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COMPACT MULTIPLICATION OPERATORS ON NEST ALGEBRAS

G. ANDREOLAS and M. ANOUSSIS

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ABSTRACT. Let $\mathcal{N}$ be a nest on a Hilbert space H and $\mathrm{Alg}\,\mathcal{N}$ the corresponding nest algebra. We obtain a characterization of the compact and weakly compact multiplication operators defined on nest algebras. This characterization leads to a description of the closed ideal generated by the compact elements of $\mathrm{Alg}\,\mathcal{N}$. We also show that there is no non-zero weakly compact multiplication operator on $\mathrm{Alg}\,\mathcal{N}/\mathrm{Alg}\,\mathcal{N}\cap\mathcal{K}(H)$.

KEYWORDS: *Nest algebra, compact multiplication operators, elementary operator, weakly compact, Calkin algebra, Jacobson radical.*

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INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra. A *multiplication operator* $M_{a,b} : \mathcal{A} \to \mathcal{A}$ corresponding to $a, b \in \mathcal{A}$ is given by $M_{a,b}(x) = axb$. Properties of compact multiplication operators have been investigated since 1964 when Vala published his work "On compact sets of compact operators" [17]. Let $\mathcal{X}$ be a normed space and $\mathcal{B}(\mathcal{X})$ the space of all bounded linear maps from $\mathcal{X}$ into $\mathcal{X}$. Vala proved that a non-zero multiplication operator $M_{a,b} : \mathcal{B}(\mathcal{X}) \to \mathcal{B}(\mathcal{X})$ is compact if and only if the operators $a \in \mathcal{B}(\mathcal{X})$ and $b \in \mathcal{B}(\mathcal{X})$ are both compact.

This concept was further investigated by Ylinen in [19] who proved a similar result for abstract C^*-algebras. An element a of a Banach algebra $\mathcal{A}$ is called *compact* if the multiplication operator $M_{a,a} : \mathcal{A} \to \mathcal{A}$ is compact. Ylinen shows that there exists an isometric $*$-representation π of a C^*-algebra $\mathcal{A}$ on a Hilbert space H such that the operator $\pi(a)$ is compact if and only if a is a compact element of $\mathcal{A}$. Ylinen showed in [20] that this is equivalent with the weak compactness of the map $\lambda_a : \mathcal{A} \to \mathcal{A}$, $\lambda_a(x) = ax$ (or equivalently of the map $\rho_a : \mathcal{A} \to \mathcal{A}$, $\rho_a(x) = xa$).

In the sequel, these results were generalized to various directions. Let H be a Hilbert space. Akemann and Wright showed in [1] that a multiplication operator

$M_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ is weakly compact if and only if either a or b is a compact operator. A map $\Phi : \mathcal{A} \to \mathcal{A}$ is called *elementary* if $\Phi = \sum_{i=1}^{m} M_{a_i,b_i}$ for some $a_i, b_i \in \mathcal{A}$, $i = 1, \dots, m$. Fong and Sourour showed that an elementary operator $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is compact if and only if there exist compact operators $c_i, d_i \in \mathcal{B}(H)$, $i = 1, \dots, m$ such that $\Phi = \sum_{i=1}^{m} M_{c_i,d_i}$ [8]. This result was expanded by Mathieu on prime C^*-algebras [11] and later on general C^*-algebras by Timoney [16]. In [11] Mathieu characterizes the weakly compact elementary operators on prime C^*-algebras as well.

From the description of the compact elementary operators by Fong and Sourour, the following conjecture arose: *If Φ is a compact elementary operator on the Calkin algebra on a separable Hilbert space, then $\Phi = 0$.* This conjecture was confirmed in [4] by Apostol and Fialkow and by Magajna in [10]. In [11] Mathieu proves that if Φ is weakly compact, then $\Phi = 0$ as well.

The weak compactness of multiplication operators has been studied in a Banach space setting by Saskmann–Tylli and Johnson–Schechtman in [15] and [9] respectively. In [15] the authors give some sufficient conditions for weak compactness of $M_{a,b} : \mathcal{B}(E) \to \mathcal{B}(E)$, where E is a Banach space. They also provide necessary and sufficient conditions for weak compactness of $M_{a,b}$ in case of some concrete Banach spaces. In [9] the authors give a classification of weakly compact multiplication operators on $\mathcal{B}(L_p(0,1))$, $1 < p < \infty$, which in particular answers a question raised in [15].

The present work is a study of the compactness properties of multiplication operators defined on nest algebras. Note that the compactness of the inner derivations defined on nest algebras, that is a special class of elementary operators, have been studied by Peligrad in [13]. He characterized the weakly compact derivations of a nest algebra and obtained necessary and sufficient conditions so that a nest algebra admits compact derivations. If $\mathcal{N}$ is a nest, we denote by $\operatorname{Alg}\mathcal{N}$ the corresponding nest algebra. In the first section of the paper, we prove a necessary and sufficient condition for the compactness of multiplication operators defined from $\operatorname{Alg}\mathcal{N}$ into $\operatorname{Alg}\mathcal{N}$. We close the section, showing by example that there exist compact multiplication operators on $\operatorname{Alg}\mathcal{N}$ that can not be written as multiplication operators with compact symbols. In the second section, we determine the closed ideal generated by the compact elements of a nest algebra. In the third section of the paper, we characterize the weakly compact multiplication operators defined on nest algebras. In the last section, we show that there are no non-zero weakly compact multiplication operators on $\operatorname{Alg}\mathcal{N}/\operatorname{Alg}\mathcal{N} \cap \mathcal{K}(H)$ exactly as in the case of Calkin algebra (i.e. when $\mathcal{N} = \{0, H\}$) [4], [10], [11].

Let us introduce some notation and definitions that will be used throughout the paper. If H is a Hilbert space, then $\mathcal{B}(H)$ is the space of all bounded linear operators and $\mathcal{K}(H)$ the space of all compact operators from H into H. Let $\mathcal{E}$ be a Banach space and r a positive number. Then, by $\mathcal{E}_r$ we denote the closed ball of centre 0 and radius r. Let e, f be elements of a Hilbert space H. We denote by $e \otimes f$ the rank one operator on H defined by $(e \otimes f)(h) = \langle h, e \rangle f$.

Nest algebras form a class of non-selfadjoint operator algebras that generalize the block upper triangular matrices to an infinite dimensional Hilbert space context. They were introduced by Ringrose in [14] and since then, they have been studied by many authors. The monograph of Davidson [5] is recommended as a reference. A nest $\mathcal{N}$ is a totally ordered family of closed subspaces of a Hilbert space H containing $\{0\}$ and H, which is closed under intersection and closed span. If H is a Hilbert space and $\mathcal{N}$ a nest on H, then the nest algebra $\operatorname{Alg}\mathcal{N}$ is the algebra of all operators T such that $T(N) \subseteq N$ for all $N \in \mathcal{N}$. If $(N_\lambda)_{\lambda \in \Lambda}$ is a family of subspaces of a Hilbert space, we denote by $\bigvee\{N_\lambda : \lambda \in \Lambda\}$ their closed linear span and by $\bigwedge\{N_\lambda : \lambda \in \Lambda\}$ their intersection. If $\mathcal{N}$ is a nest and $N \in \mathcal{N}$, then $N_- = \bigvee\{N' \in \mathcal{N} : N' < N\}$. Similarly we define $N_+ = \bigwedge\{N' \in \mathcal{N} : N' > N\}$. The subspaces $N \cap N_-^\perp$ are called the *atoms* of $\mathcal{N}$. For any $N \in \mathcal{N}$, we denote by P_N the orthonormal projection corresponding to N. We endow $\mathcal{N}$ with the order topology and $\{P_N : N \in \mathcal{N}\}$ with the strong operator topology and denote these spaces by $(\mathcal{N}, <)$ and $(P_\mathcal{N}, \mathrm{SOT})$ respectively. The natural map taking N to P_N is an order preserving homeomorphism of the compact Hausdorff space $(\mathcal{N}, <)$ onto $(P_\mathcal{N}, \mathrm{SOT})$, ([5], Theorem 2.13). We shall identify the subspaces of a nest with the corresponding orthogonal projections. In this paper we do not distinguish between these subspaces and projections. We shall frequently use the fact that a rank one operator $e \otimes f$ belongs to a nest algebra, $\operatorname{Alg}\mathcal{N}$, if and only if there exists an element N of $\mathcal{N}$ such that $e \in N_-^\perp$ and $f \in N$, ([5], Lemmas 2.8 and 3.7). Note that the nest algebras are WOT-closed subalgebras of $\mathcal{B}(H)$ ([5], Proposition 2.2). Throughout the paper we denote by $\mathcal{N}$ a nest acting on a Hilbert space H and by $\mathcal{K}(\mathcal{N})$ the ideal of compact operators of $\operatorname{Alg}\mathcal{N}$.

