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CONTINUOUS FIELDS OF PROJECTIONS AND ORTHOGONALITY RELATIONS

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ABSTRACT. From a continuous field of Fourier invariant projections of the continuous field of rotation C^*-algebras, we obtain a characteristic equation which fully determines the orthogonality of naturally arising projections from the field. The continuous field turns out to be the support projection of a noncommutative version of a 2-dimensional Theta function. Further, we compute the K-theoretical topological invariants of the projection field. The noncommutative Fourier transform is the canonical order 4 automorphism σ of the rotation C^*-algebra A_θ defined by the relations $\sigma(U) = V^{-1}, \sigma(V) = U$, where U, V are the canonical unitary generators of A_θ satisfying $VU = \mathrm{e}^{2\pi\mathrm{i}\theta}UV$.

KEYWORDS: *C^*-algebra, automorphism, projection, topological invariant, K-theory, continuous field, Jacobi–Theta function.*

MSC (2010): 46L80, 46L40, 46L35.

1. INTRODUCTION

There is a natural continuous section $\mathcal{E}$ of the continuous field of rotation C^*-algebras $\{A_t\}_{t\in(0,1)}$ such that $\mathcal{E}(t)$ is a smooth Fourier invariant projection for each t. The following concrete continuous section $\mathbb{X}$ of smooth positive operators

$$\mathbb{X}(t) = t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)}U_t^m V_t^n \tag{1.1}$$

has support projection $\mathcal{E}(t)$, where U_t, V_t are canonical unitaries of the rotation C^*-algebra A_t satisfying the commutation relation $V_tU_t = e(t)U_tV_t$, where in this paper we use the notation $e(t) := \mathrm{e}^{2\pi\mathrm{i}t}$. (Indices m, n run over the integers.) This infinite series is a noncommutative analogue of a 2-dimensional Theta function which is defined by quite similar rapidly decreasing exponential coefficients involving quadratics of the integer parameters. (One might say that the projection field $\mathcal{E}$ is the support projection of a noncommutative Theta function.) Further, their canonical traces are

$$\tau(\mathcal{E}(t)) = \tau(\mathbb{X}(t)) = t$$

for each t. In addition, the norm $\|\mathcal{E}(s) - \mathbb{X}(s)\|$ has limit 0 as $s \to 0^+$. As demonstrated in [13], this result is a useful tool for quickly and efficiently computing the topological invariants of the projection $\mathcal{E}$ by means of the limit

$$\psi^{\mathrm{t}}(\mathcal{E}(t)) = \lim_{s\to 0^+} \psi^s(\mathbb{X}(s)) \tag{1.2}$$

(being a constant function of t) where ψ^{t} is any one of the noncanonical (unbounded) trace maps that gives the topological K-theoretic invariants. A natural feature of these noncanonical traces is that they are defined in a concrete way across the continuous field, as can be seen from equations (1.13) and (1.15) below. (Further, computation of the above limit makes use of some classical Theta functions identities.) Indeed, in [13] we have carried out this computation for the analogous field of projections associated to the Cubic/Hexic transform (the canonical order 6 automorphism) of A_t.

DEFINITION 1.1. By the *coordinate* of a projection e in a C^*-algebra we understand a positive invertible element x in its corner algebra eAe.

Of course, such coordinate element is not unique, but for our practical purposes coordinate elements have a rather concrete form similar to the infinite series for $\mathbb{X}(t)$ above.

Another interesting feature of coordinates such as $\mathbb{X}(t)$ is that the complement projection $1 - \mathcal{E}(t)$ has coordinate that is of the same form (see Corollary 2.4).

We denote by $t \to U_t$ and $t \to V_t$ a pair of canonical generating sections of unitaries of the continuous field of rotation C^*-algebras $\{A_t\}_{t\in(0,1)}$ satisfying the unitary Heisenberg commutation relation

$$V_t U_t = e(t) U_t V_t. \tag{1.3}$$

The (noncommutative) Fourier transform is the canonical order 4 automorphism $\sigma = \sigma_t$ of the field of rotation algebras given by

$$\sigma(U_t) = V_t^{-1}, \quad \sigma(V_t) = U_t. \tag{1.4}$$

(We omit the subscript and simply write σ for the Fourier transform on all rotation C^*-algebras, the canonical unitaries being understood.) Its square is the flip automorphism $U_t \to U_t^{-1}, V_t \to V_t^{-1}$ (thoroughly studied in [2], [3], and [4]). The order 2 automorphism γ given by

$$\gamma(U_t) = -U_t, \quad \gamma(V_t) = -V_t$$

has the useful property that it commutes with the Fourier transform. (And it is the only toral action automorphism that does so.) The symmetry γ will be necessary to use in our orthogonality theorem below. A further feature of this symmetry is that it flips the sign of two of the five topological invariants for the Fourier transform as we shall see.

