$, set

$$c_{k,n} = \sqrt{\frac{n}{\pi}} \int_{\mathbb{R}} \alpha_t(b_k) \mathrm{e}^{-nt^2} \, \mathrm{d}t.$$

Then $c_{k,n}$ is analytic for α and $\psi(c_{k,n}^2) \leqslant \|c_{k,n}\|\psi(c_{k,n}) \leqslant \|c_{k,n}\|\psi(b_k) < \infty$ for all k, n. It follows therefore from (3.1) that $\psi(pc_{k,n}^2 p) \leqslant \psi(c_{k,n}^2) < \infty$ for all k, n. Note that

$$\lim_{k\to\infty} \lim_{n\to\infty} c_{k,n}^2 = \lim_{k\to\infty} b_k^2 = p^2 = p.$$

It follows that there are k, n such that $\|p - pc_{k,n}^2 p\| \leqslant 1/2$, and then spectral theory tells us that $pc_{k,n}^2 p \geqslant (1/2)p$. Hence $\psi(p) \leqslant 2\psi(pc_{k,n}^2 p) < \infty$. ∎

Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and $c : \mathcal{G} \to \mathbb{R}$ a continuous homomorphism. Let μ be a regular Borel measure on $\mathcal{G}^{(0)}$ and $\beta \in \mathbb{R}$ a real number. We say that μ is *$(\mathcal{G}, c)$-conformal with exponent β*, as in [14], or that μ is *quasi-invariant with Radon–Nikodym cocycle* $\mathrm{e}^{-\beta c}$, as in [7], when

$$\mu(s(W)) = \int_{r(W)} \mathrm{e}^{\beta c(r_W^{-1}(x))} \, \mathrm{d}\mu(x) \tag{3.2}$$

for every open bi-section $W \subseteq \mathcal{G}$, where r_W^{-1} denotes the inverse $r : W \to r(W)$. For each $x \in \mathcal{G}^{(0)}$ we can consider the full group C^*-algebra $C^*(\mathcal{G}_x^x)$ of the discrete group $\mathcal{G}_x^x$, the isotropy group at x. As in [7] we denote for $g \in \mathcal{G}_x^x$ by u_g the characteristic function of the element g when we consider $C^*(\mathcal{G}_x^x)$ as a completion of $C_\mathrm{c}(\mathcal{G}_x^x)$. Thus $u_g, g \in \mathcal{G}_x^x$, are the canonical unitary generators of $C^*(\mathcal{G}_x^x)$. Following [7] we say that a collection $\varphi_x, x \in \mathcal{G}^{(0)}$, of states on $C^*(\mathcal{G}_x^x)$ is a *μ-measurable field* when the function

$$\mathcal{G}^{(0)} \ni x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$$

is μ-measurable for all $f \in C_\mathrm{c}(\mathcal{G})$. We identify two μ-measurable fields $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ and $\{\varphi'_x\}_{x\in\mathcal{G}^{(0)}}$ when $\varphi_x = \varphi'_x$ for μ-almost every x.

The following theorem is a version for weights of Theorem 1.3 in [7]. Note that it deals with the full groupoid C^*-algebra $C^*(\mathcal{G})$ which is an extension of the reduced groupoid $C_\mathrm{r}^*(\mathcal{G})$. We refer to [9] for the definition of the full groupoid C^*-algebra. To understand the following theorem and its proof it suffices to know that $C^*(\mathcal{G})$, like $C_\mathrm{r}^*(\mathcal{G})$, is a completion of $C_\mathrm{c}(\mathcal{G})$ and that a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ also defines a continuous one-parameter group σ^c on $C^*(\mathcal{G})$ via the formula (2.3).

THEOREM 3.2 (Neshveyev's theorem for weights). *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected.*

There is a bijective correspondence between the β-KMS weights for σ^c on $C^*(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a regular Borel measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states φ_x on $C^*(\mathcal{G}_x^x)$ such that:

(i) *μ is quasi-invariant with Radon–Nikodym cocycle* $\mathrm{e}^{-\beta c}$,

(ii) $\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$, $h \in \mathcal{G}_x$, and*

(iii) $\varphi_x(u_g) = 0$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x \backslash c^{-1}(0)$.*

The β-KMS weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ has the properties that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and

$$\phi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x) \tag{3.3}$$

when $f \in C_c(\mathcal{G})$.

Proof. Let ϕ be a β-KMS weight for σ^c. Since $\mathcal{G}^{(0)}$ is totally disconnected by assumption there is a sequence $p_1 \leqslant p_2 \leqslant p_3 \leqslant \cdots$ of projections in $C_c(\mathcal{G}^{(0)})$ with the property that $\{p_n\}$ is an approximate unit for $C^*(\mathcal{G})$. It follows from Lemma 3.1 that $\phi(p_n) < \infty$ for all n. Since $\phi \neq 0$ we can assume, without loss of generality, that $\phi(p_n) > 0$ for all n. Since $\phi(f) < \infty$ for every non-negative function in $C_c(\mathcal{G}^{(0)})$ it follows that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and from the Riesz representation theorem that there is a unique regular Borel measure μ on $\mathcal{G}^{(0)}$ such that

$$\phi(f) = \int_{\mathcal{G}^{(0)}} f(x)\, \mathrm{d}\mu(x)$$

for all $f \in C_c(\mathcal{G}^{(0)})$. Let U_n be the compact and open support of p_n, and set

$$\mathcal{G}^n = \mathcal{G}|_{U_n} = \{\xi \in \mathcal{G} :\ r(\xi), s(\xi) \in U_n\}$$

and $c_n = c|_{\mathcal{G}^n}$. Note that $\phi(p_n)^{-1}\phi$ restricts to a β-KMS state on $p_n C^*(\mathcal{G}) p_n = C^*(\mathcal{G}^n)$. It follows from Neshveyev's theorem [7] that there is a probability measure μ_n on U_n, and a μ_n-measurable field $\{\varphi_x^n\}_{x\in U_n}$ of states such that:

(an) μ_n is quasi-invariant on $\mathcal{G}^n$ with cocycle $\mathrm{e}^{-\beta c_n}$,

(bn) $\varphi_x^n(u_g) = \varphi_{r(h)}^n(u_{hgh^{-1}})$ for μ_n-almost every $x \in U_n$ and all $g \in \mathcal{G}_x^x$, $h \in (\mathcal{G}^n)_x$,

(cn) $\varphi_x^n(u_g) = 0$ for μ_n-almost every $x \in U_n$ and all $g \in \mathcal{G}_x^x \backslash c_n^{-1}(0)$,

and

$$\phi(p_n)^{-1}\phi(f) = \int_{U_n} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\, \mathrm{d}\mu_n(x)$$

when $f \in C_c(\mathcal{G}^n)$. For every $f \in C_c(U_n)$ we get that:

$$\phi(p_n)^{-1} \int_{U_n} f(x)\, \mathrm{d}\mu(x) = \phi(p_n)^{-1} \int_{\mathcal{G}^{(0)}} f(x)\, \mathrm{d}\mu(x) = \phi(p_n)^{-1}\phi(f)$$
$$= \int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\, \mathrm{d}\mu_n(x) = \int_{U_n} f(x)\, \mathrm{d}\mu_n(x)$$

so $\mu|_{U_n} = \phi(p_n)\mu_n$. Notice that since $\phi(p_n) > 0$, being a μ null set in U_n is the same as being a μ_n null set. For a Borel set $V \subseteq U_n \subseteq U_{n+1}$ we have that:

$$\phi(p_{n+1})\mu_{n+1}(V) = \mu(V) = \phi(p_n)\mu_n(V).$$

So $\mu_n = \phi(p_{n+1})/\phi(p_n)\mu_{n+1}|_{U_n}$. For every $f \in C_c(\mathcal{G}^n)$ we get that:

$$\int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)\, \mathrm{d}\mu_n(x) = \phi(p_n)^{-1}\phi(f) = \frac{\phi(p_{n+1})}{\phi(p_n)}\phi(p_{n+1})^{-1}\phi(f)$$
$$= \frac{\phi(p_{n+1})}{\phi(p_n)} \int_{U_{n+1}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g)\, \mathrm{d}\mu_{n+1}(x)$$
$$= \int_{U_n} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^{n+1}(u_g)\, \mathrm{d}\mu_n(x).$$

