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FACTORIZATIONS OF CHARACTERISTIC FUNCTIONS

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Dedicated to Professor Gadadhar Misra on the occasion of his 60th birthday

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ABSTRACT. Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be row contractions on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and L be a contraction from $\mathcal{D}_B = \overline{\operatorname{ran}} D_B$ to $\mathcal{D}_{A^*} = \overline{\operatorname{ran}} D_{A^*}$ where $D_B = (I - B^*B)^{1/2}$ and $D_{A^*} = (I - AA^*)^{1/2}$. Let Θ_T be the characteristic function of $T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix}$. Then Θ_T coincides with the product of the characteristic function Θ_A of A, the Julia–Halmos matrix corresponding to L and the characteristic function Θ_B of B. More precisely, Θ_T coincides with

$$\begin{bmatrix} \Theta_B & 0 \\ 0 & I \end{bmatrix} \left(I_\Gamma \otimes \begin{bmatrix} L^* & (I - L^*L)^{1/2} \\ (I - LL^*)^{1/2} & -L \end{bmatrix} \right) \begin{bmatrix} \Theta_A & 0 \\ 0 & I \end{bmatrix},$$

where Γ is the full Fock space. Similar results hold for constrained row contractions.

KEYWORDS: *Row contractions, Fock space, invariant subspaces, characteristic functions, factorizations of analytic functions, upper triangular block operator matrices.*

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INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and T be a contraction (that is, $I - TT^* \geqslant 0$) on $\mathcal{H}$. Suppose $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$ are the defect operators, and $\mathcal{D}_T = \overline{\operatorname{ran}} D_T$ and $\mathcal{D}_{T^*} = \overline{\operatorname{ran}} D_{T^*}$ are the defect spaces of T. Then the *characteristic function* of T is an operator valued bounded analytic function $\Theta_T \in H^\infty_{\mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})}(\mathbb{D})$ defined by

$$\Theta_T(z) = (-T + zD_{T^*}(I - zT^*)^{-1}D_T)|_{\mathcal{D}_T} \quad (z \in \mathbb{D}).$$

The notion of characteristic function plays an important role in many areas of operator theory and function theory (see [14]). In particular, characteristic functions are one of the central objects of study in noncommutative operator theory and noncommutative function theory (see Popescu [12] and references therein).

On the other hand, the notion of Julia–Halmos matrix is important in the construction of isometric and unitary dilation maps for contractions (cf. [14]). Recall that the *Julia–Halmos matrix* corresponding to a contraction L from $\mathcal{H}$ to $\mathcal{K}$ is the unitary matrix

$$J_L = \begin{bmatrix} L^* & (I - L^*L)^{1/2} \\ (I - LL^*)^{1/2} & -L \end{bmatrix} = \begin{bmatrix} L^* & D_L \\ D_{L^*} & -L \end{bmatrix}.$$

This is also directly related to analytic or functional models for contractions in the sense of Sz.-Nagy and Foias (see Timotin [15]). However, one of the most striking results along these lines is due to Sz.-Nagy and Foias [13].

THEOREM 0.1 (Sz.-Nagy and Foias). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$ be a contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then there exist a contraction $L \in \mathcal{B}(\mathcal{D}_{T_2}, \mathcal{D}_{T_1^*})$ and (canonical) unitary operators $\tau \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T_1} \oplus \mathcal{D}_L)$ and $\tau_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{T_2^*} \oplus \mathcal{D}_{L^*})$ such that $X = D_{T_1^*} L D_{T_2}$ and*

$$\Theta_T(z) = \tau_*^{-1} \begin{bmatrix} \Theta_{T_2}(z) & 0 \\ 0 & I_{\mathcal{D}_{L^*}} \end{bmatrix} \begin{bmatrix} L^* & D_L \\ D_{L^*} & -L \end{bmatrix} \begin{bmatrix} \Theta_{T_1}(z) & 0 \\ 0 & I_{\mathcal{D}_L} \end{bmatrix} \tau \quad (z \in \mathbb{D}).$$

In this paper we first generalize the above factorization result to noncommuting tuples of row contractions. For the class of constrained row contractions, we obtain a similar result to the main factorization result.

The paper is organized as follows: in Section 1 we give a brief introduction of characteristic functions and multi-analytic functions in the noncommutative setup and fix some notations. In Section 2 we present the Sz.-Nagy and Foias type factorization results for noncommuting tuples of row contractions. In the final section we obtain similar factorization results for constrained row contractions.

