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ON THE ANDO–HIAI–OKUBO TRACE INEQUALITY

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ABSTRACT. Let A and B be positive semidefinite matrices. It is shown that $|\mathrm{Tr}(A^w B^z A^{1-w} B^{1-z})| \leqslant \mathrm{Tr}(AB)$ for all complex numbers w, z for which $|\mathrm{Re}\, w - \frac{1}{2}| + |\mathrm{Re}\, z - \frac{1}{2}| \leqslant \frac{1}{2}$. This is a generalization of a trace inequality due to T. Ando, F. Hiai, and K. Okubo for the special case when w, z are real numbers, and a recent trace inequality proved by T. Bottazzi, R. Elencwajg, G. Larotonda, and A. Varela when $w = z$ with $\frac{1}{4} \leqslant \mathrm{Re}\, z \leqslant \frac{3}{4}$.

As a consequence of our new trace inequality, we prove that $\|A^w B^z + B^{1-\overline{z}} A^{1-\overline{w}}\|_2 \leqslant \|A^w B^z + A^{1-\overline{w}} B^{1-\overline{z}}\|_2$ for all complex numbers w, z for which $|\mathrm{Re}\, w - \frac{1}{2}| + |\mathrm{Re}\, z - \frac{1}{2}| \leqslant \frac{1}{2}$. This is a generalization of a recent norm inequality proved by M. Hayajneh, S. Hayajneh, and F. Kittaneh when w, z are real numbers.

KEYWORDS: *Unitarily invariant norm, Hilbert–Schmidt norm, Schatten p-norm, trace, positive semidefinite matrix, inequality.*

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1. INTRODUCTION

In their investigation of trace inequalities for multiple products of powers of two matrices, T. Ando, F. Hiai, and K. Okubo [1] proved that

$$|\,\mathrm{tr}(A^{\mu} B^{\nu} A^{1-\mu} B^{1-\nu})| \leqslant \mathrm{tr}(AB), \tag{1.1}$$

where A, B are positive semidefinite matrices and μ, ν are positive real numbers for which

$$\Big|\mu - \frac{1}{2}\Big| + \Big|\nu - \frac{1}{2}\Big| \leqslant \frac{1}{2}. \tag{1.2}$$

In this paper, using complex interpolation related to the Hadamard three lines theorem, we generalize the inequality (1.1) by proving that the inequality

$$|\,\mathrm{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \mathrm{tr}(AB) \tag{1.3}$$

holds for all complex numbers w, z for which

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2}. \tag{1.4}$$

This generalization gives a new way of proving the Ando–Hiai–Okubo trace inequality [1] without using the method of log majorization and the technique of antisymmetric tensor products.

A special case of the inequality (1.3) when $w = z$ is the inequality

$$|\operatorname{tr}(A^z B^z A^{1-z} B^{1-z})| \leqslant \operatorname{tr}(AB). \tag{1.5}$$

Recently, T. Bottazzi et al. [5] proved the inequality (1.5) under the condition that

$$\frac{1}{4} \leqslant \operatorname{Re} z \leqslant \frac{3}{4}. \tag{1.6}$$

It is important to see that the inequality (1.3) under the condition (1.4) is a generalization of the inequality (1.5) under the condition (1.6).

This type of trace inequalities has many applications in the spectra of functional calculus of matrices and operators, including a question of J.C. Bourin and a conjecture posed by S. Hayajneh and F. Kittaneh.

In [7], and in his work on subadditivity of concave functions of positive semidefinite matrices, J.C. Bourin asked whether the unitarily invariant norm inequality

$$|||A^p B^q + B^p A^q||| \leqslant |||A^{p+q} + B^{p+q}|||$$

holds true for any positive semidefinite matrices A, B and any positive real numbers p, q. The related well-known Heinz inequality says that

$$|||A^p B^q + A^q B^p||| \leqslant |||A^{p+q} + B^{p+q}|||.$$

For an equivalent version and a generalization of this inequality, we refer to p. 265 of [4].