Since μ_n by choice satisfy (an), and since it is easily seen that $\{\varphi_x^{n+1}\}_{x \in U_n}$ satisfy (bn) and (cn), the uniqueness statement in Neshveyev's theorem gives that $\varphi_x^n = \varphi_x^{n+1}$ for a.e. $x \in U_n$. Hence for a.e. $x \in \mathcal{G}^{(0)}$ we can define a state on $C^*(\mathcal{G}_x^x)$ by:

$$\varphi_x(d) = \lim_{n \to \infty} \varphi_x^n(d).$$

For every $f \in C_c(\mathcal{G})$ there is a $N \in \mathbb{N}$ such that $f \in C_c(\mathcal{G}^N)$, and hence:

$$\phi(f) = \phi(P_N) \int_{U_N} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^N(u_g)\, \mathrm{d}\mu_N(x) = \int_{\mathcal{G}^{(0)}} \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x).$$

The properties (i)–(iii) follow from (an)–(cn), and measurability of $x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$ follows from measurability of $x \mapsto \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x^n(u_g)$.

For the converse, assume we are given a pair $(\mu, \{\varphi_x\}_{x \in \mathcal{G}^{(0)}})$ for which (i), (ii) and (iii) hold. As shown by Neshveyev in the proof of Theorem 1.1 in [7] every x gives rise to a state ψ_x on $C^*(\mathcal{G})$ such that

$$\psi_x(f) = \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$$

when $f \in C_c(\mathcal{G})$. Note that $x \to \sum_{g \in \mathcal{G}_x^x} f(g)\varphi_x(u_g)$ is μ-measurable by assumption, and then $x \to \psi_x(a)$ is also for each $a \in C^*(\mathcal{G})$. For $a \geqslant 0$ we can therefore define

$$\phi(a) = \int_{\mathcal{G}^{(0)}} \psi_x(a)\, \mathrm{d}\mu(x).$$

ϕ is a lower semi-continuous weight by Fatous lemma and by regularity

$$\phi(p_n a p_n) = \int_{U_n} \psi_x(a)\, \mathrm{d}\mu(x) \leqslant \|a\|\mu(U_n) < \infty$$

for all n, so it is also densely defined. Note that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and that (3.3) holds by construction. Since the pair $(\phi(p_n)^{-1}\mu, \{\varphi_x\}_{x\in U_n})$ represents $\phi(p_n)^{-1}\phi$ in the sense of Theorem 1.1 in [7] it follows from Theorem 1.3 in [7] that ϕ is a bounded β-KMS weight on $p_n C^*(\mathcal{G}) p_n$. Since

$$\psi_x(p_n a p_n) = \begin{cases} \psi_x(a) & x \in U_n, \\ 0 & x \notin U_n, \end{cases}$$

we find that $\lim_{n\to\infty} \phi(p_n a p_n) = \lim_{n\to\infty} \int_{U_n} \psi_x(a)\, \mathrm{d}\mu(x) = \phi(a)$ for all $a \geqslant 0$ in $C^*(\mathcal{G})$. Now note that for every a in the domain of $\sigma^c_{-\mathrm{i}\beta/2}$,

$$\phi(p_n a^* a p_n) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a p_n)\sigma^c_{-\mathrm{i}\beta/2}(a p_n)^*) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) p_n \sigma^c_{-\mathrm{i}\beta/2}(a)^*)$$

since ϕ is a bounded β-KMS weight on $p_n C^*(\mathcal{G}) p_n$. Since

$$\lim_{n\to\infty} \phi(\sigma^c_{-\mathrm{i}\beta/2}(a) p_n \sigma^c_{-\mathrm{i}\beta/2}(a)^*) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a)\sigma^c_{-\mathrm{i}\beta/2}(a)^*)$$

by the lower semi-continuity of ϕ, we conclude that

$$\phi(a^* a) = \phi(\sigma^c_{-\mathrm{i}\beta/2}(a)\sigma^c_{-\mathrm{i}\beta/2}(a)^*),$$

showing that ϕ is indeed a β-KMS weight for σ^c.

If $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ and $(\mu', \{\varphi'_x\}_{x\in\mathcal{G}^{(0)}})$ represent the same β-KMS weight it follows from the uniqueness part of the Riesz representation theorem that $\mu = \mu'$. By using (3.3) we find that

$$\int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}^x_x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x) = \int_{\mathcal{G}^{(0)}} k(x) \sum_{g\in\mathcal{G}^x_x} f(g)\varphi'_x(u_g)\, \mathrm{d}\mu(x) \tag{3.4}$$

when $f \in C_c(\mathcal{G})$ and $k \in C_c(\mathcal{G}^{(0)})$. It follows from this that

$$\sum_{g\in\mathcal{G}^x_x} f(g)\varphi_x(u_g) = \sum_{g\in\mathcal{G}^x_x} f(g)\varphi'_x(u_g)$$

for μ-almost all $x \in \mathcal{G}^{(0)}$ and all $f \in C_c(\mathcal{G})$. Thanks to the separability of $C^*(\mathcal{G})$ we conclude that $\varphi_x = \varphi'_x$ for μ-almost all x. ∎

COROLLARY 3.3. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected and that the isotropy groups $\mathcal{G}^x_x$, $x \in \mathcal{G}^{(0)}$, are all amenable.*

There is a bijective correspondence between the β-KMS weights for σ^c on $C^*_r(\mathcal{G})$ and the pairs $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$, where μ is a regular Borel measure on $\mathcal{G}^{(0)}$ and $\{\varphi_x\}_{x\in\mathcal{G}^{(0)}}$ is a μ-measurable field of states φ_x on $C^*_r(\mathcal{G}^x_x)$ such that:

(i) *μ is quasi-invariant with cocycle* $e^{-\beta c}$,

(ii) $\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x$, $h \in \mathcal{G}_x$, and*

(iii) $\varphi_x(u_g) = 0$ *for μ-almost every $x \in \mathcal{G}^{(0)}$ and all $g \in \mathcal{G}_x^x \backslash c^{-1}(0)$.*

The β-KMS weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x\in\mathcal{G}^{(0)}})$ has the properties that $C_c(\mathcal{G}) \subseteq \mathcal{M}_\phi$ and

$$\phi(f) = \int_{\mathcal{G}^{(0)}} \sum_{g\in\mathcal{G}_x^x} f(g)\varphi_x(u_g)\, \mathrm{d}\mu(x) \tag{3.5}$$

when $f \in C_c(\mathcal{G})$.

