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COMMON HYPERCYCLIC VECTORS FOR FAMILIES OF BACKWARD SHIFT OPERATORS

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ABSTRACT. We provide necessary and sufficient conditions on the existence of common hypercyclic vectors for multiples of the backward shift operator along sparse powers. Our main result strongly generalizes corresponding results which concern the full orbit of the backward shift. Some of our results are valid in a more general context, in the sense that they apply for a wide class of hypercyclic operators.

KEYWORDS: *Backward shift, hypercyclic operator, common hypercyclic vectors, uniform distribution mod* 1.

MSC (2010): 47A16.

1. INTRODUCTION

We consider the space ℓ^2 of square summable sequences over the field of complex numbers $\mathbb{C}$ endowed with the topology that is induced by the ℓ^2-norm $\|\cdot\|_2 : \ell^2 \to \mathbb{R}^+$, where

$$\|x\|_2 := \Big(\sum_{j=1}^{+\infty} |x_j|^2 \Big)^{1/2} \quad \text{for every } x = (x_1, x_2, \dots, x_n, \dots) \in \ell^2.$$

We write $\|\cdot\| := \|\cdot\|_2$ for simplicity. Let B be the unweighted backward shift operator on ℓ^2, that is

$$B((x_1, x_2, x_3, \dots)) = (x_2, x_3, \dots), \quad \text{for } (x_1, x_2, \dots) \in \ell^2.$$

Let $\lambda \in \mathbb{C}$. We define the iterates of the operator λB as follows: $(\lambda B)^1 := \lambda B$ and $(\lambda B)^{n+1} = (\lambda B)^n \circ (\lambda B)$, for $n = 1, 2, \dots$, where we denote $(\lambda B)^n \circ (\lambda B)$ the usual composition of the operators $(\lambda B)^n$ and λB. Let $\lambda \in \mathbb{C}$, and consider the set of hypercyclic vectors for λB, that is

$$\mathcal{HC}(\lambda B) := \{x = (x_1, x_2, \dots) \in \ell^2 : \overline{\{(\lambda B)^n(x),\ n = 1, 2, \dots\}} = \ell^2\}.$$

A comprehensive treatment on hypercyclicity can be found in the books [5], [11]. For the reader's convenience we include the relevant definitions. A sequence of continuous operators (T_n) acting on a Frechet space X is called hypercyclic provided there exists a vector $x \in X$ so that the set $\{T_n(x) : n = 1, 2, \dots\}$ is dense in X. Such a vector is called hypercyclic for (T_n) and the set of hypercyclic vectors for (T_n) is denoted by $\mathcal{HC}(T_n)$. Let $T : X \to X$ be an operator. We define the iterates of T as follows:

$$T^1 := T \quad \text{and} \quad T^{n+1} := T^n \circ T \quad \text{for } n = 1, 2, \dots$$

where we denote $T^n \circ T$ the usual composition of the operators T^n and T.

When T_n comes from the iterates of a single operator we say that T is hypercyclic and $\mathcal{HC}(T)$ denotes the set of hypercyclic vectors for T, i.e.

$$\mathcal{HC}(T) = \{x \in X : \{T^n(x) : n = 1, 2, \dots\} \text{ is dense in } X\}.$$

It is well known that for every fixed $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ the set $\mathcal{HC}(\lambda B)$ is a dense, G_δ subset of $(\ell^2, \|\cdot\|)$ and as the reader may guess Baire's theorem should be involved in the arguments. The following question arises naturally. If we fix an uncountable subset $J \subset \{z \in \mathbb{C} : |z| > 1\}$, is it true that $\bigcap\limits_{\lambda \in J} \mathcal{HC}(\lambda B) \neq \emptyset$? In this direction, Abakumov and Gordon [1] proved that:

$$\bigcap_{|\lambda|>1} \mathcal{HC}(\lambda B) \neq \emptyset,$$

the best possible result one can expect concerning the existence of common hypercyclic vectors for multiples of the backward shift. Later on, Costakis and Sambarino [9] gave a different proof of this result, which, roughly speaking, is based on the so called common hypercyclicity criterion. In this criterion, Baire's category theorem appears. Actually, Costakis and Sambarino showed that $\bigcap\limits_{|\lambda|>1} \mathcal{HC}(\lambda B)$ is a G_δ and dense subset of $(\ell^2, \|\cdot\|)$; hence non-empty. What is interesting here is the uncountable range of λ's, which makes things harder if one wishes to apply Baire's theorem.

One can refine the above problem as follows. Let (k_n) be a fixed strictly increasing subsequence of natural numbers. It is known, and very easy to prove, that the sequence $((\lambda B)^{k_n})$ is also hypercyclic, that is, there exists $x \in \ell^2$ such that the set $\{(\lambda B)^{k_n}(x) : n = 1, 2, \dots\}$ is dense in ℓ^2. Such a vector is called *hypercyclic* for $((\lambda B)^{k_n})$ and the set of these vectors is denoted by $\mathcal{HC}((\lambda B)^{k_n})$. From the above it should be also clear, or at least expected, that $\mathcal{HC}((\lambda B)^{k_n})$ is G_δ and dense subset of $(\ell^2, \|\cdot\|)$.

Now we are ready to ask the following.

QUESTION 1.1. Fix a strictly increasing sequence (k_n) of natural numbers. For which uncountable sets $J \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$,

$$\bigcap_{\lambda \in J} \mathcal{HC}((\lambda B)^{k_n}) \neq \emptyset \ ?$$

It turns out that the answer to this question depends heavily on the sequence (k_n). In particular, what matters is how sparse the sequence (k_n) has been chosen. Our main result is the following.