1. PREPARATORY RESULTS

In this section we recall and study some basic tools of operator theory such as characterizations of upper triangular operator matrices, characteristic functions and multi-analytic functions which appear in all later investigation. A general theory of characteristic operators and (multi-)analytic models for row contractions on Hilbert spaces was developed by G. Popescu in [5], [6] and [9] (also see [11] and references therein).

Let $\mathcal{H}$ be a Hilbert space and $\{T_j\}_{j=1}^n \subseteq \mathcal{B}(\mathcal{H})$ where $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. Then the n-tuple $T = (T_1, \ldots, T_n)$ is called a row contraction if $T : \bigoplus_{i=1}^{n} \mathcal{H} \to \mathcal{H}$ is a contraction, that is, $\sum_{j=1}^{n} T_j T_j^* \leqslant I_{\mathcal{H}}$ or, equivalently, if $\Big\| \sum_{j=1}^{n} T_j h_j \Big\|^2 \leqslant \sum_{j=1}^{n} \|h_j\|^2$, $h_1, \ldots, h_n \in \mathcal{H}$.

The defect operators and defect spaces of a row contraction T on $\mathcal{H}$ are given by

$$D_T = (I - T^*T)^{1/2} \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{H}), \quad D_{T^*} = (I - TT^*)^{1/2} \in \mathcal{B}(\mathcal{H}), \quad \text{and}$$

$$\mathcal{D}_T = \overline{\operatorname{ran}} D_T \subseteq \bigoplus_{i=1}^n \mathcal{H}, \quad \mathcal{D}_{T^*} = \overline{\operatorname{ran}} D_{T^*} \subseteq \mathcal{H},$$

respectively.

The class of row contractions with which we are concerned has the following characterization (see [13] or Lemma 2.1, Chapter IV in [4]).

THEOREM 1.1 (Sz.-Nagy and Foias). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $A = (A_1, \ldots, A_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_1, \mathcal{H}_1)$, $B = (B_1, \ldots, B_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_2)$ and $X = (X_1, \ldots, X_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_1)$ are row operators. Then the row operator*

$$T = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathcal{B}((\bigoplus_1^n \mathcal{H}_1) \oplus (\bigoplus_1^n \mathcal{H}_2), \mathcal{H}_1 \oplus \mathcal{H}_2)$$

is a row contraction if and only if A and B are row contractions and

$$X = D_{A^*} L D_B,$$

for some contraction $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$.

We now recall the following result of Sz.-Nagy and Foias about unitary operators between defect spaces (see [13] or Corollary 2.2, Chapter IV in [4]).

THEOREM 1.2 (Sz.-Nagy and Foias). *In the setting of Theorem 1.1, let T be a row contraction. Then there exist unitary operators $\sigma : \mathcal{D}_T \to \mathcal{D}_A \oplus \mathcal{D}_L$ and $\sigma_* : \mathcal{D}_{T^*} \to \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*}$ such that*

$$\sigma D_T = \begin{bmatrix} D_A & -A^*LD_B \\ 0 & D_L D_B \end{bmatrix} \quad \text{and} \quad \sigma_* D_{T^*} = \begin{bmatrix} -BL^*D_{A^*} & D_{B^*} \\ D_{L^*}D_{A^*} & 0 \end{bmatrix}. \tag{1.1}$$

The full *Fock space* over $\mathbb{C}^n$, denoted by Γ, is the Hilbert space

$$\Gamma := \bigoplus_{m=0}^{\infty} (\mathbb{C}^n)^{\otimes^m} = \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^{\otimes^2} \oplus \cdots \oplus (\mathbb{C}^n)^{\otimes^m} \oplus \cdots.$$

The *vacuum vector* $1 \oplus 0 \oplus \cdots \in \Gamma$ is denoted by $e_\emptyset$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$ and $\mathbb{F}_n^+$ be the unital free semi-group with generators $1, \ldots, n$ and the identity $\emptyset$. For $\alpha = \alpha_1 \cdots \alpha_m \in \mathbb{F}_n^+$ we denote the vector $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m}$ by e_α. Then $\{e_\alpha : \alpha \in \mathbb{F}_n^+\}$ forms an orthonormal basis of Γ. For each $j = 1, \ldots, n$, the left creation operator L_j and the right creation operator R_j on Γ are defined by

$$L_j f = e_j \otimes f, \quad R_j f = f \otimes e_j \quad (f \in \Gamma),$$

respectively. Moreover, $R_j = U^* L_j U$ where U, defined by $U(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}) = e_{i_m} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$, is the flipping operator on Γ. The *noncommutative* *disc algebra* $\mathcal{A}_n^\infty$ is the norm closed algebra generated by $\{I_\Gamma, L_1, \ldots, L_n\}$ and the *noncommutative analytic Toeplitz algebra* $\mathcal{F}_n^\infty$ is the WOT-closure of $\mathcal{A}_n^\infty$ (see [7]).

Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $M \in \mathcal{B}(\Gamma \otimes \mathcal{E}, \Gamma \otimes \mathcal{E}_*)$. Then M is said to be a *multi-analytic operator* if

$$M(L_j \otimes I_{\mathcal{E}}) = (L_j \otimes I_{\mathcal{E}_*})M \quad (j = 1, \ldots, n).$$

In this case the bounded linear map $\theta \in \mathcal{B}(\mathcal{E}, \Gamma \otimes \mathcal{E}_*)$ defined by

$$\theta(\eta) = M(e_\emptyset \otimes \eta) \quad (\eta \in \mathcal{E}),$$

is said to be the *symbol* of M and we denote $M = M_\theta$. Moreover, define $\theta_\alpha \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, $\alpha \in \mathbb{F}_n^+$ by

$$\langle \theta_\alpha \eta, \eta_* \rangle := \langle \theta\eta, e_{\overline{\alpha}} \otimes \eta_* \rangle = \langle M(e_\emptyset \otimes \eta), e_{\overline{\alpha}} \otimes \eta_* \rangle, \quad (\eta \in \mathcal{E}, \eta_* \in \mathcal{E}_*)$$

where $\overline{\alpha}$ is the reverse of α. The Fourier type representation for multi-analytic operators was considered first in [8] by Popescu and from this representation we have a unique formal Fourier expansion

$$M \sim \sum_{\alpha \in \mathbb{F}_n^+} R_\alpha \otimes \theta_\alpha \quad \text{and} \quad M = \text{SOT-}\lim_{r \to 1^-} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} r^{|\alpha|} R_\alpha \otimes \theta_\alpha$$

where $|\alpha|$ is the length of α.

A multi-analytic operator $M_\theta \in \mathcal{B}(\Gamma \otimes \mathcal{E}, \Gamma \otimes \mathcal{E}_*)$ is said to be *purely contractive* if M_θ is a contraction and

$$\|P_{e_\emptyset \otimes \mathcal{E}_*} \theta \eta\| < \|\eta\| \quad (\eta \in \mathcal{E}, \eta \neq 0).$$

We say that M_θ coincides with a multi-analytic operator $M_{\theta'} \in \mathcal{B}(\Gamma \otimes \mathcal{E}', \Gamma \otimes \mathcal{E}'_*)$ if there exist unitary operators $W : \mathcal{E} \to \mathcal{E}'$ and $W_* : \mathcal{E}_* \to \mathcal{E}'_*$ such that

$$(I_\Gamma \otimes W_*)M_\theta = M_{\theta'}(I_\Gamma \otimes W).$$

Let $\mathcal{H}$ be a Hilbert space and $T = (T_1, \ldots, T_n)$ be a row operator on $\mathcal{H}$. For simplicity of the notations we will denote by $\widetilde{T}$ and $\widetilde{R}$ the row operators $(I_\Gamma \otimes T_1, \ldots, I_\Gamma \otimes T_n)$ and $(R_1 \otimes I_{\mathcal{H}}, \ldots, R_n \otimes I_{\mathcal{H}})$ on $\Gamma \otimes \mathcal{H}$, respectively.

Among multi-analytic operators, characteristic functions [10] play an important role in multivariable operator theory and noncommutative function theory (see [12] and other references therein). The *characteristic function* of a row contraction T on $\mathcal{H}$ is a purely contractive multi-analytic operator $\Theta_T \in \mathcal{B}(\Gamma \otimes \mathcal{D}_T, \Gamma \otimes \mathcal{D}_{T^*})$ defined by

$$\Theta_T \sim -\widetilde{T} + D_{\widetilde{T}^*}(I_{\Gamma \otimes \mathcal{H}} - \widetilde{R}\widetilde{T}^*)^{-1}\widetilde{R}D_{\widetilde{T}}.$$

Hence

$$\Theta_T = \text{SOT-}\lim_{r \to 1} \Theta_T(rR),$$

where for each $r \in [0, 1)$,

$$\Theta_T(rR) := -\widetilde{T} + D_{\widetilde{T}^*}(I_{\Gamma \otimes \mathcal{H}} - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}D_{\widetilde{T}}.$$

Therefore

$$\Theta_T = \text{SOT-}\lim_{r\to 1} \Theta_T(rR) = \text{SOT-}\lim_{r\to 1}[-\widetilde{T} + D_{\widetilde{T}^*}(I_{\Gamma\otimes\mathcal{H}} - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R} D_{\widetilde{T}}]. \tag{1.2}$$

2. FACTORIZATIONS OF CHARACTERISTIC FUNCTIONS OF NONCOMMUTING TUPLES

In this section we prove the main theorem on factorizations of characteristic functions of upper triangular operator matrices. We begin with the following simple lemma.

