J. OPERATOR THEORY
77:2(2017), 333–376
doi: 10.7900/jot.2016apr08.2115

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ON THE ESSENTIAL SPECTRUM OF N-BODY HAMILTONIANS WITH ASYMPTOTICALLY HOMOGENEOUS INTERACTIONS

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Communicated by Marius Dădârlat

ABSTRACT. We determine the essential spectrum of Hamiltonians with N-body type interactions that have radial limits at infinity, which extends the classical HVZ-theorem for potentials that tend to zero at infinity. Let $\mathcal{E}(X)$ be the algebra generated by functions of the form $v \circ \pi_Y$, where $Y \subset X$ is a subspace, $\pi_Y : X \to X/Y$ is the projection, and $v : X/Y \to \mathbb{C}$ is continuous with *uniform radial limits at infinity.* We consider Hamiltonians affiliated to $\mathscr{E}(X) := \mathcal{E}(X) \rtimes X$. We determine the characters of $\mathcal{E}(X)$ and then we describe the quotient of $\mathscr{E}(X)/\mathcal{K}$ with respect to the ideal of compact operators, which in turn gives a formula for the essential spectrum of any self-adjoint operator affiliated to $\mathscr{E}(X)$.

KEYWORDS: *Self-adjoint operator, essential spectrum, compact operator, C^*-algebra, limit operator, character, radial compatification, N-body problem.*

MSC (2010): Primary 81Q10; Secondary 35P05, 47L65, 47L90, 58J40.

1. INTRODUCTION

1.1. Let X be a real, finite dimensional vector space and let X^* denote its dual. If $Y \subset X$ is a subspace, $\pi_Y : X \to X/Y$ will denote the canonical projection. Let $\mathcal{E}(X)$ be the closure in norm of the algebra of functions on X generated by all functions of the form $u \circ \pi_Y$, where Y runs over the set of all linear subspaces of X and $u : X/Y \to \mathbb{C}$ runs over the set of continuous functions that have uniform radial limits at infinity. Since X acts continuously by translations on $\mathcal{E}(X)$, we can define the crossed product C^*-algebra $\mathscr{E}(X) := \mathcal{E}(X) \rtimes X$, which will be regarded as an algebra of operators on $L^2(X)$. Our main result on essential spectra gives a description of the essential spectrum of any self-adjoint operator H on $L^2(X)$ that is affiliated to $\mathscr{E}(X)$, i.e. such that $(H + \mathrm{i})^{-1} \in \mathscr{E}(X)$. To state this result, we need first to introduce some notation. Thus, for $x \in X$, we let T_x denote the translation operator on $L^2(X)$, defined by $(T_x f)(y) := f(y-x)$. Let $\mathbb{S}_X$ be the set of half-lines in X, that is

$$\mathbb{S}_X := \{\widehat{a} : a \in X, a \neq 0\} \quad \text{where} \quad \widehat{a} := \{ra : r > 0\}. \tag{1.1}$$

We let $\overline{\bigcup} S_\alpha$ denote the *closure of the union* of a family of sets S_α.

THEOREM 1.1. *If H is a self-adjoint operator affiliated to $\mathscr{E}(X)$, then for each $a \in \alpha \in \mathbb{S}_X$, the limit $\tau_\alpha(H) := \alpha \cdot H := \operatorname*{s-lim}_{r\to+\infty} T^*_{ra} H T_{ra}$ exists and $\sigma_{\text{ess}}(H) = \overline{\bigcup}_{\alpha\in\mathbb{S}_X} \sigma(\alpha \cdot H)$.*

For the proof, see Subsection 6.4 (Theorem 6.20). The meaning of the limit above is discussed in Remark 2.10. Here we note only that it is slightly more general than the strong resolvent limit since the $\alpha \cdot H$ could be not densely defined, cf. Remark 5.5.

Theorem 1.1 is a consequence of Theorem 6.17 that is our main technical result since it gives a description of the quotient C^*-algebra of $\mathscr{E}(X)$ with respect to the ideal of compact operators, cf. Corollary 6.19. More precisely, let $[\alpha]$ denote the one dimensional linear subspace generated by $\alpha \in \mathbb{S}_X$ and let $\mathcal{E}(X/[\alpha])$ be the subalgebra of $\mathcal{E}(X)$ generated by the functions $u \circ \pi_Y$ with $Y \supset \alpha$ and u as before. Then $\mathcal{E}(X/[\alpha])$ is stable under translations, so the crossed product $\mathcal{E}(X/[\alpha]) \rtimes X$ is well defined, and we have a canonical embedding

$$\mathscr{E}(X)/\mathscr{K}(X) \hookrightarrow \prod_{\alpha\in\mathbb{S}_X} \mathcal{E}(X/[\alpha]) \rtimes X \tag{1.2}$$

defined as follows. For any $A \in \mathscr{E}(X)$ the limit $\operatorname*{s-lim}_{r\to+\infty} T^*_{ra} A T_{ra} =: \tau_\alpha(A)$ exists, the map τ_α is a $*$-algebra morphism and a linear projection of $\mathscr{E}(X)$ onto its subalgebra $\mathscr{E}(X/[\alpha])$, and an operator $A \in \mathscr{E}(X)$ is compact if and only if $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$. Then the injective morphism (1.2) is induced by the map $\tau(A) := (\tau_\alpha(A))_{\alpha\in\mathbb{S}_X}$.

1.2. In the next few subsections we give some concrete examples of self-adjoint operators on $L^2(X)$ affiliated to $\mathscr{E}(X)$.

We recall first some facts concerning the spherical compactification of X (see Section 3). The set $\mathbb{S}_X$ is thought of as the sphere at infinity of X and $\overline{X} := X \cup \mathbb{S}_X$, equipped with a certain compact space topology, is the spherical compactification of X. If f is a complex valued function on X and $\alpha \in \mathbb{S}_X$, then $\lim_{x\to\alpha} f(x) = c$ (or $\lim_\alpha f = c$) in the sense of the topology of $\overline{X}$ means the following: "for any $\varepsilon > 0$ there is an open truncated cone C such that α is eventually in C and $|f(x) - c| < \varepsilon$ if $x \in C$". A subset of X is a cone if it is a union of half-lines. A truncated cone C is the intersection of a cone with the complement of a bounded set. A half-line α is *eventually* in such a C if there is $a \in \alpha$ such that $ra \in C\ \forall r > 1$. Functions f with values in an arbitrary topological spaces are treated in exactly the same way.

The algebra $\mathcal{C}(\overline{X})$ of continuous functions on $\overline{X}$ can be identified with the set of continuous functions u on X such that $\lim_{\alpha} u$ exists for all $\alpha \in \mathbb{S}_X$ (this is equivalent to the existence of the uniform radial limits at infinity). Then we can regard $\mathcal{C}(\overline{X/Y})$ as an algebra of continuous functions on X/Y. Indeed, when there is no danger of confusion, we will identify a function u on X/Y with the function $u \circ \pi_Y$ on X. Thus, we shall think of $\mathcal{C}(\overline{X/Y})$ as an algebra of continuous functions on X. Then $\mathcal{E}(X)$ is the C^*-algebra generated by these algebras when Y runs over the set of all subspaces of X.

The following notion of *convergence in the mean* at points $\alpha \in \mathbb{S}_X$ is natural in our context (see Section 3). If u is a complex function on X and c is a complex number then we write $\text{m-}\lim_{\alpha} u = c$, or $\text{m-}\lim_{x\to\alpha} u(x) = c$, if $\lim_{a\to\alpha} \int_{a+\Lambda} |u(x) - c| \mathrm{d}x = 0$ for some (hence any) compact neighborhood of the origin Λ in X. This also makes sense if $u \in L^1_{\text{loc}}(X)$.

Let $\mathcal{B}(\overline{X})$ be the set of functions $u \in L^\infty(X)$ such that $\text{m-}\lim_{\alpha} u$ exists for any $\alpha \in \mathbb{S}_X$. This is clearly a C^*-subalgebra of $L^\infty(X)$. Then $\mathcal{B}(\overline{X/Y})$ is well defined for any subspace $Y \subset X$ and we have an obvious C^*-algebra embedding $\mathcal{B}(\overline{X/Y}) \subset L^\infty(X)$.

Finally, let $\mathcal{E}^\sharp(X) \subset L^\infty(X)$ be the C^*-subalgebra generated by the algebras $\mathcal{B}(\overline{X/Y})$ when Y runs over the set of all linear subspaces of X. The algebra $\mathcal{E}^\sharp(X)$ is, in some sense, a natural extension of $\mathcal{E}(X)$, cf. Proposition 6.24.

1.3. We now give the several examples of operators affiliated to $\mathscr{E}(X)$. First, consider pseudo-differential operators of the form

$$H = h(p) + v \tag{1.3}$$

where $h : X^* \to \mathbb{R}$ is a continuous proper function and $v \in \mathcal{E}^\sharp(X)$ is a real function. Here

$$h(p) := \mathcal{F}^{-1} m_h \mathcal{F}, \tag{1.4}$$

where $\mathcal{F}$ is a Fourier transform $L^2(X) \to L^2(X^*)$ and m_h denotes the operator of multiplication by h. Recall that a function $h : X^* \to \mathbb{R}$ is said to be *proper* if $|h(k)| \to \infty$ for $k \to \infty$. It is clear that the operator H given by (1.3) is self-adjoint on the domain of $h(p)$, since $h(p)$ is self-adjoint by spectral theory and v is a bounded operator.

As a second example of affiliated operators, we consider differential operators on $X = \mathbb{R}^n$ of the form

$$L = \sum_{|\mu|,|\nu| \leqslant m} p^\mu g_{\mu\nu} p^\nu \tag{1.5}$$

where $m \geqslant 1$ is an integer and $g_{\mu\nu} \in \mathcal{E}^\sharp(X)$. The notations are standard: $p_j = -\mathrm{i}\partial_j$, where ∂_j is the derivative with respect to the j-th variable, and for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ we set $p^\mu = p_1^{\mu_1} \cdots p_n^{\mu_n}$ and $|\mu| = \mu_1 + \cdots + \mu_n$. For real s, let $\mathcal{H}^s$ be the usual Sobolev space on X, in particular $\mathcal{H}^0 \equiv \mathcal{H} = L^2(X)$. Then $L : \mathcal{H}^m \to \mathcal{H}^{-m}$ is a well defined operator and we assume that there exist $\gamma, \delta > 0$ such that

$$\langle u|Lu\rangle + \gamma\|u\|^2 \geqslant \delta\|u\|_m^2, \quad \text{for all } u \in \mathcal{H}^m. \tag{1.6}$$

Here $\|\cdot\|$ and $\|\cdot\|_m$ denote the usual norms on $\mathcal{H}$ and $\mathcal{H}^m$. Note that, since the $g_{\mu\nu}$ are bounded, this is a condition only on the principal part of L (i.e. the part corresponding to $|\mu| = |\nu| = m$). Then $L + \gamma : \mathcal{H}^m \to \mathcal{H}^{-m}$ is a symmetric isomorphism, and hence the restriction of L to $(L+\gamma)^{-1}\mathcal{H}$ is a self-adjoint operator in $\mathcal{H}$ that we will denote by H:

$$H := L : (L+\gamma)^{-1}\mathcal{H} \to \mathcal{H}. \tag{1.7}$$

THEOREM 1.2. *Both operators H defined above in equations* (1.5) *and* (1.7) *are affiliated to $\mathscr{E}(X)$.*

We make some remarks in connection with the Theorem 1.2.

(i) If H is given by (1.3) then the strong limit in Theorem 1.1 exists in the usual sense of pointwise convergence on the domain of H. (Let us notice that in this case, the domain of H is invariant for the action of T_{ra}.) If H is associated to the operator L from (1.5), then the limit holds in the strong topology of $B(\mathcal{H}^m, \mathcal{H}^{-m})$.

(ii) The union that gives $\sigma_{\mathrm{ess}}(H)$ in Theorem 1.1 may contain an infinite number of distinct terms even in simple N-body type cases. Indeed, an example can be obtained by choosing $X = \mathbb{R}^2$, $\mathcal{Y}$ to be a countable set of lines (whose union could be dense in X), and $H := \Delta + \sum_Y v_Y$ for some conveniently chosen $v_Y \in \mathcal{C}_{\mathrm{c}}(X/Y)$ satisfying $\sum_Y \sup|v_Y| < \infty$.

(iii) The coefficients $g_{\mu\nu}$ in the principal part of L are bounded Borel functions, and locally this cannot be improved. But the other coefficients $g_{\mu\nu}$ and the potential v are assumed bounded only for the sake of simplicity, see Remark 6.25 for more general results. Later on, we shall also treat unbounded, not necessarily local perturbations. See for example Theorems 1.6 and 1.10.

(iv) We stated the applications of the abstract theorems in a way adapted to elliptic operators, but the extension to hypoelliptic operators is easy: it suffices to consider functions $h \in C^m$ with bounded derivatives of order m and to replace the Sobolev spaces by spaces associated to weights of the form $\sum_{|\mu|\leqslant m} |h^{(\mu)}(k)|$.

(v) Theorem 1.2 and the other results of the same nature remain true, with essentially no change in the proof, if the space $L^2(X)$ is replaced by $L^2(X) \otimes E$ with E a finite dimensional complex Hilbert space and $g_{\mu\nu}$ and v are $\mathcal{B}(E)$-valued functions. For this it suffices to work with the algebra $\mathscr{E}(X) \otimes \mathcal{B}(E)$ or the more general and natural object $\mathscr{E}(X) \otimes \mathcal{K}(E)$ where E can be an infinite dimensional Hilbert space. This covers matrix differential operators, e.g. the Dirac operator, which are not semi-bounded, and hence the general affiliation criterion Theorem 5.7 has to be used.

Operators of the form (1.3) (and hence also Theorem 1.2 and its generalizations) cover many of the most interesting (from a physical point of view) Hamiltonians of N-body systems. Here are two typical examples. First, in the non-relativistic case, X is equipped with a Euclidean structure and a typical choice for h is $h(\xi) = |\xi|^2$, which gives $h(p) = \Delta$. Second, in the case of N relativistic particles of spin zero and masses $m_1, \ldots, m_N$, we take $X = (\mathbb{R}^3)^N$ and, writing the momentum p as $p = (p_1, \ldots, p_N)$ where $p_j = -\mathrm{i}\nabla_j$ acts in $L^2(\mathbb{R}^3)$, we have

$$h(p) = \sum_{j=1}^{N} (p_j^2 + m_j^2)^{1/2}.$$

We refer to [12] for a thorough study of the spectral and scattering theory of the non-relativistic N-body Hamiltonians with k-body potentials that tend to zero at infinity. We also note that second order perturbations with an N-body structure of the Laplacian, i.e. operators L of second order with non trivial $g_{\mu\nu}$ in the principal part, are of physical interest in the context of pluristratified media [13].

1.4. The structure of the potential v and of the coefficients $g_{\mu\nu}$ considered above is more complicated than in the usual case of N-body hamiltonians because it can contain products of the form $v_E \circ \pi_E \cdot v_F \circ \pi_F$, which cannot be written as $v_G \circ \pi_G$ as in the usual N-body case (here E, F, G are subspaces of X and in the usual N-body situation one may take $G = E \cap F$). If v has a simpler structure, similar to that of the standard N-body potentials, then Theorem 1.2 may be reformulated in a way that stresses the similarity with the usual HVZ theorem. Moreover, in this case we will be able to treat a considerably more general class of nonlocal potentials v_Y (see Subsection 1.5).

Assume that, for each subspace $Y \subset X$, a real function $v_Y \in \mathcal{B}(\overline{X/Y})$ is given such that $v_Y = 0$ for all but a finite number of subspaces Y and let $v = \sum_Y v_Y \in \mathcal{E}^\sharp(X)$ (recall the identification $v_Y \equiv v_Y \circ \pi_Y$). If $\alpha \not\subset Y$ then $\pi_Y(\alpha) \in \mathbb{S}_{X/Y}$ is a well defined half-line in the quotient and we may define $v_Y(\alpha) = \operatorname*{m-lim}_{\pi_Y(\alpha)} v_Y$ (see Subsection 1.2 page 335).

PROPOSITION 1.3. *For each $\alpha \in \mathbb{S}_X$, let*

$$H_\alpha := h(p) + \sum_{Y \supset \alpha} v_Y + \sum_{Y \not\supset \alpha} v_Y(\alpha). \tag{1.8}$$

Then $\alpha \cdot H = H_\alpha$ and hence $\sigma_{\mathrm{ess}}(H) = \overline{\bigcup_{\alpha \in \mathbb{S}_X}} \sigma(H_\alpha)$.

REMARK 1.4. The usual N-body type Hamiltonians are characterized by the condition that all the $v_Y : X/Y \to \mathbb{R}$ vanish at infinity. Then we obtain $\alpha \cdot H = h(P) + \sum_{Y \supset \alpha} v_Y$, so Proposition 1.3 becomes the usual version of the HVZ theorem.

1.5. We give now further examples of self-adjoint operators with nonlocal and unbounded interaction affiliated to $\mathscr{E}(X)$. Let M_k be the multiplication operator $(M_k f)(x) = \mathrm{e}^{\mathrm{i}k(x)} f(x)$ for $k \in X^*$. Later we shall use the notation $\langle x|k\rangle := k(x)$.

If the perturbation V is bounded, then there are no restrictions on h besides being proper and continuous. Indeed, we have the following result.

THEOREM 1.5. *Let $H = h(p) + V$, where $h : X^* \to \mathbb{R}$ is a continuous, proper function and $V = \sum_Y V_Y$ is a finite sum with V_Y bounded symmetric linear operators on $L^2(X)$ satisfying:*

(i) $\lim_{k\to 0} \|[M_k, V_Y]\| = 0$*;*

(ii) $[T_y, V_Y] = 0$ *for all* $y \in Y$*;*

(iii) $\operatorname{s-lim}_{a\in X/Y, a\to\alpha} T_a^* V_Y T_a$ *exists for each* $\alpha \in \mathbb{S}_{X/Y}$*.*

Then H is affiliated to $\mathscr{E}(X)$.

Note that in the above Theorem 1.5, the operator $T_x^* V_Y T_x$ depends a priori on the point x in X, but if condition (ii) is satisfied, then it depends only on the class $\pi_Y(x)$ of x in X/Y. Therefore we may set $T^*_{\pi_Y(x)} V_Y T_{\pi_Y(x)} = T_x^* V_Y T_x$, which gives a meaning to $T_a^* V_Y T_a$ for any $a \in X/Y$ in condition (iii) above.

In order to treat unbounded interactions, we have to require more regularity on the function h. We denote by $|\cdot|$ a Euclidean norm on X^*.

THEOREM 1.6. *Let $h : X^* \to [0, \infty)$ be locally Lipschitz with derivative h' such that for some real numbers $c, s > 0$ and all $k \in X^*$ with $|k| > 1$*

$$c^{-1}|k|^{2s} \leqslant h(k) \leqslant c|k|^{2s} \quad \text{and} \quad |h'(k)| \leqslant c|k|^{2s}. \tag{1.9}$$

Let $V = \sum V_Y$ be a finite sum with $V_Y : \mathcal{H}^s \to \mathcal{H}^{-s}$ symmetric operators satisfying:

(i) *there are numbers γ, δ with $\gamma < 1$ such that $V \geqslant -\gamma h(p) - \delta$;*

(ii) $\lim_{k\to 0} \|[M_k, V_Y]\|_{\mathcal{H}^s \to \mathcal{H}^{-s}} = 0$*;*

(iii) $[T_y, V_Y] = 0$ *for all* $y \in Y$*;*

(iv) $\operatorname{s-lim}_{a\in X/Y, a\to\alpha} T_a^* V_Y T_a$ *exists in $B(\mathcal{H}^s, \mathcal{H}^{-s})$ for all* $\alpha \in \mathbb{S}_{X/Y}$*.*

Then $h(p) + V$ is a symmetric operator $\mathcal{H}^s \to \mathcal{H}^{-s}$, which induces a self-adjoint operator H in $L^2(X)$ affiliated to $\mathscr{E}(X)$.

REMARK 1.7. If V_Y is the operator of multiplication by a measurable function, then condition (ii) of Theorem 1.6 is automatically satisfied. On the other hand, condition (iii) gives that $V_Y(x + y) = V_Y(x)$ for all $x \in X$ and $y \in Y$. This means that $V_Y = v_Y \circ \pi_Y$ for a measurable function $v_Y : X/Y \to \mathbb{R}$, which has to be such that the operator of multiplication by V_Y is a continuous map $\mathcal{H}^s(X) \to \mathcal{H}^{-s}(X)$. For this it suffices that the operator $v_Y(q_Y)$ of multiplication by v_Y be a continuous map of $\mathcal{H}^s(X/Y)$ into $\mathcal{H}^{-s}(X/Y)$ (here q_Y is the position observable in $L^2(X/Y)$). Then the last condition means that $\lim_{a\to\alpha} v_Y(q_Y + a)$ exists strongly in $\mathcal{B}(\mathcal{H}^s(X/Y), \mathcal{H}^{-s}(X/Y))$.