The construction of the fields $\mathcal{E}(t)$ and $\mathbb{X}(t)$ comes straight from our earlier paper [11] — though $\mathcal{E}(t)$ was first constructed by Boca [1] — in which a Fourier invariant projection of trace $q(q\theta - p)$ is constructed (where θ is a given real number and $\frac{p}{q}$ is a rational approximation to it) that is the support projection of a certain positive element which comes down to $\mathbb{X}(t)$ when we specialize the situation to $p = 0, q = 1, \theta = t$. Indeed, in this latter case, $\mathbb{X}(t)$ has the C^*-inner product form $\mathbb{X}(t) = \langle f, f \rangle_D$ given near the end of Section 4 of [11], and $\mathcal{E}(t) = \langle \xi, \xi \rangle_D$ where $\xi = fb$ where, as was shown in Section 6 of [11], the inner product $b^{-2} := \langle f, f \rangle_{D^\perp}$ is invertible. It is straightforward to check that $\langle f, f \rangle_D \langle \xi b^2, \xi \rangle_D = \langle \xi, \xi \rangle_D = \mathcal{E}(t)$, showing that $\mathbb{X}(t)$ has $\mathcal{E}(t)$ as its support projection. Furthermore, as $t \to 0$ the operator $b^{-2} := \langle f, f \rangle_{D^\perp}$ converges to the identity (as is clear from equation (5.7) of [11] (page 164) because the "β" therein is just $\frac{1}{t}$ which goes to infinity). This shows that $\|\mathcal{E}(t) - \mathbb{X}(t)\| \to 0$ as $t \to 0$ (as stated above).

For each integer $n \geqslant 1$ there is a canonical $*$-morphism $\zeta_{n,t} : A_{n^2t-k} \to A_t$ given by

$$\zeta_{n,t}(U_{n^2t-k}) = U_t^n, \quad \zeta_{n,t}(V_{n^2t-k}) = V_t^n. \tag{1.5}$$

It "commutes" with the Fourier transform according to the equation

$$\sigma_t \zeta_{n,t} = \zeta_{n,t} \sigma_{n^2t-k}. \tag{1.6}$$

Likewise, for the "negative label" case, one has the canonical $*$-morphism $\zeta'_{n,t} : A_{k-n^2t} \to A_t$ given by

$$\zeta'_{n,t}(U_{k-n^2t}) = V_t^n, \quad \zeta'_{n,t}(V_{k-n^2t}) = U_t^n. \tag{1.7}$$

It "anticommutes" with the Fourier transform

$$\sigma_t \zeta'_{n,t} = \zeta'_{n,t} \sigma^{-1}_{k-n^2t}. \tag{1.8}$$

These morphisms enable one to obtain various projections in a given rotation algebra A_t by suitably "evaluating" continuous sections such as the section of projections $\mathcal{E}(t)$ — namely, by forming the projections

$$\zeta_{n,t}(\mathcal{E}(n^2t - k)), \quad \zeta'_{n,t}(\mathcal{E}(k - n^2t))$$

in A_t.

For the purpose of making the projections mentioned in the following two theorems well-defined, we make the standing hypothesis that the integers a, b, c, d, with $a, c \geqslant 1$, satisfy the condition

$$\max\left\{\frac{b}{a^2}, \frac{d-1}{c^2}\right\} < \min\left\{\frac{1+b}{a^2}, \frac{d}{c^2}\right\}.$$

For simplicity, we shall let J denote the open interval whose endpoints are these max-min values.

THEOREM 1.2. *Let $a, c \geqslant 1$ be integers, $\delta = \gcd(a,c)$, and let b, d be integers such that $a^2d - bc^2 = \delta^2$. Then for $t \in J$ we have the following orthogonality relations of projections in the rotation C^*-algebra A_t:*

$$\zeta_{a,t}\mathcal{E}(a^2t - b) \cdot \zeta'_{c,t}\gamma\mathcal{E}(d - c^2t) = 0, \quad \textit{when } \frac{a}{\delta} \textit{ is odd,} \tag{1.9}$$

$$\zeta_{a,t}\gamma\mathcal{E}(a^2t - b) \cdot \zeta'_{c,t}\mathcal{E}(d - c^2t) = 0, \quad \textit{when } \frac{c}{\delta} \textit{ is odd.} \tag{1.10}$$

This theorem has a converse.

THEOREM 1.3. *If either of the following orthogonality relations of projections holds:*

$$\zeta_{a,t}\mathcal{E}(a^2t - b) \cdot \zeta'_{c,t}\gamma\mathcal{E}(d - c^2t) = 0, \tag{1.11}$$

$$\zeta_{a,t}\gamma\mathcal{E}(a^2t - b) \cdot \zeta'_{c,t}\mathcal{E}(d - c^2t) = 0, \tag{1.12}$$

for some $t \in J$ and some integers a, b, c, d, then

$$a^2d - bc^2 = \delta^2$$

where $\delta = \gcd(a,c)$, and $\frac{a}{\delta}$ is odd is the first case, and $\frac{c}{\delta}$ is odd in the second case.

We call any of the projections appearing in Theorem 1.2 *canonical projections* in view of how they arise canonically from the continuous field $\mathcal{E}(t)$.

REMARK 1.4. One can extend slightly the morphisms $\zeta_{n,t}$ to canonical (Fourier compatible) morphisms $\zeta_{a,b;t} : A_{(a^2+b^2)t-k} \to A_t$ where the label (coefficient of t) is a sum of two squares $a^2 + b^2$, but we shall not need to do so here. It is also possible to thereby extend the orthogonality results of Theorems 1.2 and 1.3 for canonical projections arising from such maps.