In a recent paper [9], and in their investigations of the Lieb–Thirring trace inequalities, and in their attempt to answer Bourin's question, S. Hayajneh and F. Kittaneh proposed the following conjecture for commuting positive semidefinite matrices.

CONJECTURE 1.1. *Let A_1, A_2, B_1, B_2 be positive semidefinite matrices such that $A_1 B_1 = B_1 A_1$ and $A_2 B_2 = B_2 A_2$. Then, for every unitarily invariant norm,*

$$|||A_1 B_2 + A_2 B_1||| \leqslant |||A_1 B_2 + B_1 A_2|||. \tag{1.7}$$

An important special case of the inequality (1.7) is the inequality

$$|||A^s B^p + B^q A^t||| \leqslant |||A^s B^p + A^t B^q|||,$$

where A, B are positive semidefinite matrices and s, t, p, q are positive real numbers. Replacing A and B by $A^{1/(s+t)}$ and $B^{1/(p+q)}$, we see that this inequality is equivalent to saying

$$|||A^{\mu}B^{\nu} + B^{1-\nu}A^{1-\mu}||| \leqslant |||A^{\mu}B^{\nu} + A^{1-\mu}B^{1-\nu}|||, \tag{1.8}$$

for $\mu, \nu \in [0,1]$.

The Hilbert–Schmidt norm version of (1.8) is the inequality

$$\|A^{\mu}B^{\nu} + B^{1-\nu}A^{1-\mu}\|_2 \leqslant \|A^{\mu}B^{\nu} + A^{1-\mu}B^{1-\nu}\|_2. \tag{1.9}$$

Recently, M. Hayajneh, S. Hayajneh, and F. Kittaneh [8] proved the inequality (1.9) under the condition (1.2). A special case of the inequality (1.9) when $\nu = 1-\mu$ is the inequality

$$\|A^{\mu}B^{1-\mu} + B^{\mu}A^{1-\mu}\|_2 \leqslant \|A^{\mu}B^{1-\mu} + A^{1-\mu}B^{\mu}\|_2. \tag{1.10}$$

In [3], R. Bhatia proved the inequality (1.10) under the condition

$$\frac{1}{4} \leqslant \mu \leqslant \frac{3}{4}, \tag{1.11}$$

which is a significant improvement on a recent result of S. Hayajneh and F. Kittaneh [9], where they proved it only for certain special values of μ.

The following norm inequality is another special case of the inequality (1.9) with $\mu = \frac{1}{2}$:

$$\|A^{1/2}B^{\nu} + B^{1-\nu}A^{1/2}\|_2 \leqslant \|A^{1/2}B^{\nu} + A^{1/2}B^{1-\nu}\|_2. \tag{1.12}$$

It has been pointed out to the authors by J.C. Bourin that the inequality (1.12) can also be concluded from Theorem 2.2 in [6]. In [9], S. Hayajneh and F. Kittaneh proved the inequality (1.12) using some number theory tools, and the proof goes in an algorithmic way.

In Section 2, as an application of the inequality (1.3) under the condition (1.4), we also generalize the inequality (1.9) to complex values under the condition (1.4). In fact, we prove that the inequality

$$\|A^{w}B^{z} + B^{1-\bar{z}}A^{1-\bar{w}}\|_2 \leqslant \|A^{w}B^{z} + A^{1-\bar{w}}B^{1-\bar{z}}\|_2 \tag{1.13}$$

holds for the complex numbers w, z under the condition (1.4).

It should be noted that the inequality (1.13) gives a partial answer of Conjecture 1.1 for the Hilbert–Schmidt norm with the matrices A^{w}, B^{z}, $B^{1-\bar{z}}$, $A^{1-\bar{w}}$, which are not necessarily positive semidefinite matrices. So, it would be interesting to find all such cases of commuting matrices for which the Hilbert–Schmidt norm version of Conjecture 1.1 holds true when dropping the condition of positive semidefiniteness.