EXAMPLE 1.8. Let us consider the case of non-relativistic Schrödinger operators. Then X is a Euclidean space (so we may identify $X/Y = Y^\perp$) and $H_0 := \Delta = p^2$ is the (positive) Laplace operator, and hence $s = 1$. The total Hamiltonian is of the form $H = \Delta + \sum_Y V_Y$ where the sum is finite and $V_Y = 1 \otimes V_Y^\circ$ where $V_Y^\circ : \mathcal{H}^1(Y^\perp) \to \mathcal{H}^{-1}(Y^\perp)$ is a symmetric linear operator whose relative form bound with respect to the Laplace operator on $Y^\perp$ is zero (this is much more than we need). Then assume $M_k V_Y^\circ = V_Y^\circ M_k$ for all $k \in Y^\perp$. For example, V_Y° could be the operator of multiplication by a function $v_Y : Y^\perp \to \mathbb{R}$ of Kato class $K_{n(Y)}$ with $n(Y) = \dim(Y^\perp)$ (see Section 1.2 in [8], especially assertion (2) page 8) but it could also be a distribution of non zero order. Indeed, we may take as v_Y the divergence of a vector field on $Y^\perp$ whose components have squares of Kato class (e.g. are bounded functions): this covers highly oscillating perturbations of potentials that have radial limits at infinity. Note that this Kato class is convenient because then $v_Y \circ \pi_Y$ is of class $K_{\dim(X)}$, see p. 8 of [8]. To get (iv) of Theorem 1.6 it suffices to assume $\lim_{a\to\alpha} v_Y(\cdot + a)$ exists strongly in $B(\mathcal{H}^1(Y^\perp), \mathcal{H}^{-1}(Y^\perp))$ for each $\alpha \in \mathbb{S}_{Y^\perp}$.

1.6. The *spherical algebra* $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X \subset \mathscr{E}(X)$ has several interesting properties. For example, it contains the ideal $\mathscr{K}(X) = \mathcal{C}_0(X) \rtimes X$ of compact operators on $L^2(X)$. It is remarkable that both $\mathscr{S}(X)$ and its quotient $\mathscr{S}(X)/\mathscr{K}(X)$ may be described in quite explicit terms. In the next theorem and in what follows, we adopt the following convention: if we write $S^{(*)}$ in a relation, then it means that relation holds for $S^{(*)}$ replaced by either S or S^*. Let $C^*(X)$ be the group C^*-algebra of X, cf. Section 2.

THEOREM 1.9. *A bounded operator S on $L^2(X)$ belongs to $\mathscr{S}(X)$ if and only if*

$$\lim_{x\to 0} \|(T_x - 1)S^{(*)}\| = 0, \quad \lim_{k\to 0} \|[M_k, S]\| = 0, \quad \textit{and}$$
$$\operatorname*{s-lim}_{a\to\alpha} T_a^* S^{(*)} T_a \quad \textit{exists for any } \alpha \in \mathbb{S}_X.$$

If $S \in \mathscr{S}(X)$ and $\alpha \in \mathbb{S}_X$, then $\tau_\alpha(S) = \operatorname*{s-lim}_{a\to\alpha} T_a^* S T_a$ belongs to $C^*(X)$. The map $\tau(S) : \alpha \mapsto \tau_\alpha(S)$ is norm continuous, so $\tau : \mathscr{S}(X) \to C(\mathbb{S}_X) \otimes C^*(X)$. This map τ is a surjective morphism and its kernel is $\mathscr{K}(X)$. Hence we have a natural identification

$$\mathscr{S}(X)/\mathscr{K}(X) \cong \mathcal{C}(\mathbb{S}_X) \otimes C^*(X) \cong \mathcal{C}_0(\mathbb{S}_X \times X^*). \tag{1.10}$$

If H is a self-adjoint operator affiliated to $\mathscr{S}(X)$, then the limit $\alpha \cdot H := \operatorname*{s-lim}_{a\to\alpha} T_a^* H T_a$ exists for each $\alpha \in \mathbb{S}_X$ and $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.

Note that in this theorem (as well as in the next), we consider the plain union, not its closure. The next result is a general criterion of affiliation to $\mathscr{S}(X)$.

THEOREM 1.10. *Let H be a bounded from below self-adjoint operator on $L^2(X)$ such that its form domain $\mathcal{G}$ satisfies the following condition: the operators T_x and M_k*

leave $\mathcal{G}$ invariant, the operators T_x are uniformly bounded in $\mathcal{G}$, and $\lim_{x\to 0}\|T_x - 1\|_{\mathcal{G}\to\mathcal{H}} = 0$. *Assume that* $\|[M_k, H]\|_{\mathcal{G}\to\mathcal{G}^*} \to 0$ *as* $k \to 0$ *and that the limit* $\alpha \cdot H := \lim_{a\to\alpha} T_a^* H T_a$ *exists strongly in* $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, *for all* $\alpha \in \mathbb{S}_X$. *Then H is affiliated to $\mathscr{S}(X)$, for each* $\alpha \in \mathbb{S}_X$ *the operator in $L^2(X)$ associated to $\alpha \cdot H$ is self-adjoint, and* $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.

Let us notice an important difference between the morphisms τ_α of Theorems 1.1 and 1.10. (These morphisms appear in the computation of the quotients in (1.2) and (1.10).) More precisely, in the definition of the first τ, we take limits over ra, with $r \to \infty$, that are limits along rays, whereas in the second one we take general limits $a \to \alpha$ (not just along the ray α). The stronger assumptions in Theorem 1.10 then lead to a stronger result (in that we do not need the closure of the union to obtain the spectrum).

1.7. Descriptions of the essential spectrum of various classes of Hamiltonians in terms of limits at infinity of translates of the operators have already been obtained before, see for example [27], [21], [43], [31], [47] (in historical order). Our approach is based on the "localization at infinity" technique developed in [21], [22] in the context of crossed-products of C^*-algebras by actions of abelian locally compact groups. This has been extended to noncommutative unimodular amenable locally compact groups in [20], cf. Proposition 6.5 and Theorem 6.8 there. The case of noncommutative groups has also been considered recently in [35]. See [8], [44] for a general introduction to the basics of the problems studied here.

A homogeneous potential of degree zero outside of a compact set models a force that is perpendicular to the line joining the particle to the origin, and hence trying to force the particle to move on a sphere. Results on operators with homogeneous potentials or similar potentials were obtained, for example, in [26], [28], [29], [46], where further physical motivation is provided.

In fact, our results shed some new light even on the classical case when the auxiliary functions v_Y that define V_Y converge to 0 at infinity, since in our case the spectra of the relevant algebras are easier to compute and then can be used to describe the spectra in the classical case. Compared to the classical approach [8], [44] to the essential spectrum of the N-body problem, our approach has the advantage that it is more conceptual, and, once a certain machinery has been developed, one can obtain rather quickly generalizations of these results to other operators. It also takes advantage of a rather well developed theory of crossed products and representations of C^*-algebras. We also develop general techniques that may be useful for the study of other types of operators and of other types of questions, such as the study of the eigenvalues and eigenfunctions of H, even in the case when the radial limits at infinity are zero. We mention that, by using the expression (1.8) for the asymptotic operators, one could prove the Mourre estimate as in Section 9.4 of [2] for the larger class of Hamiltonians considered in Proposition 1.3.

1.8. Let us briefly describe the contents of the paper. In Section 2, we recall some facts concerning crossed products with X of translation invariant C^*-algebras of bounded uniformly continuous functions on X and the role of operators with the "position-momentum limit property" in this context. Then we discuss the question of the computation of the quotient with respect to the compacts of such crossed products. In Section 3, we briefly describe the topology and the continuous functions on the spherical compactification $\overline{X}$ of a real vector space X. This allows us to introduce and study in Section 4 the spherical algebra $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$. We obtain an explicit description of the operators that belong to $\mathscr{S}(X)$ (Theorem 4.2) and we give an explicit description of the quotient $\mathscr{S}(X)/\mathscr{K}(X)$ (Theorem 4.3). The canonical composition series of this algebra leads to Fredholm conditions, and hence to a determination of the essential spectrum for the operators affiliated to it. In Section 5, we give some general criteria for a self-adjoint operator to be affiliated to a general C^*-algebra and apply them to the case of $\mathscr{S}(X)$. The algebras $\mathcal{E}(X)$ and $\mathscr{E}(X)$ are studied in Section 6. Subsections 6.3 and 6.4 contain the main technical results. At a technical level, the main result in this section is the description of the spectrum of $\mathcal{E}(X)$ (Theorem 6.13). Subsection 6.4 is devoted to the study of the Hamiltonian algebra $\mathscr{E}(X)$ and we prove in this section two of our main results, Theorems 6.17 and 6.20. In Subsection 6.5 we prove Theorems 1.2, 1.5 and 1.6, which describe a general class of operators affiliated to $\mathscr{E}(X)$ for which we obtain explicit descriptions of the essential spectrum. We note that Theorem 6.26 gives descriptions of the algebras $\mathcal{C}(\overline{X/Y}) \rtimes X$ generating $\mathscr{E}(X)$ that are not relying on their definition as crossed products.

This paper contains the full proofs of the results announced in [23], as well as several extensions of those results.

2. CROSSED PRODUCTS AND LOCALIZATIONS AT INFINITY

In this section, we review some needed results from [22] relating essential spectra of operators and the spectrum (or character space) of some algebras.

If X is a finite dimensional vector space, we denote by $\mathcal{C}_{\mathrm{b}}(X)$ the algebra of bounded continuous functions on X, by $\mathcal{C}_0(X)$ its ideal consisting of functions vanishing at infinity, and by $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$ the subalgebra of bounded uniformly continuous functions. Let $\mathscr{B}(X) := \mathcal{B}(L^2(X))$ be the algebra of bounded operators on $L^2(X)$ and $\mathscr{K}(X) := \mathcal{K}(L^2(X))$ the ideal of compact operators.

If Y is a subspace of X, we identify a function u on X/Y with the function $u \circ \pi_Y$ on X. In other terms, we think of a function on X/Y as being a function on X that is invariant under translations by elements of Y. This clearly gives an embedding $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X/Y) \subset \mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$. The subalgebras of $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X/Y)$ can then be thought of as subalgebras of $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$. Thus $\mathcal{C}_0(X/Y)$ and the algebra $\mathcal{C}(\overline{X/Y})$ that we shall introduce below are both embedded in $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$.

For any function u, we shall denote by m_u the operator of multiplication by u on suitable L^2 spaces. If $u : X \to \mathbb{C}$ and $v : X^* \to \mathbb{C}$ are measurable functions, then $u(q)$ and $v(p)$ are the operators on $L^2(X)$ defined as follows: $u(q) = m_u$, the multiplication operator by u, and $v(p) = \mathcal{F}^{-1} m_v \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform $L^2(X) \to L^2(X^*)$. If $x \in X$ and $k \in X^*$, then the unitary operators T_x and M_k are defined on $L^2(X)$ by

$$(T_x f)(y) := f(y - x) \quad \text{and} \quad (M_k f)(y) := \mathrm{e}^{\mathrm{i}\langle y|k\rangle} f(y), \tag{2.1}$$

and can alternatively be written in terms of p and q as $T_x = \mathrm{e}^{-\mathrm{i}xp}$ and $M_k = \mathrm{e}^{\mathrm{i}kq}$.

We shall denote by $C^*(X)$ the *group C^*-algebra* of X: this is the closed subspace of $\mathscr{B}(X)$ generated by the operators of convolution with continuous, compactly supported functions. The map $v \mapsto v(p)$ establishes an isomorphism between $\mathcal{C}_0(X^*)$ and $C^*(X)$.

We shall need the following general result about commutative C^*-algebras. Let $\mathcal{A}$ be a commutative C^*-algebra and $\widehat{\mathcal{A}}$ be its spectrum (or character space), consisting of non-zero algebra morphisms $\chi : \mathcal{A} \to \mathbb{C}$. If $\mathcal{A}$ is unital, then $\widehat{\mathcal{A}}$ is a compact topological space for the weak topology. In general, it is locally compact and the Gelfand transform $\Gamma_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}_0(\widehat{\mathcal{A}})$, $\Gamma_{\mathcal{A}}(u)(\chi) := \chi(u)$, defines an isometric algebra isomorphism. In particular, any commutative C^*-algebra is of the form $\mathcal{C}_0(\Omega)$ for some locally compact space (up to isomorphism). The characters of $\mathcal{C}_0(\Omega)$ are of the form χ_ω, $\omega \in \Omega$, where

$$\chi_\omega(u) := u(\omega) \quad u \in \mathcal{C}_0(\Omega). \tag{2.2}$$

If X acts continuously on a C^*-algebra $\mathcal{A}$ by automorphisms, we shall denote by $\mathcal{A} \rtimes X$ the resulting crossed product algebra, see [41], [51]. Here the real vector space X is regarded as a locally compact, abelian group in the obvious way. Recall [21] that if $\mathcal{A}$ is a translation invariant C^*-subalgebra of $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$, then an isomorphic realization of the cross-product algebra $\mathcal{A} \rtimes X$ is the norm closed linear subspace of $\mathscr{B}(X)$ generated by the operators of the form $u(q)v(p)$, where $u \in \mathcal{A}$ and $v \in \mathcal{C}_0(X^*)$. As a rule, we shall denote by τ_a the action of $a \in X$ by translations on our algebras of functions.

DEFINITION 2.1. Let $A \in \mathscr{B}(X)$. We say that *A has the position-momentum limit property* if $\lim_{x\to 0} \|(T_x - 1)A^{(*)}\| = 0$ and $\lim_{k\to 0} \|[M_k, A]\| = 0$.

A characterization of operators having the position-momentum limit property in terms of crossed products was given in [21]: it is shown that *A has the position-momentum limit property if and only if $A \in \mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X) \rtimes X$.*

If A is an operator on $L^2(X)$, then its translation by $x \in X$ is defined by the relation

$$\tau_x(A) := T_x^* A T_x. \tag{2.3}$$

The notation $x \cdot A := \tau_x(A)$ will often be more convenient. If u is a function on X we also denote $\tau_x(u) \equiv x \cdot u$ its translation given by $(x \cdot u)(y) = u(x + y)$.

The notations are naturally related: $\tau_x(u(q)) = (x \cdot u)(q) \equiv u(x+q)$. Note that $\tau_x(v(p)) = v(p)$.

By a "point at infinity" of X, we shall mean a point in the boundary of X in a certain compactification of it. We shall next define *the translation by a point at infinity* χ for certain functions u and operators A. This construction will be needed for the description of the essential spectrum of operators of interest for us.

Let us fix a translation invariant C^*-algebra $\mathcal{A}$ of bounded uniformly continuous functions on X containing the functions that have a limit at infinity: $\mathcal{C}_0(X) + \mathbb{C} \subset \mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$. To every $x \in X$, there is an associated character χ_x, defined by $\chi_x(u) := u(x)$ for $u \in \mathcal{A}$, cf. (2.2). Since $\mathcal{A} \supset \mathcal{C}_0(X)$, X is naturally embedded as an open dense subset in $\widehat{\mathcal{A}}$. Thus $\widehat{\mathcal{A}}$ is a compactification of X and

$$\delta(\mathcal{A}) := \widehat{\mathcal{A}} \setminus X, \tag{2.4}$$

the *boundary* of X in this compactification, is a compact set that can be characterized as the set of characters χ of $\mathcal{A}$ whose restriction to $\mathcal{C}_0(X)$ is equal to zero.

Let us recall that if $x, y \in X$, then $(x \cdot u)(y) = u(x+y) = \chi_x(y \cdot u)$. If $u \in \mathcal{A}$, we extend the definition of $x \cdot u$ by replacing in this relation χ_x with a character $\chi \in \widehat{\mathcal{A}}$.

DEFINITION 2.2. Let $u \in \mathcal{A}$ and $\chi \in \widehat{\mathcal{A}}$. Then we define $(\chi \cdot u)(y) := \chi(y \cdot u)$ for all $y \in X$.

Since u is uniformly continuous, it is easy to check that $\tau_\chi(u) := \chi \cdot u \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ and that $\tau_\chi : \mathcal{A} \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ is a unital morphism. We will say that τ_χ is the *morphism associated to the character* χ. We note that if the character χ corresponds to $x \in X$, then $\tau_\chi = \tau_x$, so our notation is consistent.

In particular, we get "translations at infinity" of $u \in \mathcal{A}$ by elements $\chi \in \delta(\mathcal{A})$. The function $\chi \mapsto \chi \cdot u \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ defined on $\widehat{\mathcal{A}}$ is continuous if $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ is equipped with the topology of local uniform convergence, and hence $\chi \cdot u = \lim_{x \to \chi} x \cdot u$ in this topology for any $\chi \in \delta(\mathcal{A})$. One has $u \in \mathcal{C}_0(X)$ if and only if $\chi \cdot u = 0$ for all $\chi \in \delta(\mathcal{A})$. We mention that a translation $\chi \cdot u$ by a point at infinity $\chi \in \delta(\mathcal{A})$ does not belong to $\mathcal{A}$ in general. However, we shall see that this is true in the case $\mathcal{A} = \mathcal{E}(X)$ of interest for us, so in this case τ_χ is an endomorphism of $\mathcal{A}$.

If $A \in \mathcal{A} \rtimes X$, then we may also consider "translations at infinity" $\tau_\chi(A)$ by elements χ of the boundary $\delta(\mathcal{A})$ of X in $\widehat{\mathcal{A}}$ and we get a useful characterization of the compact operators. The following facts are proved in Subsection 5.1 of [22].

PROPOSITION 2.3. *For each $\chi \in \widehat{\mathcal{A}}$, there is a unique morphism $\tau_\chi : \mathcal{A} \rtimes X \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ such that*

$$\tau_\chi(u(q)v(p)) = (\chi \cdot u)(q)v(p), \quad \textit{for all } u \in \mathcal{A}, v \in \mathcal{C}_0(X).$$

If $A \in \mathcal{A} \rtimes X$, then $\chi \mapsto \tau_\chi(A)$ is a strongly continuous map $\widehat{\mathcal{A}} \to \mathscr{B}(X)$.

As before, we often abbreviate $\tau_\chi(A) = \chi \cdot A$. This gives a meaning to the translation by χ of any operator $A \in \mathcal{A} \rtimes X$ and any character $\chi \in \widehat{\mathcal{A}}$. Observe that $\chi \mapsto \chi \cdot A$ is just *the continuous extension to $\widehat{\mathcal{A}}$ of the strongly continuous map* $X \ni x \mapsto x \cdot A$. In particular,

$$\tau_\chi(A) = \operatorname*{s-lim}_{x \to \chi} T_x^* A T_x \quad \text{for all } A \in \mathcal{A} \rtimes X \text{ and } \chi \in \delta(\mathcal{A}). \tag{2.5}$$

We have $\mathscr{K}(X) = \mathcal{C}_0(X) \rtimes X \subset \mathcal{A} \rtimes X$. Then Theorem 1.15 of [22] gives the following.

THEOREM 2.4. *An operator $A \in \mathcal{A} \rtimes X$ is compact if and only if $\tau_\chi(A) = 0$ for all $\chi \in \delta(\mathcal{A})$. In other terms: $\bigcap_{\chi \in \delta(\mathcal{A})} \ker \tau_\chi = \mathscr{K}(X)$. The map $\tau(A) = (\tau_\chi(A))_{\chi \in \delta(\mathcal{A})}$ induces an injective morphism*

$$\mathcal{A} \rtimes X / \mathscr{K}(X) \hookrightarrow \prod_{\chi \in \delta(\mathcal{A})} \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X. \tag{2.6}$$

REMARK 2.5. We emphasize the relation between this result and some facts from the theory of crossed products. The operation of taking the crossed product by the action of an amenable group transforms exact sequences in exact sequences ([51], Proposition 3.19), and hence we have an exact sequence

$$0 \to \mathcal{C}_0(X) \rtimes X \to \mathcal{A} \rtimes X \to (\mathcal{A}/\mathcal{C}_0(X)) \rtimes X \to 0. \tag{2.7}$$

Since $\mathcal{C}_0(X) \rtimes X \simeq \mathscr{K}(X)$, we get $\mathcal{A} \rtimes X/\mathscr{K}(X) \simeq (\mathcal{A}/\mathcal{C}_0(X)) \rtimes X$, which reduces the computation of the quotient $\mathcal{A} \rtimes X/\mathscr{K}(X)$ to a description of $\mathcal{A}/\mathcal{C}_0(X)$. This is convenient since $\mathcal{A}/\mathcal{C}_0(X) \simeq \mathcal{C}(\delta(\mathcal{A}))$. Moreover, we have $\tau_\chi = \tau_\chi \rtimes \mathrm{id}_X$, where the morphisms τ_χ on the right hand side are those corresponding to $\mathcal{A}$. We complete this remark by noticing that if χ and χ_1 are obtained from each other by a translation by $x \in X$, then the corresponding morphisms τ_χ and τ_{χ_1} are unitarily equivalent by the unitary corresponding to x. In particular, in the above theorem and in the following corollary, it suffices to use one χ from each orbit of X acting on $\delta(\mathcal{A})$.