THEOREM 1.2. *Let (k_n) be a strictly increasing sequence of positive integers.*

(i) *If $\sum_{n=1}^{+\infty}(1/k_n) < +\infty$ then $\bigcap_{\lambda \in I} \mathcal{HC}((\lambda B)^{k_n}) = \emptyset$ for every non-degenerate interval I in $(1, +\infty)$.*

(ii) *If $\sum_{n=1}^{+\infty}(1/k_n) = +\infty$ then the set $\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}((\lambda B)^{k_n})$ is residual in ℓ^2, i.e. it contains a G_δ and dense set in ℓ^2; hence $\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}((\lambda B)^{k_n}) \neq \emptyset$.*

(iii) *If $\sum_{n=1}^{+\infty}(1/k_n) = +\infty$ there exists a G_δ and dense subset $\mathcal{P}$ in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ with full 2-dimensional Lebesgue measure in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ such that $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}((\lambda B)^{k_n})$ is residual in ℓ^2. In particular, $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}((\lambda B)^{k_n}) \neq \emptyset$.*

Unfortunately we are unable to show whether $\mathcal{P}$ in item (iii) of Theorem 1.2 can be replaced by $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$. So, this remains an open problem. On the other hand, both items (i) and (iii) hold in a more general setting (there is nothing special if one choses to work with the backward shift) and this is evident if one follows the relevant proofs, see Sections 2 and 4. For instance, the interested readers will have no difficulties in formulating general statements for items (i) and (iii) that involve operators T so that for a given sequence of positive integers (k_n), the sequence$((\lambda T)^{k_n})$ is hypercyclic for every λ lying in some interval or annulus, possibly with infinite length or infinite area. We mention that a kind of similar line of research is pursued in [2], [10], [18], [19], [20], where questions similar to the above one are studied for translation type operators acting on the space of entire functions. Results on the existence of common hypercyclic vectors for uncountable families of operators and, in particular, of backward shift operators can be found in [1], [4], [6], [7], [8], [9], [14], [15], [17]. Our paper is organized as follows. Each one of the following Sections 2, 3, 4, is devoted to the proof of items (i), (ii), (iii) of Theorem 1.2, respectively.

The proof of item (i) relies on an estimate which concerns the size (in terms of Lebesgue measure) of the set$\{z \in \mathbb{C} : |z^n \cdot w - 1| < \varepsilon\} \cap [m, M]$, for given $w \in \mathbb{C}$, $\varepsilon > 0$, $1 < m < M$, n positive integer. This approach is implicit in [4], [5], [17] and refines an idea of Borichev. The common hypercyclicity criterion due to Costakis and Sambarino cannot be applied in order to conclude item (ii).

What we do, is to refine in a sense this criterion in the particular case of backward shift. It seems plausible that our method will possibly work for other operators as well. We mention that there are quite a few, relatively new and powerful, criteria establishing the existence of common hypercyclic vectors for uncountable families of operators, see [4], [5], [17]. However, it is not clear to us whether these criteria can be used in our case. Finally, the proof of item (iii) relies on the following three ingredients: (1) item (ii), (2) a metric result of Weyl which says that, if (k_n) is a given sequence of distinct integers then the sequence $(k_n x)$ is uniformly distributed mod 1, see Theorem 4.1, Chapter 1, page 32 in [13], and (3) Fubini's theorem, see page 149 in [12]. Actually, to prove item (iii) we elaborate on the proof of Proposition 5.2 from [3].

Recently F. Bayart gave a new light in the subject of common hypercyclic vectors for multiples of an operator in this magnificent article [2].

2. A NEGATIVE RESULT

Fix a subsequence (k_n) of natural numbers such that $\sum\limits_{n=1}^{+\infty}(1/k_n) < +\infty$. Our goal, in this section, is to show that $\bigcap\limits_{\lambda\in J}\mathcal{H}C((\lambda B)^{k_n}) = \emptyset$, where J is a non-degenerate closed interval of the set of positive numbers.

We proceed with the following.

LEMMA 2.1. *Let $z_0 \in \mathbb{C}$, $N_0 \in \mathbb{N}$, $N_0 > 1$, $\varepsilon_0 \in (0,1)$, $1 < \mu_0 < M_0 < +\infty$ be fixed. We consider the open in $[\mu_0, M_0]$ set*

$$G := G(\mu_0, M_0, z_0, N_0, \varepsilon_0) := \{\lambda \in [\mu_0, M_0] : |\lambda^{N_0} z_0 - 1| < \varepsilon_0\}.$$

We denote m_1 the 1-dimension Lebesgue measure on the real line. We have the following estimation:

$$m_1(G) \leqslant M_0 \cdot \Big(\sqrt[N_0]{\frac{1+\varepsilon_0}{1-\varepsilon_0}} - 1 \Big).$$

Proof. Suppose that $G \neq \emptyset$, (the other case is trivial). We denote $\delta(G)$ the diameter of G. Because G is a bounded and open subset of $[\mu_0, M_0]$ we get $G \subseteq [\inf G, \sup G]$ and then

$$m_1(G) \leqslant m_1([\inf G, \sup G]) = \sup G - \inf G = \delta(G). \tag{2.1}$$

In order to prove the desired inequality it suffices to prove

$$\delta(G) \leqslant M_0 \Big(\sqrt[N_0]{\frac{1+\varepsilon_0}{1-\varepsilon_0}} - 1 \Big)$$

by (2.1). Observe that the set G has two different elements s, t. Suppose $s < t$. Of course $s > 0$. We write $z_0 = x_0 + iy_0$, where $x_0, y_0 \in IR$, $x_0 = \mathrm{Re}(z_0)$. Because

$s, t \in G$ we get

$$|s^{N_0} z_0 - 1| < \varepsilon_0, \tag{2.2}$$

$$|t^{N_0} z_0 - 1| < \varepsilon_0. \tag{2.3}$$

Considering only the real parts we get by (2.2) and (2.3)

$$1 - \varepsilon_0 < s^{N_0} x_0 < 1 + \varepsilon_0 \quad \text{and} \tag{2.4}$$

$$1 - \varepsilon_0 < t^{N_0} x_0 < 1 + \varepsilon_0. \tag{2.5}$$

Since $s, t > 0$, this forces $x_0 > 0$, and it follows immediately by (2.4) and (2.5) that $(t/s)^{N_0} < (1+\varepsilon_0)/(1-\varepsilon_0)$, which gives the required bound (2.1) on the diameter, that completes the proof. ∎

THEOREM 2.2. *Let (k_n) be a subsequence of natural numbers such that $\sum_{n=1}^{+\infty}(1/k_n) < +\infty$. Then*

$$\bigcap_{\lambda \in I} \mathcal{HC}((\lambda B)^{k_n}) = \emptyset,$$

where I is a nondegenerate interval in the set of positive numbers.