Proof. It suffices to show that the assumption on the isotropy groups implies that every β-KMS weight ϕ on $C^*(\mathcal{G})$ factorises through $C^*_r(\mathcal{G})$. To this end note that it follows from Lemma 2.1 in [13] that for each $n \in \mathbb{N}$ there is a bounded β-KMS weight $\widetilde{\phi}_n$ on $p_n C^*_r(\mathcal{G}) p_n$ such that $\widetilde{\phi}_n(p_n\pi(a)p_n) = \phi(p_n a p_n)$ for all $a \in C^*(\mathcal{G})$ where $\pi : C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ is the canonical surjection. Then $\widetilde{\phi}_n(p_n b p_n) \leqslant \widetilde{\phi}_{n+1}(p_{n+1} b p_{n+1})$ for all $b \geqslant 0$ in $C^*_r(\mathcal{G})$ and we can define a lower semi-continuous weight $\widetilde{\phi}$ on $C^*_r(\mathcal{G})$ such that $\widetilde{\phi}(b) = \lim_{n\to\infty} \widetilde{\phi}_n(p_n b p_n)$. It follows that $\widetilde{\phi} \circ \pi = \phi$. ∎

It is an interesting problem if Corollary 3.3 remains true without the amenability assumption on the isotropy groups. For the proof of our main result the following suffices.

COROLLARY 3.4. *Let $\mathcal{G}$ be a locally compact second countable Hausdorff étale groupoid and let $c : \mathcal{G} \to \mathbb{R}$ be a continuous homomorphism. Assume that the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is totally disconnected. If there is a β-KMS weight for σ^c on $C^*_r(\mathcal{G})$ there is also one which is diagonal.*

Proof. Let ϕ be a β-KMS weight for σ^c on $C^*_r(\mathcal{G})$ and let $\pi : C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ be the canonical surjection. Then $\phi \circ \pi$ is a β-KMS weight for σ^c on $C^*(\mathcal{G})$ and we can consider the corresponding regular Borel measure μ. Since μ is quasi-invariant with cocycle $e^{-\beta c}$ it follows from Proposition 2.1 in [14] that μ defines a diagonal β-KMS weight by the formula (2.2). ∎

4. CONDITIONS ON A KMS WEIGHT THAT IMPLY DIAGONALITY OF THE ACTION

4.1. WHEN KMS WEIGHTS FACTOR THROUGH THE CONDITIONAL EXPECTATION ONTO AN ABELIAN SUBALGEBRA.

A weight ω is *faithful* when $a \geqslant 0$, $\omega(a) = 0 \Rightarrow a = 0$.

PROPOSITION 4.1. *Let A be a C^*-algebra and γ a continuous one-parameter group of automorphisms on A. Let $D \subseteq A$ be an abelian C^*-subalgebra and $P : A \to D$ a conditional expectation.*

Assume that ω is a faithful β-KMS weight for γ, $\beta \neq 0$, such that $\omega \circ P = \omega$. It follows that $\gamma_t(d) = d$ for all $t \in \mathbb{R}$ and all $d \in D$.

Proof. Let $f \in D$, $f \geqslant 0$. Since ω is densely defined there is a sequence $\{a_n\}$ of positive elements in A such that $\lim_{n\to\infty} a_n = f$ and $\omega(a_n) < \infty$ for all n. Then $\lim_{n\to\infty} P(a_n) = f$ and $\omega(P(a_n)) = \omega(a_n) < \infty$. It suffices therefore to consider $f \in D$, $f \geqslant 0$ such that $\omega(f) < \infty$ and show that $\gamma_t(f) = f$ for all $t \in \mathbb{R}$.

We find that

$$\omega(af) = \omega(P(a)f) = \omega(fP(a)) = \omega(fa) \tag{4.1}$$

for all $a \in \mathcal{M}_\omega$. Since $f \in \mathcal{M}_\omega$ and this is a subalgebra, the desired conclusion follows from Result 6.29 in [5]. ∎

COROLLARY 4.2. *Let A be a simple C^*-algebra and γ a continuous one-parameter group of automorphisms on A. Let $D \subseteq A$ be an abelian C^*-subalgebra and $P : A \to D$ a conditional expectation.*

Assume that ω is a β-KMS weight for γ, $\beta \neq 0$, such that $\omega \circ P = \omega$. It follows that $\gamma_t(d) = d$ for all $t \in \mathbb{R}$ and all $d \in D$.

Proof. It suffices to show that ω is faithful. For $a \in A$ and $k \in \mathbb{N}$, set:

$$Q_k(a) = \sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-kt^2} \gamma_t(a) \mathrm{d}t.$$

Note that $Q_k(a)$ is analytic for γ and that $\lim_{k\to\infty} Q_k(a) = a$. Standard approximation arguments establish the following orservation: Assume that $a \in \mathcal{M}_\omega$. It follows that

$$\sqrt{\frac{k}{\pi}} \int_{\mathbb{R}} \mathrm{e}^{-k(t+\mathrm{i}s)^2} \gamma_t(a) \mathrm{d}t \in \mathcal{M}_\omega$$

for all $s \in \mathbb{R}$.

This will be used to show that ω is faithful in the following way: Assume that $b = b^* \in A$ and that $\omega(b^2) = 0$. For a $c \in \mathcal{M}_\omega$ it follows from the observation that $Q_k(c), \gamma_{\mathrm{i}\beta}(Q_k(c)^*) \in \mathcal{M}_\omega$, hence by an application of the Cauchy–Schwarz inequality

$$|\omega(Q_k(c)^* b^2 Q_k(c))|^2 = |\omega(b^2 Q_k(c) \gamma_{\mathrm{i}\beta}(Q_k(c)^*))|^2 \leqslant 0.$$

Lower semi-continuity now implies that $\omega(c^* b^2 c) = 0$ and by using Cauchy–Schwarz again we deduce that

$$\omega(\operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega) = \{0\}. \tag{4.2}$$

Since $\mathcal{M}_\omega$ is dense in A the closure of $\operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ is a (closed two sided) ideal in A. If $b \neq 0$ this ideal must be all of A because we assume that A is simple. But then we reach a contradiction the following way: let $a \geqslant 0$. Choose a sequence $\{x_n\} \subseteq \operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ such that $\lim_{n\to\infty} x_n = \sqrt{a}$. Since

$x_n x_n^* \in \operatorname{Span} \mathcal{M}_\omega b^2 \mathcal{M}_\omega$ and $\lim_{n\to\infty} x_n x_n^* = a$, it follows from (4.2) and the lower semi-continuity of ω that $\omega(a) = 0$. This is a contradiction because $\omega \neq 0$. Hence $b = 0$. ∎

4.2. ONE-PARAMETER GROUPS TRIVIAL ON THE DIAGONAL. The following result has a predecessor in the von Neumann algebra setting in Theorem 2 of [3].

PROPOSITION 4.3. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid and $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ a continuous one-parameter group of automorphisms on $C^*_{\mathrm{r}}(\mathcal{G})$ such that*

$$\alpha_t(f) = f$$

for all $f \in C_0(\mathcal{G}^{(0)})$ and all $t \in \mathbb{R}$. Assume that the elements of $\mathcal{G}^{(0)}$ with trivial isotropy group in $\mathcal{G}$ are dense in $\mathcal{G}^{(0)}$. There is a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that

$$\alpha_t(g)(\xi) = \mathrm{e}^{\mathrm{i}tc(\xi)} g(\xi)$$

for all $t \in \mathbb{R}$, all $g \in C_{\mathrm{c}}(\mathcal{G})$ and all $\xi \in \mathcal{G}$.

Proof. We shall use the continuous linear embedding $j : C^*_{\mathrm{r}}(\mathcal{G}) \to C_0(\mathcal{G})$ introduced by Renault in Proposition 4.2 in [9].