REMARK 2.6. Let us notice that in view of the results in [14], [51], the above theorem provides nontrivial information on the cross-product algebra $\mathcal{C}(\delta(\mathcal{A})) \rtimes X$, and hence on the action of X on $\delta(\mathcal{A})$. It would be interesting to study the corresponding properties for a general Lie group G acting on $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(G)$ [35]. Morphisms analogous to the τ_χ can be defined also in a groupoid framework [32], [38], but they do not have a similar, simple interpretation as strong limits. It would be interesting to understand the connections between the above theorem and the representation theory of groupoids [6], [7], [15], [30], [45]. Similar structures arise also in the representation theory of solvable Lie groups [4]. Moreover, several important examples of non-compact manifolds that arise in other problems lead to groupoids that are *locally* of the form studied in this paper (but possibly replacing X by a general Lie group G, see [24], [25], [36] and many other papers).

Let A be a bounded operator. By definition, $\lambda \notin \sigma_{\text{ess}}(A)$ if and only if $A - \lambda$ is Fredholm. For a self-adjoint operator A, this is equivalent to the usual definition: "$\lambda \in \sigma_{\text{ess}}(A)$ if and only if λ is an accumulation point of $\sigma(A)$ or an isolated eigenvalue of infinite multiplicity". The advantage of this second definition is that it extends right away to unbounded, normal operators (see, for instance, Remark 2.9). A crucial observation then is that $\lambda \notin \sigma_{\text{ess}}(A)$ if and only if the image $\widehat{A-\lambda}$ of $A - \lambda$ in the quotient $\mathscr{B}(X)/\mathscr{K}(X)$ is invertible, by Atkinson's theorem. So $\sigma_{\text{ess}}(A) = \sigma(\widehat{A})$. On the other hand, the spectrum of a normal operator in a product of C^*-algebras is equal to the closure of the union of the spectra of its components. Thus the theorem above gives right away the following corollary.

COROLLARY 2.7. *If $A \in \mathcal{A} \rtimes X$ is normal, then* $\sigma_{\text{ess}}(A) = \overline{\bigcup}_{\chi \in \delta(\mathcal{A})} \sigma(\tau_\chi(A))$.

If $A \in \mathscr{B}(X)$, then the element $\widehat{A} \in \mathscr{B}(X)/\mathscr{K}(X)$ may be called the *localization at infinity* of A. If $A \in \mathcal{A} \rtimes X$, then its localization at infinity can be identified with the element $\tau(A) = (\tau_\chi(A))_{\chi \in \delta(\mathcal{A})}$. Then the component $\tau_\chi(A) \in \mathcal{C}_{\text{b}}^{\text{u}}(X) \rtimes X$ is called *localization of A at $\chi \in \delta(\mathcal{A})$*. Thus *the essential spectrum of $A \in \mathcal{A} \rtimes X$ is the closure of the union of the spectra of all its localizations at infinity*, where the "infinity" is determined by $\mathcal{A}$.

We extend now the notion of localization at infinity and the formula for the essential spectrum to certain unbounded self-adjoint operators related to $\mathcal{A} \rtimes X$. Recall that a self-adjoint operator H on a Hilbert space $\mathcal{H}$ is *affiliated* to a C^*-algebra $\mathscr{C} \subset \mathcal{B}(\mathcal{H})$ if $(H - z)^{-1} \in \mathscr{C}$ for some number z outside the spectrum of H [9]. Clearly, this implies $\varphi(H) \in \mathscr{C}$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$. We shall make some more comments on this notion after the next corollary.

COROLLARY 2.8. *If H is a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{A} \rtimes X$, then, for each $\chi \in \delta(\mathcal{A})$, the limit $\tau_\chi(H) := \operatorname{s-lim}_{x \to \chi} T_x^* H T_x$ exists and $\sigma_{\text{ess}}(H) = \overline{\bigcup}_{\chi \in \delta(\mathcal{A})} \sigma(\tau_\chi(H))$.*

The meaning of the limit above will be discussed below.

REMARK 2.9. Corollary 2.8 is an immediate consequence of Theorem 2.4 if one thinks in terms of the functional calculus associated to H. Indeed, a real λ does not belong to $\sigma_{\text{ess}}(H)$ if and only if there is $\varphi \in \mathcal{C}_0(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ such that $\varphi(H)$ is compact.

For a detailed discussion of the notion of affiliation that we use in this paper we refer to Section 8.1 of [2] (or Appendix A of [9]). This notion is inspired by the quantum mechanical concept of observable as introduced by J. von Neumann in the 1930s (see e.g. Section 3.2 of [49] for a general and precise mathematical formulation) and later (1940s) developed in the von Neumann algebra setting. A notion of affiliation in the C^*-algebra setting has also been introduced by S. Baaj and S.L. Woronowicz [3], [52] *but it is different from that we use here:* the contrary was erroneously stated in p. 534 of [21], but has been corrected in p. 278 of [9]. For example, any self-adjoint operator on a Hilbert space $\mathcal{H}$ is affiliated to the algebra of compact operators $\mathcal{K}(\mathcal{H})$ in the sense of Baaj–Woronowicz, but a self-adjoint operator is affiliated to $\mathcal{K}(\mathcal{H})$ in our sense if and only if it has purely discrete spectrum.

According to our definition, a self-adjoint operator affiliated to an "abstract" C^*-algebra $\mathscr{C}$ is the same thing as a real valued observable affiliated to $\mathscr{C}$, i.e. it is just a morphism $\Phi : \mathcal{C}_0(\mathbb{R}) \to \mathscr{C}$. If $\mathscr{C} \subset \mathcal{B}(\mathcal{H})$, then a densely defined self-adjoint operator H defines an observable by $\Phi(\varphi) = \varphi(H)$ for $\varphi \in \mathcal{C}_0(\mathbb{R})$, and we say that H is affiliated to $\mathscr{C}$ if this observable is affiliated to $\mathscr{C}$. But there are observables affiliated to $\mathscr{C}$ that are not of this form: they are associated to self-adjoint operators K acting in closed subspaces $\mathcal{K} \subset \mathcal{H}$ as explained in the next remark. See Section 8.1.2 of [2] for a precise statement and proof.

We now explain the meaning of $\operatorname*{s-lim}_{x\to\chi} T_x^* H T_x$ for an arbitrary self-adjoint operator H.

REMARK 2.10. Let Y be a topological space, z a point in Y, and let $\{H_y\}$ be a set of self-adjoint operators (possibly unbounded) on a Hilbert space $\mathcal{H}$, parametrized by $Y \setminus \{z\}$. The example that we have in mind is $H_x := T_x^* H T_x$, $x \in X$, and Y obtained from X by adding some point of a compactification. We say that $\operatorname*{s-lim}_{y\to z} H_y$ *exists* if the strong limit $\Phi(\varphi) := \operatorname*{s-lim}_{y\to z} \varphi(H_y)$ exists for each function $\varphi \in \mathcal{C}_0(\mathbb{R})$. It is easy to see that this is equivalent to the existence of $\operatorname*{s-lim}_{y\to z}(H_y - \lambda)^{-1}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. But we emphasize that *this does not mean that there is a self-adjoint operator K on $\mathcal{H}$ such that $\Phi(\varphi) = \varphi(K)$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$* if the notion of self-adjointness is interpreted in the usual sense, which requires the domain to be dense in $\mathcal{H}$. However, the following is true: there is a closed subspace $\mathcal{K} \subset \mathcal{H}$ and a self-adjoint operator (in the usual sense) K in $\mathcal{K}$ such that $\Phi(\varphi)\Pi_{\mathcal{K}} = \varphi(K)\Pi_{\mathcal{K}}$ and $\Phi(\varphi)\Pi_{\mathcal{K}}^{\perp} = 0$, where $\Pi_{\mathcal{K}}$ is the projection onto $\mathcal{K}$. The couple $(\mathcal{K}, K)$ is uniquely defined and we write $\operatorname*{s-lim}_{y} H_y = K$. One may have $\mathcal{K} = \{0\}$, in which case we write $\operatorname*{s-lim}_{y} H_y = \infty$. See the Remark 5.5 for an example.

3. SPHERICAL COMPACTIFICATION

As before, X is a finite dimensional real vector space. We now briefly discuss the definition of the spherical compactification $\overline{X}$ of X, its topology, and the definition of continuous functions on $\overline{X}$. Recall that the sphere at infinity $\mathbb{S}_X$ of X is the set of all half-lines $\alpha = \widehat{a} := \mathbb{R}_+ a$, with $\mathbb{R}_+ = (0, \infty)$ and $a \in X \setminus \{0\}$, equipped with the following topology: the open sets in $\mathbb{S}_X$ are the sets of the form $\{\widehat{a} : a \in O\}$ with O open in $X \setminus \{0\}$. Let us denote by $\overline{X}$ the disjoint union

$X \cup \mathbb{S}_X$. If $|\cdot|$ is an arbitrary norm on X, then $\mathbb{S}_X$ is homeomorphic to the unit sphere $S_X := \{|\xi| = 1\}$ in X and $\overline{X} = X \cup \mathbb{S}_X$ can be endowed with a natural topology that makes it homeomorphic to the closed unit ball in X. The resulting topological space $\overline{X}$ will be referred to as *the spherical compactification of* X and is discussed in detail in this subsection, since we need a good understanding of the continuous functions on $\overline{X}$.

It is convenient to have an explicit description of the topology of $\overline{X}$ independent of the choice of a norm. A *cone* C (in X) is a subset of X stable under the action of $\mathbb{R}_+$ by multiplication. Put differently, C is a union of half-lines. A *truncated cone* (in X) is the intersection of a cone with the complement of a bounded set. A half-line α *is eventually in the truncated cone* C if there is $a \in \alpha$ such that $\lambda a \in C$ if $\lambda \geqslant 1$. Let $C^\dagger \subset \mathbb{S}_X$ be the set of half-lines that are eventually in C. Then the sets of the form $C^\dagger$, with C an open truncated cone, form a base of the topology of $\mathbb{S}_X$. For any open truncated cone C in X, we denote $C^\ddagger := C \cup C^\dagger$. Then the open sets of X and the sets of the form $C^\ddagger$ form a base of the topology of $\overline{X}$. It is easy to see that $\overline{X}$ is a compact topological space in which X is densely and homeomorphically embedded. Moreover, $\overline{X}$ induces on $\mathbb{S}_X$ the (compact) topology we defined before.

By definition, a neighborhood of $\alpha \in \mathbb{S}_X$ in $\overline{X}$ is a set that contains a subset of the form $C^\ddagger$. We denote by $\widetilde{\alpha}$ the set of traces on X of the neighborhoods of α in $\overline{X}$. Thus, a set belongs to $\widetilde{\alpha}$ if and only if it contains an open truncated cone that eventually contains α. Let Y be a topological space and let $u : X \to Y$. If $\alpha \in \mathbb{S}_X$ and $y \in Y$, then the limit $\lim_{x \to \alpha} u(x)$ (or $\lim_\alpha u$) exists and is equal to y if and only if for each neighborhood V of y, there is a truncated cone C that eventually contains α such that $u(x) \in V$ if $x \in C$. We shall need the following simple lemma.

LEMMA 3.1. *Let $u : X \to \mathbb{C}$ be such that the limit $U(\alpha) := \lim_{x \to \alpha} u(x)$ exists for each $\alpha \in \mathbb{S}_X$. Then U is a continuous function on $\mathbb{S}_X$. If u is continuous on X, then its extension by U on $\mathbb{S}_X$ is continuous on $\overline{X}$.*

Proof. Let us notice first that $\lim_{\lambda \to \infty} u(\lambda a) = U(\alpha)$ for each $a \in \alpha \in \mathbb{S}_X$. Fix $\alpha \in \mathbb{S}_X$ and $\varepsilon > 0$. There is an open truncated cone C with $\alpha \in C^\dagger$ such that $|u(x) - U(\alpha)| < \varepsilon$ for all $x \in C$. If $\beta \in C^\dagger$, then, for each $b \in \beta$, we have $\lim_{\lambda \to \infty} u(\lambda b) = U(\beta)$. Since $\lambda b \in C$ for large λ, we get that $|U(\beta) - U(\alpha)| < \varepsilon$. Since the sets $C^\dagger$ form a basis of the topology of $\mathbb{S}_X$, we see that U is continuous.

To prove the last statement and thus to complete the proof, let us extend u to $\overline{X}$ to be equal to U on $\mathbb{S}_X$. Then the argument used in the first half of the proof implies that $|u(x) - u(\alpha)| < \varepsilon$ for all $x \in C^\ddagger$. Since the sets of the form $C^\ddagger$ form a basis for the system of neighborhoods of $\alpha \in \mathbb{S}_X$ in $\mathbb{S}_X$ and α is arbitrary, the extension of u to $\overline{X}$ by U is continuous on $\mathbb{S}_X$. Hence if u is continuous on X, then its extension to $\overline{X}$ is continuous everywhere. ∎

Since X is a dense subset of $\overline{X}$, we may identify the algebra $\mathcal{C}(\overline{X})$ of continuous functions on $\overline{X}$ with a subalgebra of $\mathcal{C}(X)$. We now give several descriptions of this subalgebra that are independent of the preceding construction of $\overline{X}$. Denote by $\mathcal{C}_{\mathrm{h}}(X)$ the subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ consisting of functions homogeneous of degree zero outside a compact set:

$$\mathcal{C}_{\mathrm{h}}(X) := \{u \in C(X) : \exists K \subset X \text{ compact with } u(\lambda x) = u(x) \text{ if } x \notin K, \lambda \geqslant 1\}. \tag{3.1}$$

LEMMA 3.2. *The algebra $\mathcal{C}(\overline{X})$ coincides with the closure of $\mathcal{C}_{\mathrm{h}}(X)$ in $\mathcal{C}_{\mathrm{b}}(X)$. A function $u \in \mathcal{C}(X)$ belongs to $\mathcal{C}(\overline{X})$ if, for any compact $A \subset X \setminus \{0\}$, the limit $\lim_{\lambda \to +\infty} u(\lambda a)$ exists uniformly in $a \in A$ and, in this case, for any compact $B \subset X$, we have*

$$\lim_{\lambda \to +\infty} u(\lambda a + b) = u(\widehat{a}) \quad \text{uniformly in } a \in A \text{ and } b \in B. \tag{3.2}$$

Moreover:

$$\mathcal{C}(\overline{X}) = \Big\{ u \in \mathcal{C}(X) : \lim_{x \to \alpha} u(x) \text{ exists for each } \alpha \in \mathbb{S}_X \Big\}. \tag{3.3}$$

The proof is an exercise. Observe that the topology we introduced on $\overline{X}$ could be introduced directly in terms of $\mathcal{C}_{\mathrm{h}}(X)$: for example, $\widetilde{\alpha}$ is the filter on X defined by the sets $\{x \in X : |u(x) - u(\alpha)| < 1\}$ when u runs over $\mathcal{C}_{\mathrm{h}}(X)$.

The space $\mathcal{C}_{\mathrm{h}}(X)$ is not stable under translations if the dimension of X is larger than one. However, equation (3.2) — or a direct argument — immediately gives that $\mathcal{C}(\overline{X})$ is invariant under translations, and hence we may consider its crossed product $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ by the action of X. This crossed product is the *spherical algebra of* X and we shall study it in the next section. Before doing that, however, let us describe an abelian C^*-algebra of the same nature, but larger than $\mathcal{C}(\overline{X})$, which is naturally involved in the construction of self-adjoint operators affiliated to $\mathscr{E}(X)$. In more technical terms, this new algebra is the Gagliardo completion of $\mathcal{C}(\overline{X})$ with respect to $B(\mathcal{H}^s, \mathcal{H})$ for some (hence for all) $s > 0$, where $\mathcal{H}^s$ is the Sobolev space of order s on X, $\mathcal{H} = L^2(X)$, and we embed $\mathcal{C}(\overline{X}) \subset B(\mathcal{H}) \subset B(\mathcal{H}^s, \mathcal{H})$ by identifying a function with the corresponding multiplication operator. One may find in Section 2.1 of [2] a discussion of the Gagliardo completion in a general setting, but this is not necessary for what follows.

The following notion of *convergence in the mean* will be useful. Let Λ be a compact neighborhood of the origin in X and $\alpha \in \mathbb{S}_X$. If $u \in L^1_{\mathrm{loc}}(X)$ and $c \in \mathbb{C}$ then $\operatorname{m-lim}_{\alpha} u = c$, or $\operatorname{m-lim}_{x \to \alpha} u(x) = c$, means $\lim_{a \to \alpha} \int_{a+\Lambda} |u(x) - c| \mathrm{d}x = 0$. Then we shall have $\lim_{a \to \alpha} \int_{a+K} |u(x) - c| \mathrm{d}x = 0 \ \ \forall K \subset X$ compact: indeed, K can be covered by a finite number of translates of Λ and the filter $\widetilde{\alpha}$ is translation invariant and coarse (see page 351).

Obviously $\mathcal{B}_0(X) = \Big\{ u \in L^\infty(X) : \operatorname{m-lim}_{\alpha} u = 0 \Big\}$ is a closed self-adjoint ideal of $L^\infty(X)$. We also have $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \cap \mathcal{B}_0(X) = \mathcal{C}_0(X)$ as a consequence of

Lemma 4.1 that will be proved later on for a general class of filters. The algebra of interest for us is:

$$\mathcal{B}(\overline{X}) = \mathcal{C}(\overline{X}) + \mathcal{B}_0(X). \tag{3.4}$$

This is a C^*-algebra because the sum of a C^*-subalgebra and a closed self-adjoint ideal is always a C^*-algebra. We have the following alternative description of $\mathcal{B}(\overline{X})$.

LEMMA 3.3. *The set $\mathcal{B}(\overline{X})$ consists of the functions $u \in L^\infty(X)$ that have the following property: for any $\alpha \in \mathbb{S}_X$, there is $c \in \mathbb{C}$ such that $\operatorname{m-lim}_\alpha u = c$.*

Proof. It is clear that the functions in $\mathcal{B}(\overline{X})$ have the required property, so it suffices to prove that a function u as in the statement of the lemma may be written as a sum $u = v + w$ with $v \in \mathcal{C}(\overline{X})$ and $w \in \mathcal{B}_0(X)$. Observe that the number c is uniquely defined by α, and hence we may define a function $V : \mathbb{S}_X \to \mathbb{C}$ by the condition $V(\alpha) = c$. Thus we have

$$\lim_{a\to\alpha} \int\limits_{a+\Lambda} |u(x) - V(\alpha)| \mathrm{d}x = 0, \quad \forall \alpha \in \mathbb{S}_X. \tag{3.5}$$

Note that (3.5) means that, for any $\varepsilon > 0$, there is an open truncated cone C that eventually contains α such that $\int_{a+\Lambda} |u(x) - V(\alpha)|\mathrm{d}x < \varepsilon$ if $a \in C$. In particular, if we fix $a \in \alpha$, then we get $\lim_{r\to+\infty} \int_{ra+\Lambda} |u(x) - V(\alpha)|\mathrm{d}x = 0$. Let us show now, as in the proof of Lemma 3.1, that V is a continuous function. Let us fix α, ε and C and consider some $\beta \in C^\dagger$. By what we just proved, we have $\lim_{r\to+\infty} \int_{rb+\Lambda} |u(x) - V(\beta)|\mathrm{d}x = 0$ for an arbitrary $b \in \beta$. On the other hand, since C is a truncated open cone and β is eventually in C, we have $rb \in C$ for all large enough r, and hence $\int_{rb+\Lambda} |u(x) - V(\alpha)|\mathrm{d}x < \varepsilon$. Then

$$|V(\alpha) - V(\beta)||\Lambda| \leqslant \int\limits_{rb+\Lambda} |V(\alpha) - u(x)|\mathrm{d}x + \int\limits_{rb+\Lambda} |u(x) - V(\beta)|\mathrm{d}x < 2\varepsilon$$

for large r, where $|\Lambda|$ is the measure of Λ. Since these $C^\dagger$ are a basis of the neighborhoods of α in $\mathbb{S}_X$, this proves the continuity of V at the point α.