Proof. Let some $\varepsilon \in (0,1)$, say $\varepsilon = 1/2$. We suppose that $I = [\mu_0, M_0]$, where μ_0, M_0 to be two real numbers such that $1 < \mu_0 < M_0$. Let also some absolute constant $a > 1$, such that

$$\frac{1+\varepsilon}{1-\varepsilon} < \mathrm{e}^a. \tag{2.6}$$

Of course we have $\lim_{n \to +\infty}(1 + a/n)^n = \mathrm{e}^a$. So by (2.1) there exists some $N_0 \in \mathbb{N}$, such that

$$\left(1 + \frac{a}{n}\right)^n > \frac{1+\varepsilon}{1-\varepsilon} \quad \text{for every } n \in \mathbb{N},\ n \geqslant N_0. \tag{2.7}$$

To arrive at a contradiction, suppose that $\bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{HC}((\lambda B)^{k_n}) \neq \emptyset$. We fix some $x_0 = (x_1, x_0, \ldots) \in \bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{HC}((\lambda B)^{k_n})$, and let $e_1 := (1, 0, 0, \ldots) \in \ell^2$, $\varepsilon_1 \in (0, \varepsilon)$ and $N_1 > N_0$.

Let arbitrary $\lambda \in [\mu_0, M_0]$. There exists a subsequence (μ_n) of (k_n) such that $(\lambda B)^{\mu_n}(x_0) \to e_1$. There exists some $n_1 \in \mathbb{N}$, $n_1 \geqslant N_1$ such that

$$\|(\lambda B)^{\mu_{n_1}}(x_0) - e_1\| < \varepsilon_1. \tag{2.8}$$

By (2.8) we get

$$|\lambda^{\mu_{n_1}} x_{\mu_{n_1}+1} - 1| < \varepsilon_1. \tag{2.9}$$

Let $\mu_{n_1} = k_{n_2}$ for some $n_2 \in \mathbb{N}$. Of course $n_2 \geqslant n_1$.

Hence, (2.9) implies that for every $\lambda \in [\mu_0, M_0]$ there exists some natural number $n \geqslant N_1$ such that:

$$|\lambda^{k_n} x_{k_n+1} - 1| < \varepsilon_1. \tag{2.10}$$

Setting

$$L := \{\lambda \in [\mu_0, M_0] \mid \exists\, n \geqslant N_1 : |\lambda^{k_n} x_{k_n+1} - 1| < \varepsilon_1\},$$

we get $L = [\mu_0, M_0]$, by (2.10).

Consider the set

$$\mathcal{N}_1 := \{v \in \mathbb{N} \mid v \geqslant N_1 \text{ and } \exists\, \lambda \in [\mu_0, M_0] : |\lambda^{k_v} x_{k_v+1} - 1| < \varepsilon_1\}.$$

Then $\mathcal{N}_1 :\neq \emptyset$ by (2.10).

Let $v \in \mathcal{N}_1$ and define the set

$$G_v := \{\lambda \in [\mu_0, M_0] : |\lambda^{k_v} x_{k_v+1} - 1| < \varepsilon_1\}.$$

It is obvious that $G_v \neq \emptyset$ for every $v \in \mathcal{N}_1$. The set G_v is open in $[\mu_0, M_0]$ for every $v \in \mathcal{N}_1$ and it is obvious that

$$[\mu_0, M_0] = \bigcup_{v \in \mathcal{N}_1} G_v. \tag{2.11}$$

By the properties of Lebesgue measure, Lemma 2.1 and (2.11) we have

$$\begin{aligned} M_0 - \mu_0 = m_1([\mu_0, M_0]) = m_1\Big(\bigcup_{v \in \mathcal{N}_1} G_v\Big) &\leqslant \sum_{v \in \mathcal{N}_1} m_1(G_v) \leqslant \sum_{v \in \mathcal{N}_1} M_0\Big(\sqrt[k_v]{\frac{1+\varepsilon_1}{1-\varepsilon_1}} - 1\Big) \\ &\leqslant M_0 \cdot \sum_{v=N_1}^{+\infty}\Big(\sqrt[k_v]{\frac{1+\varepsilon_1}{1-\varepsilon_1}} - 1\Big). \end{aligned} \tag{2.12}$$

Observe that

$$\frac{1+\varepsilon_1}{1-\varepsilon_1} < \frac{1+\varepsilon}{1-\varepsilon}, \quad \text{since } \varepsilon_1 \in (0, \varepsilon). \tag{2.13}$$

By (2.7) and (2.13) we get

$$\sqrt[k_n]{\frac{1+\varepsilon_1}{1-\varepsilon_1}} - 1 < \frac{a}{k_n} \quad \text{for every } n \geqslant N_1. \tag{2.14}$$

By (2.12) and (2.14) we take

$$M_0 - \mu_0 \leqslant M_0 \cdot \sum_{n=N_1}^{+\infty} \frac{a}{k_n}. \tag{2.15}$$

The facts that (2.15) holds for every $m \geqslant N_1$ instead of N_1 and $\sum\limits_{n=1}^{+\infty}(1/k_n) < +\infty$ gives that $M_0 - m_0 \leqslant 0$ that is false of course. ∎

3. THE POSITIVE CASE IN THE HALF LINE $(1,+\infty)$

Throughout this section we fix a strictly increasing subsequence (k_n) of natural numbers such that $\sum_{n=1}^{+\infty}(1/k_n)=+\infty$. We shall prove the following.