Bottazzi et al. [5] gave a counterexample to the following special case of the inequality (1.7):

$$|||A^{\mu}B^{1-\mu} + B^{\mu}A^{1-\mu}||| \leqslant |||A^{\mu}B^{1-\mu} + A^{1-\mu}B^{\mu}|||, \tag{1.14}$$

where $0 \leqslant \mu \leqslant 1$. They answered it in the negative for just the spectral (or the usual operator) norm by giving a pair of positive semidefinite matrices such that the claim does not hold. Though the inequality (1.14) is not true for the spectral norm, in view of the inequality (1.10), which is valid under the condition (1.11), it would be interesting to discuss the inequality (1.14) for other unitarily invariant norms like the Schatten p-norms.

2. MAIN RESULTS

We introduce some notations regarding vertical strips in the complex plane. Let

$$\mathcal{S} = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant 1\}, \quad \mathcal{S}_1 = \left\{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant \frac{1}{2}\right\}, \quad \text{and}$$
$$\mathcal{S}_2 = \left\{z \in \mathbb{C} : \frac{1}{2} \leqslant \operatorname{Re} z \leqslant 1\right\}.$$

For $0 \leqslant s \leqslant 1$, we define the following strips:

$$\mathcal{A}_s = \left\{z \in \mathbb{C} : \frac{1}{2} - s \leqslant \operatorname{Re} z \leqslant \frac{1}{2} + s\right\} = \left\{z \in \mathbb{C} : \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant s\right\} \quad \text{and}$$
$$\mathcal{B}_s = \left\{z \in \mathbb{C} : s - \frac{1}{2} \leqslant \operatorname{Re} z \leqslant \frac{3}{2} - s\right\} = \left\{z \in \mathbb{C} : \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant 1 - s\right\}.$$

Recall that a norm $|||\cdot|||$ on the space of all complex square matrices of a fixed order is called unitarily invariant if $|||UXV||| = |||X|||$ for all X and for all unitary matrices U, V. An important example of unitarily invariant norms is the Schatten p-norm, denoted by $\|\cdot\|_p$ and defined for $1 \leqslant p \leqslant \infty$ as

$$\|X\|_p = (\operatorname{tr}|X|^p)^{1/p},$$

where $|X| = (X^*X)^{1/2}$. The values $p = 1,\ p = 2$, and $p = \infty$ correspond to the trace norm, the Hilbert–Schmidt norm, and the spectral norm, respectively.

A basic property of unitarily invariant norms says that for any matrices X, Y, Z, we have

$$|||XYZ||| \leqslant \|X\|_\infty |||Y||| \|Z\|_\infty \tag{2.1}$$

(see, e.g., p. 94 of [4]).

The generalized Hölder inequality for the Schatten p-norm will be frequently used in proving our main results. This inequality says that for any matrices X, Y, Z and any real numbers $p, q, r \geqslant 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, we have

$$|\operatorname{tr}(XYZ)| \leqslant \|XYZ\|_1 \leqslant \|X\|_p \|Y\|_q \|Z\|_r. \tag{2.2}$$

For more details about the inequality (2.2), see Theorem 2.8 of [12]. The following lemma, taken from [5], is an immediate consequence of the famous Araki–Lieb–Thirring trace inequality. For more information about this inequality, we refer to [2].

LEMMA 2.1. *Let A, B be positive semidefinite matrices, and let $r \geqslant 2$. Then*

$$\|A^{1/r}B^{1/r}\|_r \leqslant (\operatorname{tr}(AB))^{1/r}.$$

The following lemma can be found in [10].

LEMMA 2.2. *Let A, B be positive semidefinite matrices, and let X be any matrix. Then, for $0 \leqslant \nu \leqslant 1$ and for every unitarily invariant norm,*

$$|||A^{\nu} X B^{1-\nu}||| \leqslant |||AX|||^{\nu} |||XB|||^{1-\nu}.$$

Another useful lemma for our purpose is the following.