1. COMPACT MULTIPLICATION OPERATORS

Let H be a Hilbert space and a, b elements of $\mathcal{B}(H)$. Vala proved in [17] that if $a, b \in \mathcal{B}(H) - \{0\}$, then the map $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$, $x \mapsto axb$ is compact if and only if the operators a and b are both compact. However, such a result does not hold for nest algebras. Let $\mathcal{N}$ be a nest containing a projection P such that $\dim(P) = \dim(P^\perp) = \infty$ and $a \in \operatorname{Alg}\mathcal{N}$ be a non-compact operator such that $a = PaP^\perp$. Then, the multiplication operator

$$\begin{aligned} M_{a,a} : \operatorname{Alg}\mathcal{N} &\to \operatorname{Alg}\mathcal{N}, \\ x &\mapsto axa \end{aligned}$$

coincides with the multiplication operator $M_{0,0}$, since

$$M_{a,a}(x) = axa = PaP^\perp xPaP^\perp = 0,$$

for $P^\perp xP = 0$.

Let $a, b \in \operatorname{Alg}\mathcal{N}$. We introduce the following projections:

$$R_a = \bigvee\{P \in \mathcal{N} : aP = 0\} \quad \text{and} \quad Q_b = \bigwedge\{P \in \mathcal{N} : P^{\perp}b = 0\}.$$

PROPOSITION 1.1. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. Then, $M_{a,b} = 0$ if and only if $Q_b \leqslant R_a$.*

Proof. We observe that if $Q_b \leqslant R_a$, then for all $x \in \operatorname{Alg}\mathcal{N}$,

$$M_{a,b}(x) = axb = aR_a^{\perp}xQ_b b = aR_a^{\perp}Q_b x Q_b b = 0,$$

since $a = aR_a^{\perp}$, $b = Q_b b$ and $R_a^{\perp}Q_b = 0$.

Now, suppose that $R_a < Q_b$. We distinguish two cases.

Case 1. There exists a projection $P \in \mathcal{N}$ such that $R_a < P < Q_b$. Then, there exist two norm one vectors $e \in P_-^{\perp}$ and $f \in P$ such that $a(f) \neq 0$ and $b^*(e) \neq 0$. It follows that, $M_{a,b}(e \otimes f) = a(e \otimes f)b = b^*(e) \otimes a(f) \neq 0$.

Case 2. There is not any projection of the nest between R_a and Q_b, i.e. $R_{a+} = Q_b$. Then, there exist two norm one vectors $e \in (R_{a+})_-^{\perp} = R_a^{\perp}$ and $f \in R_{a+} = Q_b$ such that $a(f) \neq 0$ and $b^*(e) \neq 0$. It follows that $M_{a,b}(e \otimes f) = a(e \otimes f)b = b^*(e) \otimes a(f) \neq 0$. ∎

The next theorem gives a necessary and sufficient condition for a non-zero multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $M_{a,b}(x) = axb$ to be compact.

THEOREM 1.2. *Let $a, b \in \operatorname{Alg}\mathcal{N}$ such that $M_{a,b} \neq 0$. The multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is compact if and only if the operators P_+aP_+ and $P_-^{\perp}bP_-^{\perp}$ are both compact for all $P \in \mathcal{N}$, $R_a < P < Q_b$ in the case that $R_{a+} \neq Q_b$ or the operators $Q_b a Q_b$ and $R_a^{\perp}bR_a^{\perp}$ are both compact in the case that $R_{a+} = Q_b$.*

Proof. Suppose that $M_{a,b}$ is a non-zero compact multiplication operator. From Proposition 1.1, it follows that $R_a < Q_b$. Let $R_{a+} \neq Q_b$. Then, for all $P \in \mathcal{N}$ such that $R_a < P < Q_b$, we see that $aP \neq 0$. Let $(e_n)_{n\in\mathbb{N}} \subseteq P_-^{\perp}$ be a bounded sequence and $f \in P$ such that $a(f) \neq 0$. The sequence $(M_{a,b}(e_n \otimes f))_{n\in\mathbb{N}} = (b^*(e_n) \otimes a(f))_{n\in\mathbb{N}}$ has a convergent subsequence and therefore the sequence $(b^*(e_n))_{n\in\mathbb{N}}$ has a convergent subsequence as well. Thus, the operator $b^*P_-^{\perp}$ is compact and equivalently the operator $P_-^{\perp}bP_-^{\perp}$ is compact. Notice that $(P_+)_-^{\perp}b \neq 0$ since $(P_+)_- \leqslant P < Q_b$. Let $(f_n)_{n\in\mathbb{N}} \subseteq P_+$ be a bounded sequence and $e \in (P_+)_-^{\perp}$ such that $b^*(e) \neq 0$. The sequence $(M_{a,b}(e \otimes f_n))_{n\in\mathbb{N}} = (b^*(e) \otimes a(f_n))_{n\in\mathbb{N}}$ has a convergent subsequence and therefore the sequence $(a(f_n))_{n\in\mathbb{N}}$ has a convergent subsequence as well. Thus, the operator $aP_+ = P_+aP_+$ is compact. Now, consider the case in which $R_{a+} = Q_b$. We see that $aQ_b \neq 0$. Let $(e_n)_{n\in\mathbb{N}} \subseteq Q_{b-}^{\perp} = R_a^{\perp}$ be a bounded sequence and $f \in Q_b$ such that $a(f) \neq 0$. The sequence $(M_{a,b}(e_n \otimes f))_{n\in\mathbb{N}} = (b^*(e_n) \otimes a(f))_{n\in\mathbb{N}}$ has a convergent subsequence and therefore the sequence $(b^*(e_n))_{n\in\mathbb{N}}$ has a convergent subsequence as well. Thus, the operator $b^*R_a^{\perp}$ is compact and equivalently the operator $R_a^{\perp}bR_a^{\perp}$ is compact. Notice that $R_a^{\perp}b \neq 0$. Let $(f_n)_{n\in\mathbb{N}} \subseteq Q_b$ be a bounded sequence and $e \in Q_{b-}^{\perp} = R_a^{\perp}$ such that $b^*(e) \neq 0$. The sequence $(M_{a,b}(e \otimes f_n))_{n\in\mathbb{N}} = (b^*(e) \otimes a(f_n))_{n\in\mathbb{N}}$ has a convergent subsequence and therefore the sequence $(a(f_n))_{n\in\mathbb{N}}$ has a convergent subsequence as well. Thus, the operator $aQ_b = Q_b a Q_b$ is compact.

Now, we prove the opposite direction. First, we suppose that $R_{a+} \neq Q_b$ and for all $P \in \mathcal{N}$ with $R_a < P < Q_b$, the operators $P_+ a P_+$ and $P_-^\perp b P_-^\perp$ are compact. The multiplication operator $M_{a,b}$ can be written as follows:

$$\begin{aligned} M_{a,b}(x) &= axb = aP_+xb + aP_+^\perp xb = aP_+xb + aP_+^\perp xP_-^\perp b + aP_+^\perp xP_- b \\ &= aP_+xb + aP_+^\perp xP_-^\perp b + aP_+^\perp P_- xP_- b = P_+aP_+xb + aP_+^\perp xP_-^\perp bP_-^\perp \\ &= M_{P_+aP_+,b}(x) + M_{aP_+^\perp, P_-^\perp bP_-^\perp}(x), \end{aligned}$$

since $aP_+^\perp P_- xP_- b = 0$. We only show that the multiplication operator $M_{P_+aP_+,b}$ is compact since the proof of the compactness of $M_{aP_+^\perp, P_-^\perp bP_-^\perp}$ is similar. We distinguish two cases.

Case 1. We suppose that $R_{a+} \neq R_a$. Let $S = R_{a+} > R_a$. Then, $R_a < S < Q_b$. Observing that $S_- = R_a$ it follows that $P_+aP_+ = P_+aP_+S_-^\perp$. For all $x \in \operatorname{Alg}\mathcal{N}$ it follows that

$$M_{P_+aP_+,b}(x) = P_+aP_+xb = P_+aP_+S_-^\perp xb = P_+aP_+xS_-^\perp bS_-^\perp = M_{P_+aP_+,S_-^\perp bS_-^\perp}(x).$$

Thus, the multiplication operator $M_{P_+aP_+,b} = M_{P_+aP_+,S_-^\perp bS_-^\perp}$ is compact since the operators P_+aP_+ and $S_-^\perp bS_-^\perp$ are both compact ([17], Theorem 3).