REMARK 1.5. Note that the conditions on the integers in the theorem imply that the parameters $a^2t - b$ and $d - c^2t$ are in $(0,1)$, the domain of $\mathcal{E}$. Further, it is easy to check that their sum $a^2t - b + d - c^2t \leqslant 1$. This can be seen by looking at the cases $a = c, a > c$ and $a < c$ separately. For example, in the case $a > c$, it follows from the inequality $\frac{d}{c^2} \leqslant \frac{1+b-d}{a^2-c^2}$ (since $t < \frac{d}{c^2}$). In the case $a < c$, it follows from the inequality $\frac{d-b-1}{c^2-a^2} \leqslant \frac{b}{a^2}$. (The case $a = c$ is trivial since then $d = b + 1$.)

REMARK 1.6. One would expect similar orthogonality theorems for the continuous field and coordinates obtained in our recent paper [13] for the Cubic and Hexic transforms. Indeed the coordinate of the projection field in this case has a form quite similar to that of equation (1.1).

We next obtain the topological invariants ψ_{jk} (to be given shortly) of the field $\mathcal{E}(t)$ of projections.

THEOREM 1.7. *The (Fourier) topological invariants of the projection section $\mathcal{E}(t)$ are the following, for any $t \in (0,1)$:*

$$\psi_{10} = \psi_{11} = \frac{1}{2}(1-\mathrm{i}), \quad \psi_{20} = \psi_{21} = \frac{1}{2}, \quad \psi_{22} = 1.$$

These topological invariants are computed in Section 3 below in the same vein as we have done in [13]. Consequently, one can obtain the topological invariants of the field $\gamma\mathcal{E}(t)$ by working out the commutation between γ and ζ maps (which is straightforward to do since γ commutes with the Fourier transform σ).

If α is an automorphism of an algebra A (usually a pre-C^*-algebra like A_θ^∞), by an *α-trace* we understand a complex-valued linear map ψ defined on A satisfying the condition $\psi(xy) = \psi(\alpha(y)x)$ for each x, y in A. (We say that ψ is α-invariant when $\psi\alpha = \psi$.)

In [9] we found that there are two basic σ-trace linear maps $\psi_{10}, \psi_{11} : A_\theta^\infty \to \mathbb{C}$ given on the basis elements by

$$\psi_{10}(U^m V^n) = e(-\tfrac{\theta}{4}(m+n)^2)\,\delta_2^{m-n}, \tag{1.13}$$

$$\psi_{11}(U^m V^n) = e(-\tfrac{\theta}{4}(m+n)^2)\,\delta_2^{m-n-1}, \tag{1.14}$$

where the divisor δ-function δ_d^m is 1 when $d|_m$ and 0 otherwise, and $e(t) := \mathrm{e}^{2\pi \mathrm{i} t}$. (Thus, any σ-trace linear map on A_θ^∞ is a linear combination of ψ_{10}, ψ_{11}.) In addition, there are three basic σ-invariant σ^2-trace linear maps $\psi_{2j}, j = 0, 1, 2$, given by

$$\psi_{20}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\,\delta_2^m \delta_2^n, \tag{1.15}$$

$$\psi_{21}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\,\delta_2^{m-1} \delta_2^{n-1}, \tag{1.16}$$

$$\psi_{22}(U^m V^n) = e(-\tfrac{\theta}{2}mn)\,\delta_2^{m-n-1}. \tag{1.17}$$

We shall sometimes write ψ_{jk}^θ to indicate its dependence on θ (as will be done in Section 3). When restricted to the fixed point smooth $*$-subalgebra $A_\theta^{\sigma,\infty} = A_\theta^\infty \cap A_\theta^\sigma$, these functionals define unbounded traces. These unbounded traces ψ_{jk} along with the canonical bounded trace τ form the Connes–Chern character group homomorphism on the K_0-group of the fixed point C^*-subalgebra (Fourier orbifold) A_θ^σ:

$$T : K_0(A_\theta^\sigma) \to \mathbb{C}^6, \quad x \to (\tau(x);\ \psi_{10}(x), \psi_{11}(x), \psi_{20}(x), \psi_{21}(x), \psi_{22}(x)). \tag{1.18}$$

This map is known to be injective. (In [10] this was shown for a dense G_δ set of θ's, but since by [6] or [7] one has $K_0(A_\theta^\sigma) \cong \mathbb{Z}^9$ for all θ, this map is injective for all irrational θ.)

As the maps ψ_{jk} induce group homomorphisms $K_0(A_\theta^\sigma) \to \mathbb{C}$, we simply write $\psi_{1k}(e) := \psi_{1k}[e]$ for any projection e in A_θ.