LEMMA 2.3. *Let A, B be positive semidefinite matrices, and let $w, z \in \mathbb{C}$ with $s = \operatorname{Re} w$ and $r = \operatorname{Re} z$. If $0 \leqslant r, s \leqslant 1$, then*

$$|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \min\{\|A\|_{\infty}\|B\|_{1}, \|A\|_{1}\|B\|_{\infty}\}.$$

Proof. Without loss of generality, we may assume that A, B are invertible. The general case follows by a continuity argument.

Let $w = s + \mathrm{i}x$ and $z = r + \mathrm{i}y$ with $0 \leqslant r, s \leqslant 1$ and $x, y \in \mathbb{R}$. Then

$$\begin{aligned}
&|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \\
&\quad \leqslant \|A^{w} B^{z} A^{1-w} B^{1-z}\|_{1} \quad \text{(by the inequality (2.2))} \\
&\quad \leqslant \|A^{w}\|_{\infty}\|B^{z} A^{1-w} B^{1-z}\|_{1} \quad \text{(by the inequality (2.1))} \\
&\quad = \|A^{s}\|_{\infty}\|B^{r} A^{1-w} B^{1-r}\|_{1} \quad (\text{since } A^{\mathrm{i}x} \text{ and } B^{\mathrm{i}y} \text{ are unitary}) \\
&\quad \leqslant \|A\|_{\infty}^{s}\|BA^{1-w}\|_{1}^{r}\|A^{1-w}B\|_{1}^{1-r} \quad \text{(by Lemma 2.2)} \\
&\quad = \|A\|_{\infty}^{s}\|BA^{1-s}\|_{1}^{r}\|A^{1-s}B\|_{1}^{1-r} \\
&\quad \leqslant \|A\|_{\infty}^{s}\|B\|_{1}^{r}\|A^{1-s}\|_{\infty}^{r}\|A^{1-s}\|_{\infty}^{1-r}\|B\|_{1}^{1-r} \quad \text{(by the inequality (2.1))} \\
&\quad = \|A\|_{\infty}^{s}\|B\|_{1}^{r}\|A\|_{\infty}^{(1-s)r}\|A\|_{\infty}^{(1-s)(1-r)}\|B\|_{1}^{1-r} = \|A\|_{\infty}\|B\|_{1}.
\end{aligned}$$

Similarly, it can be shown that

$$|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \|A\|_{1}\|B\|_{\infty}.$$

Thus, $|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \min\{\|A\|_{\infty}\|B\|_{1}, \|A\|_{1}\|B\|_{\infty}\}$. ∎

Now, we are ready to state our first main result. In the proof of this result, we use the Hadamard three lines theorem (see, e.g., p. 33 of [11] or p. 387 of [13]).

THEOREM 2.4. *Let A, B be positive semidefinite matrices, and let $w, z \in \mathbb{C}$ with $s = \operatorname{Re} w$. Then the following hold:*

(i) *If $w \in \mathcal{S}_1$ and $z \in \mathcal{A}_s$, then*

$$|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \operatorname{tr}(AB).$$

(ii) *If $w \in \mathcal{S}_2$ and $z \in \mathcal{B}_s$, then*

$$|\operatorname{tr}(A^{w} B^{z} A^{1-w} B^{1-z})| \leqslant \operatorname{tr}(AB).$$

Proof. Without loss of generality, we may assume that A, B are invertible. The general case follows by a continuity argument. To prove part (i), let $w = s + \mathrm{i}x \in \mathcal{S}_1, x \in \mathbb{R}$. Then $0 \leqslant s \leqslant \frac{1}{2}$.