Case 2. Now, we suppose that $R_{a+} = R_a$. Then, there exists a net $(S_i)_{i\in I} \subseteq \mathcal{N}$ which is SOT-convergent to the projection R_a and for all $i \in I$ the inequality $R_a < S_i$ is satisfied ([5], Theorem 2.13). The compactness of the operator P_+aP_+ implies that the net $(P_+aP_+S_i)_{i\in I}$ converges to zero ([5], Proposition 1.18). It follows that for some $\varepsilon > 0$, we can choose a projection $S \in \mathcal{N}$, with $R_a < S < Q_b$ so that $\|P_+aP_+S_-\| < \varepsilon/\|b\|$. We write the multiplication operator $M_{P_+aP_+,b}$ as follows:

$$M_{P_+aP_+,b} = M_{P_+aP_+S_-^\perp,b} + M_{P_+aP_+S_-,b}.$$

Given that $\|M_{P_+aP_+S_-,b}\| < \varepsilon$, it suffices to show that the multiplication operator $M_{P_+aP_+S_-^\perp,b}$ is compact. For all $x \in \operatorname{Alg}\mathcal{N}$, we deduce that:

$$M_{P_+aP_+S_-^\perp,b}(x) = P_+aP_+S_-^\perp xb = P_+aP_+S_-^\perp xS_-^\perp bS_-^\perp = M_{P_+aP_+S_-^\perp,S_-^\perp bS_-^\perp}(x).$$

Therefore, the multiplication operator $M_{P_+aP_+S_-^\perp,S_-^\perp bS_-^\perp}$ is compact since the operators P_+aP_+ and $S_-^\perp bS_-^\perp$ are both compact.

Finally, we consider the case where $R_{a+} = Q_b$ and the operators Q_baQ_b and $R_a^\perp bR_a^\perp$ are both compact. Seeing that $a = aR_a^\perp$ and $b = Q_b b$, the multiplication operator $M_{a,b}$ can be written in the following form:

$$M_{a,b}(x) = axb = aR_a^\perp xQ_b b = Q_baQ_bR_a^\perp xQ_bR_a^\perp bR_a^\perp = M_{Q_baQ_bR_a^\perp, Q_bR_a^\perp bR_a^\perp}(x),$$

and therefore $M_{a,b} = M_{Q_baQ_bR_a^\perp, Q_bR_a^\perp bR_a^\perp}$ is a compact multiplication operator as the operators Q_baQ_b and $R_a^\perp bR_a^\perp$ are both compact. ∎

REMARK 1.3. Consider the nest $\mathcal{N} = \{\{0\}, H\}$ and let $a, b \in \operatorname{Alg}\mathcal{N} = \mathcal{B}(H)$ with $a, b \neq 0$. From Theorem 1.2 it follows that the multiplication operator $M_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ is compact if and only if the operators a and b are both compact. In that case the result coincides with Vala's theorem.

COROLLARY 1.4. *Let $a, b \in \operatorname{Alg}\mathcal{N}$ such that $M_{a,b} \neq 0$. Then, the multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is compact if and only if the multiplication operator $M_{a,b}|_{\mathcal{K}(\mathcal{N})} : \mathcal{K}(\mathcal{N}) \to \mathcal{K}(\mathcal{N})$ is compact.*

Proof. The forward direction is immediate. For the opposite direction we observe that the proof is the same as the proof of the forward direction of Theorem 1.2. Therefore, we deduce that the compactness of $M_{a,b}|_{\mathcal{K}(\mathcal{N})}$ is equivalent with the assertions of Theorem 1.2. ∎

COROLLARY 1.5. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of finite rank projections that increase to the identity and $\mathcal{N}$ the nest $\{P_n\}_{n=1}^{\infty} \cup \{\{0\}, H\}$. Let $a, b \in \operatorname{Alg}\mathcal{N}$ such that $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is a non-zero multiplication operator. Then, b is a compact operator if and only if $M_{a,b}$ is a compact multiplication operator. The set of compact elements of $\operatorname{Alg}\mathcal{N}$ is the ideal $\mathcal{K}(\mathcal{N})$.*

Let $\mathcal{A}$ be a C^*-algebra and Φ an elementary operator on $\mathcal{A}$. Timoney proved in Theorem 3.1 of [16] that Φ is compact if and only if Φ can be expressed as $\Phi(x) = \sum_{i=1}^{m} a_i x b_i$ for a_i and b_i compact elements of $\mathcal{A}$ ($1 \leqslant i \leqslant m$). The question that arises is whether a compact multiplication operator defined on a nest algebra can always be written as an elementary operator with compact symbols i.e., if $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is a compact multiplication operator, then are there an $l \in \mathbb{N}$ and compact operators $c_i, d_i \in \mathcal{B}(H)$, $i \in \{1, \ldots, l\}$, (where H is the underlying Hilbert space of the nest) such that $M_{a,b} = \sum_{i=1}^{l} M_{c_i,d_i}$? Another question is whether a compact multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ can be written as an elementary operator $\sum_{i=1}^{l} M_{c_i,d_i}$ such that the operators $c_i, d_i \in \operatorname{Alg}\mathcal{N}$ $i \in \{1, \ldots, l\}$ are compact elements of the nest algebra. The following example shows that both questions have a negative answer.

EXAMPLE 1.6. Let H be a Hilbert space, $\{e_i\}_{i\in\mathbb{N}}$ an orthonormal sequence of H, $\mathcal{N} = \{[\{e_i : i \in \mathbb{N}, i \leqslant n\}] : n \in \mathbb{N}\} \cup \{\{0\}, H\}$ and $b = \sum_{n\in\mathbb{N}} (1/n) e_n \otimes e_n$ a compact operator of $\operatorname{Alg}\mathcal{N}$. Then, the multiplication operator $M_{I,b}$ is compact (Corollary 1.5). We suppose that there exist compact operators $c_i, d_i \in \mathcal{B}(H)$, $i = 1, \ldots, l$ such that $M_{I,b} = \sum_{i=1}^{l} M_{c_i,d_i}$ and we shall arrive at a contradiction. We consider the following family of rank one operators,

$$\{x_{r,s}\}_{r\in\mathbb{N},\, s\in\mathbb{N}\cup\{0\},\, s<r} = \{e_r \otimes e_{r-s}\}_{r\in\mathbb{N},\, s\in\mathbb{N}\cup\{0\},\, s<r} \subseteq \operatorname{Alg}\mathcal{N}.$$

Then, $M_{I,b}(x_{r,s}) = \sum\limits_{i=1}^{l} M_{c_i,d_i}(x_{r,s})$ i.e.,

$$\sum_{n\in\mathbb{N}} \frac{1}{n} e_n \otimes x_{r,s}(e_n) = \sum_{i=1}^{l} c_i x_{r,s} d_i$$

or

$$\frac{1}{r} e_r \otimes e_{r-s} = \sum_{i=1}^{l} d_i^*(e_r) \otimes c_i(e_{r-s}). \tag{1.1}$$

The relation (1.1) implies that

$$\left\langle e_{r-s}, \frac{1}{r} e_r \otimes e_{r-s}(e_r) \right\rangle = \sum_{i=1}^{l} \langle e_{r-s}, d_i^*(e_r) \otimes c_i(e_{r-s})(e_r) \rangle$$

or

$$\frac{1}{r} = \sum_{i=1}^{l} \langle e_r, d_i^*(e_r) \rangle \langle e_{r-s}, c_i(e_{r-s}) \rangle. \tag{1.2}$$

For all $r \in \mathbb{N}$ and $i \in \{1, \ldots, l\}$, we set $D_{r,i} = \langle e_r, d_i^*(e_r) \rangle$ and $C_{r,i} = \langle e_r, c_i(e_r) \rangle$. We denote the vectors $(D_{r,1}, \ldots, D_{r,l}) \in \mathbb{C}^l$ and $(C_{r,1}, \ldots, C_{r,l}) \in \mathbb{C}^l$ by D_r and C_r respectively for all $r \in \mathbb{N}$. Now, we can write equation (1.2) in the form

$$\frac{1}{r} = \sum_{i=1}^{l} D_{r,i} C_{r-s,i}. \tag{1.3}$$

This implies

$$0 = \sum_{i=1}^{l} D_{r,i}(C_{r-s,i} - C_{1,i}). \tag{1.4}$$

The sequence $(\mathcal{V}_n)_{n\in\mathbb{N}} = (\mathrm{span}\{C_2 - C_1, \ldots, C_n - C_1\})_{n\in\mathbb{N}}$ of subspaces of $\mathbb{C}^l$ is increasing and therefore there exists an $n_0 \in \mathbb{N}$ such that $\mathcal{V}_{n_0} = \mathcal{V}_n$ for all $n \geqslant n_0$. Therefore, the following holds for all $n \in \mathbb{N}$.

$$0 = \sum_{i=1}^{l} D_{n_0,i}(C_{n,i} - C_{1,i}). \tag{1.5}$$

Since the operators c_i, $i = 1, \ldots, l$ are compact, the sequence $(C_n)_{n\in\mathbb{N}}$ converges to 0. Taking limits in equation (1.5) as $n \to \infty$ we obtain $0 = -1/n_0$ which is a contradiction.