We make free use of the divisor delta function δ_m^n defined to be 1 when m divides n, and 0 otherwise. We have the summation formula

$$\sum_{k=0}^{m-1} e(\tfrac{nk}{m}) = m\delta_m^n.$$

We shall also use the usual Kronecker delta $\delta_{m,n} = 1$ when $m = n$, and $\delta_{m,n} = 0$ when $m \neq n$. The classical Jacobi–Theta functions that will arise in our computations are

$$\vartheta_2(z,t) = \sum_n e^{\pi i t(n+(1/2))^2} e^{i2z(n+(1/2))}, \quad \vartheta_3(z,t) = \sum_n e^{\pi i t n^2} e^{i2zn},$$
$$\vartheta_4(z,t) = \sum_n (-1)^n e^{\pi i t n^2} e^{i2zn},$$

for $z, t \in \mathbb{C}$ and $\mathrm{Im}(t) > 0$, where all summations range over the integers $\mathbb{Z}$. (Following the classic treatment of [15].)

2. ORTHOGONALITY OF CANONICAL PROJECTIONS

We begin with a lemma introducing one of my favorite zeros.

LEMMA 2.1. *One has*

$$\sum_{p,q} (-1)^{p+q+pq} e^{-(\pi/2)(p^2+q^2)} = 0.$$

Proof. Let S denote this sum. Using the relation $(-1)^{p+q+pq} = 2\delta_2^p \delta_2^q - 1$, the sum S can be split up as follows

$$S = \sum_{p,q} ((2\delta_2^p \delta_2^q - 1) e^{-(\pi/2)(p^2+q^2)} = 2\Big(\sum_p \delta_2^p e^{-(\pi/2)p^2}\Big)^2 - \Big(\sum_p e^{-(\pi/2)p^2}\Big)^2$$
$$= 2\Big(\sum_p e^{-2\pi p^2}\Big)^2 - \Big(\sum_p e^{-(\pi/2)p^2}\Big)^2 = 2\vartheta_3(0,2i)^2 - \vartheta_3(0,\tfrac{i}{2})^2 = 0$$

in view of the Theta function identity $\vartheta_3(0, \frac{i}{2}) = \sqrt{2}\vartheta_3(0,2i)$. ∎

REMARK 2.2. The series in the lemma actually contains the following generalization for all real positive numbers $z > 0$:

$$\sum_{p,q} (-1)^{p+q+pq} e^{-(\pi/2)(zp^2+(1/z)q^2)} = 0.$$

This, in fact, holds for all complex numbers z with positive real part — as the series defines an analytic function on the right half plane that vanishes on the positive real axis.

We will also need the following lemma.

LEMMA 2.3. *Let $D \geqslant 1$ be a real number and k an integer. Then*

$$\sum_{m,n} (-1)^{k(m+n)} e(\tfrac{1}{2}Dmn) e^{-(\pi/2)D(m^2+n^2)} = 0$$

if and only if $D = 1$ and k is odd.

Proof. One direction was already proved by Lemma 2.1. Conversely, let S denote the sum in the lemma and assume $S = 0$. If k is even, then the sum can be written

$$\begin{aligned} S &= \sum_{m,n} e(\tfrac{1}{2}Dmn)\mathrm{e}^{-(\pi/2)D(m^2+n^2)} = \sum_m \mathrm{e}^{-(\pi/2)Dm^2} \sum_n e(\tfrac{1}{2}Dmn)\mathrm{e}^{-(\pi/2)Dn^2} \\ &= \sum_m \mathrm{e}^{-(\pi/2)Dm^2} \vartheta_3(\tfrac{\pi}{2}Dm, \mathrm{i}\tfrac{D}{2}) \end{aligned}$$

where each term here is positive since $\vartheta_3(x, \mathrm{i}\frac{D}{2}) \geqslant \vartheta_3(\frac{\pi}{2}, \mathrm{i}\frac{D}{2}) > 0$ for any real x; thus, $S > 0$. Therefore, k has to be odd, in which case S can be written

$$S = \sum_{m,n} (-1)^{m+n} e(\tfrac{1}{2}Dmn)\mathrm{e}^{-(\pi/2)D(m^2+n^2)} = \sum_m (-1)^m \mathrm{e}^{-(\pi/2)Dm^2} \vartheta_4(\tfrac{\pi}{2}Dm, \tfrac{1}{2}\mathrm{i}D).$$

Breaking this down according to parity of m gives

$$\begin{aligned} S &= \sum_m \mathrm{e}^{-2\pi Dm^2} \vartheta_4(\pi Dm, \tfrac{1}{2}\mathrm{i}D) - \sum_m \mathrm{e}^{-2\pi D(m+(1/2))^2} \vartheta_4(\pi D(m+\tfrac{1}{2}), \tfrac{1}{2}\mathrm{i}D) \\ &\geqslant \sum_m \mathrm{e}^{-2\pi Dm^2} \vartheta_3(\tfrac{\pi}{2}, \tfrac{1}{2}\mathrm{i}D) - \sum_m \mathrm{e}^{-2\pi D(m+(1/2))^2} \vartheta_3(0, \tfrac{1}{2}\mathrm{i}D) \\ &= \vartheta_3(0, 2\mathrm{i}D)\vartheta_3(\tfrac{\pi}{2}, \tfrac{1}{2}\mathrm{i}D) - \vartheta_2(0, 2\mathrm{i}D)\vartheta_3(0, \tfrac{1}{2}\mathrm{i}D) \end{aligned}$$