Now let $\theta : X \to \mathbb{R}$ be a continuous function such that $\theta(x) = 0$ on a neighborhood of zero and $\theta(x) = 1$ for large x. Then the function defined by $v(x) = \theta(x)V(\widehat{x})$ for $x \neq 0$ and $v(0) = 0$ belongs to $\mathcal{C}_{\mathrm{h}}(X)$, and hence it belongs to $\mathcal{C}(\overline{X})$ as well. On the other hand, $w := u - v \in L^\infty$ and, if we set $W(a) = \int_{a+\Lambda} |w|\mathrm{d}x$, then $\lim_{a\to\alpha} W(a) = 0$ for each $\alpha \in \mathbb{S}_X$. This is because

$$W(\alpha) \leqslant \int\limits_{a+\Lambda} |u(x) - V(\alpha)|\mathrm{d}x + \sup_{x\in a+\Lambda} |v(x) - V(\alpha)||\Lambda|$$

and the function v extended by V on $\mathbb{S}_X$ is continuous on $\overline{X}$, and hence the last term above tends to zero when $a \to \alpha$. If $\varepsilon > 0$, then for each $\alpha \in \mathbb{S}_X$ there is an open truncated cone C_α such that $W(a) < \varepsilon$ if $a \in C_\alpha$. Since $\{C_\alpha^\dagger\}_{\alpha \in \mathbb{S}_X}$ is an open cover of the compact $\mathbb{S}_X$, there is a finite set $A \subset \mathbb{S}_X$ such that $\mathbb{S}_X = \bigcup_{\alpha \in A} C_\alpha^\dagger$. Finally, it is clear that $\bigcup_{\alpha \in A} C_\alpha$ is a neighborhood of infinity in X on which we have $W(a) < \varepsilon$, so $\lim_{a \to \infty} W(a) = 0$. ∎

Now we give a description of $\mathcal{B}(\overline{X})$ as a Gagliardo completion of $\mathcal{C}(\overline{X})$. Recall that $u(q)$ is the operator of multiplication by the function u and $\mathcal{H}^s$ are Sobolev spaces.

PROPOSITION 3.4. *The set $\mathcal{B}(\overline{X})$ consists of the functions $u \in L^\infty(X)$ with the following property: there is a sequence of functions $u_n \in \mathcal{C}(\overline{X})$ such that*

(3.6)
$$\sup_n \|u_n\|_{L^\infty} < \infty \quad \textit{and} \quad \lim_n \|u_n(q) - u(q)\|_{\mathcal{H}^s \to \mathcal{H}} = 0 \quad \textit{for some real } s > 0.$$

Proof. We begin by noticing that the condition (3.6) is independent of s. In fact, if it holds for some s then clearly it remains true if we replace s by any $t \geqslant s$ and it will also hold for $0 < t < s$ because if we set $T = u_n(q) - u(q)$ then we have

$$\|T\|_{\mathcal{H}^t \to \mathcal{H}} \leqslant \|T\|^\mu_{\mathcal{H}^s \to \mathcal{H}} \|T\|^\nu_{\mathcal{H} \to \mathcal{H}} \quad \text{with } \mu = \frac{t}{s}, \nu = 1 - \frac{t}{s}.$$

It is clear that what we really have to prove is the same assertion, but with $\mathcal{B}(\overline{X})$ replaced by $\mathcal{B}_0(X)$ and $\mathcal{C}(\overline{X})$ replaced by $\mathcal{C}_0(X)$. Assume first that $u \in L^\infty$ can be approximated with functions $u_n \in \mathcal{C}_0$ as in (3.6) and let $\varepsilon_n = \|u_n(q) - u(q)\|_{\mathcal{H}^s \to \mathcal{H}}$. Let $\eta \in \mathcal{C}_c^\infty$ such that $\eta(x)$ is a constant, $c \neq 0$ on Λ and $\|\eta\|_{\mathcal{H}^s} = 1$ and let us denote $\eta_a(x) = \eta(x-a)$. Then $\|(u - u_n)\eta_a\|_{\mathcal{H}} \leqslant \varepsilon_n \|\eta_a\|_{\mathcal{H}^s} = \varepsilon_n$, and hence $\|u\eta_a\|_{\mathcal{H}} \leqslant \varepsilon_n + \|u_n \eta_a\|_{\mathcal{H}}$. Since $u_n \in \mathcal{C}_0$, there is a neighborhood U_n of infinity in X such that $\|u_n \eta_a\|_{\mathcal{H}} \leqslant \varepsilon_n$ if $a \in U_n$, and then we get $\int_{a+\Lambda} |u|^2 \mathrm{d}x \leqslant 4c^{-2}\varepsilon_n^2$ for $a \in U_n$. This clearly implies $u \in \mathcal{B}_0$.

Reciprocally, let $u \in \mathcal{B}_0$. Choose a positive function $\theta \in \mathcal{C}_c^\infty(X)$ with $\int \theta(x)\mathrm{d}x = 1$ and let $\theta_\varepsilon(x) = \theta(x/\varepsilon)/\varepsilon^d$ if the dimension of X is d. Then it is clear that the convolution product $= u * \theta_\varepsilon$ belongs to $\mathcal{C}_0(X)$ and $\|u * \theta_\varepsilon\|_{L^\infty} \leqslant \|u\|_{L^\infty}$. Hence it suffices to prove that there is $s > 0$ such that $\|u * \theta_\varepsilon(q) - u(q)\|_{\mathcal{H}^s \to \mathcal{H}} \to 0$ if $\varepsilon \to 0$. But this is easy because, for $s > d/2$, we have an estimate

$$\|f(q)\|^2_{\mathcal{H}^s \to \mathcal{H}} \leqslant C \sup_a \int_{a+\Lambda} |f|^2 \mathrm{d}x \leqslant C \|f\|_{L^\infty} \sup_a \int_{a+\Lambda} |f| \mathrm{d}x.$$

We take here $f = u * \theta_\varepsilon - u$ and note that $\int_K |u * \theta_\varepsilon - u| \mathrm{d}x \to 0$ for any compact K while, for large a, we use the relation $\lim_{a \to \infty} \int_{a+\Lambda} |u| \mathrm{d}x = 0$. ∎

4. THE SPHERICAL ALGEBRA

We study now the spherical algebra $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ defined in the Introduction. We begin with a lemma that will be needed in the proof of Theorem 4.2. In order to clarify the statement of the following lemma and in order to prepare the ground for the use of filters in other proofs, we recall now some facts about filters [5].

A *filter on* X is a set ξ of subsets of X such that: (1) $X \in \xi$, (2) $\emptyset \notin \xi$, (3) if $\xi \ni F \subset G$, then $G \in \xi$, and (4) if $F, G \in \xi$ then $F \cap G \in \xi$. If Y is a topological space and $u : X \to Y$, then $\lim_{\xi} u = y$, or $\lim_{x \to \xi} u(x) = y$, means that $u^{-1}(V) \in \xi$ for any neighborhood V of y. The filter ξ on X is called *translation invariant* if, for each $F \in \xi$ and $x \in X$, we have $x + F \in \xi$. We say that ξ is *coarse* if, for each $F \in \xi$ and each compact K in X, there is $G \in \xi$ such that $G + K \subset F$. Recall that we have denoted by $\widetilde{\alpha}$ the set of traces on X of the neighborhoods of α in $\overline{X}$. Clearly $\widetilde{\alpha}$ is a translation invariant and coarse filter on X for each $\alpha \in \mathbb{S}_X$.

LEMMA 4.1. *Let ξ be a translation invariant filter in X, let Λ be a compact neighborhood of the origin, and $u \in \mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$. Then*

$$\lim_{\xi} u = 0 \Leftrightarrow \lim_{a \to \xi} \int_{a+\Lambda} |u(x)| \mathrm{d}x = 0 \Leftrightarrow \operatorname*{s-lim}_{a \to \xi} u(q + a) = 0. \tag{4.1}$$

Proof. Recall that $u(q)$ denotes the operator of multiplication by u and $u(q + a)$ is its translation by a. We have $\operatorname*{s-lim}_{a \to \xi} u(q + a) = 0$ if and only if

$$\int |u(x + a) f(x)|^2 \mathrm{d}x \to 0 \quad \text{as } a \to \xi \text{ for all } f \in L^2(X),$$

by the definition of the strong limit. By taking f to be the characteristic function of the compact set Λ and by using the Cauchy–Schwartz inequality, we obtain $\lim_{a \to \xi} \int_{a+\Lambda} |u(x)| \mathrm{d}x = 0$. Reciprocally, if this relation is satisfied then it is also satisfied with Λ replaced by any of its translates because ξ is translation invariant. By summing a finite number of such relations, we get $\lim_{a \to \xi} \int_{a+K} |u(x)| \mathrm{d}x = 0$ for any compact K. Since u is bounded, we also obtain $\lim_{a \to \xi} \int_{a+K} |u(x)|^2 \mathrm{d}x = 0$ and so $\lim_{a \to \xi} \int |u(x + a) f(x)|^2 \mathrm{d}x = 0$, for any simple function f. Using again the boundedness of u, we then obtain

$$\lim_{a \to \xi} \int |u(x + a) f(x)|^2 \mathrm{d}x = 0 \quad \text{for } f \in L^2(X)$$

.

We now show that $\lim_{\xi} u = 0$ is equivalent to $\lim_{a \to \xi} \int_{a+\Lambda} |u(x)| \mathrm{d}x = 0$. We may assume $u \geqslant 0$, and since u and $a \mapsto \int_{a+\Lambda} u(x) \mathrm{d}x$ are bounded uniformly continuous functions, we may also assume that ξ is coarse. (This follows from Lemma 2.2 of [22] and a simple argument, which shows that the round envelope of a translation invariant filter is coarse. We do not include the details since in our applications $\xi = \widetilde{\alpha}$, which is coarse.) If $\lim_{\xi} u = 0$, then $\{u < \varepsilon\} \in \xi$, for any $\varepsilon > 0$. Since ξ is coarse, there is $F \in \xi$ such that $F + \Lambda \subset \{u < \varepsilon\}$, and hence, if $a \in F$, then $\int_{a+\Lambda} u(x)\mathrm{d}x \leqslant \varepsilon|a + \Lambda| = \varepsilon|\Lambda|$. Thus we have $\lim_{a\to\xi} \int_{a+\Lambda} u(x)\mathrm{d}x = 0$. Conversely, assume that this last condition is satisfied and let $\varepsilon > 0$. Since u is uniformly continuous, there is a compact symmetric neighborhood $L \subset \Lambda$ of zero such that $|u(x) - u(y)| < \varepsilon$ if $x, y \in L$. Then

$$u(a)|L| = \int_{a+L} (u(a) - u(x))\mathrm{d}x + \int_{a+L} u(x)\mathrm{d}x \leqslant \varepsilon|L| + \int_{a+L} u(x)\mathrm{d}x,$$

and hence $\limsup_{a\to\xi} u(a) \leqslant \varepsilon|L|$. ∎

Recall that $\tau_a(S) = T_a^* S T_a$, where the unitary translation operators T_a are defined in (2.1).

THEOREM 4.2. *The algebra $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ consists of the $S \in \mathscr{B}(X)$ that have the position-momentum limit property and are such that* $\operatorname{s-lim}_{a\to\alpha} \tau_a(S)^{(*)}$ *exists* $\forall \alpha \in \mathbb{S}_X$.

Proof. Let $\mathscr{A}$ be the set of bounded operators that have the properties in the statement of the theorem. We first show that $\mathscr{S}(X) \subset \mathscr{A}$. Recall that in the concrete realization we mentioned above, $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ is identified with the norm closed linear space generated by the operators $S = u(q)v(p)$ with $u \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ and $v \in \mathcal{C}_0(X^*)$, while $\mathcal{C}(\overline{X}) \rtimes X$ is the norm closed subspace generated by the same type of operators, but with $u \in \mathcal{C}(\overline{X})$. It follows that an operator $S = u(q)v(p)$, with $u \in \mathcal{C}(\overline{X})$, has the position-momentum limit property and

$$\operatorname{s-lim}_{a\to\alpha} T_a^* S T_a = \operatorname{s-lim}_{a\to\alpha} u(q + a)v(p) = u(\alpha)v(p), \tag{4.2}$$

because of relation (3.2). Thus $\mathscr{S}(X) \subset \mathscr{A}$ and it remains to prove the opposite inclusion.

It is clear that $\mathscr{A}$ is a C^*-algebra. From Theorem 3.7 of [22] it follows that $\mathscr{A}$ is a crossed product $\mathscr{A} = \mathcal{A} \rtimes X$ with $\mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ if and only if $\mathscr{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ and

$$x \in X, k \in X^*, S \in \mathscr{A} \Rightarrow T_x S \in \mathscr{A} \quad \text{and} \quad M_k S M_k^* \in \mathscr{A}. \tag{4.3}$$

By the definition of $\mathscr{A}$, the condition $\mathscr{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ is obviously satisfied. Moreover, we have $T_a^* T_x S T_a = T_x T_a^* S T_a$ and $T_a^* M_k S M_k^* T_a = M_k T_a^* S T_a M_k^*$, and hence the last two conditions in (4.3) are also satisfied. Therefore $\mathscr{A}$ is a crossed product. Theorem 3.7 form [22] gives more: the unique translation invariant C^*-subalgebra $\mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ such that $\mathscr{A} = \mathcal{A} \rtimes X$ is the set of $u \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ such that

$u(q)v(p)$ and $\overline{u}(q)v(p)$ belong to $\mathscr{A}$ if $v \in \mathcal{C}_0(X^*)$. In our case, we see that $\mathcal{A}$ is the set of all $u \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ such that $\operatorname*{s-lim}_{a\to\alpha} T_a^* u(q)^{(*)} T_a v(p)$ exists for all $\alpha \in \mathbb{S}_X$ and $v \in \mathcal{C}_0(X^*)$. But the operators $T_a^* u(q)^{(*)} T_a = u^{(*)}(q+a)$ are normal and uniformly bounded and the union of the ranges of the operators $v(p)$ is dense in $L^2(X)$, and hence

$$\mathcal{A} = \{u \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X) : \exists \operatorname*{s-lim}_{a\to\alpha} u(q+a)\ \forall \alpha \in \mathbb{S}_X\}.$$

Let us fix α and let $u \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ be such that the limit $\operatorname*{s-lim}_{a\to\alpha} u(q+a)$ exists. This limit is a function, but since the filter $\widetilde{\alpha}$ is translation invariant, this function must be in fact a constant c. Applying Lemma 4.1 to $u - c$ we get $\lim_\alpha u = c$. Lemma 3.2 then gives the following which proves the theorem:

$$\mathcal{A} = \left\{u \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X) : \exists \lim_{x\to\alpha} u(x)\ \forall \alpha \in \mathbb{S}_X\right\} = \mathcal{C}(\overline{X}). \quad \blacksquare$$

For each $\alpha \in \mathbb{S}_X$ and $S \in \mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$, we then define

$$\tau_\alpha(S) := \operatorname*{s-lim}_{a\to\alpha} T_a^* S T_a. \tag{4.4}$$

THEOREM 4.3. *If $S \in \mathscr{S}(X)$ and $\alpha \in \mathbb{S}_X$, then $\tau_\alpha(S) \in C^*(X)$ and the map $\tau(S) : \alpha \mapsto \tau_\alpha(S)$ is norm continuous, and hence $\tau : \mathscr{S}(X) \to C(\mathbb{S}_X) \otimes C^*(X)$. The resulting morphism τ is a surjective morphism and its kernel is the set $\mathscr{K}(X) = \mathcal{C}_0(X) \rtimes X$ of compact operators on $L^2(X)$. Hence we have a natural identification*

$$\mathscr{S}(X)/\mathscr{K}(X) \cong \mathcal{C}(\mathbb{S}_X) \otimes C^*(X) \cong \mathcal{C}_0(\mathbb{S}_X \times X^*). \tag{4.5}$$

Proof. If $S = u(q)v(p)$, then, from (4.2), we get $\tau_\alpha(u(q)v(p)) = u(\alpha)v(p)$, and thus $\tau(S) = \widetilde{u} \otimes v(p)$, where $\widetilde{u}$ is the restriction of $u : \overline{X} \to \mathbb{C}$ to $\mathbb{S}_X$. The first assertion of the theorem then follows from the density in $\mathscr{S}(X)$ of the linear space generated by the operators of the form $u(q)v(p)$. The fact that τ_α are morphisms follows from their definition as strong limits, and it implies the fact that τ is a morphism. Since the range of a morphism is closed and $u \mapsto \widetilde{u}$ is a surjective map $\mathcal{C}(\overline{X}) \to C(\mathbb{S}_X)$, we get the surjectivity of τ. It remains to show that $\ker\tau = \mathcal{C}_0(X) \rtimes X$. By what we have proved, $\mathscr{A}_0 = \ker\tau$ is the set of operators S that have the position-momentum property and are such that $\operatorname*{s-lim}_{a\to\alpha} T_a S T_a = 0$ for all $\alpha \in \mathbb{S}_X$. The argument of the proof of Theorem 4.2 with $\mathscr{A}$ replaced by $\mathscr{A}_0$ shows that $\mathscr{A}_0 = \mathcal{A}_0 \rtimes X$, with $\mathcal{A}_0$ equal to the set of all $u \in \mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ such that $\lim_{x\to\alpha} u(x) = 0$ for all $\alpha \in \mathbb{S}_X$. Therefore $\mathcal{A}_0 = \mathcal{C}_0(X)$. $\blacksquare$

REMARK 4.4. The fact that $\tau_\alpha(S)$ belongs to $C^*(X)$ can be understood more generally as follows. Since the filter $\widetilde{\alpha}$ is translation invariant, if S is an arbitrary bounded operator such that the limit $S_\alpha := \operatorname*{s-lim}_{a\to\alpha} T_a^* S T_a$ exists, then S_α commutes with all the T_x, and hence S is of the form $v(p)$, for some $v \in L^\infty(X^*)$. If S has the position-momentum limit property, then it is clear that S_α also has the position-momentum limit property, which forces $v \in \mathcal{C}_0(X^*)$.

We have the following consequences of the above theorem. We notice that we do not need closure in the union in the following results since $\tau_\alpha(S)$ depends *norm* continuously on α. See [39] for a general discussion of the need of closures of the unions in results of this type.

COROLLARY 4.5. *If $S \in \mathscr{S}(X)$ is a normal element, then $\sigma_{\mathrm{ess}}(S) = \bigcup_\alpha \sigma(\tau_\alpha(S))$.*

Similarly, we have the following.

COROLLARY 4.6. *Let H be a self-adjoint operator affiliated to $\mathscr{S}(X)$. Then for each $\alpha \in \mathbb{S}_X$ the limit $\alpha \cdot H := \operatorname{s-lim}_{a\to\alpha} T_a^* H T_a$ exists and $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.*

For the proof and for the meaning of the limit above, see Remark 2.10.

We give now the simplest concrete application of Corollary 4.6. The boundedness condition on V can be eliminated, but this requires some technicalities, which will be discussed in the next section.

PROPOSITION 4.7. *Let $H = h(p) + V$, where $h : X^* \to \mathbb{R}$ is a continuous proper function and V is a bounded symmetric linear operator on $L^2(X)$ satisfying:*
(i) $\lim_{k\to 0} \|[M_k, V]\| = 0$;
(ii) $\alpha \cdot V := \operatorname{s-lim}_{a\to\alpha} T_a^* V T_a$ *exists for each* $\alpha \in \mathbb{S}_X$.
Then H is affiliated to $\mathscr{S}(X)$, we have $\alpha \cdot H = h(p) + \alpha \cdot V$, and $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$. Moreover, for each $\alpha \in \mathbb{S}_X$, there is a function $v_\alpha \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X^*)$ such that $\alpha \cdot V = v_\alpha(p)$.