2. THE IDEAL GENERATED BY THE COMPACT ELEMENTS

The set of compact elements of a nest algebra does not form an ideal in general. Let $\mathcal{N}$ be a continuous nest and $P, Q \in \mathcal{N} - \{0, I\}$. Then, from Proposition 1.1 we can easily see that there exist non-compact operators but compact elements a, b of $\operatorname{Alg}\mathcal{N}$ such that $a = PaP^{\perp}$, $b = QbQ^{\perp}$, while $M_{a+b,a+b}$ is non-compact.

The next proposition characterizes the nests for which the compact elements form an ideal.

PROPOSITION 2.1. *The set of compact elements of* $\operatorname{Alg}\mathcal{N}$ *is an ideal if and only if for all* $P, S \in \mathcal{N} - \{0, I\}$, *with* $P < S$, *the dimension of* $S - P$ *is finite. In that case, the set of compact elements of* $\operatorname{Alg}\mathcal{N}$ *is the ideal* $\mathcal{K}(\mathcal{N}) + Q\operatorname{Alg}\mathcal{N}Q^{\perp}$, *for some* $Q \in \mathcal{N} - \{0, I\}$.

Proof. Suppose that there exist $P, S \in \mathcal{N} - \{0, I\}$, with $P < S$, and $\dim(S - P) = \infty$. Let a, b be compact elements of $\operatorname{Alg}\mathcal{N}$ such that $R_a < Q_a \leqslant P < S \leqslant R_b < S_b$ and the operator $S_+ a S_+$ is not compact. We observe that $R_{a+b} = R_a$ and $Q_{a+b} = Q_b$. The operator $S_+ b S_+$ is compact while the operator $S_+ a S_+$ is not compact. It follows that the operator $S_+(a + b)S_+$ is not compact and therefore the element $a + b$ is non-compact since $R_{a+b} < S < Q_{a+b}$ (Theorem 1.2). Thus, the set of compact elements of $\operatorname{Alg}\mathcal{N}$ is not an ideal.

Now, suppose that for all $P, S \in \mathcal{N}$, with $P < S$, the dimension of $S - P$ is finite. Let a be a compact element of $\operatorname{Alg}\mathcal{N}$. Then, then exists a projection $R \in \mathcal{N} - \{0, I\}$ such that the operators RaR and $R^{\perp}aR^{\perp}$ are compact from Theorem 1.2. Let $Q \in \mathcal{N} - \{0, I\}$, with $Q > R$. Then, the operator $QaQ = aR + a(Q - R)$ is compact since $\dim(Q - R) < \infty$ and the operator $aR = RaR$ is compact. Similarly, we observe that the operator $Q^{\perp}aQ^{\perp}$ is compact since $Q > R$ and the operator $R^{\perp}a = R^{\perp}aR^{\perp}$ is compact. If $Q < R$, it is immediate that the operator QaQ is compact. The operator $Q^{\perp}aQ^{\perp} = R^{\perp}a + (Q^{\perp} - R^{\perp})a$ is compact as well, since $\dim(R^{\perp} - Q^{\perp}) = \dim(Q - R) < \infty$. It follows that the set of compact elements of $\operatorname{Alg}\mathcal{N}$ is the ideal $\mathcal{K}(\mathcal{N}) + Q\operatorname{Alg}\mathcal{N}Q^{\perp}$, for some $Q \in \mathcal{N} - \{0, I\}$. ∎

As we have seen, the set of compact elements of $\operatorname{Alg}\mathcal{N}$ does not form an ideal. However, the norm closed ideal generated by the compact elements of $\operatorname{Alg}\mathcal{N}$ has a nice description. Let $\mathbf{A}$ be the set of atoms of $\mathcal{N}$. The map $\Delta_{\mathcal{N}} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $x \mapsto \sum_{A_\alpha \in \mathbf{A}} A_\alpha x A_\alpha$ is a projection to the atomic part of the diagonal of $\operatorname{Alg}\mathcal{N}$. The Jacobson radical of $\operatorname{Alg}\mathcal{N}$ is denoted by $\operatorname{Rad}(\mathcal{N})$.

THEOREM 2.2. *The ideal* $\mathcal{J}_{\mathrm{c}}$, *generated by the compact elements of the nest algebra* $\operatorname{Alg}\mathcal{N}$, *is equal to* $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$.

Proof. From Theorem 11.6 of [5] it follows that the set $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$ is a closed ideal.

Let a be a compact element of $\operatorname{Alg}\mathcal{N}$. From Proposition 1.1 and Theorem 1.2 it follows that a is the sum of a compact operator and an operator of the form $PaP^{\perp}$ for some $P \in \mathcal{N}$. Therefore, the operator a belongs to the set $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N})$.

Now, we prove that $\mathcal{K}(\mathcal{N}) + \operatorname{Rad}(\mathcal{N}) \subseteq \mathcal{J}_{\mathrm{c}}$. It suffices to show that $\operatorname{Rad}(\mathcal{N}) \subseteq \mathcal{J}_{\mathrm{c}}$, since the compact operators of $\operatorname{Alg}\mathcal{N}$ are compact elements of the nest algebra. Let a be an element of $\operatorname{Rad}(\mathcal{N})$ and ε a strictly positive number. Then, there is a finite subnest $\mathcal{F} = \{0 = P_1 < P_2 < \cdots < P_n = I\}$ of $\mathcal{N}$ such that $\|\Delta_{\mathcal{F}}(a)\| < \varepsilon$ ([5], Theorem 6.7). We thus have:

$$\Delta_{\mathcal{F}}(a) + \sum_{i=2}^{n} P_i P_{i-1}^{\perp} a P_i^{\perp} = a.$$

It follows that a can be written as a sum of the compact elements of $\operatorname{Alg}\mathcal{N}$, $P_i P_{i-1}^{\perp} a P_i^{\perp}$, and an operator $\Delta_{\mathcal{F}}(a)$ of norm less than ε. Therefore, $a \in \mathcal{J}_{\mathrm{c}}$. ∎

COROLLARY 2.3. *If $\mathcal{N}$ is a continuous nest, then $\mathcal{J}_{\mathrm{c}} = \operatorname{Rad}(\mathcal{N})$.*

Proof. It is immediate from the fact that a compact operator $c \in \operatorname{Alg}\mathcal{N}$ belongs to the radical if and only if $\Delta_{\mathcal{N}}(c) = 0$ ([5], Corollary 6.9). ∎

3. WEAKLY COMPACT MULTIPLICATION OPERATORS

Akemann and Wright give a characterization of certain weakly compact maps on $\mathcal{B}(H)$ in Proposition 2.1 of [1]. We adjust that result to the case of nest algebras.

PROPOSITION 3.1. *Let $\varphi : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ be a bounded linear map which is* w*-*continuous and maps $\mathcal{K}(\mathcal{N})$ into $\mathcal{K}(\mathcal{N})$. Then, $\varphi = (\varphi|_{\mathcal{K}(\mathcal{N})})^{**}$ and φ is weakly compact if and only if $\varphi(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(\mathcal{N})$.*

Proof. The steps of the proof are very similar to those of Proposition 2.1 in [1]. Note that the dual space of $\mathcal{K}(\mathcal{N})$ is $\mathcal{L}^1(H)/\mathcal{A}_0$, where H is the the underlying Hilbert space of $\mathcal{N}$, $\mathcal{L}^1(H)$ the space of the trace class operators on H and $\mathcal{A}_0 = \{T \in \mathcal{L}^1(H) : P_-^{\perp} T P = 0,\ \forall P \in \mathcal{N}\}$. The second dual of $\mathcal{K}(\mathcal{N})$ is $\operatorname{Alg}\mathcal{N}$ ([5], Theorem 16.6). Note that $(\varphi|_{\mathcal{K}(\mathcal{N})})^{**} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is w*-continuous as a dual operator and it agrees with the w*-continuous map φ on the w*-dense set $\mathcal{K}(\mathcal{N}) \subseteq \operatorname{Alg}\mathcal{N}$, ([5], Corollary 3.13). Therefore $\varphi = (\varphi|_{\mathcal{K}(\mathcal{N})})^{**}$ since φ and $(\varphi|_{\mathcal{K}(\mathcal{N})})^{**}$ are w*-continuous and $\overline{\mathcal{K}(\mathcal{N})}^{\mathrm{w}^*} = \operatorname{Alg}\mathcal{N}$.