since $\vartheta_4(x, \frac{1}{2}\mathrm{i}D) = \vartheta_3(x + \frac{\pi}{2}, \frac{1}{2}\mathrm{i}D)$ is always in the interval $[\vartheta_3(\frac{\pi}{2}, \mathrm{i}\frac{D}{2}),\ \vartheta_3(0, \mathrm{i}\frac{D}{2})]$ (see page 165 in [11]). Using the identity (see, for example, equation (4.4) of [9]) $\vartheta_3(w, u) = \vartheta_3(2w, 4u) + \vartheta_2(2w, 4u)$ (and also $\vartheta_2(\pi, u) = -\vartheta_2(0, u)$, $\vartheta_3(\pi, u) = \vartheta_3(0, u)$), one gets

$$S \geqslant A(A-B) - B(A+B) = (A - (\sqrt{2}+1)B)(A + (\sqrt{2}-1)B)$$

where $A := \vartheta_3(0, 2\mathrm{i}D) > 0,\ B := \vartheta_2(0, 2\mathrm{i}D) > 0$. Since the factor $A + (\sqrt{2}-1)B$ is already positive, it is enough to check that the first factor

$$A - (\sqrt{2}+1)B = \vartheta_3(0, 2\mathrm{i}D) - (\sqrt{2}+1)\vartheta_2(0, 2\mathrm{i}D)$$

is positive for $D > 1$. This, however, was already established near the end of the proof of Lemma 6.2 (page 168) in [11]. Therefore, $S > 0$ if $D > 1$. ∎

We now prove Theorems 1.2 and 1.3.

Proof. Let $A = a^2t - b, C = d - c^2t$ (both positive numbers by hypothesis). Then

$$\gamma^r \mathbb{X}(A) = A \sum_{m,n} (-1)^{rm+rn} e(\tfrac{A}{2}mn)\mathrm{e}^{-(\pi/2)A(m^2+n^2)} U_A^m V_A^n \quad \text{and}$$

$$\zeta_{a,t}\gamma^r \mathbb{X}(A) = A \sum_{m,n} (-1)^{rm+rn} e(\tfrac{A}{2}mn)\mathrm{e}^{-(\pi/2)A(m^2+n^2)} U_t^{am} V_t^{an}.$$

Similarly,

$$\zeta'_{c,t}\gamma^s \mathbb{X}(C) = C \sum_{k,\ell} (-1)^{sk+s\ell} e(\tfrac{C}{2}k\ell)\mathrm{e}^{-(\pi/2)C(k^2+\ell^2)} V_t^{ck} U_t^{c\ell}.$$

We have

$$\tau_0 := \tau(\zeta_{a,t}\gamma^r\mathbb{X}(A)\cdot\zeta'_{c,t}\gamma^s\mathbb{X}(C))$$
$$= AC\sum_{m,n,k,\ell}(-1)^{rm+rn+sk+s\ell}e(\tfrac{A}{2}mn+\tfrac{C}{2}k\ell)\mathrm{e}^{-(\pi/2)A(m^2+n^2)-(\pi/2)C(k^2+\ell^2)}$$
$$\cdot\,\tau(U_t^{am}V_t^{an}V_t^{ck}U_t^{c\ell}).$$

The trace in the sum necessitates that $an = -ck$ and $am = -c\ell$, which can be written in terms of $a = \delta a_1, c = \delta c_1$ where $\delta = \gcd(a, c)$ and $\gcd(a_1, c_1) = 1$. This gives

$$n = c_1 p, \quad m = c_1 q, \quad k = -a_1 p, \quad \ell = -a_1 q$$

where $p, q \in \mathbb{Z}$. Therefore, we obtain

$$\tau_0 = AC\sum_{p,q}(-1)^{rc_1q+rc_1p-sa_1p-sa_1q}e(\tfrac{1}{2}(Ac_1^2+Ca_1^2)pq)$$
$$\cdot\,\mathrm{e}^{-(\pi/2)Ac_1^2(p^2+q^2)-(\pi/2)Ca_1^2(p^2+q^2)}$$
$$= AC\sum_{p,q}(-1)^{(rc_1-sa_1)(p+q)}e(\tfrac{1}{2}Dpq)\mathrm{e}^{-(\pi/2)D(p^2+q^2)}$$

where we have written $D = Ac_1^2 + Ca_1^2 > 0$. Observe that

$$D = \frac{1}{\delta^2}(Ac^2+Ca^2) = \frac{1}{\delta^2}((a^2t-b)c^2+(d-c^2t)a^2) = \frac{1}{\delta^2}(a^2d-bc^2) = a_1^2d-bc_1^2$$

hence $D \geqslant 1$ is an integer.

If the underlying projections are orthogonal, so that $\tau_0 = 0$, then by Lemma 2.3, $D = 1$ and $rc_1 - sa_1$ is an odd integer. Thus, $a^2d - bc^2 = \delta^2$ and a_1 is odd if equation (1.11) holds, and c_1 is odd when the second equation (1.12) holds. (For instance, in the first orthogonality equation in Theorem 1.3 we have $r = 0, s = 1$, meaning that a_1 is odd.) This proves Theorem 1.3.