Proof. First we have to check that the self-adjoint operator H is affiliated to $\mathscr{S}(X)$. For this, it suffices to prove that there is a number z such that the operator $S = (H - z)^{-1}$ satisfies the conditions of Theorem 4.2. To check the position-momentum limit property we have to prove that $(T_x - 1)S$ and $[M_k, S]$ tend to zero in norm when $x \to 0$ and $k \to 0$ (the condition involving S^* will then also be satisfied since S^* is of the same form as S). Since the range of S is the domain of $h(p)$, the first condition is clearly satisfied. If we denote $S_0 = (h(p) - z)^{-1}$ and choose z such that $\|VS_0\| < 1$, then we have $S = S_0(1 + VS_0)^{-1}$ and $S_0 \in C^*(X)$, and hence $[M_k, S_0]$ tends to zero in norm as $k \to 0$. It remains to be shown that $(1 + VS_0)^{-1}$ also satisfies this condition: but this is clear because the set of bounded operators A such that $\|[M_k, A]\| \to 0$ is a C^*-algebra, and hence a full subalgebra of $\mathscr{B}(X)$.

The fact that $\operatorname{s-lim}_{a\to\alpha} T_a^* S T_a$ exists and is equal to $\alpha \cdot S = (\alpha \cdot H - z)^{-1}$ for each $\alpha \in \mathbb{S}_X$ is an easy consequence of the relation $T_a^* S T_a = S_0(1 + T_a^* V T_a S_0)^{-1}$.

Finally, to show that $\alpha \cdot V = v_\alpha(p)$, for some $v_\alpha \in \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X^*)$, we use the argument of Remark 4.4. Indeed, we shall have this representation for some bounded Borel function v_α, which is uniformly continuous because $\lim_{k\to 0} \|[M_k, \alpha \cdot V]\| = 0$. ∎

EXAMPLE 4.8. A typical example is when V is the operator of multiplication by a bounded Borel function $V : X \to \mathbb{R}$ such that $V(\alpha) := \lim_{x\to\alpha} V(x)$ exists for each $\alpha \in \mathbb{S}_X$. Then $\alpha \cdot V$ is the operator of multiplication by the number $V(\alpha)$. Note that by Lemma 3.1 the limit function $\alpha \mapsto V(\alpha)$ is continuous on $\mathbb{S}_X$, even if V is not continuous on X.

REMARK 4.9. The constructions in this section are related to the ones involving the so called "SG-calculus" or "scattering calculus", see [10], [11], [26], [37], [40], [42] and the references therein. In fact, the closure in norm of the algebra of order -1, SG-pseudodifferential operators coincides with the algebra $\mathcal{C}(\overline{X}) \rtimes X$.

5. AFFILIATION CRITERIA

We now recall, for the benefit of the reader, a little bit of the formalism that we shall use below. If H is a self-adjoint operator on a Hilbert space $\mathcal{H}$, then the domain of $|H|^{1/2}$ equipped with the graph topology is called the *form domain* of H. If we denote it $\mathcal{G}$, then we have a natural continuous embedding $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$, where $\mathcal{G}^*$ is the space adjoint of $\mathcal{G}$ (space of conjugate linear continuous forms on $\mathcal{G}$). The operator $H : D(H) \to \mathcal{H}$ extends to a continuous symmetric operator $\widehat{H} \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, which has the following property: a complex number z belongs to the resolvent set of H if and only if $\widehat{H} - z$ is a bijective map $\mathcal{G} \to \mathcal{G}^*$. In this case, $(H - z)^{-1}$ coincides with the restriction of $(\widehat{H} - z)^{-1}$ to $\mathcal{H}$. Conversely, let $\mathcal{G}$ be a Hilbert space densely and continuously embedded in $\mathcal{H}$. If $L : \mathcal{G} \to \mathcal{G}^*$ is a symmetric operator, then the *operator induced by L in $\mathcal{H}$* is the operator H in $\mathcal{H}$ whose domain is the set of $u \in \mathcal{G}$ such that $Lu \in \mathcal{H}$ given by $H = L|_{D(H)}$. If $L - z : \mathcal{G} \to \mathcal{G}^*$ is a bijective map for some complex z, then $D(H)$ is a dense subspace of $\mathcal{H}$, the operator H is self-adjoint, and $\widehat{H} = L$. If L is bounded from below, then $\mathcal{G}$ coincides with the form domain of H. From now on, we shall drop the "hat" from the notation $\widehat{H}$ and write simply H for the extended operator when there is no danger of confusion.

LEMMA 5.1. *Let $\mathcal{G}$ be a Hilbert space densely and continuously embedded in $L^2(X)$. Then the following conditions are equivalent:*

(i) *The operators T_x and M_k leave invariant $\mathcal{G}$, we have $\|T_x\|_{\mathcal{B}(\mathcal{G})} \leqslant C$ for a number C independent of x, and $\lim_{x \to 0} \|T_x - 1\|_{\mathcal{G} \to \mathcal{H}} = 0$.*

(ii) *$\mathcal{G} = D(w(p))$ for some proper Borel function $w : X^* \to [1, \infty)$ that has the following property: there exists a compact neighborhood Λ of zero in X^* and a number $c > 0$ such that $\sup_{\ell \in \Lambda} w(k + \ell) \leqslant c w(k)$ for all $k \in X^*$.*

Proof. We discuss only the nontrivial implication. Denote $\mathcal{H} = L^2(X)$. Since $\{T_x\}_{x \in X}$ is a strongly continuous unitary group in $\mathcal{H}$ that leaves $\mathcal{G}$ invariant, the restrictions $T_x|_{\mathcal{G}}$ form a C_0-group in $\mathcal{G}$, which by assumption is (uniformly) bounded. It is well known that this implies that there is a Hilbert structure on $\mathcal{G}$, equivalent to the initial one, for which the operators $T_x|_{\mathcal{G}}$ are unitary (indeed,

$\mathbb{R}$ is amenable). Thus, from now on, we may assume that the operators T_x are unitary in $\mathcal{G}$. Then, by the Friedrichs theorem, there exists a unique self-adjoint operator G on $\mathcal{H}$ with the following properties:

(i) $G \geqslant c > 0$ for some number c;
(ii) $\mathcal{G} = D(G)$;
(iii) for all $g \in \mathcal{G}$, we have $\|g\|_{\mathcal{G}} = \|Gg\|$.

By hypothesis, the unitary operator T_x leaves invariant the domain of G and $\|g\|_{\mathcal{G}} = \|GT_xg\| = \|T_x^*GT_xg\|$ for all $g \in D(G)$ and $x \in X$. By the uniqueness of G, we have $T_x^*GT_x = G$, and hence G commutes with all translations. It follows that there is a Borel function $w : X^* \to [c, \infty)$ such that $G = w(p)$. We have

$$\begin{aligned}\|(T_x - 1)\|_{\mathcal{G}\to\mathcal{H}} &= \|(T_x - 1)G^{-1}\|_{\mathcal{H}\to\mathcal{H}} = \|(\mathrm{e}^{\mathrm{i}xP} - 1)w^{-1}(P)\|_{\mathcal{H}\to\mathcal{H}} \\ &= \operatorname*{esssup}_{p\in X^*} |(\mathrm{e}^{\mathrm{i}xp} - 1)w^{-1}(p)|\end{aligned}$$

and w^{-1} is a bounded Borel function. It follows that w^{-1} tends to zero at infinity.

Now we shall use the fact that the operators M_k also leave invariant $\mathcal{G}$. Then the group induced by $\{M_k\}$ in $\mathcal{G}$ is of class C_0. In particular, $\|w(p)M_\ell g\| \leqslant C\|w(p)g\|$ if $\ell \in \Lambda$ and $g \in \mathcal{G}$. Since $M_\ell^* w(p)M_\ell = w(p + \ell)$, we get $\|w(p + \ell)w(p)^{-1}f\| \leqslant C\|f\|$ for $\ell \in \Lambda$ and $f \in \mathcal{H}$, which means that $w(k + \ell)w(k)^{-1} \leqslant C$ for all $k \in X^*$ and $\ell \in \Lambda$. Thus for each fixed k, w is bounded on $k + \Lambda$, and hence w is bounded on any compact. ∎

The next result is a general criterion of affiliation to $\mathscr{S}(X)$ for semi-bounded operators.

THEOREM 5.2. *Let H be a self-adjoint operator on $L^2(X)$ that is bounded from below and its form domain $\mathcal{G}$ satisfies the conditions of Lemma* 5.1. *Assume that we have $\|[M_k, H]\|_{\mathcal{G}\to\mathcal{G}^*} \to 0$ as $k \to 0$ and that the limit $\alpha \cdot H := \lim_{a\to\alpha} T_a^* HT_a$ exists strongly in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, for all $\alpha \in \mathbb{S}_X$. Then H is affiliated to $\mathscr{S}(X)$, for each $\alpha \in \mathbb{S}_X$ the operator in $L^2(X)$ associated to $\alpha \cdot H$ is self-adjoint, and $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.*

Proof. We shall use Theorem 4.2 and then Corollary 4.6. We first check that the first condition of Theorem 4.2 is satisfied. Denote $R = (H + \mathrm{i})^{-1}$. We have $\|(T_x - 1)\|_{\mathcal{G}\to\mathcal{H}} = \|(T_x - 1)|R|^{1/2}\|$, and hence $\lim_{x\to 0} \|(T_x - 1)R\| = 0$. As explained above, R extends uniquely to an operator $\widehat{R} \in \mathcal{B}(\mathcal{G}^*, \mathcal{G})$. The operators M_k leave $\mathcal{G}$ invariant and thus extend continuously to $\mathcal{G}^*$. Consequently, we have $[M_k, \widehat{R}] = \widehat{R}[H, M_k]\widehat{R}$. Hence we get $\lim_{k\to 0} \|[M_k, \widehat{R}]\|_{\mathcal{G}^*\to\mathcal{G}} = 0$, which is more than enough to show that H has the position-momentum limit property.

To finish the proof of the proposition, it is enough to check the last condition of Theorem 4.2 and then use Corollary 4.6. Clearly $\alpha \cdot H : \mathcal{G} \to \mathcal{G}^*$ satisfies $\langle g|\alpha \cdot Hg\rangle = \lim_{a\to\alpha}\langle T_a g|HT_a g\rangle$ for each $g \in \mathcal{G}$. Note that since we assumed H

bounded from below, we may assume that $H \geqslant 1$ (otherwise we add to it a sufficiently large number). Then, if w is as in Lemma 5.1, the norm $\|w(p)g\|$ defines the topology of $\mathcal{G}$, and hence $\langle u|Hu\rangle \geqslant c\|w(p)u\|^2$ for some number c and all $u \in \mathcal{G}$. This implies $\langle T_a g|HT_a g\rangle \geqslant c\|w(p)T_a g\|^2 = c\|w(p)g\|^2$. Thus we get $\langle g|\alpha \cdot Hg\rangle \geqslant c\|w(p)g\|^2$, and hence $\alpha \cdot H$ is a bijective map $\mathcal{G} \to \mathcal{G}^*$. Next, to simplify the notation, we set $H_a = T_a^* H T_a$, $H_\alpha = \alpha \cdot H$, and note that since these operators are isomorphisms $\mathcal{G} \to \mathcal{G}^*$, we have $H_a^{-1} - H_\alpha^{-1} = H_a^{-1}(H_\alpha - H_a)H_\alpha^{-1}$ as operators $\mathcal{G}^* \to \mathcal{G}$, which clearly implies $\operatorname{s-lim}_{a\to\alpha} T_a^* H^{-1} T_a = H_\alpha^{-1}$ in $\mathcal{B}(\mathcal{G}^*, \mathcal{G})$, which is more than enough to prove the convergence of the self-adjoint operators $T_a^* H T_a$ to the self-adjoint operator $\alpha \cdot H$ in $L^2(X)$ in the sense required in Corollary 4.6. ∎

In the next theorem, we consider operators of the form $h(p) + V$, with V unbounded, and impose on h the simplest conditions that ensure that the form domain of $h(p)$ is stable under the operators M_k. Obviously, much more general conditions could have been used to obtain the same result, however, these conditions are well adapted to elliptic operators with non-smooth coefficients. For any real number s, let $\mathcal{H}^s \equiv \mathcal{H}^s(X)$ be the Sobolev space of order s on X. Also, let $|\cdot|$ be any norm on X^*.

THEOREM 5.3. *Let $h : X^* \to [0, \infty)$ be a locally Lipschitz function with derivative h' such that, for some real numbers $c, s > 0$ and all $k \in X^*$ with $|k| > 1$, we have:*

$$c^{-1}|k|^{2s} \leqslant h(k) \leqslant c|k|^{2s} \quad \text{and} \quad |h'(k)| \leqslant c|k|^{2s}. \tag{5.1}$$

Let $V : \mathcal{H}^s \to \mathcal{H}^{-s}$ symmetric such that $V \geqslant -\gamma h(p) - \delta$, for some numbers γ, δ, with $\gamma < 1$. We assume that V satisfies the following two conditions:

(i) $\lim_{k\to 0} \|[M_k, V]\|_{\mathcal{H}^s \to \mathcal{H}^{-s}} = 0$;

(ii) $\forall \alpha \in \mathbb{S}_X$ *the limit* $\alpha \cdot V := \operatorname{s-lim}_{a\to\alpha} T_a^* V T_a$ *exists strongly in* $B(\mathcal{H}^s, \mathcal{H}^{-s})$.

Then $h(p) + V$ and $h(p) + \alpha \cdot V$ are symmetric operators $\mathcal{H}^s \to \mathcal{H}^{-s}$ and the operators H and $\alpha \cdot H$ associated to them in $L^2(X)$ are self-adjoint and affiliated to $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$. Moreover, the essential spectrum of H is given by the relation $\sigma_{\text{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.

Proof. If we denote $w = \sqrt{1+h}$, then the form domain $\mathcal{G}$ of $h(p)$ is $\mathcal{G} = D(w(p)) = \mathcal{H}^s$. The second condition of Lemma 5.1 is satisfied if $\sup_{|\ell|<1} h(k+\ell) \leqslant c(1 + h(k))$ for some number $c > 0$, which is clearly true under our assumptions on h. Then note that we have $h(p) + V + \delta + 1 \geqslant (1-\gamma)h(p) + 1$ as operators $\mathcal{G} \to \mathcal{G}^*$ and this estimate remains true if V is replaced by $\alpha \cdot V$. It follows that $h(p) + V + \delta + 1 : \mathcal{G} \to \mathcal{G}^*$ is bijective, and hence the operator H induced by $h(p) + V$ in $L^2(X)$ is self-adjoint. The same method applies to $\alpha \cdot H$. Thus the conditions of Theorem 5.2 are satisfied and we may use it to get the results of the present theorem. ∎

EXAMPLE 5.4. The simplest examples that are covered by the preceding result are the usual elliptic symmetric operators $\sum_{|\mu|,|\nu|\leqslant m} p^\mu g_{\mu\nu} p^\nu$ with bounded measurable coefficients $g_{\mu\nu}$ such that $\lim_{a\to\alpha} g_{\mu\nu}(x+a) = g^\alpha_{\mu\nu}$ exists for each $x \in X$ and $\alpha \in \mathbb{S}_X$. Here $X = \mathbb{R}^n$ and the notations are as in (1.5) and we assume (1.6). Then the localizations at infinity will be the operators $\alpha \cdot H$, which are of the same form, but with the functions $g_{\mu\nu}$ replaced by the numbers $g^\alpha_{\mu\nu}$. Note that $\alpha \mapsto g^\alpha_{\mu\nu}$ are continuous functions. We can also allow the lower order coefficients $g_{\mu\nu}$ to be suitable singular functions or even suitable non-local operators.

REMARK 5.5. The examples considered above could give the wrong impression that the localizations at infinity $\alpha \cdot H$ are self-adjoint operators in the usual sense on $L^2(X)$. The following example shows that this is not true even in simple situations. Let $H = p^2 + v(q)$ in $L^2(\mathbb{R})$ with $v(x) = 0$ if $x < 0$ and $v(x) = x$ if $x \geqslant 0$. It is clear that H has the position-momentum limit property and, if $R = (H+1)^{-1}$, it is not difficult to check that $\operatorname{s-lim}_{a\to+\infty} T_a^* R T_a = 0$ and $\operatorname{s-lim}_{a\to-\infty} T_a^* R T_a = (p^2+1)^{-1}$. Indeed, the translated potentials $v_a(x) = (T_a^* v(q) T_a)(x) = v(x+a)$ form an increasing family, i.e. $v_a \leqslant v_b$ if $a \leqslant b$, such that $v_a(x) \to +\infty$ if $a \to +\infty$ and $v_a(x) \to 0$ if $a \to -\infty$. Thus $H_{+\infty} = \infty$, in the sense that its domain is equal to $\{0\}$, and $H_{-\infty} = p^2$.

REMARK 5.6. In view of the Remark 5.5, it is tempting to see what happens in the case of the *Stark Hamiltonian* $H = p^2 + q$. In fact, the situation in the case of the Stark Hamiltonian is much worse: *H has not the position-momentum limit property (both conditions of Definition* 2.1 *are violated by the resolvent of H)* and we have $\operatorname{s-lim}_{|a|\to\infty} T_a^* H T_a = \infty$ and $\operatorname{s-lim}_{|k|\to\infty} M_k^* H M_k = \infty$, while the essential spectrum of H is $\mathbb{R}$. So the localizations of H in the regions $|p| \sim \infty$ and $|q| \sim \infty$ say nothing about the essential spectrum of H.

We now recall some definitions and a result that can be used for operators that are not semi-bounded and that will be especially useful in the general context of N-body Hamiltonians.

Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with form domain $\mathcal{G}$. We say that a continuous sesquilinear form V on $\mathcal{G}$ (i.e. a symmetric linear map $V : \mathcal{G} \to \mathcal{G}^*$) is a *standard form perturbation* of H_0 if there are positive numbers γ, δ with $\gamma < 1$ such that either $\pm V \leqslant \gamma |H_0| + \delta$ or H_0 is bounded from below and $V \geqslant -\gamma H_0 - \delta$. In this case, the operator H in $\mathcal{H}$ associated to $H_0 + V : \mathcal{G} \to \mathcal{G}^*$ is self-adjoint (see the comments at the beginning of this section).

We do not recall the definition of strict affiliation, but we use the following fact: a self-adjoint operator H is *strictly affiliated* to a C^*-algebra $\mathscr{C}$ of operators on $\mathcal{H}$ if and only if there is $\theta \in \mathcal{C}_0(\mathbb{R})$ with $\theta(0) = 1$ such that $\lim_{\varepsilon\to 0} \|\theta(\varepsilon H) C - C\| = 0$, for all $C \in \mathscr{C}$. The following is a consequence of Theorem 2.8 and Lemma 2.9 in [9].

THEOREM 5.7. *Let H_0 be a self-adjoint operator, V a standard form perturbation of H_0, and $H = H_0 + V$ the self-adjoint operator defined above. Assume that H_0 is strictly affiliated to a C^*-algebra $\mathscr{C}$ of operators on $\mathcal{H}$. If there is $\phi \in \mathcal{C}_0(\mathbb{R})$ with $\phi(x) \sim |x|^{-1/2}$ for large x such that $\phi(H_0)^2 V\phi(H_0) \in \mathscr{C}$, then H is also strictly affiliated to $\mathscr{C}$.*

We may of course replace $\phi(H_0)^2 V\phi(H_0) \in \mathscr{C}$ by the more symmetric and simpler looking condition $\phi(H_0)V\phi(H_0) \in \mathscr{C}$, but this will not cover in the applications the case when the operator V is of the same order as H_0. For operators bounded from below we have:

THEOREM 5.8. *Let H_0 be a positive operator strictly affiliated to a C^*-algebra of operators $\mathscr{C}$ on a Hilbert space $\mathcal{H}$. Let V be a continuous sesquilinear form on $D(H_0^{1/2})$ such that $V \geqslant -\gamma H_0 - \delta$ with $\gamma < 1$. If $\varphi(H_0)V(H_0 + 1)^{-1/2} \in \mathscr{C}$ for any $\varphi \in \mathcal{C}_{\mathrm{c}}(\mathbb{R})$, then the form sum $H = H_0 + V$ is a self-adjoint operator strictly affiliated to $\mathscr{C}$.*

The next proposition is an immediate consequence of Theorem 5.8. Note that below the form domain of $h(p)$ is the domain of $k(p)$, where k is the function $|h|^{1/2}$. It is clear that if h is a proper continuous function, then $h(p)$ is strictly affiliated to $\mathscr{S}(X)$.

PROPOSITION 5.9. *Let $H = h(p) + V$, where $h : X^* \to \mathbb{R}$ is a continuous proper function and V is a standard form perturbation of $h(p)$. If $(1 + |h(p)|)^{-1}V(1 + |h(p)|)^{-1/2}$ belongs to $\mathscr{S}(X)$, then H is strictly affiliated to $\mathscr{S}(X)$.*

We may replace above $(1 + |h|)^{-1/2}$ by any function of the form $\theta \circ h$ with θ as in Theorem 5.8. Indeed, $\mathcal{C}_{\mathrm{b}}(X^*)$ is obviously included in the multiplier algebra of $\mathscr{S}(X)$.