THEOREM 3.1. *The set* $\bigcap_{\lambda\in(1,+\infty)}\mathcal{H}C((\lambda B)^{k_n})$ *is a residual subset of* $(\ell^2,\|\cdot\|)$.

The proof of this theorem will be completed at the end of Section 4. Firstly, we need the two following lemmas.

LEMMA 3.2. *Let a_0, b_0 be two positive real numbers such that $1<a_0<b_0<+\infty$. Let (k_n) be a subsequence of natural numbers such that $\sum_{n=1}^{+\infty}(1/k_n)=+\infty$. Then for every positive number $\varepsilon>0$ there exist two natural numbers n_0 and i_0 and a finite number of terms $k_{n_0},k_{n_0+1},\dots,k_{n_0+i_0}$ of (k_n) and positive numbers $\beta_1,\beta_2,\dots,\beta_{i_0+1}$ such that: for every $\lambda\in[a_0,b_0]$, there exists some $j\in\{0,1,\dots,i_0\}$ such that*

$$|\lambda^{k_{n_0+j}}\beta_{j+1}-1|<\varepsilon.$$

Proof. We fix some positive number

$$\varepsilon_0\in(0,\min\{1,(b_0/a_0)^{2/3}-1\}).$$

After we fix some natural number n_0 such that:

$$k_{n_0}>\frac{\log(1+\varepsilon_0)}{\log(b_0/a_0)}. \tag{3.1}$$

Of course $\sum_{j=0}^{+\infty}(1/k_{n_0+j})=+\infty$. So by the fact that the sequence (k_n), $n=1,2,\dots$ is strictly increasing and the choice of the number ε_0 there exists the unique natural number $i_0\in\{2,3,\dots\}$ such that

$$(1+\varepsilon_0)^{\sum_{j=0}^{i_0}(1/k_{n_0+j})}\leqslant\frac{b_0}{a_0}\quad\text{and}\quad(1+\varepsilon_0)^{\sum_{j=0}^{i_0+1}(1/k_{n_0+j})}>\frac{b_0}{a_0}. \tag{3.2}$$

We set $\beta_1:=1/a_0^{k_{n_0}}$. Then for every $\lambda\in[a_0,b_0]$ such that $\lambda<a_0\cdot\sqrt[k_{n_0}]{1+\varepsilon_0}$, we get $|\lambda^{k_{n_0}}\cdot\beta_1-1|<\varepsilon_0$.

After we set $a_1:=a_0\cdot\sqrt[k_{n_0}]{1+\varepsilon_0}$ and $\beta_2:=1/a_1^{k_{n_0+1}}$.

Then, for every $\lambda\in[a_1,b_0]$ with $\lambda<a_1\cdot\sqrt[k_{n_0+1}]{1+\varepsilon_0}$ we have $|\lambda^{k_{n_0+1}}\beta_2-1|<\varepsilon_0$.

We continue inductively.

We suppose that we have defined the number $a_j=a_0(1+\varepsilon_0)^{\sum_{i=0}^{j-1}(1/k_{n_0+i})}$ for some $j\in\{1,2,\dots,i_0-1\}$.

After we define $\beta_{j+1} := 1/a_j^{k_{n_0+j}}$ and for every $\lambda \in [a_i, b_0]$ with $\lambda < a_j \cdot \sqrt[k_{n_0+j}]{1+\varepsilon_0}$ we get $|\lambda^{k_{n_0+j}} \cdot \beta_{j+1} - 1| < \varepsilon_0$.

Finally, after a finite number of steps we get the conclusion of this lemma with the following data.

We have

$$a_j = a_0 \cdot \prod_{i=0}^{j-1} (1+\varepsilon_0)^{1/k_{n_0+i}} \quad \text{for every } j = 1, 2, \dots, i_0,$$

$$\beta_{j+1} = \frac{1}{a_j^{k_{n_0+j}}} \quad \text{for every } j = 0, 1, \dots, i_0.$$

It follows that for every $\lambda \in [a_j, b_0]$ where $\lambda < a_j \cdot \sqrt[k_{n_0+j}]{1+\varepsilon_0}$ we have:

$$|\lambda^{k_{n_0}+j} \beta_{j+1} - 1| < \varepsilon_0 \quad \text{for every } j = 0, 1, \dots, i_0.$$

With the above data the proof of this lemma is completed. ∎

LEMMA 3.3. *Let (k_n) be a strictly increasing subsequence of natural numbers such that $\sum\limits_{n=1}^{+\infty} (1/k_n) = +\infty$.*

Then for every positive number $M > 0$ there exists a subsequence (μ_n) of (k_n) such that:

(i) $\mu_{n+1} - \mu_n > M$ *for every* $n = 1, 2, \dots$ *and*

(ii) $\sum\limits_{n=1}^{+\infty} (1/\mu_n) = +\infty$.

Proof. Let some positive number $M > 1$. Let $N := [M] + 1$ (where $[x]$ is the integer part of the real number x).

We consider the subsequences of (k_n), $\mu_\rho^j := k_{\rho N + j}$, $\rho = 1, 2, \dots$, for $j = 0, 1, \dots, N-1$. We get $\sum\limits_{n=1}^{+\infty} (1/k_n) = \sum\limits_{j=0}^{N-1} \sum\limits_{\rho=1}^{+\infty} (1/\mu_\rho^j)$. By this equality and the fact that $\sum\limits_{n=1}^{+\infty} (1/k_n) = +\infty$ we conclude easily that there exists a subsequence of $(\mu_\rho^j)_{\rho \in \mathbb{N}}$, for some $j \in \{0, 1, \dots, N-1\}$ at least that satisfies (i) and (ii). ∎