LEMMA 2.1. *Let T be a row contraction on $\mathcal{H}$. Then for each $r \in [0, 1)$*

$$\Theta_T(rR)D_{\widetilde{T}} = D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1}(r\widetilde{R} - \widetilde{T}) \quad \textit{and}$$
$$I + \Theta_T(rR)\widetilde{T}^* = D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1}D_{\widetilde{T}^*}.$$

Proof. Let $r \in [0, 1)$. Since $\widetilde{T}D_{\widetilde{T}} = D_{\widetilde{T}^*}\widetilde{T}$ (see equation (3.4) in Chapter I, [14]), we have

$$\begin{aligned}
\Theta_T(rR)D_{\widetilde{T}} &= [-\widetilde{T} + D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}D_{\widetilde{T}}]D_{\widetilde{T}} \\
&= -D_{\widetilde{T}^*}\widetilde{T} + D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}D^2_{\widetilde{T}} \\
&= D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1}(-(I - r\widetilde{R}\widetilde{T}^*)\widetilde{T} + r\widetilde{R}D^2_{\widetilde{T}}) \\
&= D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1}(r\widetilde{R} - \widetilde{T}).
\end{aligned}$$

For the second equality we compute

$$\begin{aligned}
D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1}D_{\widetilde{T}^*} &= D_{\widetilde{T}^*}(I + (I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}\widetilde{T}^*)D_{\widetilde{T}^*} \\
&= D^2_{\widetilde{T}^*} + D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}\widetilde{T}^* D_{\widetilde{T}^*} \\
&= I - \widetilde{T}\widetilde{T}^* + D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}D_{\widetilde{T}}\widetilde{T}^* \\
&= I + (-\widetilde{T} + D_{\widetilde{T}^*}(I - r\widetilde{R}\widetilde{T}^*)^{-1} r\widetilde{R}D_{\widetilde{T}})\widetilde{T}^* \\
&= I + \Theta_T(rR)\widetilde{T}^*.
\end{aligned}$$

This completes the proof. ∎

We are now ready to prove the main result of this section.

THEOREM 2.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and*

$$T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix} : (\textstyle\bigoplus_1^n \mathcal{H}_1) \oplus (\bigoplus_1^n \mathcal{H}_2) \to \mathcal{H}_1 \oplus \mathcal{H}_2,$$

be a row contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$ where $A = (A_1, \ldots, A_n)$ on $\mathcal{H}_1$ and $B = (B_1, \ldots, B_n)$ on $\mathcal{H}_2$ are row contractions and $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$ is a contraction. Then

$$\Theta_T = (I_\Gamma \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_L} \end{bmatrix} (I_\Gamma \otimes \sigma),$$

where $\sigma \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_L)$ and $\sigma_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*})$ are unitary operators as in Theorem 1.2 *and J_L is the Julia–Halmos matrix corresponding to L.*

Proof. For each $r \in [0,1)$, Theorem 1.2, Lemma 2.1 and the fact that $(I_\Gamma \otimes \sigma_*)D_{\widetilde{T}^*} = I_\Gamma \otimes \sigma_* D_{T^*}$ yield

$$(I_\Gamma \otimes \sigma_*)\Theta_T(rR)D_{\widetilde{T}} = \begin{bmatrix} -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} & D_{\widetilde{B}^*} \\ D_{\widetilde{L}^*} D_{\widetilde{A}^*} & 0 \end{bmatrix} (I_{\Gamma\otimes\mathcal{H}} - r\widetilde{R}\widetilde{T}^*)^{-1}(r\widetilde{R} - \widetilde{T}).$$