For $z = \frac{1}{2} - s + \mathrm{i}y$, using the inequality (2.2) for the partition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = s + (\frac{1}{2} - s) + \frac{1}{2} = 1$ and noting that with $p = \frac{1}{s}, q = \frac{1}{1/2-s}, r = 2 \geqslant 2$, we have

$$
\begin{aligned}
&|\operatorname{tr}(A^w B^z A^{1-w} B^{1-z})| \\
&\quad = |\operatorname{tr}(A^{s+\mathrm{i}x} B^{1/2-s+\mathrm{i}y} A^{1-s-\mathrm{i}x} B^{1/2+s-\mathrm{i}y})| \\
&\quad = |\operatorname{tr}(A^s A^{\mathrm{i}x} B^{\mathrm{i}y} B^{1/2-s} A^{1/2-s} A^{-\mathrm{i}x} A^{1/2} B^{1/2} B^{-\mathrm{i}y} B^s)| \\
&\quad = |\operatorname{tr}(B^s A^s A^{\mathrm{i}x} B^{\mathrm{i}y} B^{1/2-s} A^{1/2-s} A^{-\mathrm{i}x} A^{1/2} B^{1/2} B^{-\mathrm{i}y})| \\
&\quad \leqslant \|B^s A^s A^{\mathrm{i}x} B^{\mathrm{i}y} B^{1/2-s} A^{1/2-s} A^{-\mathrm{i}x} A^{1/2} B^{1/2} B^{-\mathrm{i}y}\|_1 \\
&\quad \leqslant \|B^s A^s A^{\mathrm{i}x}\|_p \|B^{\mathrm{i}y} B^{1/2-s} A^{1/2-s} A^{-\mathrm{i}x}\|_q \|A^{1/2} B^{1/2} B^{-\mathrm{i}y}\|_r \\
&\quad = \|B^s A^s\|_p \|B^{1/2-s} A^{1/2-s}\|_q \|A^{1/2} B^{1/2}\|_r \quad (\text{since } A^{\mathrm{i}x} \text{ and } B^{\mathrm{i}y} \text{ are unitary}) \\
&\quad \leqslant (\operatorname{tr}(AB))^{s+(1/2-s)+1/2} \quad (\text{by Lemma 2.1}) \\
&\quad = \operatorname{tr}(AB).
\end{aligned}
$$

For $z = \frac{1}{2} + s + \mathrm{i}y$, using the inequality (2.2) for the partition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = s + \frac{1}{2} + (\frac{1}{2} - s) = 1$ and noting that with $p = \frac{1}{s}, q = 2, r = \frac{1}{1/2-s} \geqslant 2$, we have

$$
\begin{aligned}
&|\operatorname{tr}(A^w B^z A^{1-w} B^{1-z})| \\
&\quad = |\operatorname{tr}(A^{s+\mathrm{i}x} B^{1/2+s+\mathrm{i}y} A^{1-s-\mathrm{i}x} B^{1/2-s-\mathrm{i}y})| \\
&\quad = |\operatorname{tr}(A^{\mathrm{i}x} A^s B^s B^{\mathrm{i}y} B^{1/2} A^{1/2} A^{-\mathrm{i}x} A^{1/2-s} B^{1/2-s} B^{-\mathrm{i}y})| \\
&\quad \leqslant \|A^{\mathrm{i}x} A^s B^s B^{\mathrm{i}y} B^{1/2} A^{1/2} A^{-\mathrm{i}x} A^{1/2-s} B^{1/2-s} B^{-\mathrm{i}y}\|_1 \\
&\quad \leqslant \|A^{\mathrm{i}x} A^s B^s B^{\mathrm{i}y}\|_p \|B^{1/2} A^{1/2} A^{-\mathrm{i}x}\|_q \|A^{1/2-s} B^{1/2-s} B^{-\mathrm{i}y}\|_r \\
&\quad = \|A^s B^s\|_p \|B^{1/2} A^{1/2}\|_q \|A^{1/2-s} B^{1/2-s}\|_r \quad (\text{since } A^{\mathrm{i}x} \text{ and } B^{\mathrm{i}y} \text{ are unitary}) \\
&\quad \leqslant (\operatorname{tr}(AB))^{s+1/2+(1/2-s)} \quad (\text{by Lemma 2.1}) \\
&\quad = \operatorname{tr}(AB).
\end{aligned}
$$