In order to prove Theorem 3.1 we assign some notations and terminology. Let $D := \{x = (x_1, x_2, \dots) \in \ell^2 : \{n \in \mathbb{N} : x_n \neq 0\}$ is a finite subset of $\mathbb{N}$ and $x_n \in \mathbb{Q} + \mathrm{i}\mathbb{Q}$ for every $n = 1, 2, \dots\}$, where $\mathbb{Q}$ is the set of rational numbers, The set D is countable and dense in $(\ell^2, \|\cdot\|_2)$. We set $\overline{0} := (0, 0, \dots) \in \ell^2$ and $D^* := D \setminus \{\overline{0}\}$. Let $\Psi := \{y_1, y_2, y_3, \dots, y_n, \dots\}$ be an enumeration of D^*. We fix a strictly decreasing sequence of positive numbers (a_n) such that $a_n \to 1$ (for example $a_n := 1 + (1/n)$, $n = 1, 2, \dots$) and we also fix a strictly increasing sequence (β_n) of positive numbers such that $\beta_n \to +\infty$ and $a_1 < \beta_1$ (for example $\beta_n = n + 2$, $n = 1, 2, \dots$). Then we set $\Delta_n := [a_n, \beta_n]$, $n = 1, 2, \dots$. Of course, the sequence of compact sets Δ_n, $n = 1, 2, \dots$ forms an exhausting family of $(1, +\infty)$, i.e. $(1, +\infty) = \bigcup_{n=1}^{+\infty} \Delta_n$.

For every $n, j, s \in \mathbb{N}$ we define the sets $E_{\Delta_n}(j,s) := \{x = (x_1, x_2, \dots) \in \ell^2 :$ for every $\lambda \in \Delta_n$ there exists some $v \in \mathbb{N}$, such that $\|(\lambda B)^{k_v}(x) - y_j\| < 1/s\}$.

We finally set

$$G := \bigcap_{n=1}^{+\infty} \bigcap_{j=1}^{+\infty} \bigcap_{s=1}^{+\infty} E_{\Delta_n}(j,s).$$

It is easy to show that the sets $E_{\Delta_n}(j,s)$ are open for every $j, s, n \in \mathbb{N}$ and that $G = \bigcap_{\lambda \in (1,+\infty)} \mathcal{H}C((\lambda B)^{k_n})$.

From this result we have that the set $\bigcap_{\lambda \in (1,+\infty)} \mathcal{H}C((\lambda B)^{k_n})$ is a G_δ subset of $(\ell^2, \|\cdot\|)$.

Now we are ready to prove the most important result of this paper.

LEMMA 3.4. *For every $n, j, s \in \mathbb{N}$ the set $E_{\Delta_n}(j,s)$ is dense in $(\ell^2, \|\cdot\|)$.*

Proof. We fix $n_0, j_0, s_0 \in \mathbb{N}$ and we will show that the set $E_{\Delta_{n_0}}(j_0, s_0)$ is dense in $(\ell^2, \|\cdot\|)$.

We set $E := E_{\Delta_{n_0}}(j_0, s_0)$ for simplicity.

Let $y_{j_0} := (q_1, q_2, \dots, q_{v_0}, 0, 0, \dots)$ where $q_{v_0} \neq 0$, $y_{j_0}(v) = q_v$, for every $v = 1, 2, \dots, v_0$, $y_{j_0}(v) = 0$ for every $v \geqslant v_0 + 1$, and $q_j \in \mathbb{Q} + i\mathbb{Q}$ for every $j = 1, 2, \dots, v_0$, for some fixed $v_0 \in \mathbb{N}$.

We fix $\varepsilon_0 > 0$ and $c_0 = (c_1, c_2, \dots, c_{v_1}, 0, 0, \dots) \in D^*$ where $c_{v_1} \neq 0$, $v_1 \in \mathbb{N}$, fixed, $c_0(v) = c_v$ for every $v = 1, 2, \dots, v_1$, $c_0(v) = 0$ for every $v \geqslant v_1 + 1$, $c_j \in \mathbb{Q} + i\mathbb{Q}$ for every $j = 1, 2, \dots, v_1$. We consider the ball $B_{\ell^2}(c_0, \varepsilon_0) := \{x \in \ell^2 : \|x - c_0\| < \varepsilon_0\}$.

We will show that

$$E \cap B_{\ell^2}(c_0, \varepsilon_0) \neq \emptyset. \tag{3.3}$$

In order to show the relation (3.3) it suffices to show that there exists some $x_0 = (x_1, x_2, \dots, x_n, \dots) \in \ell^2$ and $m_0 \in \mathbb{N}$ such that:

(i) $\|x_0 - c_0\| < \varepsilon_0$ and

(ii) for every $\lambda \in \Delta_{n_0}$ there exists some $v \in \mathbb{N}$, such that

$$\|(\lambda B)^{k_v}(x_0) - y_{j_0}\| < \frac{1}{s_0}. \tag{3.4}$$

We will achieve (i) and (ii) above as follows.

From the data of the problem we define a finite number of complex numbers x_j, $j = 1, 2, \dots, \ell_0$ for some fixed $\ell_0 \in \mathbb{N}$.

Afterwards, we define the sequence $x_0 := (x_1, x_2, \dots, x_{\ell_0}, 0, 0)$ where $x_0(j) = x_j$ for every $j = 1, 2, \dots, \ell_0$ and $x_0(j) = 0$ for every $j \geqslant \ell_0 + 1$. So we have $x_0 \in \ell^2$.

Finally, we show that x_0 satisfies properties (i) and (ii) of (3.4) as above.

Without loss of generality let $\Delta_{n_0} := [a_0, b_0]$, where $1 < a_0 < b_0 < +\infty$. We set

$$M_1 := \max\{|q_j|,\ j = 1,2,\dots,v_0\} > 0.$$

We also set

$$M_2 := \frac{1}{2\log a_0} \cdot \log\Big(\frac{2s_0^2 \cdot M_1^2 \cdot v_0}{1-(1/a_0)}\Big).$$

By Lemma 3.3 we choose a subsequence (μ_n) of (k_n) such that the following two properties hold:

(i) $\mu_{n+1} - \mu_n > \max\{M_2, v_0\}$ for every $n = 1,2,\dots$ and

(ii) $\sum\limits_{n=1}^{+\infty}(1/\mu_n) = +\infty$.