For $0 \leqslant s \leqslant 1$, let $\mathcal{G}^s := D(|h(p)|^s)$ equipped with the graph topology and let $\mathcal{G}^{-s}$ be its adjoint space. So $\mathcal{G}^1 = \mathcal{G}$, $\mathcal{G}^0 = \mathcal{H}$, and $\mathcal{G}^{-1} = \mathcal{G}^*$. If V is a continuous symmetric form on $\mathcal{G}$ such that $V\mathcal{G}^1 \subset \mathcal{G}^{-s}$ for some $s < 1$, then for each $\gamma > 0$ there is a real δ such that $\pm V \leqslant \gamma|h(p)| + \delta$, and hence V is a standard form perturbation of $h(p)$ and H is well defined.

COROLLARY 5.10. *Let $H = h(p) + V$, where $h : X^* \to \mathbb{R}$ is a continuous proper function, and let V be a continuous symmetric form on $\mathcal{G}$ such that $V\mathcal{G}^1 \subset \mathcal{G}^{-s}$ with $s < 1$. Let ϕ be a smooth function such that $\phi(x) \sim |x|^{-1/2}$ for large x and denote $L = \phi(H_0)V\phi(H_0)$. If $\lim_{k\to 0} \|[M_k, L]\| = 0$ and $\alpha \cdot V = \operatorname{s-lim}_{a\to\alpha} T_a^* V T_a$ exists in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$ for each $\alpha \in \mathbb{S}_X$, then H is affiliated to $\mathscr{S}(X)$, we have $\alpha \cdot H = h(p) + \alpha \cdot V$, and $\sigma_{\mathrm{ess}}(H) = \bigcup_\alpha \sigma(\alpha \cdot H)$.*

6. N-BODY TYPE INTERACTIONS

In this section we introduce and study the algebra of potentials (or elementary interactions) in the N-body case. We will implicitly assume X of dimension

$\geqslant 2$, because in the one dimensional case the algebra $\mathscr{E}(X)$ defined in (6.14) coincides with $\mathscr{S}(X)$.

6.1. N-BODY FRAMEWORK. The framework that we introduce here allows us to define and classify N-body Hamiltonians in terms of the complexity of the interactions inside subsets of particles.

Assume that for each finite dimensional real vector space E a translation invariant C^*-subalgebra $\mathcal{P}(E)$ of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(E)$ has been specified (the letter $\mathcal{P}$ should suggest "potentials"). Then, for each subspace $Y \subset X$, we get a translation invariant subalgebra $\mathcal{P}(X/Y) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$. Let us denote by $\langle A_\alpha, \alpha \in I\rangle$ the norm closed subalgebra generated by a family $\{A_\alpha\}_{\alpha\in I}$ of sets $A_\alpha \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$. Then we let

$$\mathcal{R}_{\mathcal{P}}(X) := \langle \mathcal{P}(X/Y), Y \subset X\rangle \quad \text{and} \quad \mathscr{R}_{\mathcal{P}}(X) := \mathcal{R}_{\mathcal{P}}(X) \rtimes X. \tag{6.1}$$

Thus $\mathcal{R}_{\mathcal{P}}(X)$ is the norm-closed subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ generated by the $\mathcal{P}(X/Y)$, where Y runs over the set of *all linear subspaces* of X. Clearly this is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$. We shall regard the crossed product $\mathscr{R}_{\mathcal{P}}(X) := \mathcal{R}_{\mathcal{P}}(X) \rtimes X$, as a C^*-subalgebra of $\mathscr{B}(X)$. Its structure will play a crucial role in what follows. For instance, for our approach, it will be convenient to assume that $\mathcal{C}_0(E) \subset \mathcal{P}(E)$ and $\mathcal{P}(0) = \mathbb{C}$. Clearly then $\mathscr{R}_{\mathcal{P}}(X)$ contains $C^*(X)$ and $\mathscr{K}(X)$.

It will be natural to call $\mathcal{R}_{\mathcal{P}}(X)$ the *algebra of elementary interactions of type* $\mathcal{P}$ and $\mathscr{R}_{\mathcal{P}}(X) := \mathcal{R}_{\mathcal{P}}(X) \rtimes X$ the *algebra of N-body type Hamiltonians with interactions of type* $\mathcal{P}$. Indeed, $\mathscr{R}_{\mathcal{P}}(X)$ is the C^*-algebra of operators on $L^2(X)$ generated by the resolvents of the self-adjoint operators of the form $h(p) + V$, with $h : X^* \to \mathbb{R}_+$ continuous and proper, and $V \in \mathcal{R}_{\mathcal{P}}(X)$ ([22], Proposition 3.3). The self-adjoint operators affiliated to $\mathscr{R}_{\mathcal{P}}(X)$ will be called *N-body Hamiltonians with interactions of type* $\mathcal{P}$.

We give three examples of possible choices for $\mathcal{P}$, in increasing order of difficulty.

First, the "standard" N-body situation, as described for example in Section 4 of [9] and Section 6.5 of [22], corresponds to the choice $\mathcal{P}(E) = \mathcal{C}_0(E)$. The algebra of elementary interactions $\mathcal{R}_{\mathcal{P}}(X) = \mathcal{R}_{\mathcal{C}_0}(X)$ in this case has a remarkable feature: it is graded by the ordered set of all linear subspaces of X, more precisely $\mathcal{R}_{\mathcal{C}_0}(X)$ is the norm closure of $\sum_{Y\subset X} \mathcal{C}_0(X/Y)$, this sum is direct, and we have $\mathcal{C}_0(X/Y)\mathcal{C}_0(X/Z) \subset \mathcal{C}_0(X/(Y\cap Z))$. Then the corresponding algebra $\mathscr{R}_{\mathcal{C}_0}(X)$ of N-body Hamiltonians with interactions of type $\mathcal{C}_0$ inherits a graded C^*-algebra structure [33], [34]. The usual N-body Hamiltonians are self-adjoint operators affiliated to $\mathscr{R}_{\mathcal{C}_0}(X)$, and their analysis is greatly simplified by the existence of the grading.

Let us now discuss the choice of the space of potential functions $\mathcal{P}(X/Y)$ that will used in this paper. Namely, for any real finite dimensional vector space E we consider the spherical compactification $\overline{E}$ of E and denote $\overline{\mathcal{C}}(E) = \mathcal{C}(\overline{E})$. Our main goal in this paper is to treat the larger class of interactions $\mathcal{P}(E) = \overline{\mathcal{C}}(E)$ and to analyze the N-body Hamiltonians associated to them. We recall the notations already used in the introduction:

$$\mathcal{E}(X) := \mathcal{R}_{\overline{\mathcal{C}}}(X) := \langle \overline{\mathcal{C}}(X/Y), Y \subset X \rangle \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X), \tag{6.2}$$

$$\mathscr{E}(X) := \mathscr{R}_{\overline{\mathcal{C}}}(X) := \mathcal{E}(X) \rtimes X \subset \mathscr{B}(X). \tag{6.3}$$

One of the main difficulties now comes from the absence of a grading of the algebra $\mathcal{E}(X)$ of elementary interactions, which requires more care in understanding its spectrum. Observe that, besides the ideal $\mathcal{C}_0(X) \rtimes X \simeq \mathscr{K}(X)$ of compact operators, $\mathscr{E}(X)$ also contains the spherical algebra $\mathscr{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ consisting of two-body type operators.

A third natural choice, which gives an even larger class of elementary interactions and of N-body type Hamiltonians, is to take $\mathcal{P}(E)$ as the algebra of slowly oscillating functions on E, a class of functions whose importance has been pointed out by H.O. Cordes (see Section 6.2 in [22] for a discussion of this point and several references). In this context, we mention M.E. Taylor's thesis [48] where hypoelliptic operators with slowly oscillating coefficients of two-body type are considered: this is one of the first papers where Fredholmness criteria are obtained in a general setting by using the comparison C^*-algebras introduced by Cordes. In fact, his C^*-algebra $\mathfrak{A}$ is just the crossed product of the C^*-algebra of slowly oscillating functions by the action of X.

6.2. THE ALGEBRA OF ELEMENTARY INTERACTIONS. The algebra $\mathcal{E}(X)$ will play a leading role in our approach. From the definition, it follows that $\mathcal{E}(X)$ is a translation invariant subalgebra since the generating subspaces $\mathcal{C}(\overline{X/Y})$ are already translation invariant. The algebra $\mathcal{E}(X)$ is not graded, as in the standard N-body framework of the algebra $\mathcal{R}_{\mathcal{C}_0}(X)$, but has a natural filtration that plays an important role in our analysis.

Let us fix a linear subspace $Z \subset X$. Then X/Z is a finite dimensional real vector space, and hence the C^*-algebra $\mathcal{E}(X/Z) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Z)$ is well defined and the embedding $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Z) \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ allows us to identify $\mathcal{E}(X/Z)$ with a C^*-subalgebra of $\mathcal{E}(X)$. If $Y \supset Z$ is another linear subspace then $Y/Z \subset X/Z$ and we may identify $X/Y = (X/Z)/(Y/Z)$. Therefore we can identify

$$\mathcal{E}(X/Z) = C^*\text{-subalgebra of } \mathcal{E}(X) \text{ generated by } \bigcup_{Z \subset Y} \mathcal{C}(\overline{X/Y}). \tag{6.4}$$

Thus, the C^*-algebra $\mathcal{E}(X)$ is equipped with a family of C^*-subalgebras $\mathcal{E}(X/Y)$, where Y runs over the set of linear subspaces of X, such that, for $0 \subset Z \subset Y \subset X$, we have

$$\mathbb{C} = \mathcal{E}(0) = \mathcal{E}(X/X) \subset \mathcal{E}(X/Y) \subset \mathcal{E}(X/Z) \subset \mathcal{E}(X). \tag{6.5}$$

Recall now that $\mathbb{S}_X$ consists of the half-lines of X. We shall denote by $[\alpha]$ the one dimensional subspace generated by a half-line $\alpha \in \mathbb{S}_X$. Observe that the algebras $\mathcal{E}(X/[\alpha])$ are maximal among the non-trivial subalgebras of $\mathcal{E}(X)$ of the form $\mathcal{E}(X/Y)$.

Translation at infinity along a direction $\alpha = \mathbb{R}_+ a \in \mathbb{S}_X$ gives us a linear projection τ_α of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\alpha])$ as follows. For $u \in \mathcal{E}(X)$ we define

$$\tau_\alpha(u)(x) := \lim_{r\to+\infty} u(ra+x). \tag{6.6}$$

LEMMA 6.1. *Let $Y \subset X$ be a real, linear subspace and $u \in \mathcal{C}(\overline{X/Y})$. Then*

$$\tau_\alpha(u) = \begin{cases} u(\pi_Y(\alpha)) \in \mathbb{C} & \text{if } \alpha \not\subset Y, \\ u & \text{if } \alpha \subset Y. \end{cases}$$

Proof. If $\alpha \not\subset Y$, $\pi_Y(\alpha)$ is a half line in X/Y, and hence $u(\pi_Y(\alpha))$ is defined. The fact that the limit is as stated follows from the definition. ∎

Note that, in the above lemma, $\tau_\alpha(u)$ is a constant if $\alpha \not\subset Y$. The lemma gives right away the following.

PROPOSITION 6.2. *If $\alpha \in \mathbb{S}_X$ and $u \in \mathcal{E}(X)$, then the limit in* (6.6) *exists for all $x \in X$, is independent of the choice of $a \in \alpha$, and $\tau_\alpha(u) \in \mathcal{E}(X)$. The map $\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X)$ is an algebra morphism with range $\mathcal{E}(X/[\alpha])$ and $\tau_\alpha(u) = u$ for all $u \in \mathcal{E}(X/[\alpha])$.*

Proof. Lemma 6.1 shows that the map τ_α maps $\mathcal{C}(\overline{X/Y})$ to itself if $\alpha \subset Y$, and maps $\mathcal{C}(\overline{X/Y})$ to $\mathbb{C}$ otherwise. The subspace of $B \subset \mathcal{E}(X)$, for which the limit $\tau_\alpha(u)(x)$ exists for any x is a norm closed, conjugation invariant subalgebra of $\mathcal{E}(X)$. Since B contains the generators of $\mathcal{E}(X)$, we obtain that $B = \mathcal{E}(X)$. Consequently, the limit $\tau_\alpha(u)(x)$ exists for all $u \in \mathcal{E}(X)$ and all $x \in X$. Also, we obtain that τ_α maps the generators of $\mathcal{E}(X)$ to a system of generators of $\mathcal{E}(X/[\alpha]) \subset \mathcal{E}(X)$, and hence τ_α maps $\mathcal{E}(X)$ onto $\mathcal{E}(X/[\alpha])$ surjectively. To complete the proof, we notice that $\tau_\alpha \circ \tau_\alpha = \tau_\alpha$ on the standard system of generators of $\mathcal{E}(X)$, and hence $\tau_\alpha = \mathrm{id}$ on the range of τ_α, that is, on $\mathcal{E}(X/[\alpha])$. ∎

REMARK 6.3. The proof of Proposition 6.2 gives that, for each $\alpha \in \mathbb{S}_X$, *the relation* (6.6) *defines a unital endomorphism τ_α of $\mathcal{E}(X)$, which is also a linear projection of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\alpha])$. We note that τ_α does not commute with τ_β in general: if a subspace Z does not contain α and β and $u \in \mathcal{C}(\overline{X/Z})$ then $\tau_\alpha\tau_\beta(u) = u(\pi_Z(\beta))$ and $\tau_\beta\tau_\alpha(u) = u(\pi_Z(\alpha))$.*

REMARK 6.4. For the purpose of this paper, the elements of $\mathcal{E}(X)$ should be thought as multiplication operators on the space $L^2(X)$. If, according to the notational conventions from the beginning of Section 2, we denote by $u(q)$ the operator of multiplication by $u \in \mathcal{E}(X)$ and, if we set $\tau_\alpha(u(q)) = \tau_\alpha(u)(q)$, then we get an expression similar to (4.4):

$$\tau_\alpha(u(q)) = \operatorname*{s-lim}_{r\to+\infty} T_{ra}^* u(q) T_{ra} = \operatorname*{s-lim}_{r\to+\infty} u(ra+q). \tag{6.7}$$

We emphasize however that $\operatorname*{s-lim}_{a\to\alpha} T_a^* u(q) T_a$ *does not exist* for general $u \in \mathcal{E}(X)$.

The next few results concern the subalgebras $\mathcal{E}(X/Y)$.

PROPOSITION 6.5. *Let $\overline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a system of half-lines, which generate a subspace Y of X. Then*

$$\mathcal{E}(X/Y) = \mathcal{E}(X/[\alpha_1]) \cap \cdots \cap \mathcal{E}(X/[\alpha_n]). \tag{6.8}$$

The morphism $\tau_{\overline{\alpha}} := \tau_{\alpha_1}\tau_{\alpha_2}\cdots\tau_{\alpha_n}$ is a linear projection of $\mathcal{E}(X)$ onto $\mathcal{E}(X/Y)$.

Proof. If $u \in \mathcal{C}(\overline{X/Z})$, for some Z, then Lemma 6.1 gives $\tau_{\overline{\alpha}}(u) = u$ if $Y \subset Z$ and $\tau_{\overline{\alpha}}(u) \in \mathbb{C}$ otherwise. In any case, $\tau_{\overline{\alpha}}(u) \in \mathcal{E}(X/Y)$. Since $\tau_{\overline{\alpha}}$ is a morphism, the range of $\tau_{\overline{\alpha}}$ is included in $\mathcal{E}(X/Y)$ and $\tau_{\overline{\alpha}}(u) = u$ if $u \in \mathcal{E}(X/Y)$. Thus $\tau_{\overline{\alpha}}$ is a linear projection of $\mathcal{E}(X)$ onto $\mathcal{E}(X/Y)$. Let $u \in \mathcal{E}(X)$. We obtain that $u \in \mathcal{E}(X/Y)$ if and only if $\tau_{\overline{\alpha}}(u) = u$. If u belongs to the right hand side of (6.8), then $\tau_{\overline{\alpha}}(u) = u$, so $u \in \mathcal{E}(X/Y)$. ∎

Note that a permutation of the $\alpha_1, \alpha_2, \dots, \alpha_n$ will give a different projection onto $\mathcal{E}(X/Y)$ (see Remark 6.3). More generally, if $\overline{\beta} = (\beta_1, \dots, \beta_m)$ is a second system of half-lines that generates Y, then $\tau_{\overline{\beta}}$ is a projection $\mathcal{E}(X) \to \mathcal{E}(X/Y)$ distinct from $\tau_{\overline{\alpha}}$ in general.

By using (6.5) and (6.8) we get

$$\mathcal{E}(X/Y) = \bigcap_{\alpha \subset Y} \mathcal{E}(X/[\alpha]) = \{u \in \mathcal{E}(X) : \tau_\alpha(u) = u \ \forall \alpha \subset Y\}, \tag{6.9}$$

from which we get

$$\mathcal{E}(X/Y) = \{u \in \mathcal{E}(X) : u(x+y) = u(x) \ \forall y \in Y\} = \mathcal{E}(X) \cap \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X/Y). \tag{6.10}$$

Indeed, if C is the middle term in (6.10), then $\mathcal{E}(X/Y) \subset C$, by the definition of $\mathcal{E}(X/Y)$ and the definition of τ_α shows that C is included in the right hand side of (6.9).

PROPOSITION 6.6. *If Y, Z are subspaces of X then $\mathcal{E}(X/(Y+Z)) = \mathcal{E}(X/Y) \cap \mathcal{E}(X/Z)$.*

Proof. Let Y', Z' be supplements of $Y \cap Z$ in Y and Z respectively. Choose a basis $a_1, \dots, a_n$ of $Y + Z$ such that $a_1, \dots, a_i$ is a basis of Y', then $a_{i+1}, \dots, a_j$ is a basis of $Y \cap Z$, and $a_{j+1}, \dots, a_n$ is a basis of Z'. Denote α_k the half-line determined by a_k. From (6.8) we get

$$\mathcal{E}(X/Y) = \bigcap_{k<j} \mathcal{E}(X/[\alpha_k]) \quad \text{and} \quad \mathcal{E}(X/Z) = \bigcap_{k>i} \mathcal{E}(X/[\alpha_k]),$$

and hence $\mathcal{E}(X/Y) \cap \mathcal{E}(X/Z) = \bigcap_{k=1}^{n} \mathcal{E}(X/[\alpha_k])$, which is equal to $\mathcal{E}(X/(Y+Z))$, by (6.8). This completes the proof. ∎

6.3. THE CHARACTER SPACE. We now turn to the study of the spectrum (or character space) of the algebra $\mathcal{E}(X)$ of elementary interactions. We begin with an elementary remark.

Let $x \in X$. Then x there corresponds to the character $\chi_x(u) = u(x)$ on $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$. The character χ_x is completely determined by its restriction to the ideal $\mathcal{C}_0(X)$ of $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$. Similarly, if $\alpha \in \overline{X}$, then α defines a character $\chi_\alpha : \mathcal{C}(\overline{X}) \to \mathbb{C}$ by $\chi_\alpha(u) = u(\alpha)$.

The following lemma and its corollary will provide a crucial ingredient in the proof of Theorem 6.13 identifying the spectrum of $\mathcal{E}(X)$, which is one of our main results.

LEMMA 6.7. *Let $Y \subset X$ be a subspace, let B be the C^*-algebra generated by $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ in $\mathcal{C}^{\mathrm{u}}_{\mathrm{b}}(X)$, and let $\alpha \in \mathbb{S}_X \setminus \mathbb{S}_Y$. Then the character χ_α of $\mathcal{C}(\overline{X})$ extends to a unique character of B. This extension is the restriction of τ_α to B.*

Proof. Recall that the canonical projection $\pi_Y : X \to X/Y$ extends to a continuous map $\pi_Y : \overline{X} \setminus \mathbb{S}_Y \to \overline{X/Y}$, which sends $\mathbb{S}_X \setminus \mathbb{S}_Y$ onto $\mathbb{S}_Y$. Thus $\beta := \pi_Y(\alpha) \in \mathbb{S}_Y$ and χ_β is a character of $\mathcal{C}(\overline{X/Y})$. Let χ be a character of B such that $\chi|_{\mathcal{C}(\overline{X})} = \chi_\alpha$. We shall verify now that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$.