Since for a positive invertible matrix X, the matrix-valued function $\psi(z) = X^z = \exp(z \ln X) = \sum_{k=0}^{\infty} \frac{z^k (\ln X)^k}{k!}$ is entire, it follows that the complex-valued function

$$\varphi(z) = \operatorname{tr}(A^w B^z A^{1-w} B^{1-z})$$

is entire. Moreover, by Lemma 2.3, the function is bounded on the vertical strip $0 \leqslant \operatorname{Re} z \leqslant 1$. Since $[\frac{1}{2} - s, \frac{1}{2} + s] \subseteq [0, 1]$, it follows that the function is bounded on the vertical strip $\frac{1}{2} - s \leqslant \operatorname{Re} z \leqslant \frac{1}{2} + s$. Since $|\varphi(z)| \leqslant \operatorname{tr}(AB)$ on the edges $\operatorname{Re} z = \frac{1}{2} - s$ and $\operatorname{Re} z = \frac{1}{2} + s$, it follows by the Hadamard three lines theorem that $|\varphi(z)| \leqslant \operatorname{tr}(AB)$ for all z in the vertical strip $\frac{1}{2} - s \leqslant \operatorname{Re} z \leqslant \frac{1}{2} + s$. This proves part (i).

To prove part (ii), let $w = s + \mathrm{i}x \in \mathcal{S}_2, x \in \mathbb{R}$. Then $\frac{1}{2} \leqslant s \leqslant 1$.

For $z = s - \frac{1}{2} + \mathrm{i}y$, using the inequality (2.2) for the partition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = (s - \frac{1}{2}) + (1-s) + \frac{1}{2} = 1$ and noting that with $p = \frac{1}{s-1/2}, q = \frac{1}{1-s}, r = 2 \geqslant 2$, we have

$$\begin{aligned}
|\mathrm{tr}(A^w B^z A^{1-w} B^{1-z})| &= |\,\mathrm{tr}(A^{s+\mathrm{i}x} B^{s-1/2+\mathrm{i}y} A^{1-s-\mathrm{i}x} B^{3/2-s-\mathrm{i}y})| \\
&= |\,\mathrm{tr}(A^s A^{\mathrm{i}x} B^{s-1/2} B^{\mathrm{i}y} A^{1-s} A^{-\mathrm{i}x} B^{3/2-s} B^{-\mathrm{i}y})| \\
&= |\,\mathrm{tr}(A^{\mathrm{i}x} A^{s-1/2} B^{s-1/2} B^{\mathrm{i}y} A^{-\mathrm{i}x} A^{1-s} B^{1-s} B^{-\mathrm{i}y} B^{1/2} A^{1/2})| \\
&\leqslant \|A^{\mathrm{i}x} A^{s-1/2} B^{s-1/2} B^{\mathrm{i}y} A^{-\mathrm{i}x} A^{1-s} B^{1-s} B^{-\mathrm{i}y} B^{1/2} A^{1/2}\|_1 \\
&\leqslant \|A^{\mathrm{i}x} A^{s-1/2} B^{s-1/2} B^{\mathrm{i}y}\|_p \|A^{-\mathrm{i}x} A^{1-s} B^{1-s} B^{-\mathrm{i}y}\|_q \|B^{1/2} A^{1/2}\|_r \\
&= \|A^{s-1/2} B^{s-1/2}\|_p \|A^{1-s} B^{1-s}\|_q \|B^{1/2} A^{1/2}\|_r \\
&\leqslant (\mathrm{tr}(AB))^{(s-1/2)+(1-s)+\frac{1}{2}} \quad \text{(by Lemma 2.1)} \\
&= \mathrm{tr}(AB).
\end{aligned}$$