OBSERVATION 4.4. Let $f \in C_{\mathrm{c}}(\mathcal{G})$ be supported in an open subset $U \subseteq \mathcal{G}$ such that $r : U \to \mathcal{G}^{(0)}$ is injective. Assume that $f(\xi) = 0$ for some $\xi \in U$. It follows that $j(\alpha_t(f))(\xi) = 0$ for all $t \in \mathbb{R}$.

To prove this, let $\varepsilon > 0$. There is an open bisection W of ξ such that $W \subseteq U$ and $|f(\mu)| \leqslant \varepsilon$ for all $\mu \in W$. Let $\varphi \in C_{\mathrm{c}}(\mathcal{G}^{(0)})$ be such that $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp} \varphi \subseteq r(W)$ and $\varphi(r(\xi)) = 1$. By use of Proposition 4.2 in [9] we find that

$$j(\alpha_t(f))(\xi) = \varphi(r(\xi)) j(\alpha_t(f))(\xi) = j(\varphi\alpha_t(f))(\xi) = j(\alpha_t(\varphi f))(\xi). \tag{4.3}$$

Note that $\operatorname{supp}(\varphi f) \subseteq W$ and that $\|\varphi f\|_\infty \leqslant \varepsilon$. It follows that

$$\|j(\alpha_t(\varphi f))\|_\infty \leqslant \|\alpha_t(\varphi f)\| = \|\varphi f\| = \|\varphi f\|_\infty \leqslant \varepsilon,$$

where the last identity follows from Lemma 2.4 in [12]. In particular, $|j(\alpha_t(\varphi f))(\xi)| \leqslant \varepsilon$, and then (4.3) shows that $|j(\alpha_t(f))(\xi)| \leqslant \varepsilon$. This proves Observation 4.4.

In the same way we obtain the following.

OBSERVATION 4.5. Let $f \in C_{\mathrm{c}}(\mathcal{G})$ be supported in an open subset $U \subseteq \mathcal{G}$ such that $s : U \to \mathcal{G}^{(0)}$ is injective. Assume that $f(\xi) = 0$ for some $\xi \in U$. It follows that $j(\alpha_t(f))(\xi) = 0$ for all $t \in \mathbb{R}$.

OBSERVATION 4.6. Let $\xi \in \mathcal{G}$, and let $h, h' \in C_{\mathrm{c}}(\mathcal{G})$ be supported in (not necessarily the same) open bisections in $\mathcal{G}$. Assume that $h(\xi) = h'(\xi) = 1$. Then

$$j(\alpha_t(h))(\xi) = j(\alpha_t(h'))(\xi) \tag{4.4}$$

for all $t \in \mathbb{R}$.

To show this, let $h \cdot h'$ be the point wise product of h and h'. It follows from Observation 4.4 that

$$j(\alpha_t(h \cdot h' - h'))(\xi) = j(\alpha_t(h \cdot h' - h))(\xi) = 0,$$

which yields (4.4): $j(\alpha_t(h))(\xi) = j(\alpha_t(h \cdot h'))(\xi) = j(\alpha_t(h'))(\xi)$.

It follows from Observation 4.6 that we can define a map $G_t : \mathcal{G} \to \mathbb{C}$ such that

$$G_t(\xi) = j(\alpha_t(h))(\xi),$$

where h is any element of $C_c(\mathcal{G})$ which is supported in an open bisection and takes the value 1 at ξ. Note that G_t is continuous by construction.

OBSERVATION 4.7. For every $f \in C_c(\mathcal{G})$ and every $\xi \in \mathcal{G}$,

$$j(\alpha_t(f))(\xi) = G_t(\xi)f(\xi). \tag{4.5}$$

To show this, we may assume that there are open bisections $U \subseteq V$ such that $\operatorname{supp} f \subseteq U$ and $\overline{U} \subseteq V$. Assume first that $\xi \notin \overline{U}$. We must show that $j(\alpha_t(f))(\xi) = 0$ in this case. By continuity and the assumption on $\mathcal{G}$ we may assume that $s(\xi)$ has trivial isotropy. If $\mu \in U$ and $r(\mu) = r(\xi)$, $s(\mu) = s(\xi)$, we see that

$$r(\mu^{-1}\xi) = s(\mu) = s(\xi) \quad \text{and} \quad s(\mu^{-1}\xi) = s(\xi)$$

which is impossible since $\xi \neq \mu$. It follows that we can write f as a finite sum

$$f = \sum_i f_i$$

such that each $f_i \in C_c(U)$ is supported in an open set $W_i \subseteq U$ such that either $s(\xi) \notin s(\overline{W_i})$ or $r(\xi) \notin r(\overline{W_i})$. It follows that $j(\alpha_t(f_i))(\xi) = 0$; in the first case thanks to Observation 4.5, in the second thanks to Observation 4.4. Hence

$$j(\alpha_t(f))(\xi) = \sum_i j(\alpha_t(f_i))(\xi) = 0,$$

as desired. Assume then that $\xi \in \overline{U} \subseteq V$. Choose $\varepsilon > 0$ such that $f(\xi) + \varepsilon \neq 0$ and a function $\varphi \in C_c(V)$ such that $\varphi(\xi) = 1$. Then

$$j(\alpha_t(f + \varepsilon\varphi))(\xi) = j\Big(\alpha_t\Big(\frac{f + \varepsilon\varphi}{f(\xi) + \varepsilon}\Big)\Big)(\xi)(f(\xi) + \varepsilon) = G_t(\xi)(f(\xi) + \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain (4.5).

Note that it follows from Observation 4.7 that $\alpha_t(C_c(\mathcal{G})) \subseteq C_c(\mathcal{G})$, and

$$\alpha_t(f)(\xi) = G_t(\xi)f(\xi)$$

for all $f \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$. Since $\|f\| = \|\alpha_t(f)\|$ this implies that $|G_t(\xi)| = 1$. Furthermore, if $h \in C_c(\mathcal{G})$ is supported in a bisection and $h(\xi) = 1$, we find that

$$\begin{aligned} G_{t+s}(\xi) &= \alpha_t(\alpha_s(h))(\xi) = \alpha_s(h)(\xi)\alpha_t\Big(\frac{\alpha_s(h)}{\alpha_s(h)(\xi)}\Big)(\xi) \\ &= \alpha_s(h)(\xi)G_t(\xi) = G_s(\xi)G_t(\xi). \end{aligned}$$

Since $t \mapsto G_t(\xi)$ is continuous, this implies that there is a unique real-valued function $c : \mathcal{G} \to \mathbb{R}$ such that

$$G_t(\xi) = \mathrm{e}^{\mathrm{i}tc(\xi)}. \tag{4.6}$$

To show that c is a homomorphism, let $\gamma_1, \gamma_2 \in \mathcal{G}$ such that $s(\gamma_1) = r(\gamma_2)$. Set $\gamma = \gamma_1\gamma_2$. Let U be an open bisection containing γ and U_i an open bisection containing γ_i, $i = 1,2$, such that $\mu_1\mu_2 \in U$ when $(\mu_1, \mu_2) \in \mathcal{G}^{(2)} \cap (U_1 \times U_2)$. Choose $h_i \in C_c(U_i)$ such that $h_i(\gamma_i) = 1$. Then $h_1h_2(\gamma) = 1$ and

$$\begin{aligned} G_t(\gamma) &= j(\alpha_t(h_1h_2))(\gamma) = \alpha_t(h_1)\alpha_t(h_2)(\gamma) \\ &= \alpha_t(h_1)(\gamma_1)\alpha_t(h_2)(\gamma_2) = G_t(\gamma_1)G_t(\gamma_2). \end{aligned} \tag{4.7}$$

Hence G_t is a homomorphism as asserted. Combining (4.6) and (4.7) and taking derivatives with respect to t, it follows that c is a homomorphism, i.e.

$$c(\gamma_1\gamma_2) = c(\gamma_1) + c(\gamma_2)$$

when $s(\gamma_1) = r(\gamma_2)$.