We set $\varepsilon_1 := 1/(\sqrt{2v_0} \cdot s_0 \cdot M_1)$.

Now we can choose some fixed natural number $v_2 \in \mathbb{N}$ such that the following three inequalities hold:

(a) $\mu_{v_2} > v_1 + 1$;

(b) $\mu_{v_2} > \log(1+\varepsilon_1)/\log(b_0/a_0)$;

(c) $\mu_{v_2} > (1/(2\log a_0)) \cdot \log\Big(\frac{v_0 \cdot M_1^2}{\varepsilon_0^2 \cdot (1-(1/a_0))}\Big)$.

Because $\sum\limits_{n=1}^{+\infty}(1/\mu_n) = +\infty$ we have $\sum\limits_{j=0}^{+\infty}(1/\mu_{v_2+j}) = +\infty$.

Let i_0 be the unique natural number $i_0 \in \{2,3,\dots\}$ such that

$$(1+\varepsilon_1)^{\sum_{i=0}^{i_0}(1/\mu_{v_2+i})} \leqslant \frac{b_0}{a_0} \quad \text{and} \quad (1+\varepsilon_1)^{\sum_{i=0}^{i_0+1}(1/\mu_{v_2+i})} > \frac{b_0}{a_0}. \tag{3.5}$$

We define the numbers

$$a_j := a_0 \cdot \prod_{i=0}^{j-1}(1+\varepsilon_1)^{1/\mu_{v_2+i}} \quad \text{for every } j = 1,\dots,i_0 \quad \text{and}$$

$$\beta_{j+1} := \frac{1}{a_j^{\mu_{v_2+j}}} \quad \text{for every } j = 0,1,\dots,i_0-1.$$

Now, we are ready to define the vector $x_0 = (x_1, x_2, \dots) \in \ell^2$ with full details.

We define $x_j = c_j$ for every $j = 1,2,\dots,v_1$. We define $x_j = 0$ for every $j \in \mathbb{N}$ such that $v_1 + 1 \leqslant j \leqslant \mu_{v_2}$.

Because $\sum\limits_{n=1}^{+\infty}(1/\mu_n) = +\infty$, we can apply Lemma 3.2 for the sequence (μ_n). By the relations (b) and (3.5) above that the numbers v_2 and ε_1, i_0 satisfy and using Lemma 3.2 for the sequence (μ_n) we take as above that for every $\lambda \in [a_0, b_0]$ there exists unique $j \in \{0,1,\dots,i_0\}$ such that

$$|\lambda^{\mu_{v_2}+j}\beta_{j+1} - 1| < \varepsilon_1. \tag{3.6}$$

We remark by the previous Lemma 3.2 that for every $\lambda \in [a_0, b_0]$ there exists unique a_j, $j \in \{0, 1, \ldots, i_0\}$ such that $a_j \leqslant \lambda \leqslant a_{j+1}$ if $i \leqslant i_0 - 1$ or $a_{i_0} \leqslant \lambda \leqslant b_0$.

So, we have defined completely the positive numbers β_{j+1}, for $j = 0, 1, \ldots, i_0$.

Now, we define $x_{\mu_{v_2+j}+i} = \beta_{j+1} q_i$ for every $j = 0, 1, \ldots, i_0$ and for every $i = 1, 2, \ldots, v_0$.

The previous terms $x_{\mu_{v_2+j}+i}$ for $j = 0, 1, \ldots, i_0$, $i = 1, 2, \ldots, v_0$ are defined because $\mu_{n+1} - \mu_n > v_0$ for every $n = 1, 2, \ldots$ by the definition of the sequence (μ_n).

Finally, we define that $x_i = 0$ for every $i \in \mathbb{N}$, $i > \mu_{v_2}$ for which there are not $j \in \{0, 1, \ldots, i_0\}$ and $i_1 \in \{1, 2, \ldots, v_0\}$ such that $i = \mu_{v_2+j} + i_1$.

By the previous procedure we have defined completely the vector $x_0 = (x_1, x_2, \ldots, x_n, \ldots)$ where $\{n \in \mathbb{N} : x_n \neq 0\}$ is finite and thus $x_0 \in \ell^2$ obviously.

Now, we show that the vector x_0 satisfies relation (3.4).

Firstly we prove that $x_0 \in B_{\ell^2}(c_0, \varepsilon_0)$.

By the definition of the vector x_0 we get:

$$\begin{aligned}
\|x_0 - c_0\|^2 := \sum_{j=1}^{+\infty} |x_j - c_j|^2 &= \sum_{j=v_1+1}^{+\infty} |x_j|^2 = \sum_{i=0}^{i_0} \sum_{j=1}^{v_0} |x_{\mu_{v_2+j}+i}|^2 \\
&= \sum_{j=1}^{v_0} \sum_{i=0}^{i_0} |x_{\mu_{v_2+j}+i}|^2 = \sum_{j=1}^{v_0} \sum_{i=0}^{i_0} |\beta_{j+1} q_i|^2 = \sum_{j=1}^{v_0} |q_i|^2 \sum_{i=0}^{i_0} |\beta_{j+1}|^2 \\
&\leqslant v_0 M_1^2 \cdot \sum_{i=0}^{i_0} |\beta_{j+1}|^2 = v_0 M_1^2 \cdot \sum_{i=0}^{i_0} \frac{1}{a_j^{2\mu_{v_2+j}}} \\
&\leqslant v_0 M_1^2 \cdot \sum_{i=0}^{i_0} \frac{1}{a_0^{2\mu_{v_2+j}}} < v_0 M_1^2 \cdot \sum_{v=2\mu_{v_2}}^{+\infty} \frac{1}{a_0^v} \\
&= v_0 M_1^2 \frac{1}{a_0^{2\mu_{v_2}}} \cdot \frac{1}{1 - (1/a_0)} < \varepsilon_0^2 \qquad (3.7)
\end{aligned}$$

by the inequality (c) above for μ_{v_2}.