For $z = \frac{3}{2} - s + \mathrm{i}y$, using the inequality (2.2) for the partition $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2} + (1-s) + (s - \frac{1}{2}) = 1$ and noting that with $p = 2, q = \frac{1}{1-s}, r = \frac{1}{s-1/2} \geqslant 2$, we have

$$\begin{aligned}
&|\,\mathrm{tr}(A^w B^z A^{1-w} B^{1-z})| \\
&\quad = |\,\mathrm{tr}(A^{s+\mathrm{i}x} B^{3/2-s+\mathrm{i}y} A^{1-s-\mathrm{i}x} B^{-1/2+s-\mathrm{i}y})| \\
&\quad = |\,\mathrm{tr}(A^s A^{\mathrm{i}x} B^{3/2-s} B^{\mathrm{i}y} A^{1-s} A^{-\mathrm{i}x} B^{s-1/2} B^{-\mathrm{i}y})| \\
&\quad = |\,\mathrm{tr}(A^{\mathrm{i}x} A^{1/2} B^{1/2} B^{\mathrm{i}y} B^{1-s} A^{1-s} A^{-\mathrm{i}x} B^{-\mathrm{i}y} B^{s-1/2} A^{s-1/2})| \\
&\quad \leqslant \|A^{\mathrm{i}x} A^{1/2} B^{1/2} B^{\mathrm{i}y} B^{1-s} A^{1-s} A^{-\mathrm{i}x} B^{-\mathrm{i}y} B^{s-1/2} A^{s-1/2}\|_1 \\
&\quad \leqslant \|A^{\mathrm{i}x} A^{1/2} B^{1/2} B^{\mathrm{i}y}\|_p \|B^{1-s} A^{1-s} A^{-\mathrm{i}x}\|_q \|B^{-\mathrm{i}y} B^{s-1/2} A^{s-1/2}\|_r \\
&\quad = \|A^{1/2} B^{1/2}\|_p \|B^{1-s} A^{1-s}\|_q \|B^{s-1/2} A^{s-1/2}\|_r \quad \text{(since } A^{\mathrm{i}x} \text{ and } B^{\mathrm{i}y} \text{ are unitary)} \\
&\quad \leqslant (\mathrm{tr}(AB))^{1/2+(1-s)+(s-1/2)} \quad \text{(by Lemma 2.1)} \\
&\quad = \mathrm{tr}(AB).
\end{aligned}$$

By Lemma 2.3, the function

$$\varphi(z) = \mathrm{tr}(A^w B^z A^{1-w} B^{1-z})$$

is bounded on the vertical strip $0 \leqslant \mathrm{Re}\, z \leqslant 1$. Since $[s - \frac{1}{2}, \frac{3}{2} - s] \subseteq [0, 1]$, it follows that the function is bounded on the vertical strip $s - \frac{1}{2} \leqslant \mathrm{Re}\, z \leqslant \frac{3}{2} - s$. Since $|\varphi(z)| \leqslant \mathrm{tr}(AB)$ on the edges $\mathrm{Re}\, z = s - \frac{1}{2}$ and $\mathrm{Re}\, z = \frac{3}{2} - s$, it follows by the Hadamard three lines theorem that $|\varphi(z)| \leqslant \mathrm{tr}(AB)$ for all z in the vertical strip $s - \frac{1}{2} \leqslant \mathrm{Re}\, z \leqslant \frac{3}{2} - s$. This proves part (ii). ∎

To understand the conditions related to the strips given in Theorem 2.4 in a more clear way, we need the following lemma.

LEMMA 2.5. *Let w,z be complex numbers with $s = \operatorname{Re} w$. Then*

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} \Longleftrightarrow (w \in \mathcal{S}_1 \text{ and } z \in \mathcal{A}_s) \text{ or } (w \in \mathcal{S}_2 \text{ and } z \in \mathcal{B}_s).$$

Proof. ($\Longrightarrow$) Suppose that $|\operatorname{Re} w - \frac{1}{2}| + |\operatorname{Re} z - \frac{1}{2}| \leqslant \frac{1}{2}$. Then $|\operatorname{Re} w - \frac{1}{2}| \leqslant \frac{1}{2}$. Therefore, $0 \leqslant \operatorname{Re} w \leqslant 1$, and so we have the following two cases.