Now assume that φ is weakly compact. Then, $(\varphi|_{\mathcal{K}(\mathcal{N})})^{**} = \varphi$ is weakly compact, whence $\varphi|_{\mathcal{K}(\mathcal{N})}$ is weakly compact ([7], Theorem 8, p. 485). This implies that $\varphi(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(\mathcal{N})$, ([7], Theorem 2, p. 482).

Conversely, assume that $\varphi(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(\mathcal{N})$. The nest algebra $\operatorname{Alg}\mathcal{N}$ is w*-closed ([5], Proposition 2.2) and therefore the closed unit ball $(\operatorname{Alg}\mathcal{N})_1$ is w*-compact. By the w*-continuity of φ the set $\varphi((\operatorname{Alg}\mathcal{N})_1)$ is w*-compact. Therefore, the set $\varphi((\operatorname{Alg}\mathcal{N})_1) \subseteq \mathcal{K}(\mathcal{N})$ is weakly compact since the relative w*-topology of $\mathcal{K}(\mathcal{N})$ coincides with the weak topology on $\mathcal{K}(\mathcal{N})$. ∎

COROLLARY 3.2. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. Then, the multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact if and only if $M_{a,b}(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(\mathcal{N})$.*

LEMMA 3.3. *Let $a, b \in \operatorname{Alg}\mathcal{N}$ and $(e_n)_{\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ orthonormal sequences in H such that $e_n \otimes f_n \in \operatorname{Alg}\mathcal{N}$ for all $n \in \mathbb{N}$. If there exists an $\varepsilon > 0$ such that $\|a(f_n)\| \geqslant \varepsilon$ and $\|b^*(e_n)\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$, then there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ such that the operator $a\Big(\sum_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}\Big)b = \sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n}) \in \operatorname{Alg}\mathcal{N}$ is not compact and for any subsequence $(k_{n_m})_{m\in\mathbb{N}}$ the operator $\sum_{n\in\mathbb{N}} b^*(e_{k_{n_m}}) \otimes a(f_{k_{n_m}}) \in \operatorname{Alg}\mathcal{N}$ is non-compact as well.*

Proof. First, we construct the sequence $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ by induction. We set $k_1 = 1$. Suppose we have determined k_i for all $i \in \{2, \dots, n-1\}$ for some $n \in \mathbb{N}$. Then, we choose $k_n \in \mathbb{N}$ such that

$$|\langle a(f_{k_m}), a(f_{k_n})\rangle| = |\langle a^*a(f_{k_m}), f_{k_n}\rangle| < \frac{\varepsilon^2}{3\cdot 2^n} \quad \text{and} \tag{3.1}$$

$$|\langle b^*(e_{k_m}), b^*(e_{k_n})\rangle| = |\langle bb^*(e_{k_m}), e_{k_n}\rangle| < \frac{\varepsilon^2}{3\cdot 2^n} \tag{3.2}$$

for all $m \in \mathbb{N}$, with $m < n$.

We suppose that the operator $\sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n})$ is compact. Then, the operator $a^*\Big(\sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n})\Big)b^*$ is compact as well and therefore, there exists a $m_0 \in \mathbb{N}$ such that,

$$\begin{aligned} \frac{\varepsilon^4}{2} &> \Big|\Big\langle a^*\Big(\sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n})\Big)b^*(e_{k_{m_0}}), f_{k_{m_0}}\Big\rangle\Big| \\ &= \Big|\Big\langle \Big(\sum_{n\in\mathbb{N}} b^*(e_{k_n}) \otimes a(f_{k_n})\Big)b^*(e_{k_{m_0}}), a(f_{k_{m_0}})\Big\rangle\Big| \\ &= \Big|\sum_{n\in\mathbb{N}} \Big\langle b^*(e_{k_{m_0}}), b^*(e_{k_n})\Big\rangle \langle a(f_{k_n}), a(f_{k_{m_0}})\Big\rangle\Big|. \end{aligned} \tag{3.3}$$

For all $n \in \mathbb{N}$, we set

$$\lambda_n = \langle b^*(e_{k_{m_0}}), b^*(e_{k_n})\rangle \mu_n = \langle a(f_{k_n}), a(f_{k_{m_0}})\rangle.$$

Note that $\lambda_{m_0} = \langle b^*(e_{k_{m_0}}), b^*(e_{k_{m_0}})\rangle = \|b^*(e_{k_{m_0}})\|^2 \geqslant \varepsilon^2$ and similarly $\mu_{m_0} \geqslant \varepsilon^2$. From the inequalities (3.1) and (3.2) it follows that

$$|\lambda_n||\mu_n| < \Big(\frac{\varepsilon^2}{3\cdot 2^{m_0}}\Big)^2$$

for all $n < m_0$ and

$$|\lambda_n||\mu_n| < \Big(\frac{\varepsilon^2}{3\cdot 2^{n}}\Big)^2$$

for all $n > m_0$. Thus, the inequality (3.3) implies that

$$\frac{\varepsilon^4}{2} > \Big|\sum_{n\in\mathbb{N}} \lambda_n\mu_n\Big| \geqslant \lambda_{m_0}\mu_{m_0} - \Big|\sum_{n\neq m_0} \lambda_n\mu_n\Big| \geqslant \varepsilon^4 - \sum_{n\neq m_0} |\lambda_n\mu_n|$$

$$= \varepsilon^4 - \sum_{n<m_0} |\lambda_n \mu_n| - \sum_{n>m_0} |\lambda_n \mu_n| > \varepsilon^4 - (m_0 - 1)\Big(\frac{\varepsilon^2}{3 \cdot 2^{m_0}}\Big)^2 - \sum_{n>m_0} \Big(\frac{\varepsilon^2}{3 \cdot 2^n}\Big)^2$$
$$> \varepsilon^4 - \frac{\varepsilon^4}{9} \sum_{n\in\mathbb{N}} \frac{1}{2^n} > \varepsilon^4 - \frac{\varepsilon^4}{9} = \frac{8\varepsilon^4}{9},$$

which is a contradiction and therefore the operator $a\Big(\sum_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}\Big)b \in \operatorname{Alg}\mathcal{N}$ is not compact. It is obvious that we can follow the above steps of the proof for all subsequences $(k_{n_m})_{m\in\mathbb{N}}$ of $(k_n)_{n\in\mathbb{N}}$. Therefore, the operator $\sum_{n\in\mathbb{N}} b^*(e_{k_{n_m}}) \otimes a(f_{k_{n_m}}) \in \operatorname{Alg}\mathcal{N}$ is non-compact as well. ∎

The following lemma provides us with a sufficient condition for the weak compactness of a multiplication operator.

LEMMA 3.4. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. If there exists a projection $P \in \mathcal{N}$ such that the operators PaP and $P^\perp bP^\perp$ are both compact, then the multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact.*

Proof. Suppose that there exists a projection $P \in \mathcal{N}$ such that the operators PaP and $P^\perp bP^\perp$ are both compact. Let $x \in \operatorname{Alg}\mathcal{N}$. Then,

$$\begin{aligned} M_{a,b}(x) &= axb = (PaP + PaP^\perp + P^\perp aP^\perp)x(PbP + PbP^\perp + P^\perp bP^\perp) \\ &= PaPxb + (PaP^\perp + P^\perp aP^\perp)xP^\perp bP^\perp \\ &= M_{PaP,b}(x) + M_{(PaP^\perp + P^\perp aP^\perp), P^\perp bP^\perp}(x). \end{aligned}$$

It follows that the multiplication operators $M_{PaP,b}$ and $M_{(PaP^\perp+P^\perp aP^\perp),P^\perp bP^\perp}$ are weakly compact since the operators PaP and $P^\perp bP^\perp$ are both compact (Corollary 3.2). ∎

The next lemma gives a necessary condition for the weak compactness of a multiplication operator.

LEMMA 3.5. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. If the multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact, then for all $P \in \mathcal{N}$, either the operator PaP is compact or the operator $P^\perp bP^\perp$ is compact.*

Proof. Let $P \in \mathcal{N}$. It follows that the multiplication operator

$$M_{a,b} : P\operatorname{Alg}\mathcal{N}P^\perp \to P\operatorname{Alg}\mathcal{N}P^\perp$$

is weakly compact or equivalently the multiplication operator

$$M_{PaP,P^\perp bP^\perp} : \mathcal{B}(H) \to \mathcal{B}(H)$$

is weakly compact. Therefore, either the operator PaP is compact or the operator $P^\perp bP^\perp$ is compact ([1], Proposition 2.3). ∎

Now, we proceed to the main theorem of this section. To do so, we introduce the following projections:

$$U_a = \bigvee\{P \in \mathcal{N} : PaP \text{ is a compact operator}\} \quad \text{and}$$
$$L_b = \bigwedge\{P \in \mathcal{N} : P^{\perp}bP^{\perp} \text{ is a compact operator}\},$$

where $a, b \in \operatorname{Alg}\mathcal{N}$.