To prove that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$, it suffices to show that the kernel of χ_β is included in that of χ, which means that for $u \in \mathcal{C}(\overline{X/Y})$ with $u(\beta) = 0$, we should have $\chi(u) = 0$. By a density argument, it suffices to assume that $u = 0$ on a neighborhood V of β in $\overline{X/Y}$. It is clear that we can find $v \in \mathcal{C}(\overline{X})$ with $v(\alpha) = 1$ with support in the $\pi_Y^{-1}(V)$, and hence $uv = 0$. Since $u, v \in B$, we have

$$0 = \chi(uv) = \chi(u)\chi(v) = \chi(u)\chi_\alpha(v) = \chi(u)v(\alpha) = \chi(u).$$

This proves that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$, as claimed.

From the relation $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$ just proved, we obtain the uniqueness of χ, since $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ generate B. To complete the proof, let us notice that the restriction of τ_α to $\mathcal{C}(\overline{X})$ is χ_α and its restriction to $\mathcal{C}(\overline{X/Y})$ is also a character, because $\alpha \not\subset Y$. Thus τ_α is a character on B and we get $\chi = \tau_\alpha|_B$ by uniqueness. This completes the proof. ∎

COROLLARY 6.8. *Let χ_1 and χ_2 be characters of $\mathcal{E}(X)$. Let us assume that there exists $\alpha \in \mathbb{S}_X$ such that $\chi_1(u) = \chi_2(u) = u(\alpha)$ for all $u \in \mathcal{C}(\overline{X})$ and that $\chi_1 = \chi_2$ on $\mathcal{E}(X/[\alpha])$. Then $\chi_1 = \chi_2$.*

Proof. It is enough to show that $\chi_1 = \chi_2$ on each of the algebras $\mathcal{C}(\overline{X/Y})$, since the later generate $\mathcal{E}(X)$, by definition. Since $\chi_1 = \chi_2 = \chi_\alpha$ on $\mathcal{C}(\overline{X})$, we obtain $\chi_1 = \chi_2$ on all $\mathcal{C}(\overline{X/Y})$ with $\alpha \not\subset Y$, by Lemma 6.7. Since $\mathcal{C}(X/[\alpha])$ contains (indeed, it is generated by) all $\mathcal{C}(\overline{X/Y})$ with $\alpha \subset Y$, the result follows. ∎

We now proceed to the construction of the characters of $\mathcal{E}(X)$. We begin with a remark concerning the simplest nontrivial case that helps to understand the general case.

REMARK 6.9. If $\alpha \in \mathbb{S}_X$ and $\beta \in \mathbb{S}_{X/[\alpha]}$, then $[\beta]$ is the one dimensional subspace generated by β in $X/[\alpha]$, and hence $\pi_{[\alpha]}^{-1}([\beta])$ is a two dimensional subspace of X that we shall denote by $[\alpha,\beta]$. Note that we may and shall identify $(X/[\alpha])/[\beta]$ with $X/[\alpha,\beta]$. Then Proposition 6.2 gives us two morphisms $\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha])$ and $\tau_\beta : \mathcal{E}(X/[\alpha]) \to \mathcal{E}(X/[\alpha,\beta])$ that are linear projections. Thus $\tau_\beta\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha,\beta])$ is a morphism and a projection, and if $a \in X/[\alpha,\beta]$, then $u \mapsto (\tau_\beta\tau_\alpha u)(a)$ is a character of $\mathcal{E}(X)$.

We now extend the construction of the above remark to an arbitrary number of half-lines. However, it will be convenient first to introduce the following notations.

Notation. Our construction involves finite sequences $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $0 \leqslant n \leqslant \dim(X)$ and linear subspaces $[\overrightarrow{\alpha}] := [\alpha_1, \alpha_2, \ldots, \alpha_n]$ of X associated to them. If $n = 0$, then we define $\overrightarrow{\alpha}$ as the empty set and we associate to it the subspace of X reduced to zero: $[\emptyset] = \{0\}$. If $n = 1$ then $\overrightarrow{\alpha} = (\alpha_1)$ with $\alpha_1 \in \mathbb{S}_X$ and, as before, $[\alpha_1]$ is the one dimensional subspace of X generated by α_1. The case $n = 2$ is treated in the Remark 6.9 and we extend the notation to $n \geqslant 3$ by induction: $\alpha_n \in \mathbb{S}_{X/[\alpha_1,\ldots,\alpha_{n-1}]}$ and $[\alpha_1, \ldots, \alpha_n] = \pi_Y^{-1}([\alpha_n])$ is an n-dimensional subspace of X (here $Y = [\alpha_1, \ldots, \alpha_{n-1}]$). Note that we may identify $X/[\alpha_1, \ldots, \alpha_n] = (X/[\alpha_1, \ldots, \alpha_{n-1}])/[\alpha_n]$. We denote $\widetilde{\Omega}_X^{(n)}$ the set of the just defined finite sequences $\overrightarrow{\alpha}$ of length n and

$$\Omega_X^{(n)} := \{(a, \overrightarrow{\alpha}) : \overrightarrow{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \widetilde{\Omega}_X^{(n)}, a \in X/[\alpha_1, \ldots, \alpha_n]\}.$$

In particular, $\Omega_X^{(0)} \equiv X$ and $\Omega_X^{(N)} \equiv \widetilde{\Omega}_X^{(N)}$ if $N = \dim(X)$, since $[\alpha_1, \ldots, \alpha_N] = X$. Let

$$\Omega_X = \bigcup_{n=0}^{\dim(X)} \Omega_X^{(n)}. \tag{6.11}$$

DEFINITION 6.10. If $(a, \overrightarrow{\alpha}) \in \Omega_X^{(n)}$, then we define

$$\tau_{\overrightarrow{\alpha}} = \tau_{\alpha_n}\tau_{\alpha_{n-1}} \cdots \tau_{\alpha_1} \quad \text{and} \quad \tau_{a,\overrightarrow{\alpha}} = \tau_a \tau_{\overrightarrow{\alpha}}, \tag{6.12}$$

which are endomorphisms of $\mathcal{E}(X)$. We agree that $\tau_\emptyset$ is the identity of $\mathcal{E}(X)$.

In particular, the range of $\tau_{\overrightarrow{\alpha}}$ is $\mathcal{E}(X/[\overrightarrow{\alpha}])$ and $\tau_{\overrightarrow{\alpha}}$ is an endomorphism of $\mathcal{E}(X)$ and a linear projection of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\overrightarrow{\alpha}])$. The morphisms of the form $\tau_{\overline{\alpha}}$ considered in Proposition 6.5 also have these properties, but they may be distinct from the $\tau_{\overrightarrow{\alpha}}$, the objects $\overline{\alpha}$ and $\overrightarrow{\alpha}$ being different in nature. Note also that, since $a \in X/[\overrightarrow{\alpha}]$, translation by a is a morphism τ_a of $\mathcal{E}(X/[\overrightarrow{\alpha}])$, and hence $\tau_{a,\overrightarrow{\alpha}}$ is well defined.

We now introduce what will turn out to be a parametrization of the characters of $\mathcal{E}(X)$.

DEFINITION 6.11. If $(a, \overrightarrow{\alpha}) \in \Omega_X$, we define the character $\chi_{a,\overrightarrow{\alpha}}$ of $\mathcal{E}(X)$ by the formula

$$\chi_{a,\overrightarrow{\alpha}}(u) := \chi_a(\tau_{\overrightarrow{\alpha}}(u)) = \tau_{\overrightarrow{\alpha}}(u)(a). \tag{6.13}$$

We need to explain what happens in the limit case $n = \dim(X)$.

REMARK 6.12. Let $n = \dim(X)$ and $(a, \overrightarrow{\alpha}) \in \Omega_X^{(n)}$. Then $[\overrightarrow{\alpha}] = X$, and hence $X/[\overrightarrow{\alpha}] = 0$, so the only possible choice for a is $a = 0$. Moreover, $\tau_{\overrightarrow{\alpha}} : \mathcal{E}(X) \to \mathbb{C}$ is already a character. Since $\tau_0 = \mathrm{id}$, we get $\chi_{0,\overrightarrow{\alpha}} = \tau_{\overrightarrow{\alpha}}$.

We are ready now to prove one of our main results, which is a description of all the characters of the algebra $\mathcal{E}(X)$. Recall that we denote by $\widehat{\mathcal{E}(X)}$ the character space of $\mathcal{E}(X)$.

THEOREM 6.13. *The map $\Omega_X \to \widehat{\mathcal{E}(X)}$ defined by $(a, \overrightarrow{\alpha}) \mapsto \chi_{a,\overrightarrow{\alpha}}$ is bijective.*

Proof. The preceding construction shows that $\chi_{a,\overrightarrow{\alpha}}$ is a character, therefore we only need to show that every character χ of $\mathcal{E}(X)$ is of this form and that the pair $(a, \overrightarrow{\alpha})$ is uniquely determined. To this end, we look at the restriction of χ to the subalgebra $\mathcal{C}(\overline{X})$ and proceed by induction on the dimension of X.

Every character of $\mathcal{C}(\overline{X})$ is of the form $u \mapsto u(x) = \chi_x$ for some $x \in \overline{X}$. Hence there is a unique $x \in \overline{X}$ such that $\chi|_{\mathcal{C}(\overline{X})} = \chi_x$. We distinguish two cases: $x \in X$ and $x \in \overline{X} \setminus X$. In the first case, we have $x = a \in X$; that is, $\chi(u) = u(a)$ for all $u \in \mathcal{E}(X)$. In our terminology, this means $\chi = \chi_{a,\emptyset}$. The characters χ of this form are characterized by the fact that the restriction of χ to $\mathcal{C}_0(X)$ is non-zero. The value of a is then determined by restriction to $\mathcal{C}_0(X)$, since there is a one-to-one correspondence between the characters of $\mathcal{C}_0(X)$ and the points of X. Thus all the characters $\chi_{a,\emptyset}$, $a \in X$, are distinct.

Now let us assume that $x \notin X$, that is, $x = \alpha \in \mathbb{S}_X := \overline{X} \setminus X$, and that the assertion of the theorem is true for all vector spaces of dimension strictly less than that of X (induction hypothesis). Then the theorem holds for the space $X/[\alpha]$, so there is $\overrightarrow{\beta} = (\beta_1, \dots, \beta_k)$ with

$$\beta_1 \in X/[\alpha], \quad \beta_2 \in X/[\alpha, \beta_1], \dots, \beta_k \in X/[\alpha, \beta_1, \dots, \beta_{k-1}],$$

such that the restriction of χ to $\mathcal{E}(X/[\alpha])$ is given by $\chi(u) = (\tau_{\overrightarrow{\beta}} u)(b)$ for some $b \in (X/[\alpha])/\overrightarrow{\beta}$. That is, $\chi = \chi_{b,\overrightarrow{\beta}}$ on $\mathcal{E}(X/[\alpha])$. Let $a = b$ and let $\overrightarrow{\alpha}$ be obtained by including α in front of the sequence $\overrightarrow{\beta}$, thus $\overrightarrow{\alpha} = (\alpha, \beta_1, \dots, \beta_k)$. Then $\chi_{a,\overrightarrow{\alpha}}(u) = (\tau_{\overrightarrow{\beta}} \circ \tau_\alpha u)(b)$ and the characters χ and $\chi_{a,\overrightarrow{\alpha}}$ coincide on $\mathcal{E}(X/[\alpha])$. On the other hand, on $\mathcal{C}(\overline{X})$, the characters χ and $\chi_{a,\overrightarrow{\alpha}}$ coincide with the character $\chi_\alpha : \mathcal{E}(X/[\alpha]) \to \mathbb{C}$. Therefore $\chi = \chi_{a,\overrightarrow{\alpha}}$ by Corollary 6.8.

The same argument can be used to show that we obtain a one-to-one parametrization of all these characters. We shall proceed once more by induction on the length of $\overrightarrow{\alpha}$. If $\chi_{a,\overrightarrow{\alpha}} = \chi_{b,\overrightarrow{\beta}}$, we have two possibilities: first that their restrictions to $\mathcal{C}_0(X)$ is non-zero and, second, that their restrictions to $\mathcal{C}_0(X)$ is zero. In the first case, we must have $\overrightarrow{\alpha} = \emptyset$ and $\overrightarrow{\beta} = \emptyset$, by the discussion earlier in the proof. By restricting to $\mathcal{C}_0(X)$, we also obtain $a = b \in X$. Let us assume that $\overrightarrow{\alpha} \neq \emptyset$, then $\chi_{a,\overrightarrow{\alpha}}$ restricts to zero on $\mathcal{C}_0(X)$, and hence $\overrightarrow{\beta} \neq \emptyset$ as well. Since the restrictions of $\chi_{a,\overrightarrow{\alpha}}$ and $\chi_{b,\overrightarrow{\beta}}$ to $\mathcal{C}(\overline{X/Y})$ are χ_{α_1} and χ_{β_1} respectively, we obtain $\alpha_1 = \beta_1$. The proof is completed by induction using the restrictions of these characters to $\mathcal{E}(X/[\alpha_1])$, as in the first part of the proof. ∎

We shall describe now the morphism τ_χ on $\mathcal{E}(X)$ defined as the translation by a character $\chi = \chi_{a,\overrightarrow{\alpha}} \in \widehat{\mathcal{E}(X)}$, see Section 2, Definition 2.2.

THEOREM 6.14. *The translation morphism associated to the character $\chi_{a,\overrightarrow{\alpha}}$ by Definition* 2.2 *is the unital endomorphism $\tau_{a,\overrightarrow{\alpha}}$ of $\mathcal{E}(X)$ introduced in Definition* 6.10.

Proof. If $\chi = \chi_a \equiv \tau_{a,\emptyset}$ for some $a \in X$, then this is just the usual translation by a, i.e. $\tau_{\chi_a}(u) = \tau_a(u) = a \cdot u$ is the function $x \mapsto u(a+x)$. In general, we have to use the definition in Definition 2.2, that is, $(\tau_\chi(u))(y) = \chi(y \cdot u)$ for all $y \in X$. Thus, if $\chi = \chi_{a,\overrightarrow{\alpha}}$ as above, then from Definition 6.11 we get

$$(\tau_\chi(u))(x) = \chi_{a,\overrightarrow{\alpha}}(x \cdot u) = \chi_a(\tau_{\overrightarrow{\alpha}}(x \cdot u)).$$

It is clear that X acts by translation on each of the algebras $\mathcal{E}(X/Y)$ and that the morphism $\tau_{\overrightarrow{\alpha}} : \mathcal{E}(X) \to \mathcal{E}(X/\overrightarrow{\alpha})$ is covariant for this action, that is, $\tau_{\overrightarrow{\alpha}}(x \cdot u) = x \cdot (\tau_{\overrightarrow{\alpha}}(u))$. Thus

$$(\tau_\chi(u))(x) = \chi_a(x \cdot (\tau_{\overrightarrow{\alpha}}(u))) = (x \cdot (\tau_{\overrightarrow{\alpha}}(u)))(a) = (\tau_{\overrightarrow{\alpha}}(u))(x+a),$$

and hence we get $\tau_\chi(u) = \tau_a \tau_{\overrightarrow{\alpha}}(u)$, which is (6.12). ∎

REMARK 6.15. Although we shall not use this here, let us mention that in view of Remark 2.5 and of Theorem 6.14, it is interesting to notice that the action of X on the space of characters of $\mathcal{E}(X)$ is given by $\tau_x(\chi_{a,\overrightarrow{\alpha}}) = \chi_{a-\pi_{\overrightarrow{\alpha}}(x),\overrightarrow{\alpha}}$, where $\pi_{\overrightarrow{\alpha}}$ is the canonical map $X \to X/[\overrightarrow{\alpha}]$. Hence, for the determination of the essential spectrum, it is enough to consider the characters $\chi_{0,\overrightarrow{\alpha}}$ and their associated translations $\tau_{\chi_{0,\overrightarrow{\alpha}}} = \tau_{0,\overrightarrow{\alpha}} = \tau_{\overrightarrow{\alpha}}$.

6.4. THE HAMILTONIAN ALGEBRA. We now apply the results we have proved to the study of essential spectra. Since $\mathcal{E}(X)$ is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ such that $\mathcal{C}_0(X) + \mathbb{C} \subset \mathcal{E}(X)$, we may take $\mathcal{A} = \mathcal{E}(X)$ in Section 2. The algebra generated by the Hamiltonians that are of interest for us is the crossed product

$$\mathscr{E}(X) := \mathcal{E}(X) \rtimes X. \tag{6.14}$$

As explained in Section 2, $\mathscr{E}(X)$ can be thought as the closed linear subspace of $\mathscr{B}(X)$ generated by the operators of the form $u(q)v(p)$ with $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. On the other hand, since $\mathcal{C}(\overline{X/Y})$ is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$, we may also consider the crossed product $\mathcal{C}(\overline{X/Y}) \rtimes X$ and we clearly have

$$\mathscr{E}(X) = C^*\text{-subalgebra of } \mathscr{B}(X) \text{ generated by } \bigcup_{Y \subset X} \mathcal{C}(\overline{X/Y}) \rtimes X. \tag{6.15}$$

Similarly, for any subspace $Y \subset X$, we may consider the crossed product $\mathscr{E}(X/Y) = \mathcal{E}(X/Y) \rtimes X$. We thus obtain a family of C^*-subalgebras of $\mathscr{E}(X)$ that, as a consequence of (6.5), has the following property: if $Z \subset Y$ then

$$C^*(X) = \mathscr{E}(0) = \mathscr{E}(X/X) \subset \mathscr{E}(X/Y) \subset \mathscr{E}(X/Z) \subset \mathscr{E}(X). \tag{6.16}$$

From the general facts described in Section 2, and by taking into account the properties of $\mathcal{E}(X)$ established in the preceding subsection, we see that for any $A \in \mathscr{E}(X)$ the map $x \mapsto \tau_x(A) = T_x^* A T_x$ extends to a strongly continuous map $\chi \mapsto \tau_\chi(A) \in \mathscr{E}(X)$ on the spectrum of $\mathcal{E}(X)$ such that

$$\tau_\chi(u(q)v(p)) = \tau_\chi(u(q))v(p) \quad \text{for all } u \in \mathcal{E}(X) \text{ and } v \in \mathcal{C}_0(X^*).$$

Here $\chi \in \widehat{\mathcal{E}(X)}$, and hence it is of the form described in Theorem 6.13 and the associated endomorphism τ_χ of $\mathcal{E}(X)$ is described in (6.12). Note that, in virtue of Theorem 2.4, we are only interested in the characters that belong to the boundary $\delta(\mathcal{E}(X))$ of X in $\widehat{\mathcal{E}(X)}$, which are those with $\overrightarrow{\alpha} \neq \emptyset$. Then Proposition 2.3 and Theorem 6.13 imply the following.

PROPOSITION 6.16. *Let $\chi = \chi_{a,\overrightarrow{\alpha}} \in \delta(\mathcal{E}(X))$. Then there is a unique continuous linear map $\tau_{a,\overrightarrow{\alpha}} : \mathscr{E}(X) \to \mathscr{E}(X)$ such that $\tau_{a,\overrightarrow{\alpha}}(u(q)v(p)) = (\tau_{a,\overrightarrow{\alpha}}u)(q)v(p)$ for all $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. This map is a morphism and a linear projection of $\mathscr{E}(X)$ onto its subalgebra $\mathcal{E}(X/[\overrightarrow{\alpha}]) \rtimes X$.*

Now we shall use the special form of the morphisms $\tau_{a,\overrightarrow{\alpha}}$ in order to improve the compactness criterion of Theorem 2.4.