Finally, to show that c is continuous, let $\xi \in \mathcal{G}$ and $\varepsilon > 0$ be given. Choose open bisections $U \subseteq V$ such that $\xi \in U \subseteq \overline{U} \subseteq V$ and $h \in C_c(V)$ a function such that $h = 1$ on $\overline{U}$. Then

$$G_t(\gamma) = \alpha_t(h)(\gamma)$$

for all $t \in \mathbb{R}$ and all $\gamma \in U$. Let $K \subseteq \mathbb{R}$ be a compact set. There are finitely many points $t_i \in K, i = 1,2,\ldots,N$, such that for every $t \in K$ there is an i such that

$$\|\alpha_t(h) - \alpha_{t_i}(h)\|_\infty = \|\alpha_t(h) - \alpha_{t_i}(h)\| \leqslant \varepsilon.$$

By continuity of $\alpha_{t_i}(h)$ there is an open neighborhood $W \subseteq U$ of ξ such that

$$|\alpha_{t_i}(h)(\gamma) - \alpha_{t_i}(h)(\xi)| \leqslant \varepsilon$$

for all $\gamma \in W$ and $i = 1,2,\ldots,N$. It follows that $|G_t(\gamma) - G_t(\xi)| \leqslant 3\varepsilon$ for all $t \in K$ and all $\gamma \in W$. By Pontryagin duality this implies that c is continuous. ∎

THEOREM 4.8. *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid such that for at least one element $x \in \mathcal{G}^{(0)}$ the isotropy $\mathcal{G}_x^x$ is trivial, i.e. $\mathcal{G}_x^x = \{x\}$, and that $\mathcal{G}$ is minimal in the sense that $s(r^{-1}(y))$ is dense in $\mathcal{G}^{(0)}$ for all $y \in \mathcal{G}^{(0)}$. Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C_r^*(\mathcal{G})$ and assume that for some $\beta \in \mathbb{R}\backslash\{0\}$ there is a diagonal β-*KMS *weight for α. Then α is diagonal, i.e. there is a continuous homomorphism $c : \mathcal{G} \to \mathbb{R}$ such that*

$$\alpha_t(g)(\xi) = \mathrm{e}^{\mathrm{i}tc(\xi)}g(\xi) \tag{4.8}$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G})$ and all $\xi \in \mathcal{G}$.

Proof. Combine Corollary 4.2 and Proposition 4.3, using that in the presence of a single unit with trivial isotropy group the minimality of $\mathcal{G}$ is equivalent to the simplicity of $C_r^*(\mathcal{G})$ by Corollary 2.18 in [12]. ∎

We can now put the pieces together for a *Proof of Theorem* 2.1. (i)⇒(iii) follows from Proposition 4.1. That (iii) is equivalent to (iv) follows from a standard argument using that $\mathcal{G}^{(0)}$ is totally disconnected. The implication (iii)⇒(v) follows from Proposition 4.3 and (v)⇒(ii) from Corollary 3.4. This gives the equivalence of all five conditions since (ii)⇒(i) is trivial.

EXAMPLE 4.9. Let $\mathcal{G} = \mathbb{F}_2$ be the free group on two generators. Then $C^*_r(\mathbb{F}_2)$ is a simple C^*-algebra and $C_0(\mathcal{G}^{(0)}) = \mathbb{C}1$. Let $A = A^* \in C^*_r(\mathbb{F}_2)$ and set

$$\alpha_t(a) = e^{itA} a e^{-itA}.$$

Note that α_t acts trivially on $C_0(\mathcal{G}^{(0)}) = \mathbb{C}1$. Let $U_x, x \in \mathbb{F}_2$, be the canonical unitaries generating $C^*_r(\mathbb{F}_2)$. Assume that there is a homomorphism $c : \mathbb{F}_2 \to \mathbb{R}$ such that $\alpha_t(U_x) = e^{itc(x)} U_x$ for all t, x. By differentiation this leads to the conclusion that $AU_x - U_x A = c(x) U_x$ and hence that $U_x^* A U_x = A + c(x)1$. The last equation implies that the spectrum $\sigma(A)$ of A satisfies $\sigma(A) = \sigma(A) + c(x)$, i.e. $c(x) = 0$. But then $\alpha_t(U_x) = U_x$ for all t, x, i.e. $\alpha_t = \mathrm{id}$ for all $t \in \mathbb{R}$. This implies by differentiation that $AX = XA$ for all $X \in C^*_r(\mathbb{F}_2)$, i.e. A is in the center of $C^*_r(\mathbb{F}_2)$. So by choosing $A \notin \mathbb{R}1$, we have an example showing that Proposition 4.3 does not always hold when there are no units with trivial isotropy in $\mathcal{G}$. In relation to Theorem 2.1 note that there are β-KMS weights for α for all $\beta \in \mathbb{R}$. Indeed, when ω is the tracial state on $C^*_r(\mathbb{F}_2)$, the functional

$$C^*_r(\mathbb{F}_2) \ni a \mapsto \omega(e^{-\beta A} a)$$

is a bounded β-KMS weight. Since condition (iii) in Theorem 2.1 holds while (v) does not, it follows that it is necessary, in Theorem 2.1, to assume the existence of a unit with trivial isotropy group.

Similarly, by considering a disjoint union $\mathbb{F}_2 \sqcup \mathcal{H}$, where $\mathcal{H}$ is an appropriate groupoid, it is easy to obtain examples showing that the implication (iv)⇒(i) in Theorem 2.1 fails in general if $\mathcal{G}$ is not minimal.

5. APPLICATIONS TO GRAPH C^*-ALGEBRAS

In this section we apply the results obtained above to the study of KMS weights on graph C^*-algebras. For this we first show how a graph C^*-algebra can be realized as the groupoid C^*-algebra of a locally defined local homeomorphism as it was introduced by Renault in [10]. Recall that graph C^*-algebras were originally introduced for row-finite graphs in [4] as the C^*-algebra of the left-shift on the space of infinite paths in the graph. We show that in general, when the graph may have infinite emitters, its C^*-algebra is still the groupoid C^*-algebra of a local homeomorphism which is generally only defined on a dense open subset of a locally compact Hausdorff space.

5.1. THE RENAULT GROUPOID OF A LOCAL HOMEOMORPHISM. Let X be a locally compact second countable Hausdorff space. Let $U \subseteq X$ be an open subset and $\varphi : U \to X$ a local homeomorphism, i.e. for every $u \in U$ there is an open subset $V \subseteq U$ such that $u \in V$, $\varphi(V)$ is open and $\varphi : V \to \varphi(V)$ is a homeomorphism. Set $\varphi^0 = \mathrm{id}_X$ (with domain $D(\varphi^0) = X$) and for $n \geqslant 1$, set

$$D(\varphi^n) = U \cap \varphi^{-1}(U) \cap \varphi^{-2}(U) \cap \cdots \cap \varphi^{-n+1}(U)$$

and let φ^n be the map

$$\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi : D(\varphi^n) \to X.$$

Set

$$\mathcal{G}_\varphi = \{(x, n-m, y) \in X \times \mathbb{Z} \times X : x \in D(\varphi^n),\ y \in D(\varphi^m),\ \varphi^n(x) = \varphi^m(y)\}$$

which is a groupoid with product $(x,k,y)(y,l,z) = (x,k+l,z)$ and inversion $(x,k,y)^{-1} = (y,-k,x)$. Sets of the form

$$\{(x, n-m, y) :\ \varphi^n(x) = \varphi^m(y),\ x \in W,\ y \in V\}$$

for some open subsets $W \subseteq D(\varphi^n)$, $V \subseteq D(\varphi^m)$, constitute a basis for a topology in $\mathcal{G}_\varphi$ which turns it into a locally compact second countable Hausdorff étale groupoid.