Case 1. If $w \in \mathcal{S}_1$, then

$$\begin{aligned}
\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} &\Longleftrightarrow \frac{1}{2} - \operatorname{Re} w + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} \\
&\Longleftrightarrow \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \operatorname{Re} w \\
&\Longleftrightarrow z \in \mathcal{A}_s.
\end{aligned}$$

Case 2. If $w \in \mathcal{S}_2$, then

$$\begin{aligned}
\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} &\Longleftrightarrow \operatorname{Re} w - \frac{1}{2} + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} \\
&\Longleftrightarrow \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant 1 - \operatorname{Re} w \\
&\Longleftrightarrow z \in \mathcal{B}_s.
\end{aligned}$$

($\Longleftarrow$) *Case* 1. If $w \in \mathcal{S}_1$ and $z \in \mathcal{A}_s$, then

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| = \frac{1}{2} - \operatorname{Re} w + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2} - \operatorname{Re} w + \operatorname{Re} w = \frac{1}{2}.$$

Case 2. If $w \in \mathcal{S}_2$ and $z \in \mathcal{B}_s$, then

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| = \operatorname{Re} w - \frac{1}{2} + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \operatorname{Re} w - \frac{1}{2} + 1 - \operatorname{Re} w = \frac{1}{2}.$$

This completes the proof. ∎

Now, using Lemma 2.5 and Theorem 2.4, we have the following corollary, which is a generalization of the Ando–Hiai–Okubo trace inequality [1].

COROLLARY 2.6. *Let A,B be positive semidefinite matrices, and let w,z be complex numbers such that*

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2}.$$

Then

$$|\operatorname{tr}(A^w B^z A^{1-w} B^{1-z})| \leqslant \operatorname{tr}(AB).$$

As an application of Corollary 2.6, we obtain our second main result, which is a generalization of the norm inequality (1.9), under the condition (1.2), to complex values.

THEOREM 2.7. *Let A, B be positive semidefinite matrices, and let w, z be complex numbers such that*

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2}. \tag{2.3}$$

Then

$$\|A^w B^z + B^{1-\bar{z}} A^{1-\overline{w}}\|_2 \leqslant \|A^w B^z + A^{1-\overline{w}} B^{1-\bar{z}}\|_2. \tag{2.4}$$

Proof. We can see that the square of the left-hand side of the inequality (2.4) is equal to

$$\operatorname{tr}\big(A^{w+\overline{w}} B^{z+\bar{z}} + B^{\bar{z}} A^{\overline{w}} B^{1-\bar{z}} A^{1-\overline{w}} + A^{1-w} B^{1-z} A^w B^z + A^{2-(w+\overline{w})} B^{2-(z+\bar{z})}\big)$$

and the square of the right-hand side is equal to

$$\operatorname{tr}\big(A^{w+\overline{w}} B^{z+\bar{z}} + 2AB + A^{2-(w+\overline{w})} B^{2-(z+\bar{z})}\big).$$

Here, we have used the fact that for all matrices X, Y, $\|X\|_2 = (\operatorname{tr} X^* X)^{1/2}$ and the cyclicity of the trace, i.e., $\operatorname{tr} XY = \operatorname{tr} YX$.

Therefore, the inequality (2.4) is equivalent to the statement

$$\operatorname{Re}\ \operatorname{tr}(A^w B^z A^{1-w} B^{1-z}) \leqslant \operatorname{tr}(AB). \tag{2.5}$$

By Corollary 2.6 and the fact that for every matrix X, $\operatorname{Re} \operatorname{tr} X \leqslant |\operatorname{tr} X|$, the inequality (2.5) holds provided

$$\left|\operatorname{Re} w - \frac{1}{2}\right| + \left|\operatorname{Re} z - \frac{1}{2}\right| \leqslant \frac{1}{2}.$$

Hence, the inequality (2.4) is valid under the condition (2.3). ∎

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