THEOREM 3.6. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. The multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$, $x \mapsto axb$ is weakly compact if and only if one of the following conditions is satisfied:*

(i) $U_a > L_b$.

(ii) $U_a = L_b = S$ *and the operators SaS and $S^{\perp}bS^{\perp}$ are both compact.*

(iii) $U_a = L_b = S$, *the operator SaS is compact, the operator $S^{\perp}bS^{\perp}$ is non-compact and for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{N}$, $P > S$ such that $\|a(P-S)\| < \varepsilon$.*

(iv) $U_a = L_b = S$, *the operator $S^{\perp}bS^{\perp}$ is compact, the operator SaS is non-compact and for any $\varepsilon > 0$, there exists a projection $P \in \mathcal{N}$, $P < S$ such that $\|(S-P)b\| < \varepsilon$.*

Proof. Suppose that $U_a > L_b$. If there exists a projection $P \in \mathcal{N}$ such that $U_a > P > L_b$, then the operators PaP and $P^{\perp}bP^{\perp}$ are both compact. If $U_a L_b^{\perp}$ is an atom, then the operators $U_a a U_a$ and $L_b^{\perp} b L_b^{\perp}$ are both compact and therefore the operator $U_a^{\perp} b U_a^{\perp}$ is compact as well since $U_a = L_{b+}$. Thus, the multiplication operator $M_{a,b}$ is weakly compact (Lemma 3.4).

If condition (ii) holds, the weak compactness of $M_{a,b}$ follows from Lemma 3.4 as well.

We suppose that condition (iii) is satisfied. Let $\varepsilon > 0$ and $P \in \mathcal{N}$, $P > S$ such that $\|a(P-S)\| < \varepsilon$. Then

$$a = aS + a(P-S) + aP^{\perp} \quad \text{and} \quad b = Sb + (S^{\perp} - P^{\perp})b + P^{\perp}b.$$

Let $x \in (\operatorname{Alg}\mathcal{N})_1$. Then

$$\begin{aligned} M_{a,b}(x) = axb &= (aS + a(P-S) + aP^{\perp})xb \\ &= aSxb + a(P-S)xb + aP^{\perp}x(Sb + (S^{\perp} - P^{\perp})b + P^{\perp}b) \\ &= aSxb + a(P-S)xb + aP^{\perp}xP^{\perp}b. \end{aligned}$$

The operator $M_{a,b}(x)$ is compact since the operators aS and $P^{\perp}b$ are both compact and $\|a(P-S)xb\| < \varepsilon$. The multiplication operator $M_{a,b}$ is weakly compact as the set of weakly compact operators is norm closed ([18], II.C Section 6).

Condition (iv) is symmetric to condition (iii) and the proof is similar.

Now, suppose that the multiplication operator $M_{a,b}$ is weakly compact. Then, if $U_a < L_b$ we distinguish two cases.

Case 1. Suppose that there exists a projection $P \in \mathcal{N}$ such that $U_a < P < L_b$. In that case, the operators PaP and $P^{\perp}bP^{\perp}$ are both non-compact which is a contradiction by Lemma 3.5.

Case 2. Suppose that $U_{a+} = L_b$. Then, the operators aL_b and $L_{b-}^{\perp}b$ are both non-compact. Let $\varepsilon > 0$ and $(e_n)_{n\in\mathbb{N}} \subseteq L_b$, $(f_n)_{n\in\mathbb{N}} \subseteq L_{b-}^{\perp}$ be orthonormal sequences such that $\|a(e_n)\| \geqslant \varepsilon$ and $\|b^*(f_n)\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$ ([6], Proposition 5.2.1). Then, there are subsequences $(e_{k_n})_{n\in\mathbb{N}}$ and $(f_{k_n})_{n\in\mathbb{N}}$ such that the operator $a\Big(\sum\limits_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}\Big)b \in M_{a,b}((\operatorname{Alg}\mathcal{N})_1)$ is not compact (Lemma 3.3). From Corollary 3.2 we conclude that the multiplication operator $M_{a,b}$ is not weakly compact, that is a contradiction.

Now, we examine the only two possible cases, $U_a > L_b$ and $U_a = L_b = S$. The first one is condition (i) of this theorem, so we study the second case. In that case, either the operator SaS is compact or the operator $S^{\perp}bS^{\perp}$ is compact (Lemma 3.5). We suppose that the operator SaS is compact. If the operator $S^{\perp}bS^{\perp}$ is compact as well, the condition (ii) is satisfied. We shall see that if the operator $S^{\perp}bS^{\perp}$ is not compact, then condition (iii) holds.

Suppose that the operator SaS is compact, the operator $S^{\perp}bS^{\perp}$ is not compact and there exists an $\varepsilon_1 > 0$ such that for all $P \in \mathcal{N}$, with $P > S$, the inequality $\|a(P-S)\| \geqslant \varepsilon_1$ holds. We observe that $S_+ = S$ (if $S_+ > S$, the operator $S_+^{\perp}bS_+^{\perp}$ would be compact and then $S = Q_b = S_+$). The operator $P^{\perp}bP^{\perp}$ is compact, for all $P \in \mathcal{N}$, with $P > S$. It follows that $\|(P-S)b\| \geqslant \varepsilon_2$ or equivalently $\|b^*(P-S)\| \geqslant \varepsilon_2$ for some $\varepsilon_2 > 0$, since the operator $S^{\perp}bS^{\perp}$ is not compact. Let $(P_n)_{n\in\mathbb{N}}$ be a decreasing sequence with $P_n > S$ for all $n \in \mathbb{N}$ such that $\text{SOT-}\lim\limits_{n\to\infty} P_n = S$ ([5], Theorem 2.13). We set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then, for all $n \in \mathbb{N}$, $\|a(P_n - S)\| \geqslant \varepsilon$ and $\|b^*(P_n - S)\| \geqslant \varepsilon$. We choose a norm one vector $e_1 \in P_1 - S$ such that $\|b^*(P_1 - S)e_1\| \geqslant 2\varepsilon/3$. The SOT-convergence of the sequence $(P_n)_{n\in\mathbb{N}}$ implies that $\lim\limits_{n\to\infty} \|b^*(P_n - S)(e_1)\| \leqslant \|b^*\| \lim\limits_{n\to\infty} \|(P_n - S)(e_1)\| = 0$ and therefore, there exists a $k_2 \in \mathbb{N}$, $k_2 > 1$ such that $\|b^*(P_{k_2} - S)(e_1)\| < \varepsilon/3$. Then,

$$\begin{aligned}\frac{2\varepsilon}{3} &\leqslant \|b^*(P_1 - S)(e_1)\| = \|b^*(P_1 - P_{k_2})(e_1) + b^*(P_{k_2} - S)(e_1)\| \\ &\leqslant \|b^*(P_1 - P_{k_2})(e_1)\| + \|b^*(P_{k_2} - S)(e_1)\| \leqslant \|b^*(P_1 - P_{k_2})(e_1)\| + \frac{\varepsilon}{3}.\end{aligned}$$

It follows that

$$\|b^*(P_1 - P_{k_2})(e_1)\| \geqslant \frac{\varepsilon}{3}.$$

We set $k_1 = 1$ and we may suppose that $e_1 \in P_{k_1} - P_{k_2}$.

Now, we choose a norm one vector $f_1 \in P_{k_2} - S$ such that $\|a(P_{k_2} - S)f_1\| \geqslant 2\varepsilon/3$. Repeating the arguments of the previous paragraph, we find a $k_3 \in \mathbb{N}$, $k_3 > k_2$, such that

$$\|a(P_{k_2} - P_{k_3})f_1\| \geqslant \frac{\varepsilon}{3},$$

while considering that $f_1 \in P_{k_2} - P_{k_3}$. Using these arguments, one can construct by induction a subsequence $(P_{k_n})_{n\in\mathbb{N}}$ and two orthonormal sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ with the following properties:

(i) $e_n \in P_{2n-1} - P_{2n}$ and $f_n \in P_{2n} - P_{2n+1}$, for all $n \in \mathbb{N}$.

(ii) $\|b^*(e_n)\| = \|b^*(P_{2n-1} - P_{2n})(e_n)\| > \varepsilon/3$ and $\|a(f_n)\| = \|a(P_{2n} - P_{2n+1})(f_n)\| > \varepsilon/3$, for all $n \in \mathbb{N}$.