Let $F : X \to \mathbb{R}$ be a function which is continuous on U. We can then define $c_F : \mathcal{G}_\varphi \to \mathbb{R}$ such that

$$c_F(x, n-m, y) = \sum_{i=0}^{n} F(\varphi^i(x)) - \sum_{i=0}^{m} F(\varphi^i(y)).$$

Note that c_F is a continuous homomorphism, and if $F' : X \to \mathbb{R}$ is a function which agrees with F on U, then $c_{F'} = c_F$.

PROPOSITION 5.1. *Let $c : \mathcal{G}_\varphi \to \mathbb{R}$ be a continuous homomorphism. There is a map $F : X \to \mathbb{R}$ which is continuous on U such that $c = c_F$.*

Proof. Define $F : X \to \mathbb{R}$ such that

$$F(x) = \begin{cases} c(x, 1, \varphi(x)) & x \in U, \\ 0 & x \notin U. \end{cases}$$

It is straightforward to verify that F is continuous on U and that $c = c_F$. ∎

It follows that the continuous homomorphisms $\mathcal{G}_\varphi \to \mathbb{R}$ are in bijective correspondence with the continuous maps $U \to \mathbb{R}$.

A point $x \in X$ is *aperiodic* under φ when

$$x \in D(\varphi^n) \cap D(\varphi^m),\ \varphi^n(x) = \varphi^m(x) \ \Rightarrow\ n = m.$$

Under the identification of X with the unit space of $\mathcal{G}_\varphi$ the aperiodic points are the elements with trivial isotropy group. We can therefore combine Proposition 5.1 with Proposition 4.3 to obtain the following.

PROPOSITION 5.2. *Let X be a locally compact second countable Hausdorff space, $U \subseteq X$ an open subset and $\varphi : U \to X$ a local homeomorphism. Assume that $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ is a continuous one-parameter group of automorphisms on $C^*_r(\mathcal{G}_\varphi)$ such that*

$$\alpha_t(f) = f$$

for all $f \in C_0(X) \subseteq C^*_r(\mathcal{G}_\varphi)$ and all $t \in \mathbb{R}$. Assume also that the aperiodic points of φ are dense in X.

There is a continuous map $F : U \to \mathbb{R}$ such that

$$\alpha_t(g)(\xi) = e^{itc_F(\xi)} g(\xi)$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G}_\varphi)$ and all $\xi \in \mathcal{G}_\varphi$.

For $n, m \in \mathbb{N} \cup \{0\}$, set

$$\varphi^{-m}(\varphi^n(x)) = \begin{cases} \emptyset & \text{when } x \notin D(\varphi^n), \\ \{y \in D(\varphi^m) : \varphi^m(y) = \varphi^n(x)\} & \text{when } x \in D(\varphi^n). \end{cases}$$

We say that φ is *minimal* when

$$\bigcup_{n,m\in\mathbb{N}\cup\{0\}} \varphi^{-m}(\varphi^n(x)) \tag{5.1}$$

is dense in X for all $x \in X$. Note that (5.1) is the orbit of x under the action of $\mathcal{G}_\varphi$ on its unit space. Thus φ is minimal if and only if $\mathcal{G}_\varphi$ is.

PROPOSITION 5.3. *Let X be a locally compact second countable Hausdorff space, $U \subseteq X$ an open subset and $\varphi : U \to X$ a local homeomorphism. Assume that φ is minimal and that there is at least one aperiodic point for φ. Let $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C^*_r(\mathcal{G}_\varphi)$.*

*If, for some $\beta \neq 0$, there is a diagonal β-*KMS *weight for α, then there is a continuous map $F : U \to \mathbb{R}$ such that*

$$\alpha_t(g)(\xi) = e^{itc_F(\xi)} g(\xi) \tag{5.2}$$

for all $t \in \mathbb{R}$, all $g \in C_c(\mathcal{G}_\varphi)$ and all $\xi \in \mathcal{G}_\varphi$.

Proof. In view of Corollary 4.2 and Proposition 5.2 it suffices to observe that $C^*_r(\mathcal{G}_\varphi)$ is simple under the present assumptions, cf. Proposition 2.5 in [10]. ∎

5.2. A LOCAL HOMEOMORPHISM FROM AN INFINITE GRAPH. Let G be a directed graph with vertexes V and edges E. We assume that G is countable in the sense that V and E are both countable sets. We let r and s denote the maps $r : E \to V$ and $s : E \to V$ which associate to an edge $e \in E$ its target vertex $r(e)$ and source vertex $s(e)$, respectively. A vertex v is an *infinite emitter* when $s^{-1}(v)$ contains infinitely many edges and a *sink* when $s^{-1}(v)$ is empty. The union of sinks and infinite emitters constitute a set which will be denoted by V_∞. The graph C^*-algebra $C^*(G)$ is by definition the universal C^*-algebra generated by a collection $S_e, e \in E$, of partial isometries and a collection $P_v, v \in V$, of mutually orthogonal projections subject to the conditions that:

(1) $S_e^*S_e = P_{r(e)}$, $\forall e \in E$,
(2) $\sum_{e\in F} S_eS_e^* \leqslant P_v$ for every finite subset $F \subseteq s^{-1}(v)$ and all $v \in V$, and
(3) $P_v = \sum_{e\in s^{-1}(v)} S_eS_e^*$, $\forall v \in V\backslash V_\infty$.

Let $P_f(G)$ and $P(G)$ denote the set of finite and infinite paths in G, respectively. The range and source maps, r and s, extend in the natural way to $P_f(G)$; the source map also to $P(G)$. Set $\Omega_G = P(G) \cup Q(G)$, where

$$Q(G) = \{p \in P_f(G) : r(p) \in V_\infty\}$$

is the set of finite paths that terminate at a vertex in V_∞. In particular, $V_\infty \subseteq Q(G)$ because vertexes are considered to be finite paths of length 0. For any $p \in P_f(G)$, let $|p|$ denote the length of p. When $|p| \geqslant 1$, set

$$Z(p) = \{q \in \Omega_G : |q| \geqslant |p|,\ q_i = p_i,\ i = 1,2,\ldots,|p|\}, \quad \text{and}$$
$$Z(v) = \{q \in \Omega_G : s(q) = v\},$$

when $v \in V$. When $\nu \in P_f(G)$ and F is a finite subset of $P_f(G)$, set

$$Z_F(\nu) = Z(\nu)\backslash\Big(\bigcup\nolimits_{\mu\in F} Z(\mu)\Big). \tag{5.3}$$