On the other hand, under the hypothesis of Theorem 1.2, it follows that $D = 1$. And if $\frac{a}{\delta} = a_1$ is odd one picks $r = 0, s = 1$, and if $\frac{c}{\delta} = c_1$ is odd one picks $r = 1, s = 0$ so that $rc_1 - sa_1$ is odd in either of these two cases. Thus the above sum becomes

$$\tau_0 = AC\sum_{p,q}(-1)^{p+q+pq}\mathrm{e}^{-(\pi/2)(p^2+q^2)} = 0$$

in view of Lemma 2.1. Since the canonical trace is faithful, the positive operators $\zeta_{a,t}\gamma^r\mathbb{X}(A)$ and $\zeta'_{c,t}\gamma^s\mathbb{X}(C)$ are orthogonal, hence so are their (respective) support projections $\zeta_{a,t}\gamma^r\mathcal{E}(A)$ and $\zeta'_{c,t}\gamma^s\mathcal{E}(C)$. ∎

Theorem 1.2 gives us the projection equation

$$\mathcal{E}(t) + \zeta'_{1,t}\gamma\mathcal{E}(1-t) = 1$$

for $0 < t < 1$ (by taking $a = c = d = 1$ and $b = 0$), since the projections are orthogonal and their traces add up to 1. It also allows us to write down coordinates of the complementary projection $1 - \mathcal{E}(t)$ as given by the following.

COROLLARY 2.4. *The coordinate operator of the projection* $1-\mathcal{E}(t)$ *is*

$$\zeta'_{1,t}\gamma\mathbb{X}(1-t) = (1-t)\sum_{m,n}(-1)^{m+n}e(-\tfrac{(1-t)}{2}mn)\mathrm{e}^{-(\pi/2)(1-t)(m^2+n^2)}U_t^m V_t^n.$$

In particular, this shows that the orthogonal sum

$$\mathbb{X}(t) + \zeta'_{1,t}\gamma\mathbb{X}(1-t)$$

is a smooth positive invertible operator in A_t^∞ that is invariant under the Fourier transform.

COROLLARY 2.5. *One has the following projection equations, for* $\frac{b}{a^2} < t < \frac{1+b}{a^2}$ *and* a, b *any integers:*

$$1-\zeta_{a,t}\mathcal{E}(a^2t-b) = \zeta'_{a,t}\gamma\mathcal{E}(1+b-a^2t),$$
$$1-\zeta_{a,t}\gamma\mathcal{E}(a^2t-b) = \zeta'_{a,t}\mathcal{E}(1+b-a^2t).$$

This corollary follows from the theorem by taking $d = 1+b, c = a$ (in which case $\delta = a$) and the fact that the orthogonal projections have traces summing to 1.

Another consequence is that some of these canonical projections do commute. This arises from the simple possibility that their complements are orthogonal. If e, f are projections with orthogonal complements, then it is easy to see that they commute and their product projection is given by $ef = e+f-1 = e-(1-f)$. The main theorem and the preceding corollary already tell us when this situation can occur.

COROLLARY 2.6. *For each* $t \in J$*, the canonical projections*

$$\zeta_{a,t}\mathcal{E}(a^2t-b), \quad \zeta'_{c,t}\gamma\mathcal{E}(d-c^2t)$$

have orthogonal complements, hence they commute, when

$$c^2(b+1)-(d-1)a^2 = \delta^2$$

where $\delta = \gcd(a,c)$.

Proof. Working out the complements of these two projections according to Corollary 2.5, one checks that the characteristic equation for their complements is satisfied by their parameters, so that the result follows from Theorem 1.2. ∎

If e, f are projections such as in Corollary 2.6, the topological invariants of their product can easily be computed from $ef = e+f-1$. (In general, computation of topological invariants of general commuting products of projections is not an easy task.)

3. TOPOLOGICAL INVARIANTS

We now prove Theorem 1.7 and show that the topological data for the continuous section of projections $\mathcal{E}(t)$ is $(\frac{1-\mathrm{i}}{2}, \frac{1-\mathrm{i}}{2}; \frac{1}{2}, \frac{1}{2}, 1)$. We also show how the topological maps ψ_{jk} (the unbounded traces) transform under the canonical embeddings $\zeta_{n,\theta}$.

We shall use the following well-known inversion formulas for Theta functions in the proof:

$$\vartheta_3(z,t) = (-\mathrm{i}t)^{-1/2}\,\mathrm{e}^{z^2/\pi\mathrm{i}t}\,\vartheta_3(\tfrac{z}{t}, -\tfrac{1}{t}), \tag{3.1}$$
$$\vartheta_2(z,t) = (-\mathrm{i}t)^{-1/2}\,\mathrm{e}^{z^2/\pi\mathrm{i}t}\,\vartheta_4(\tfrac{z}{t}, -\tfrac{1}{t}).$$

(See, for example, equations (5.4) in [13].)