Lemma 3.3 shows that there exist subsequences $(e_{k_n})_{n\in\mathbb{N}}$ and $(f_{k_n})_{n\in\mathbb{N}}$ such that the operator $a\Big(\sum_{n\in\mathbb{N}} e_{k_n} \otimes f_{k_n}\Big)b \in M_{a,b}((\operatorname{Alg}\mathcal{N})_1)$ is not compact and Proposition 3.1 leads us to a contradiction. Therefore, condition (iii) is satisfied.

The proof in the last case (i.e. $S = U_a = L_b$, $S^\perp bS^\perp$ is compact and SaS is not compact) is similar to the proof of the previous case, and we omit the details. ∎

REMARK 3.7. Observe that if we suppose that the multiplication operator $M_{a,b}$ is not weakly compact, then the conditions of Lemma 3.3 are satisfied. We shall use this fact in the proof of Theorem 4.1.

The next theorem provides another characterization of weakly compact multiplication operators.

THEOREM 3.8. *Let $a, b \in \operatorname{Alg}\mathcal{N}$. The multiplication operator $M_{a,b} : \operatorname{Alg}\mathcal{N} \to \operatorname{Alg}\mathcal{N}$ is weakly compact if and only if for all $\varepsilon > 0$ there exist two projections $P_1, P_2 \in \mathcal{N}$, with $P_1 \leqslant P_2$, such that the operators P_1aP_1 and $P_2^\perp bP_2^\perp$ are both compact and $\|a(P_2 - P_1)\| < \varepsilon$ or $\|(P_2 - P_1)b\| < \varepsilon$.*

Proof. Let $M_{a,b}$ be a weakly compact multiplication operator. Suppose that $U_a > L_b$ (condition (i) of Theorem 3.6). Then, either there exists a projection $P_1 = P_2 \in \mathcal{N}$ such that $U_a > P_1 = P_2 > L_b$ or the operators U_aaU_a and $U_a^\perp bU_a^\perp$ are both compact. In the second case we set $P_1 = P_2 = U_a$. In any case, the inequality $\|a(P_2 - P_1)\| < \varepsilon$ is satisfied for all $\varepsilon > 0$, while the operators P_1aP_1 and $P_2^\perp bP_2^\perp$ are both compact. If $U_a = L_b = S$ and the operators SaS and $S^\perp bS^\perp$ are both compact (condition (ii) of Theorem 3.6), then for $P_1 = P_2 = S$ it follows that $\|a(P_2 - P_1)\| = 0$. If condition (iii) of Theorem 3.6 holds, then for all $P_2 \in \mathcal{N}$ with $P_2 > S$ the operator $P_2^\perp bP_2^\perp$ is compact. Then, for all $\varepsilon > 0$ and $P_1 = S$, there exists $P_2 > S$ such that $\|a(P_2 - P_1)\| < \varepsilon$. If condition (iv) of Theorem 3.6 is satisfied the proof is similar.

For the opposite direction, let $\varepsilon > 0$ and $x \in \operatorname{Alg}\mathcal{N}$. Without loss of generality, we suppose that $\|a\| \leqslant 1$ and $\|b\| \leqslant 1$. Then, there exist two projections $P_1, P_2 \in \mathcal{N}$, with $P_1 < P_2$ that satisfy our hypothesis. It follows that:

$$M_{a,b}(x) = axb = (aP_1 + aP_1^\perp)x(P_2^\perp b + P_2b) = aP_1x(P_2^\perp b + P_2b) + aP_1^\perp xP_2^\perp b + aP_1^\perp xP_2b.$$

The operators $aP_1 = P_1aP_1$ and $P_2^\perp b = P_2^\perp bP_2^\perp$ are both compact and $\|aP_1^\perp xP_2b\| = \|a(P_2 - P_1)x(P_2 - P_1)b\| \leqslant \|a(P_2 - P_1)\|\|x\|\|(P_2 - P_1)b\| < \varepsilon\|x\|$. Therefore, the operator $M_{a,b}$ is weakly compact since the space of weakly compact operators is closed ([18], Theorem 6, p. 52). ∎

COROLLARY 3.9. *Let $\mathcal{N} = \{P_n\}_{n\in\mathbb{N}} \cup \{\{0\}, H\}$ be a nest consisting of a sequence of finite rank projections that increase to the identity, and let $a, b \in \operatorname{Alg}\mathcal{N}$. The* *multiplication operator* $M_{a,b} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$, $x \mapsto axb$ *is weakly compact if and only if either the operator* a *is compact or the operator* b *is compact.*

Proof. Suppose that neither a nor b is a compact operator. Then, $U_a = L_b = I$, where I is the identity operator. The operator $I^{\perp} b I^{\perp} = 0$ is compact, the operator $IaI = a$ is non-compact and there exists an $\varepsilon > 0$ such that the inequality $\|(I-P)b\| \geqslant \varepsilon$ is satisfied for all $P \in \mathcal{N}$, $P < I$. The last inequality follows from the non-compactness of the operator b. Thus, the multiplication operator $M_{a,b}$ is not weakly compact (Theorem 3.6, case (iv)).

The opposite direction is immediate from Proposition 2.3 of [1]. ∎

If $\mathcal{S}$ is a nonempty subset of the unit ball of a normed space $\mathcal{A}$, then the *contractive perturbations* of $\mathcal{S}$ are defined as $\mathrm{cp}(\mathcal{S}) = \{x \in \mathcal{A} : \|x \pm s\| \leqslant 1 \ \forall s \in \mathcal{S}\}$. We shall write $\mathrm{cp}(a)$ instead of $\mathrm{cp}(\{a\})$ for $a \in \mathcal{A}$. One may define contractive perturbations of higher order by using the recursive formula $\mathrm{cp}^{n+1}(\mathcal{S}) = \mathrm{cp}(\mathrm{cp}^n(\mathcal{S}))$, $n \in \mathbb{N}$. The second contractive perturbations, $\mathrm{cp}^2(a)$, were introduced in [2] to characterize the compact elements of a C^*-algebra. Let $\mathcal{N}$ be a nest as in Corollary 3.9. The second author and Katsoulis proved in Theorem 2.7 of [3] that $a \in (\mathrm{Alg}\,\mathcal{N})_1$ is a compact operator if and only if the set of its second contractive perturbations, $\mathrm{cp}^2_{\mathrm{Alg}\,\mathcal{N}}(a)$, is compact. The next corollary of Theorem 3.6 complements that result.

COROLLARY 3.10. *Let* $\mathcal{N}$ *be a nest as in Corollary* 3.9 *and* $a \in \mathrm{Alg}\,\mathcal{N}$. *The following are equivalent:*

(i) *The set* $\mathrm{cp}^2(a)$ *is compact.*
(ii) *The set* $\mathrm{cp}^2(a)$ *is weakly compact.*
(iii) *The operator* a *is compact.*

Proof. The implication (i)⇒(ii) is obvious.

Now, suppose that the set $\mathrm{cp}^2(a)$ is weakly compact. From Proposition 1.2 of [2] we know that $M_{a,a}(\mathrm{Alg}\,\mathcal{N}_{1/2}) \subseteq \mathrm{cp}^2_{\mathrm{Alg}\,\mathcal{N}}(a)$ and therefore $M_{a,a} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is a weakly compact multiplication operator. Therefore, Corollary 3.9 implies that a is a compact operator.

Let a be a compact operator. Then, the set $\mathrm{cp}^2(a)$ is compact ([3], Theorem 2.7). ∎

REMARK 3.11. Let $\mathcal{N}$ be a nest as in Corollary 3.9 and $a, b \in \mathrm{Alg}\,\mathcal{N}$. From Corollary 1.5 and Corollary 3.9 it follows that the multiplication operator $M_{a,b} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is weakly compact while being non-compact if and only if the operator a is compact and the operator b is non-compact.

REMARK 3.12. Let $a, b \in \mathrm{Alg}\,\mathcal{N} + \mathcal{K}(H)$. The algebra $\mathrm{Alg}\,\mathcal{N} + \mathcal{K}(H)$ is called the quasitriangular algebra of $\mathcal{N}$. The multiplication operator

$$M^{QT}_{a,b} : \mathrm{Alg}\,\mathcal{N} + \mathcal{K}(H) \to \mathrm{Alg}\,\mathcal{N} + \mathcal{K}(H)$$

is compact (weakly compact) if and only if the operators a and b are both compact (either a or b is compact).