The sets $Z_F(\nu)$ form a basis of compact and open subsets for a locally compact Hausdorff topology on Ω_G. When $\mu \in P_f(G)$ and $x \in \Omega_G$, we can define the concatenation $\mu x \in \Omega_G$ in the obvious way when $r(\mu) = s(x)$. The groupoid $\mathcal{G}_G$ consists of the elements in $\Omega_G \times \mathbb{Z} \times \Omega_G$ of the form

$$(\mu x, |\mu| - |\mu'|, \mu' x),$$

for some $x \in \Omega_G$ and some $\mu, \mu' \in P_f(G)$. The product in $\mathcal{G}_G$ is defined by

$$(\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x,\ |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y),$$

when $\mu' x = \nu y$, and the involution by $(\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)$. To describe the topology on $\mathcal{G}_G$, let $Z_F(\mu)$ and $Z_{F'}(\mu')$ be two sets of the form (5.3) with $r(\mu) = r(\mu')$. The topology we shall consider has as a basis the sets of the form

$$\{(\mu x, |\mu| - |\mu'|, \mu' x) : \mu x \in Z_F(\mu),\ \mu' x \in Z_{F'}(\mu')\}. \tag{5.4}$$

With this topology $\mathcal{G}_G$ becomes an étale locally compact second countable Hausdorff groupoid and we can consider the reduced C^*-algebra $C_r^*(\mathcal{G}_G)$ as in [9]. As shown by Paterson in [8] there is an isomorphism $C^*(G) \to C_r^*(\mathcal{G}_G)$ which sends S_e to 1_e, where 1_e is the characteristic function of the compact and open set

$$\{(ex, 1, r(e)x) : x \in \Omega_G\} \subseteq \mathcal{G}_G,$$

and P_v to 1_v, where 1_v is the characteristic function of the compact and open set

$$\{(vx, 0, vx) : x \in \Omega_G\} \subseteq \mathcal{G}_G.$$

In the following we use the identification $C^*(G) = C^*_{\mathrm{r}}(\mathcal{G}_G)$ and identify Ω_G with the unit space of $\mathcal{G}_G$ via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$.

Note that $\Omega_G \backslash V_\infty$ is an open subset of Ω_G and that we can define a local homeomorphism

$$\sigma : \Omega_G \backslash V_\infty \to \Omega_G$$

such that σ is the usual left shift on $P(G)$, defined such that $\sigma(x)_i = x_{i+1}$, while $\sigma(e_1 e_2 \cdots e_n)$ is defined as follows when $e_1 e_2 \cdots e_n \in Q(G)$:

$$\sigma(e_1 e_2 \cdots e_n) = \begin{cases} e_2 e_3 \cdots e_n & \text{when } n \geqslant 2, \\ r(e_1) & \text{when } n = 1. \end{cases}$$

It is straightforward to check that there is an identification

$$\mathcal{G}_G = \mathcal{G}_\sigma,$$

as topological groupoids. In particular, it follows that any continuous function $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ defines a continuous homomorphism $c_F : \mathcal{G}_G \to \mathbb{R}$ such that

$$c_F(\mu x, |\mu| - |\mu'|, \mu' x) = \sum_{n=0}^{|\mu|} F(\sigma^n(\mu x)) - \sum_{n=0}^{|\mu'|} F(\sigma^n(\mu' x)).$$

To simplify notation the one-parameter group σ^{c_F} defined from c_F will be denoted by σ^F. It follows from Proposition 5.1 that every continuous homomorphism $\mathcal{G}_G \to \mathbb{R}$ arises from a continuous function $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ as above. We can therefore formulate Corollary 3.3 in the following way for graph C^*-algebras.

THEOREM 5.4. *Let $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ be a continuous function. There is a bijective correspondence between the β-KMS weights for σ^F on $C^*(G)$ and the pairs $(\mu, \{\varphi_x\}_{x \in \Omega_G})$, where μ is a regular Borel measure on Ω_G and $\{\varphi_x\}_{x \in \Omega_G}$ is a μ-measurable field of states φ_x on $C^*_{\mathrm{r}}((\mathcal{G}_G)^x_x)$ such that:*

(i) *μ is $\mathrm{e}^{\beta F}$-conformal for σ,*

(ii) *$\varphi_x(u_g) = \varphi_{r(h)}(u_{hgh^{-1}})$ for μ-almost every $x \in \Omega_G$ and all $g \in (\mathcal{G}_G)^x_x$, $h \in (\mathcal{G}_G)_x$, and*

(iii) *$\varphi_x(u_g) = 0$ for μ-almost every $x \in \Omega_G$ and all $g \in (\mathcal{G}_G)^x_x \backslash c_F^{-1}(0)$.*

The β-KMS weight ϕ corresponding to the pair $(\mu, \{\varphi_x\}_{x \in \Omega_G})$ has the properties that $C_{\mathrm{c}}(\mathcal{G}_G) \subseteq \mathcal{M}_\phi$ and

$$\phi(f) = \int_{\Omega_G} \sum_{g \in (\mathcal{G}_G)^x_x} f(g) \varphi_x(u_g) \, \mathrm{d}\mu(x) \tag{5.5}$$

when $f \in C_{\mathrm{c}}(\mathcal{G}_G)$.

Similarly, for graph C^*-algebras our main result takes the following form.

THEOREM 5.5. *Let G be a countable directed graph such that $C^*(G)$ is simple. Let $\alpha = (\alpha_t)_{t \in \mathbb{R}}$ be a continuous one-parameter group of automorphisms on $C^*(G)$ and assume that for some $\beta_0 \neq 0$ there is a β_0-KMS weight for α.*

The following are equivalent:

(i) *There is a* $\beta_1 \neq 0$ *and a diagonal* β_1-KMS *weight for* α.

(ii) *Whenever* $\beta \neq 0$ *and there is a* β-KMS *weight for* α*, there is also a diagonal* β-KMS *weight for* α.

(iii) $\alpha_t(f) = f$ *for all* $t \in \mathbb{R}$ *and all* $f \in C_0(\Omega_G)$.

(iv) *There is a continuous function* $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ *such that* $\alpha = \sigma^F$.

It follows from Theorem 5.4 (and Proposition 5.1) that all KMS weights for a diagonal action on the C^*-algebra of a graph without loops are diagonal. This is not true in general; not even for finite strongly connected graphs as shown in [15]. However, we can now show that it holds for strongly connected graphs when the function F has bounded local variation in the a sense we now make precise.

Let v be a vertex in G and set

$$\mathrm{Var}_{n,v}(F) = \sup_{x,y} \Big| \sum_{j=0}^{n-1} F(\sigma^j(x)) - \sum_{j=0}^{n-1} F(\sigma^j(y)) \Big|$$

where we take the supremum over all pairs $x, y \in P(G)$ with the property that $x_i = y_i$, $i = 1, 2, \dots, n$, and $s(x_1) = s(y_1) = v$. The following condition (5.6) should be compared with *Bowen's condition* used by Walters, cf. [16].

PROPOSITION 5.6. *Let* G *be a countable directed graph such that* $C^*(G)$ *is simple and let* $F : \Omega_G \backslash V_\infty \to \mathbb{R}$ *be a continuous function such that for some vertex* v,

$$\sup_n \mathrm{Var}_{n,v}(F) < \infty. \tag{5.6}$$

Then every KMS *weight for* σ^F *is diagonal.*

Proof. The assumption that $C^*(G)$ is simple means that G is cofinal in the sense used (e.g.) in [14] and that every minimal loop in G has an exit, cf. [11]. It is easily seen that the set of vertexes v for which (5.6) holds is both hereditary and saturated. Under the present assumptions it will therefore hold for all v. Consider a β-KMS weight ϕ and the pair $(\mu, \{\varphi_x\}_{x \in \Omega_G})$ associated to it by Theorem 5.4. It suffices to show that the elements $x \in \Omega_G$ for which the isotropy group $(\mathcal{G}_G)_x^x$ is non-trivial is a null set with respect to μ. The isotropy group of a point $x \in \Omega_G$ is non-trivial if and only if x is an infinite pre-periodic path in G, and there are at most countably many such points. It suffices therefore to show that $\mu(\{x\}) = 0$ for any infinite pre-periodic path x. There is an $m \in \mathbb{N}$ such that $x_0 = \sigma^m(x)$ is periodic. It follows from (3.2) that