THEOREM 6.17. *Let $A \in \mathscr{E}(X)$. Then for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$ the limit $\tau_\alpha(A) \equiv \alpha \cdot A := \operatorname{s-lim}_{r \to +\infty} T_{ra}^* A T_{ra}$ exists and is independent of the choice of a. The map τ_α is a morphism and a linear projection of $\mathscr{E}(X)$ onto its subalgebra $\mathcal{E}(X/[\alpha]) \rtimes X$. The operator A is compact if and only if $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$.*

Proof. The first assertion follows from the preceding results, but it is easier to prove it directly. Indeed, it suffices to consider A of the form $A = u(q)v(p)$ with $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. Then $T_{ra}^* A T_{ra} = \tau_{ra}(A) = \tau_{ra}(u(q))v(p)$, which converges to $(\alpha \cdot u)(q)v(p)$ by Proposition 6.2 (or see Remark 6.4). The properties of the endomorphism τ_α are consequences of the same proposition. Everything follows also by using general properties of crossed products and the fact that at the abelian level $\tau_\alpha : \mathcal{E}(X) \to \mathcal{E}(X/[\alpha])$ is a covariant morphism. To prove the compactness assertion, note first that $\tau_\alpha(A) = 0$ if A is compact because $T_{ra} \to 0$ weakly as $r \to \infty$. Then if $A \in \mathscr{E}(X)$ and $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$ then it is clear by (6.12) that $\tau_{a,\overrightarrow{\alpha}}(A) = 0$ if $\overrightarrow{\alpha} \neq \emptyset$, and hence $\tau_\chi(A) = 0$ for all $\chi \in \delta(\mathcal{E}(X))$, and so A is compact by Theorem 2.4. ∎

REMARK 6.18. If Y is a linear subspace of X, then the algebras $\mathcal{E}(X/Y)$ and $\mathscr{E}(X/Y)$ are a priori defined by our formalism as algebras of operators on $L^2(X/Y)$. In Section 6.2, we have defined $\mathcal{E}(X/Y)$ as a subalgebra of $\mathcal{C}_\mathrm{b}^\mathrm{u}(X)$ satisfying the relation (6.10); this definition is natural because of our general convention to identify subalgebras of $\mathcal{C}_\mathrm{b}^\mathrm{u}(X/Y)$ with subalgebras of $\mathcal{C}_\mathrm{b}^\mathrm{u}(X)$. On the other hand, we note that the algebras $\mathscr{E}(X/Y) = \mathcal{E}(X/Y) \rtimes (X/Y)$ and $\mathcal{E}(X/Y) \rtimes X$ are quite different objects: indeed

$$\mathcal{E}(X/Y) \rtimes X \simeq \mathscr{E}(X/Y) \otimes C^*(Y) \tag{6.17}$$

by a general fact from the theory of crossed products, namely

$$(\mathcal{A} \otimes \mathcal{B}) \rtimes (G \times H) \simeq (\mathcal{A} \rtimes G) \otimes (\mathcal{B} \rtimes H) \tag{6.18}$$

if $(\mathcal{A}, G)$ and $(\mathcal{B}, H)$ are amenable C^*-dynamical systems. In particular:

$$\mathcal{E}(X/[\alpha]) \rtimes X \simeq \mathscr{E}(X/[\alpha]) \otimes C^*([\alpha]). \tag{6.19}$$

COROLLARY 6.19. *The map $\tau(A) = (\tau_\alpha(A))_{\alpha \in \mathbb{S}_X}$ induces an injective morphism*

$$\mathscr{E}(X)/\mathscr{K}(X) \hookrightarrow \prod_{\alpha \in \mathbb{S}_X} \mathcal{E}(X/[\alpha]) \rtimes X. \tag{6.20}$$

The following theorem is an immediate consequence of the preceding corollary.

THEOREM 6.20. *Let H be a self-adjoint operator on $L^2(X)$ affiliated to $\mathscr{E}(X)$. Then for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$ the limit $\tau_\alpha(H) \equiv \alpha \cdot H = \operatorname{s-lim}_{r \to +\infty} T_{ra}^* H T_{ra}$ exists and is independent of the choice of a. We have $\sigma_{\mathrm{ess}}(H) = \overline{\bigcup_{\alpha \in \mathbb{S}_X}} \sigma(\alpha \cdot H)$.*

The question whether the union $\bigcup_{\alpha \in \mathbb{S}_X} \sigma(\alpha \cdot H)$ is closed or not will not be treated in this paper (see [39] for related results). That the union is closed if $\mathscr{E}(X)$ is replaced by the standard N-body algebra $\mathscr{E}_{\mathcal{C}_0}(X)$ is shown in Theorem 6.27 of [22] and is a consequence of the fact that $\{\tau_\alpha(A) : \alpha \in \mathbb{S}_X\}$ is a compact subset of $\mathscr{E}_{\mathcal{C}_0}(X)$ for each $A \in \mathscr{E}_{\mathcal{C}_0}(X)$. Unfortunately this is not true in the present case.

LEMMA 6.21. *If $A \in \mathscr{E}(X)$, then $\{\tau_\alpha(A) : \alpha \in \mathbb{S}_X\}$ is a relatively compact subset of $\mathscr{E}(X)$, but is not compact in general.*

Proof. We first show that $\{\tau_\alpha(A) : \alpha \in \mathbb{S}_X\}$ is a relatively compact set in $\mathscr{E}(X)$. Since the product and the sum of two relatively compact subsets is relatively compact, it suffices to prove this for A in a generating subset of the algebra $\mathscr{E}(X)$, so we may assume that $A = u(q)v(p)$ with $u \in \mathcal{C}(\overline{X/Y})$ and $v \in \mathcal{C}_0(X^*)$ for some subspace Y. Then $\tau_\alpha(A) = A$ if $\alpha \subset Y$ and $\tau_\alpha(A) = \tau_\alpha(u)v(p)$ if $\alpha \not\subset Y$.

In the second case we have $\tau_\alpha(u) \in \mathbb{C}$ and $|\tau_\alpha(u)| \leqslant \|u\|$, so it is clear that the set of the $\tau_\alpha(A)$ is relatively compact.

We shall give now an example when this set is not closed. Let $X = \mathbb{R}^2$, $Y = \{0\} \times \mathbb{R}$, and let us identify $X/Y = \mathbb{R} \times \{0\}$. The operator A will be of the form $A = u(q)v(p)$ so that $\tau_\alpha(A) = \tau_\alpha(u)(q)v(p)$ with $u = u_0 + u_Y$ for some $u_0 \in \mathcal{C}(\overline{X})$ and $u_Y \in \mathcal{C}(\overline{X/Y})$. We have $\overline{X/Y} = \overline{\mathbb{R} \times \{0\}} \equiv [-\infty, +\infty]$, and hence $\mathbb{S}_{\overline{X/Y}}$ consists of two points $\pm\infty$. If $\alpha \in \mathbb{S}_X$ then $\tau_\alpha(u) = u_0(\alpha) + \tau_\alpha(u_Y)$ where $\tau_\alpha(u_Y) = u_Y$ if $\alpha \subset Y$ and $\tau_\alpha(u_Y) = u_Y(\pi_Y(\alpha))$ if $\alpha \not\subset Y$. In the last case we have only two possibilities: $\tau_\alpha(u_Y) = u_Y(+\infty)$ if α is in the open right half-plane and $\tau_\alpha(u_Y) = u_Y(-\infty)$ if α is in the open left half-plane.

Let β be the upper half-axis, i.e. $\beta = \{(0, y) : y > 0\}$, and let us choose u_0 such that $u_0(\gamma) \neq u_0(\beta)$ for all $\gamma \in \mathbb{S}_X, \gamma \neq \beta$. Then choose u_Y such that $u_Y(+\infty) - u_Y(-\infty)$ be strictly larger than $u_0(\gamma) - u_0(\beta)$ for all $\gamma \in \mathbb{S}_X$. Then $\{\tau_\alpha(u) : \alpha \in \mathbb{S}_X\}$ consists of the following elements:

$$u_0(\beta) + u_Y, \quad u_0(-\beta) + u_Y, \quad u_0(\alpha) + u_Y(+\infty),$$

if α is in the open right half-plane, and

$$u_0(\alpha) + u_Y(-\infty)$$

if α is in the open left half-plane. We shall prove that this set is not closed. Indeed, let $\{\alpha_n\}$ be a sequence of rays in the open right half-plane that converges to β. Then $\tau_{\alpha_n}(u) = u_0(\alpha_n) + u_Y(+\infty)$ is a sequence of complex numbers that converges to $u_0(\beta) + u_Y(+\infty)$. This number cannot be of the form $\tau_\gamma(u)$ for some $\gamma \in \mathbb{S}_X$ because, if $\gamma \subset Y$, then $\tau_\gamma(u) = u_0(\gamma) + u_Y$ is not a number. If γ is in the open right half-plane, then $\tau_\gamma(u) = u_0(\gamma) + u_Y(+\infty)$, which cannot be equal to $u_0(\beta) + u_Y(+\infty)$, because $u_0(\gamma) \neq u_0(\beta)$. On the other hand, if γ is in the open left half-plane, then $\tau_\gamma(u) = u_0(\gamma) + u_Y(-\infty)$, which cannot be equal to $u_0(\beta) + u_Y(+\infty)$ because $u_0(\gamma) - u_0(\beta) < u_Y(+\infty) - u_Y(-\infty)$. ∎

REMARK 6.22. It is important to notice that finding good compactifications of X related to the N-body problem is useful for the problem of approximating numerically the eigenvalues and eigenfunctions of N-body Hamiltonians [1], [16], [17], [18], [19], [50]. In particular, this gives a further justification for trying to find the structure of the character space of $\mathcal{E}(X)$.

6.5. SELF-ADJOINT OPERATORS AFFILIATED TO $\mathscr{E}(X)$. Our purpose here is to show that the class of self-adjoint operators affiliated to $\mathscr{E}(X)$ is quite large. As mentioned at the beginning of Section 6, we may and shall assume $\dim X \geqslant 2$.

We first prove Theorem 1.2. We recall the definition of $\mathcal{E}^\sharp(X)$ in terms of the algebras $\mathcal{B}(\overline{X/Y})$ defined as in (3.4) and Lemma 3.3. Note that, according to our notational conventions, $\mathcal{B}(\overline{X/Y})$ is identified with a C^*-algebra of functions on X.

DEFINITION 6.23. $\mathcal{E}^\sharp(X)$ is the C^*-subalgebra of $L^\infty(X)$ generated by the functions of the form $v \circ \pi_Y$, where Y runs over the set of linear subspaces of X and $v \in \mathcal{B}(\overline{X/Y})$.

PROPOSITION 6.24. *$u \in \mathcal{E}^\sharp(X)$ if and only if there is a sequence of functions $u_n \in \mathcal{E}(X)$ such that $\sup_n \|u_n\|_{L^\infty(X)} < \infty$ and $\lim_n \|u_n(q) - u(q)\|_{\mathcal{H}^s(X)\to\mathcal{H}(X)} = 0$ for some $s > 0$.*

Proof. We need the following consequence of Proposition 3.4: $u \in \mathcal{B}(\overline{X/Y})$ if and only if there is a sequence of functions $u_n \in \mathcal{C}(\overline{X/Y})$ such that $\|u_n\|_{L^\infty} \leqslant C$ with C independent of n and $\lim_n \|u_n(q) - u(q)\|_{\mathcal{H}^s(X)\to\mathcal{H}(X)} = 0$ for some $s > 0$. For the proof, it is useful to distinguish between the function u on X and the function u' on X/Y related to it by $u = u' \circ \pi_Y$. Then Proposition 3.4 gives us functions $u'_n : X/Y \to \mathbb{C}$ of class $\mathcal{C}(\overline{X/Y})$ such that $\|u'_n\|_{L^\infty(X/Y)} \leqslant C$ and $u'_n(q) \to u'(q)$ in norm in the space of bounded operators $\mathcal{H}^s(X/Y) \to \mathcal{H}(X/Y)$. Thus, if we set $u_n = u'_n \circ \pi_Y$, it suffices to show that $\lim_n \|u_n(q) - u(q)\|_{\mathcal{H}^s(X)\to\mathcal{H}(X)} = 0$. But this is clear because, if Z is a subspace supplementary to Y in X, then we have $X/Y \simeq Z$, $\mathcal{H}(X) \simeq \mathcal{H}(Y) \otimes \mathcal{H}(Z)$ and $\mathcal{H}^s(X) \simeq (\mathcal{H}^s(Y) \otimes \mathcal{H}(Z)) \cap (\mathcal{H}(Y) \otimes \mathcal{H}^s(Z))$.

Since $\mathcal{E}^\sharp$ is the norm closure in L^∞ of the space of linear combinations of products of functions in $\mathcal{B}(\overline{X/Y})$ with Y running over all subspaces of X, it remains to prove that if u is a finite product $u = u^1 \cdots u^k$ of functions $u^i \in \mathcal{B}(\overline{X/Y_i})$, then one may construct a sequence $\{u_n\}$ as in the statement of the proposition. By what we have proved, such a sequence $\{u^i_n\}$ exists for each i and clearly it suffices to take $u_n = u^1_n \cdots u^k_n$. ∎

Proof of Theorem 1.2. We consider first the operator H defined in (1.3). Since $\dim X \geqslant 2$, the function h is either lower or upper semi-bounded, and hence we may assume $h \geqslant 0$. In Theorem 5.8 we take $H_0 = h(p)$, so H_0 is a positive operator strictly affiliated to $\mathscr{E}(X)$. Then, according to Theorem 5.8, if V is a bounded self-adjoint operator on $\mathcal{H}$ such that

$$\varphi(H_0)V(H_0+1)^{-1/2} \in \mathscr{E}(X) \quad \text{for all } \varphi \in \mathcal{C}_c(\mathbb{R}) \tag{6.21}$$

then $H = H_0 + V$ is a self-adjoint operator strictly affiliated to $\mathscr{E}(X)$. Since h is a proper, continuous function we have $\varphi \circ h \in \mathcal{C}_c(\mathbb{R})$, and hence for any $s > 0$ the function $\psi(k) = \varphi(h(k))\langle k\rangle^s$ also belongs to $\mathcal{C}_c(\mathbb{R})$, so $\psi(p) \in \mathscr{E}(X)$. Then $\varphi(H_0) = \psi(p)\langle p\rangle^{-s}$ hence (6.21) is satisfied if $\langle p\rangle^{-s}V \in \mathscr{E}(X)$. This last fact is clearly true if $V = u(q)$ with $u \in \mathcal{E}(X)$ and remains true if $u \in \mathcal{E}^\sharp(X)$ by Proposition 6.24.

Now let H be the self-adjoint operator associated to the operator $L : \mathcal{H}^m \to \mathcal{H}^{-m}$ defined by (1.5). Then we take $H_0 = 1 + \sum_{|\mu|=m} p^{2\mu}$ and equip $\mathcal{H}^m$ with the scalar product $\langle u|H_0u\rangle$. If we set $V = L - H_0$, then (1.6) implies $V \geqslant -(1 -$

$\delta)H_0 - \gamma$. Since $\delta > 0$ we see that the conditions on V in Theorem 5.8 are satisfied (with a change of notation). The second condition on V is clearly satisfied if $H_0^{-1}VH_0^{-1/2} \in \mathscr{E}(X)$. Observe that the operator V has the same form as L, only the coefficients $g_{\mu\nu}$ in the principal part being changed in an irrelevant manner (replaced by $g_{\mu\nu} - \delta_{\mu\nu}$ and $g_{00} - 1$ respectively). Thus it remains only to check that $p^\mu H_0^{-1} g_{\mu\nu} p^\nu H_0^{-1/2}$ belongs to $\mathscr{E}(X)$ if $|\mu|, |\nu| \leqslant m$. Since $p^\nu H_0^{-1/2}$ belongs to the multiplier algebra of $\mathscr{E}(X)$, it suffices to have $p^\mu H_0^{-1} g_{\mu\nu} \in \mathscr{E}(X)$. Since H_0 is of order $\langle p \rangle^{2m}$ and p^μ is of order at most m, this follows from what we proved before in the case $H = h(p) + V$. ∎

REMARK 6.25. One may treat, by the technique of the preceding proof, operators L with unbounded coefficients in the terms of lower order. Assume that for each μ, ν the operator of multiplication by $g_{\mu\nu}$ maps $\mathcal{H}^{m-|\nu|}$ into $\mathcal{H}^{|\mu|-m}$. Then $L : \mathcal{H}^m \to \mathcal{H}^{-m}$ is well defined and the condition (1.6) ensures the existence of the self-adjoint operator H associated to it. It has been shown in Example 4.13 of [22] that this operator is affiliated to the crossed product $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$, and hence its essential spectrum can be described in terms of localizations at infinity of H. However, its affiliation to the smaller algebra $\mathscr{E}(X)$ would give a much more precise characterization of the essential spectrum. For this, by the argument of the preceding proof, it suffices to have $p^\mu H_0^{-1} g_{\mu\nu} p^\nu H_0^{-1/2} \in \mathscr{E}(X)$ for all μ, ν. And this is satisfied if the operator $g_{\mu\nu}(q)$ is the norm limit in $\mathcal{B}(\mathcal{H}^{m-|\nu|}, \mathcal{H}^{|\mu|-m-1})$ of a sequence of operators $g_{\mu\nu}^k(q)$ with $g_{\mu\nu}^k \in \mathcal{E}(X)$.

In the rest of this section we consider only potentials that have a simpler N-body type structure, as explained in Subsection 1.4 (page 337), and we shall prove Theorems 1.5 and 1.6. We will be able to cover a large class of such interactions by using a more explicit description of the algebras $\mathcal{C}(\overline{X/Y}) \rtimes X$ that we describe now.

Observe first that if Z is a supplement of Y in X, so Z is a linear subspace of X such that $Y \cap Z = \{0\}$ and $Y + Z = X$, then:

$$\mathcal{C}(\overline{X/Y}) \rtimes X = C^*(Y) \otimes \mathscr{S}(Z) \quad \text{relatively to } L^2(X) = L^2(Y) \otimes L^2(Z). \tag{6.22}$$

Indeed, $\mathcal{C}(\overline{X/Y}) \rtimes X$ is the norm closed subspace generated by the operators of the form $u(q)v(p)$ with $u \in \mathcal{C}(\overline{X/Y})$ and $v(p) \in C^*(X)$. But once Z is chosen, we may identify $\mathcal{C}(\overline{X/Y}) = 1 \otimes \mathcal{C}(\overline{Z})$ and $C^*(X) = C^*(Y) \otimes C^*(Z)$, and hence (6.22). Of course, this is a particular case of the relation (6.18) from Remark 6.18.

It is useful to express (6.22) in an intrinsic way, independent of the choice of Z. This is in fact an extension of Theorem 4.2 to the present setting.

Observe first that if A is a bounded operator on $L^2(X)$ and $[A, T_y] = 0$ for all $y \in Y$, then $T_x^* A T_x$ depends only on the class $z = \pi_Y(x)$ of x in X/Y. Thus we have an action τ of X/Y on the set of operators A in the commutant of $\{T_y\}_{y \in Y}$ such that $\tau_z(A) = T_x^* A T_x$ if $\pi_Y(x) = z$. Later on we shall keep the notation

$\tau_a(A) = T_a^* A T_a$ for $a \in X/Y$ since the correct interpretation should be clear from the context.

THEOREM 6.26. *The set* $\mathcal{C}(\overline{X/Y}) \rtimes X$ *consists of the operators* $A \in \mathscr{B}(X)$ *that have the position-momentum limit property and are such that:*

(i) $[A, T_y] = 0$ *for all* $y \in Y$;

(ii) *for each* $\alpha \in \mathbb{S}_{X/Y}$ *the limit* s-lim $\tau_a(A)^{(*)}$ *with* $a \to \alpha$ *in* X/Y *exists.*

Proof. Let $\check{\alpha} = \pi_Y^{-1}(\widetilde{\alpha})$ be the inverse image of the filter $\widetilde{\alpha}$ through the map π_Y, i.e. the set of subsets of X of the form $\pi_Y^{-1}(F)$ with $F \in \widetilde{\alpha}$. This is a translation invariant filter of subsets of X and, if f is a function defined on X/Y with values in a topological space $\mathcal{B}$, then $\lim_{z\to\alpha} f(z) = b$ if and only if $\lim_{x\to\check{\alpha}} f \circ \pi_Y(x) = b$. It is then clear that the condition (ii) above is equivalent to the fact that $\operatorname{s-lim}_{x\to\check{\alpha}} T_x^* A T_x$ exists for each $\alpha \in \mathbb{S}_{X/Y}$. Now the proof is essentially a repetition of the proof of Theorem 4.2, the filter $\widetilde{\alpha}$ on X/Y being replaced by the translation invariant filter $\check{\alpha}$ on X. ∎

There is no simple analogue of Theorem 5.2 in the present context, but one can extend Proposition 4.7 and Theorem 5.3. Indeed, both Theorems 1.5 and 1.6 follow from Theorems 5.7, 5.8 and 6.26. To prove Theorem 1.6 for example, let us set $\langle p \rangle = (1 + |p|^2)^{1/2}$. Since we have $1 + h(p) \sim \langle p \rangle^{2s}$, it suffices to prove that for each Y the operator $\langle p \rangle^{-2s} V_Y \langle p \rangle^{-s}$ is in $\mathcal{C}(\overline{X/Y}) \rtimes X$. This clearly follows from Theorem 6.26.

Acknowledgements. We thank B. Ammann and N. Prudhon for useful discussions. We also thank the referee for carefully reading the paper and for useful comments. Vladimir Georgescu has been partially supported by Labex MME-DII (ANR11-LBX-0023-01) and Victor Nistor has been partially supported by ANR-14-CE25-0012-01.

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Received April 8, 2016; revised August 6, 2016.