Proof. If the operators a and b are both compact (either a or b is compact), the result follows from Proposition 2.3 of [1]. If the multiplication operator $M_{a,b}^{QT}$ is compact (weakly compact), the restriction $M_{a,b}^{QT}|_{\mathcal{K}(H)}$ is compact (weakly compact) and therefore, the second dual $(M_{a,b}^{QT}|_{\mathcal{K}(H)})^{**} = M_{a,b}$ defined on $\mathcal{B}(H)$ is compact (weakly compact ([7], Theorem 8, p. 485)). Therefore, the operators a and b are both compact (either a or b is compact) ([1], Proposition 2.3). Note that these arguments apply to any operator algebra containing the compact operators. ∎

4. MULTIPLICATION OPERATORS ON $\mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})$

In this section, we show that there is no non-zero weakly compact multiplication operator on $\mathrm{Alg}\,\mathcal{N}$.

THEOREM 4.1. *Let $a, b \in \mathrm{Alg}\,\mathcal{N}$ and $\pi : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})$ be the quotient map. The multiplication operator $M_{\pi(a),\pi(b)} : \mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N}) \to \mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})$ is weakly compact if and only if $M_{\pi(a),\pi(b)} = 0$.*

Proof. We suppose that $M_{\pi(a),\pi(b)} \neq 0$, or equivalently $M_{a,b}(\mathrm{Alg}\,\mathcal{N}) \nsubseteq \mathcal{K}(\mathcal{N})$. This is also equivalent to the fact that the multiplication operator $M_{a,b} : \mathrm{Alg}\,\mathcal{N} \to \mathrm{Alg}\,\mathcal{N}$ is not weakly compact (Corollary 3.2). We can see that Remark 3.7 and Proposition 5.2.1 of [6] ensure the existence of some orthonormal sequences $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ that satisfy the conditions of Lemma 3.3 for the operators a and b, i.e. $e_n \otimes f_n \in \mathrm{Alg}\,\mathcal{N}$, $\|a(f_n)\| \geqslant \varepsilon$ and $\|b^*(e_n)\| \geqslant \varepsilon$, for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. The subsequences of $(e_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ that Lemma 3.3 provides are denoted again by the same symbols.

Let $(A_n)_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ such that A_n be an infinite set for all $n \in \mathbb{N}$. We show that the following map is an isomorphic embedding:

$$u : \ell^\infty \to M_{\pi(a),\pi(b)}(\mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})),$$
$$(x_n)_{n\in\mathbb{N}} \mapsto M_{\pi(a),\pi(b)}\Big(\pi\Big(\sum_{n\in\mathbb{N}} x_n \sum_{i\in A_n} e_i \otimes f_i\Big)\Big).$$

First of all we see that u is bounded (we assume that $\|a\| \leqslant 1$ and $\|b\| \leqslant 1$). Indeed, for all $(x_n)_{n\in\mathbb{N}} \subseteq \ell^\infty$,

$$\begin{aligned}\|u((x_n)_{n\in\mathbb{N}})\|_{\mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})} &= \inf_{K\in\mathcal{K}(\mathcal{N})} \Big\|a\Big(\sum_{n\in\mathbb{N}} x_n \sum_{i\in A_n} e_i \otimes f_i\Big)b + K\Big\| \\ &\leqslant \Big\|a\Big(\sum_{n\in\mathbb{N}} x_n \sum_{i\in A_n} e_i \otimes f_i\Big)b\Big\|\end{aligned}$$

$$\leqslant \|a\|\|b\|\Big\|\Big(\sum_{n\in\mathbb{N}} x_n \sum_{i\in A_n} e_i\otimes f_i\Big)\Big\| \leqslant \|(x_n)_{n\in\mathbb{N}}\|_\infty.$$

Then, it suffices to prove that u is bounded below, i.e. there is a positive number δ such that $\|u((x_n)_{n\in\mathbb{N}}\|_{\mathrm{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})} \geqslant \delta\|(x_n)_{n\in\mathbb{N}}\|_\infty$, $((x_n)_{n\in\mathbb{N}}\in\ell^\infty)$. Let $(x_n)_{n\in\mathbb{N}}$ be a non-zero element of ℓ^∞ and $n_0\in\mathbb{N}$ such that $|x_{n_0}|\geqslant(3/4)\|(x_n)_{n\in\mathbb{N}}\|_\infty$. Then,

$$\begin{aligned}
\|u((x_n)_{n\in\mathbb{N}})\| &= \inf_{K\in\mathcal{K}(\mathcal{N})}\Big\|a\Big(\sum_{n\in\mathbb{N}} x_n \sum_{i\in A_n} e_i\otimes f_i\Big)b+K\Big\| \\
&\geqslant \Big\|\sum_{n\in\mathbb{N}} x_n\sum_{i\in A_n} b^*(e_i)\otimes a(f_i)+K_\varepsilon\Big\| - \frac{\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\| \\
&\qquad \text{(for some } K_\varepsilon\in\mathcal{K}(\mathcal{N})) \\
&\geqslant \Big|\Big\langle \sum_{n\in\mathbb{N}} x_n\sum_{i\in A_n} b^*(e_i)\otimes a(f_i)(b^*(e_{i_0})), a(f_{i_0})\Big\rangle\Big| \qquad (4.1)\\
&\qquad - |\langle K_\varepsilon(b^*(e_{i_0})), a(f_{i_0})\rangle| - \frac{\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\| \\
&= \Big|\sum_{n\in\mathbb{N}} x_n\sum_{i\in A_n}\langle b^*(e_{i_0}, b^*(e_i)\rangle\langle a(f_i), a(f_{i_0})\rangle\Big| \\
&\qquad - |\langle a^*K_\varepsilon b^*(e_{i_0}), f_{i_0}\rangle| - \frac{\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\|,
\end{aligned}$$

where $i_0\in A_{n_0}\subseteq\mathbb{N}$ satisfies $|\langle a^*K_\varepsilon b^*(e_{i_0}), f_{i_0}\rangle| < (\varepsilon^4/9)\|(x_n)_{n\in\mathbb{N}}\|$. Such an i_0 exists since the operator K_ε is compact and the set $A_{n_0}\subseteq\mathbb{N}$ is infinite. Before we continue our calculations, we set

$$\lambda_i = \langle b^*(e_{i_0}), b^*(e_i)\rangle \mu_i = \langle a(f_i), a(f_{i_0})\rangle,$$

for all $i\in\mathbb{N}$. Now, from the estimation of the formula (4.1) and the proof of Lemma 3.3 we can write

$$\begin{aligned}
\|u((x_n)_{n\in\mathbb{N}})\| &\geqslant \Big|\sum_{n\in\mathbb{N}} x_n\sum_{i\in A_n}\lambda_i\mu_i\Big| - \frac{2\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\| \\
&\geqslant \Big|x_{n_0}\sum_{i\in A_{n_0}}\lambda_i\mu_i\Big| - \Big|\sum_{n\neq n_0} x_n\sum_{i\in A_n}\lambda_i\mu_i\Big| - \frac{2\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\| \\
&\geqslant \frac{3}{4}\|(x_n)_{n\in\mathbb{N}}\|\frac{8\varepsilon^4}{9} - \|(x_n)_{n\in\mathbb{N}}\|\frac{\varepsilon^4}{9} - \frac{2\varepsilon^4}{9}\|(x_n)_{n\in\mathbb{N}}\| = \frac{\varepsilon^4}{3}\|(x_n)_{n\in\mathbb{N}}\|.
\end{aligned}$$

Thus, the map u is an isomorphism. Then, the closed unit ball of the space $u(\ell^\infty)$ is not weakly compact and therefore the multiplication operator $M_{\pi(a),\pi(b)}$ is not weakly compact. ∎

REMARK 4.2. Let $a, b\in\mathrm{Alg}\,\mathcal{N}$. Then the following are equivalent:

(i) The multiplication operator $M_{\pi(a),\pi(b)}:\mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})\to\mathrm{Alg}\,\mathcal{N}/\mathcal{K}(\mathcal{N})$ is compact.

(ii) The multiplication operator $M_{\pi(a),\pi(b)} : \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N}) \to \operatorname{Alg}\mathcal{N}/\mathcal{K}(\mathcal{N})$ is weakly compact.

(iii) $M_{\pi(a),\pi(b)} = 0$.

(iv) $M_{a,b}(\operatorname{Alg}\mathcal{N}) \subseteq \mathcal{K}(H)$.

(v) The multiplication operator $M_{a,b}$ is weakly compact.

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G. ANDREOLAS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE AEGEAN, 832 00 KARLOVASSI, SAMOS, GREECE
E-mail address: gandreolas@aegean.gr

M. ANOUSSIS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF THE AEGEAN, 832 00 KARLOVASSI, SAMOS, GREECE
E-mail address: mano@aegean.gr

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