$$\mu(\{x\}) = e^{-\beta \sum_{j=0}^{m-1} F(\sigma^j(x))} \mu(\{x_0\}),$$

so it suffices to show that $\mu(\{x_0\}) = 0$. Since x_0 is periodic there is a finite loop δ in G such that $x_0 = \delta^\infty$, and since G is cofinal and every loop in G has an exit there is also a finite loop δ' in G such that $\delta' \nsubseteq x_0$ and $s(\delta') = s(\delta)$. By prolonging δ and δ' if necessary we may assume that the length of δ and δ' are the same, say p. For each $k \in \mathbb{N}$ set

$$y_k = \delta^k \delta' x_0.$$

Since x_0 is p-periodic it follows from (3.2) that

$$\mu(\{x_0\}) = \mathrm{e}^{-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))} \mu(\{x_0\}),$$

for all $k \in \mathbb{N}$, and the desired conclusion follows if $-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))$ is not zero for some k. Consider therefore now the case where $-\beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0)) = 0$ for all $k \in \mathbb{N}$. Since (5.6) holds we find then that

$$\left|\beta \sum_{j=0}^{kp-1} F(\sigma^j(y_k))\right| = \left|\beta \sum_{j=0}^{kp-1} F(\sigma^j(y_k)) - \beta \sum_{j=0}^{kp-1} F(\sigma^j(x_0))\right| \leqslant |\beta| K \tag{5.7}$$

for all k, where $K = \sup_n \mathrm{Var}_{n,v}(F)$ and $v = s(\delta)$ is the source of δ. Now apply (3.2) again to find that

$$\mu(\{y_k\}) = \mathrm{e}^{-\beta \sum_{j=0}^{(k+1)p-1} F(\sigma^j(y_k))} \mu(\{x_0\}).$$

Inserting (5.7) this leads to the conclusion that

$$\mu(\{y_k\}) \geqslant \mathrm{e}^{-|\beta| K} \mathrm{e}^{-\beta \sum_{j=0}^{p-1} F(\sigma^j(z))} \mu(\{x_0\}),$$

where $z = \delta' x_0 = \sigma^{kp}(y_k)$. Since

$$\sum_{k=1}^{\infty} \mu(\{y_k\}) \leqslant \mu(Z(v)) < \infty,$$

we conclude that $\mu(\{x_0\}) = 0$, as desired. ∎

It follows from Proposition 5.6 that a generalized gauge action on a graph C^*-algebra, considered for example in [14], where F only depends on the first edge only has gauge-invariant KMS weights, at least as long as the algebra is simple.

REMARK 5.7. It should be emphasized that the conclusion in Proposition 5.6 does not hold without some condition on F. To see this observe that the example presented in [15] shows that already for the canonical finite graph G for which $C^*(G)$ is a copy of the Cuntz algebra O_2, namely the graph consisting of one vertex and two arrows, there are continuous non-negative functions $F : \Omega_G \to \mathbb{R}$ such that σ^F admits non-diagonal KMS states. In the example from [15] there is at least a single extremal KMS state which is diagonal, namely the extremal KMS-state corresponding to the lowest inverse temperature β_0. Here we want to indicate how to modify the example in [15] to get an example where no *extremal* KMS state is diagonal. The basis for this is a sequence $\{b_n\}_{n=1}^{\infty}$ of positive numbers with the following properties:

(a) $b_n \geqslant b_{n+1}$ $\forall n$,

(b) $\lim\limits_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$,

(c) $\sum\limits_{n=1}^{\infty} b_n < 1$, and

(d) $\sum\limits_{n=1}^{\infty} b_n^s = \infty$ for all $s < 1$.

We leave the reader to verify the existence of such a sequence. Set $a_1 = -\log b_1$ and $a_k = \log b_{k-1} - \log b_k$, $k \geqslant 2$, and identify the infinite path space Ω_G with $\{0,1\}^{\mathbb{N}}$ by labelling the two arrows in G by 0 and 1. Define then $T : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ such that $T((x_i)_{i=1}^{\infty}) = a_k$ where $k = \min\{i : x_i = 0\}$ when $(x_i)_{i=1}^{\infty} \neq 1^{\infty}$, and set $T(1^{\infty}) = 0$. (As in [15] 1^{∞} is the infinite string of 1's.) This is a continuous non-negative function. By using Theorem 2.2 in [15] and arguing exactly as in Section 3 of [15], but with the sequence $\{n^{-1}\}$ replaced by $\{a_n\}$, it follows that there are β-KMS states for σ^T if and only if $\beta \geqslant 1$, and for each $\beta \geqslant 1$ the extremal KMS states are parametrised by the circle, and none are diagonal. As guaranteed by Theorem 5.5 there are for each $\beta \geqslant 1$ also one which is diagonal. As explained in [15] it arises by integrating the extremal ones with respect to Lebesgue measure on the circle.

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REFERENCES

[1] O. BRATTELI, D.W. ROBINSON, *Operator Algebras and Quantum Statistical Mechanics*. I, II, Texts Monogr. Phys., Springer-Verlag, New York-Heidelberg-Berlin 1979 and 1981.

[2] F. COMBES, Poids associé à une algèbre hilbertienne à gauche, *Compos. Math.* **23**(1971), 49–77.

[3] J. FELDMAN, C.C. MOORE, Ergodic equivalence relations, cohomology, and von Neumann algebras. II, *Trans. Amer. Math. Soc.* **234**(1977), 325–359.

[4] A. KUMJIAN, D. PASK, I. RAEBURN, J. RENAULT, Graphs, groupoids, and Cuntz–Krieger algebras, *J. Funct. Anal.* **144**(1997), 505–541.

[5] J. KUSTERMANS, KMS-weights on C^*-algebras, arXiv:funct-an/9704008.

[6] J. KUSTERMANS, S. VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup. (4)* **33**(2000), 837–934.

[7] S. NESHVEYEV, KMS states on the C^*-algebras of non-principal groupoids, *J. Operator Theory* **70**(2013), 513–530.

[8] A.L.T. PATERSON, Graph inverse semigroups, groupoids and their C^*-algebras, *J. Operator Theory* **48**(2002), 645–662.

[9] J. RENAULT, *A Groupoid Approach to C^*-Algebras*, Lecture Notes in Math., vol. 793, Springer-Verlag, Berlin-Heidelberg-New York 1980.

[10] J. RENAULT, Cuntz-like algebras, in *Operator Theoretical Methods (Timisoara, 1998)*, Theta Foundation, Bucharest 2000, pp. 371–286.

[11] W. SZYMANSKI, Simplicity of Cuntz–Krieger algebras of infinite matrices, *Pacific J. Math.* **122**(2001), 249–256.

[12] K. THOMSEN, Semi-étale groupoids and applications, *Ann. Inst. Fourier* **60**(2010), 759–800.

[13] K. THOMSEN, KMS-states and conformal measures, *Comm. Math. Phys.* **316**(2012), 615–640.

[14] K. THOMSEN, KMS weights on groupoid and graph C^*-algebras, *J. Funct. Anal.* **266**(2014), 2959–2988.

[15] K. THOMSEN, Phase transition in O_2, arXiv:1510.07513[math.OA]

[16] P. WALTERS, Convergence of the Ruelle operator for a function satisfying Bowen's condition, *Trans. Amer. Math. Soc.* **353**(2001), 327–347.

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