Inequality (3.7) gives that $x_0 \in B_{\ell^2}(c_0, \varepsilon_0)$, so property (i) of (3.4) holds.

We show now that property (ii) also holds.

We fix some $\lambda \in [a_0, b_0]$. Then there exists unique $\rho_0 \in \{1, 2, \ldots, i_0 - 1\}$ such that $a_{\rho_0} \leqslant \lambda < a_{\rho_0} \cdot (1 + \varepsilon_0)^{1/(\mu_{v_2} + \rho_0)}$ or $a_{i_0} \leqslant \lambda \leqslant b_0$.

We show that

$$\|(\lambda B)^{\mu_{v_2+\rho_0}}(x_0) - y_{j_0}\| < \frac{1}{s_0}.$$

We have:

$$\|(\lambda B)^{\mu_{v_2+\rho_0}}(x_0) - y_{j_0}\|^2 = \sum_{i=1}^{v_0} |\lambda^{\mu_{v_0+\rho_0}} x_{\mu_{v_2+\rho_0}+i} - q_j|^2 + \sum_{i=v_0+1}^{+\infty} |\lambda^{\mu_{v_2+\rho_0}} x_{\mu_{v_2+\rho_0}+i}|^2. \tag{3.8}$$

By definition we have for $i = 1, 2, \ldots, v_0$ $x_{\mu_{v_2+\rho_0}+i} = \beta_{\rho_0+1} q_i$. So, for $j = 1, 2, \ldots, v_0$ and $\lambda \in [a_{\rho_0}, a_{\rho_0+1}]$, where $a_{\rho_0+1} = a_{\rho_0} \cdot (1 + \varepsilon_1)^{1/(\mu_{v_2} + \rho_0)}$ or $\lambda \in [a_{\rho_0}, b_0]$ if $\rho_0 = i_0$

we have

$$|\lambda^{\mu_{v_2+\rho_0}} x_{\mu_{v_2+\rho_0}+i} - q_i|^2 = |\lambda^{\mu_{v_2+\rho_0}} \beta_{\rho_0+1} q_i - q_i|^2 = |\lambda^{\mu_{v_2+\rho_0}} \beta_{\rho_0+1} - 1|^2 |q_i|^2$$
$$\leqslant |\lambda^{\mu_{v_2+\rho_0}} \beta_{\rho_0+1} - 1|^2 \cdot M_1^2 < \varepsilon_1^2 M_1^2 = \frac{1}{2v_0 s_0^2}.$$

So we have:

$$\sum_{i=1}^{v_0} |\lambda^{\mu_{v_2+\rho_0}} x_{\mu_{v_2+\rho_0}+i} - q_i|^2 < \frac{1}{2s_0^2}. \tag{3.9}$$

If $\rho_0 = i_0$ the second member of (3.8) is 0 and the conclusion holds by (3.9).

So for the sequel we suppose that $\rho_0 \leqslant i_0 - 1$. In this case we get

$$\begin{aligned}
&\sum_{i=v_0+1}^{+\infty} |\lambda^{\mu_{v_2+\rho_0}} x_{\mu_{v_2+\rho_0}+i}|^2 \\
&= \lambda^{2\mu_{v_2+\rho_0}} \sum_{i=1}^{v_0} \sum_{j=1}^{j_0-\rho_0} |x_{\mu_{v_2+\rho_0+j}+i}|^2 = \lambda^{2\mu_{v_2+\rho_0}} \sum_{i=1}^{v_0} |q_j|^2 \sum_{j=1}^{i_0-\rho_0} |\beta_{\rho_0+j+1}|^2 \\
&\leqslant \lambda^{2\mu_{v_2+\rho_0}} v_0 M_1^2 \sum_{j=1}^{j_0-\rho_0} |\beta_{\rho_0+j+1}|^2 = \lambda^{2\mu_{v_2+\rho_0}} v_0 M_1^2 \sum_{j=1}^{i_0-\rho_0} \frac{1}{(a_{\rho_0+j}^{\mu_{v_2+\rho_0+j}})^2} \\
&< \lambda^{2\mu_{v_2+\rho_0}} v_0 M_1^2 \sum_{v=2\mu_{v_2+\rho_0+1}}^{+\infty} \frac{1}{a_{\rho_0+1}^{v}} = \lambda^{2\mu_{v_2+\rho_0}} v_0 M_1^2 \frac{1}{a_{\rho_0+1}^{2\mu_{v_2+\rho_0+1}}} \cdot \frac{1}{1-(1/a_{\rho_0+1})} \\
&< a_{\rho_0+1}^{2\mu_{v_2+\rho_0}} v_0 M_1^2 \frac{1}{a_{\rho_0+1}^{2\mu_{v_2+\rho_0+1}}} \cdot \frac{1}{1-(1/a_0)} = \frac{v_0 M_1^2}{1-(1/a_0)} \cdot \frac{1}{a_{\rho_0+1}^{2(\mu_{v_2+\rho_0+1}-\mu_{v_2+\rho_0})}}
\end{aligned}$$

$$< \frac{v_0 M_1^2}{1-(1/a_0)} \cdot \frac{1}{a_0^{2(\mu_{v_2+\rho_0+1}-\mu_{v_2+\rho_0})}} < \frac{1}{2s_0^2} \tag{3.10}$$

because $\mu_{v+1} - \mu_v > M_2$ for every $v \geqslant v_2$ by the hypothesis (i) for the sequence (μ_n) and the definition of M_2.

By (3.8), (3.9) and (3.10) we get that $\|(\lambda B)^{\mu_{v_2+\rho_0}}(x_0) - y_{j_0}\| < 1/s_0$ for the arbitrary $\lambda \in [a_0, b_0]$. This completes property (ii) of (3.4) and the proof of this Lemma 3.4 is completed. ∎

4. A RESULT IN MEASURE AND CATEGORY

In this section we prove item (iii) of Theorem 1.2. Actually, we shall prove the following, more general, result. As we already mentioned in the Introduction, its proof elaborates on the proof of Proposition 5.2 from [3].