THEOREM 3.1. *The topological numbers of the projection section $\mathcal{E}(t)$ are the following, for any $t \in (0,1)$:*

$$\psi_{10} = \psi_{11} = \frac{1}{2}(1-\mathrm{i}), \quad \psi_{20} = \psi_{21} = \frac{1}{2}, \quad \psi_{22} = 1.$$

Proof. We compute the topological numbers of $\mathcal{E}$ by the quicker method employed in [13] in terms of the unbounded traces of its canonical coordinate $\mathbb{X}(t)$ according to equation (1.2). We have

$$\begin{aligned}
\psi_{1j}^t(\mathbb{X}(t)) &= t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)}\psi_{1j}^t(U_t^n V_t^m) \\
&= t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)} e(-\tfrac{t}{4}(m+n)^2)\delta_2^{m-n-j} \\
&= t\sum_{m,n} \mathrm{e}^{-(\pi/2)tm^2}\mathrm{e}^{-(\pi/2)tn^2} e(-\tfrac{t}{4}m^2)e(-\tfrac{t}{4}n^2)\delta_2^{m-n-j} \\
&= t\sum_{m,n} \mathrm{e}^{-\pi\nu tm^2}\mathrm{e}^{-\pi\nu tn^2}\delta_2^{m-n-j}
\end{aligned}$$

where $\nu = \frac{1}{2}(1+\mathrm{i})$; now make the replacement $n \to n-j$

$$= t\sum_{m,n} \mathrm{e}^{-\pi\nu tm^2}\mathrm{e}^{-\pi\nu t(n-j)^2}\delta_2^{m-n}$$

and break the sum into two sums, one over m,n even and the second over m,n odd

$$\begin{aligned}
&= t\sum_{m,n} \mathrm{e}^{-4\pi\nu tm^2}\mathrm{e}^{-4\pi\nu t(n-(j/2))^2} + t\sum_{m,n} \mathrm{e}^{-4\pi\nu t(m+(1/2))^2}\mathrm{e}^{-4\pi\nu t(n+(1-j)/2)^2} \\
&= t\vartheta_3(0,\mathrm{i}4\nu t)\vartheta_{3-j}(0,\mathrm{i}4\nu t) + t\vartheta_2(0,\mathrm{i}4\nu t)\vartheta_{2+j}(0,\mathrm{i}4\nu t).
\end{aligned}$$

Now use the inversion formulas for Theta functions (3.1) to get

$$\psi_{1j}^{\mathrm{t}}(\mathbb{X}(t)) = t(4\nu t)^{-1}\vartheta_3(0,\tfrac{\nu}{2t})\vartheta_{3+j}(0,\tfrac{\nu}{2t}) + t(4\nu t)^{-1}\vartheta_4(0,\tfrac{\nu}{2t})\vartheta_{4-j}(0,\tfrac{\nu}{2t}).$$

Here $\vartheta_3(0,\frac{\nu}{2t}) \to 1$ and $\vartheta_4(0,\frac{\nu}{2t}) \to 1$ as $t \to 0$ (for $j = 0,1$), so taking the limit as $t \to 0$ one gets

$$\psi^{\mathrm{t}}(\mathcal{E}) = \lim_{t\to 0}\psi^{\mathrm{t}}(\mathbb{X}(t)) = \frac{2}{4\nu} = \frac{1}{2}(1-\mathrm{i}).$$

For ψ_{2k} for $k = 0, 1$ one has

$$\begin{aligned}\psi^{\mathrm{t}}_{2k}(\mathbb{X}(t)) &= t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)}\psi^t_{2k}(U_t^n V_t^m)\\ &= t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)}e(-\tfrac{t}{2}mn)\delta_2^{m-k}\delta_2^{n-k}\\ &= t\Big(\sum_n \mathrm{e}^{-(\pi/2)tn^2}\delta_2^{n-k}\Big)^2\end{aligned}$$

now make the replacement $n \to 2n + k$

$$= t\Big(\sum_n \mathrm{e}^{-2\pi t(n+(k/2))^2}\Big)^2 = t\vartheta_{3-k}(0, 2\mathrm{i}t)^2$$

and inversion gives

$$= t(2t)^{-1}\,\vartheta_{3+k}(0, \tfrac{\mathrm{i}}{2t})^2 = \frac{1}{2}\vartheta_{3+k}(0, \tfrac{\mathrm{i}}{2t})^2$$

which goes to $\frac{1}{2}$ as $t \to 0^+$. Therefore, ψ_{20} and ψ_{21} are both equal to $\frac{1}{2}$.

Finally, for ψ_{22}:

$$\begin{aligned}\psi^t_{22}(\mathbb{X}(t)) &= t\sum_{m,n} e(\tfrac{t}{2}mn)\mathrm{e}^{-(\pi/2)t(m^2+n^2)}\psi^t_{22}(U_t^n V_t^m)\\ &= t\sum_{m,n} \mathrm{e}^{-(\pi/2)t(m^2+n^2)}\delta_2^{n-m-1}\end{aligned}$$

here m, n have to have opposite parity hence

$$\begin{aligned}&= 2t\sum_n \mathrm{e}^{-2\pi tn^2}\sum_m \mathrm{e}^{-2\pi t(m+(1/2))^2} = 2t\vartheta_3(0, 2\mathrm{i}t)\vartheta_2(0, 2\mathrm{i}t)\\ &= 2t(2t)^{-1}\,\vartheta_3(0, \tfrac{\mathrm{i}}{2t})\vartheta_4(0, \tfrac{\mathrm{i}}{2t})\end{aligned}$$

which has limit 1 as $t \to 0^+$. ∎

As the Chern character of Connes is constant for a continuous field of projections, which in our case is simply the integer coefficient of t in its trace when t is irrational, the Connes–Chern number of $\mathcal{E}(t)$ is 1.