THEOREM 4.1. *Let* (k_n) *be a strictly increasing sequence of positive integers. Let* T *be a bounded linear operator acting on a (complex) Banach space* X *such that* $((\lambda T)^{k_n})$ *is hypercyclic for every* $\lambda \in \mathbb{C}$ *with* $|\lambda| > 1$ *and assume in addition that*

$$\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda T)^{k_n}\}) \neq \emptyset.$$

Then, there exists a G_δ *and dense subset* $\mathcal{P}$ *in* $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ *with full* 2*-dimensional Lebesgue measure in* $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ *such that* $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda T)^{k_n}\})$ *is residual in* X. *In particular,* $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda T)^{k_n}\}) \neq \emptyset$.

Proof. By performing a change of variables it suffices to prove the following.

Claim. Fix $x \in \bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda T)^{k_n}\})$. *Then there exists a* G_δ *and dense subset* A *of* $(1,+\infty) \times \mathbb{R}$ *with full (2-dimensional) Lebesgue measure such that the set* $\{((rT)^{k_n}x, e^{2\pi i k_n \theta}) : n = 1, 2, \dots\}$ *is dense in* $X \times \mathbb{T}$ for every $(r, \theta) \in A$.

Here $\mathbb{T}$ denotes the unit circle, i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proof of Claim. Let $\{x_j : j \in \mathbb{N}\}$, $\{t_l : l \in \mathbb{N}\}$ be dense subsets of X, $\mathbb{T}$ respectively. For every $j, l, s, n \in \mathbb{N}$ define the set

$$A_{j,l,s,n} := \Big\{(r,\theta) \in (1,+\infty) \times \mathbb{R} : \|(rT)^{k_n}x - x_j\| < \frac{1}{s}, |e^{2\pi i k_n \theta} - t_l| < \frac{1}{s}\Big\}.$$

We shall prove that the set $A := \bigcap_{j,l,s} \bigcup_n A_{j,l,s,n}$ has the desired properties. Since $A_{j,l,s,n}$ is open we conclude that A is G_δ. Let us show that A is dense in $(1,+\infty) \times \mathbb{R}$. In view of Baire's theorem it suffices to prove that for any fixed $j, l, s \in \mathbb{N}$ the set $\bigcup_n A_{j,l,s,n}$ is dense in $(1,+\infty) \times \mathbb{R}$. To this end, fix $j, l, s \in \mathbb{N}$ and let $b > 1$, $a \in \mathbb{R}$ and $\varepsilon > 0$. We seek $r > 0$, $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ such that

$$|b - r| < \varepsilon, \quad |a - \theta| < \varepsilon, \quad |t_l - e^{2\pi i k_n \theta}| \quad \text{and} \quad \|(rT)^{k_n}x - x_j\| < \frac{1}{s}.$$

Define the set $B := \{k_n : \|(bT)^{k_n}x - x_j\| < 1/s\}$ and consider its elements in an increasing order, say $k_{\rho_1} < k_{\rho_2} < \cdots$. Of course, we have $B = \{k_{\rho_n} : n \in \mathbb{N}\}$. Now we use Weyl's theorem, see Theorem 4.1, Chapter 1, in page 32 from [13], to conclude that the sequence $(k_{\rho_n}\theta)$ is uniformly distributed modulo 1 for almost all θ in $\mathbb{R}$. Hence, there exists $\theta \in \mathbb{R}$ such that the set $\{e^{2\pi i k_{\rho_n}\theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$ and $|a - \theta| < \varepsilon$. Finally, setting $r := b$ and from all the above we conclude that there exists $n := \rho_m$ for some $m \in \mathbb{N}$ such that

$$|b - r| < \varepsilon, \quad |a - \theta| < \varepsilon, \quad |t_l - e^{2\pi i k_n \theta}| \quad \text{and} \quad \|(rT)^{k_n}x - x_j\| < \frac{1}{s},$$

which is what we wanted to prove. It remains to show that A has full measure in $(1,+\infty) \times \mathbb{R}$. Actually, it is enough to prove that the set $\bigcup_n A_{j,l,s,n}$ has full measure in $(1,+\infty) \times \mathbb{R}$ for every $j, l, s \in \mathbb{N}$. Fix $j, l, s \in \mathbb{N}$ and take any four numbers d_1, d_2, d_3, d_4 with $d_1 < d_2$, $1 < d_3 < d_4$. For any subset B of $(1,+\infty) \times \mathbb{R}$ the symbol B_r stands for its section, i.e. $B_r := \{\theta \in \mathbb{R} : (r, \theta) \in E\}$ and for simplicity reasons we set $E := \bigcup_n A_{j,l,s,n}$. Observe that the proof of denseness result implies that for every $r \in [d_3, d_4]$ we have $(r, \theta) \in E$ for almost every θ in $\mathbb{R}$ (of course the set of such θ's depends on r). It now follows that

$$\mu((E \cap ([d_3, d_4] \times [d_1, d_2]))_r) = d_2 - d_1 = \mu(([d_3, d_4] \times [d_1, d_2])_r) \quad \text{for every } r \in [d_3, d_4],$$

where μ denotes the Lebesgue measure, and by Fubini theorem, see page 149 (5), in [12], we conclude that

$$\mu \times \mu(E \cap ([d_3, d_4] \times [d_1, d_2])) = (d_2 - d_1)(d_4 - d_3).$$

Thus, E has full measure in $(1, +\infty) \times \mathbb{R}$. This completes the proof of the Claim and hence that of Theorem 4.1. ∎

Item (ii) of Theorem 1.2 and Theorem 4.1 directly imply item (iii) of Theorem 1.2.

Now by Lemmas 3.4 and the facts that the space $(\ell^2, \|\cdot\|)$ is a complete metric space and Baire's category theorem the proof of Theorem 3.1 is completed.

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