Recall that with $\theta_n = n^2\theta - k_n$, we have the (smooth) unital $*$-embedding

$$\zeta_{n,\theta} : A_{\theta_n} \to A_\theta, \quad \zeta_{n,\theta}(U_{\theta_n}) = U_\theta^n, \quad \zeta_{n,\theta}(V_{\theta_n}) = V_\theta^n.$$

It intertwines with the Fourier transform: $\sigma\zeta_{n,\theta} = \zeta_{n,\theta}\sigma$.

LEMMA 3.2. *With $\theta_n = n^2\theta - k$ and k an integer, the unbounded traces on A_{θ_n} and A_θ are related to the morphism $\zeta_{n,\theta} : A_{\theta_n} \to A_\theta$ by the transformation equations:*

$$\begin{aligned}\psi^\theta_{10}\zeta_{n,\theta} &= \psi^{\theta_n}_{10} + \mathrm{i}^{-k}\delta_2^n\psi^{\theta_n}_{11},\\ \psi^\theta_{11}\zeta_{n,\theta} &= \mathrm{i}^{-k}\delta_2^{n-1}\psi^{\theta_n}_{11},\end{aligned}$$

$$\begin{aligned}
\psi_{20}^{\theta}\zeta_{n,\theta} &= \psi_{20}^{\theta_n} + (-1)^k\delta_2^n\psi_{21}^{\theta_n} + \delta_2^n\psi_{22}^{\theta_n},\\
\psi_{21}^{\theta}\zeta_{n,\theta} &= (-1)^k\delta_2^{n-1}\psi_{21}^{\theta_n},\\
\psi_{22}^{\theta}\zeta_{n,\theta} &= \delta_2^{n-1}\psi_{22}^{\theta_n}.
\end{aligned}$$

Proof. It is easy to check that $\psi_{10}^{\theta}\zeta_{n,\theta}$ is a σ-trace on A_{θ_n} (of course, we mean on its smooth dense $*$-subalgebra $A_{\theta_n}^{\infty}$). But as $\psi_{10}^{\theta_n}, \psi_{11}^{\theta_n}$ are a basis for such, it must be a linear combination of them:

$$\psi_{10}^{\theta}\zeta_{n,\theta} = \psi_{10}^{\theta_n} + C\psi_{11}^{\theta_n},$$

for some complex number C. (The first coefficient of $\psi_{10}^{\theta_n}$ is clearly 1 upon evaluation of both sides at the identity.) One now simply evaluates both sides at, say, U_{θ_n} to obtain C:

$$\psi_{10}^{\theta}\zeta_{n,\theta}(U_{\theta_n}) = C\psi_{11}^{\theta_n}(U_{\theta_n}) = Ce(-\tfrac{\theta_n}{4})$$

so

$$C = e(\tfrac{\theta_n}{4})\psi_{10}^{\theta}\zeta_{n,\theta}(U_{\theta_n}) = e(\tfrac{\theta_n}{4})\psi_{10}^{\theta}(U_{\theta}^{n}) = e(\tfrac{\theta_n}{4})e(-\tfrac{\theta}{4}n^2)\delta_2^n$$

from which $C = \mathrm{i}^{-k_n}\delta_2^n$. The relations for ψ_{11}, ψ_{2j} are similar — in fact they can be checked more directly by evaluation at the generic unitaries $U_{\theta_n}^{a}V_{\theta_n}^{b}$. ∎

In the following lemma we recall that the adjoint ψ^* of a linear functional ψ is given by $\psi^*(x) = \overline{\psi(x^*)}$.

LEMMA 3.3. *The unbounded traces on $A_{\theta_n'}$ and A_θ are related to the morphism $\zeta_{n,\theta}' : A_{\theta_n'} \to A_\theta$, where $\theta_n' = k - n^2\theta$ and k is an integer, by the transformation equations:*

$$\begin{aligned}
\psi_{10}^{\theta}\zeta_{n,\theta}' &= (\psi_{10}^{\theta_n'})^* + \mathrm{i}^{-k}\delta_2^n(\psi_{11}^{\theta_n'})^*,\\
\psi_{11}^{\theta}\zeta_{n,\theta}' &= \mathrm{i}^{-k}\delta_2^{n-1}(\psi_{11}^{\theta_n'})^*,\\
\psi_{20}^{\theta}\zeta_{n,\theta}' &= \psi_{20}^{\theta_n'} + (-1)^k\delta_2^n\psi_{21}^{\theta_n'} + \delta_2^n\psi_{22}^{\theta_n'},\\
\psi_{21}^{\theta}\zeta_{n,\theta}' &= (-1)^k\delta_2^{n-1}\psi_{21}^{\theta_n'},\\
\psi_{22}^{\theta}\zeta_{n,\theta}' &= \delta_2^{n-1}\psi_{22}^{\theta_n'}.
\end{aligned}$$

These are verified by evaluation of both sides on the generic unitaries $V_{\theta_n'}^{b}U_{\theta_n'}^{a}$ (and note that the twisted traces $\psi_{2j}^{\theta_n'}$ are self-adjoint).

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