Now setting $X = D_{A^*}LD_B$, we get

$$\begin{aligned}
&(I_{\Gamma\otimes\mathcal{H}} - r\widetilde{R}\widetilde{T}^*)^{-1} \\
&\quad= \left(\begin{bmatrix} I_{\Gamma\otimes\mathcal{H}_1} & 0 \\ 0 & I_{\Gamma\otimes\mathcal{H}_2} \end{bmatrix} - r \begin{bmatrix} \widetilde{R} & 0 \\ 0 & \widetilde{R} \end{bmatrix} \begin{bmatrix} \widetilde{A}^* & 0 \\ \widetilde{X}^* & \widetilde{B}^* \end{bmatrix} \right)^{-1} \\
&\quad= \begin{bmatrix} I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^* & 0 \\ -r\widetilde{R}\widetilde{X}^* & I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^* \end{bmatrix}^{-1} \\
&\quad= \begin{bmatrix} (I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1} & 0 \\ (I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1}(r\widetilde{R}\widetilde{X}^*)(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1} & (I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1} \end{bmatrix} \\
&\quad= \begin{bmatrix} F & 0 \\ G & H \end{bmatrix},
\end{aligned}$$

where $F = (I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}$, $H = (I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1}$ and $G = H(r\widetilde{R}\widetilde{X}^*)F = (I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1}(r\widetilde{R}\widetilde{X}^*)(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}$. Therefore

$$\begin{aligned}
(I_\Gamma \otimes \sigma_*)\Theta_T(rR)D_{\widetilde{T}} &= \begin{bmatrix} -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} & D_{\widetilde{B}^*} \\ D_{\widetilde{L}^*} D_{\widetilde{A}^*} & 0 \end{bmatrix} \begin{bmatrix} F & 0 \\ G & H \end{bmatrix} \begin{bmatrix} r\widetilde{R} - \widetilde{A} & -\widetilde{X} \\ 0 & r\widetilde{R} - \widetilde{B} \end{bmatrix} \\
&= \begin{bmatrix} -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} & D_{\widetilde{B}^*} \\ D_{\widetilde{L}^*} D_{\widetilde{A}^*} & 0 \end{bmatrix} \begin{bmatrix} F(r\widetilde{R} - \widetilde{A}) & -F\widetilde{X} \\ G(r\widetilde{R} - \widetilde{A}) & -G\widetilde{X} + H(r\widetilde{R} - \widetilde{B}) \end{bmatrix} \\
&= \begin{bmatrix} C_{11}(r) & C_{12}(r) \\ C_{21}(r) & C_{22}(r) \end{bmatrix} \in \mathcal{B}((\Gamma \otimes \mathcal{H}_1) \oplus (\Gamma \otimes \mathcal{H}_2)),
\end{aligned}$$

where $C_{11}(r) = -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) + D_{\widetilde{B}^*} G(r\widetilde{R} - \widetilde{A})$, $C_{12}(r) = \widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} F\widetilde{X} - D_{\widetilde{B}^*} G\widetilde{X} + D_{\widetilde{B}^*} H(r\widetilde{R} - \widetilde{B})$, $C_{21}(r) = D_{\widetilde{L}^*} D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A})$ and $C_{22}(r) = -D_{\widetilde{L}^*} D_{\widetilde{A}^*} F\widetilde{X}$. Further, we compute

$$\begin{aligned}
C_{11}(r) &= -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) + D_{\widetilde{B}^*}(Hr\widetilde{R}\widetilde{X}^* F)(r\widetilde{R} - \widetilde{A}) \\
&= -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) + D_{\widetilde{B}^*} H(r\widetilde{R}\widetilde{X}^*) F(r\widetilde{R} - \widetilde{A}) \\
&= -\widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) + D_{\widetilde{B}^*} H(r\widetilde{R} D_{\widetilde{B}} \widetilde{L}^* D_{\widetilde{A}^*}) F(r\widetilde{R} - \widetilde{A}) \\
&= [-\widetilde{B} + D_{\widetilde{B}^*} H(r\widetilde{R} D_{\widetilde{B}})]\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) \\
&= [-\widetilde{B} + D_{\widetilde{B}^*}(I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1} r\widetilde{R} D_{\widetilde{B}}]\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A}) \\
&= \Theta_B(rR)\widetilde{L}^* D_{\widetilde{A}^*} F(r\widetilde{R} - \widetilde{A})
\end{aligned}$$

$$= \Theta_B(rR)\widetilde{L}^* D_{\widetilde{A}^*}(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}(r\widetilde{R} - \widetilde{A})$$
$$= \Theta_B(rR)\widetilde{L}^*\Theta_A(rR)D_{\widetilde{A}},$$

where the last equality follows from Lemma 2.1. Also

$$\begin{aligned}
C_{12}(r) &= \widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*}F\widetilde{X} - D_{\widetilde{B}^*}G\widetilde{X} + D_{\widetilde{B}^*}H(r\widetilde{R} - \widetilde{B}) \\
&= \widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*}F\widetilde{X} - D_{\widetilde{B}^*}H(r\widetilde{R}\widetilde{X}^*)F\widetilde{X} + D_{\widetilde{B}^*}H(r\widetilde{R} - \widetilde{B}) \\
&= \widetilde{B}\widetilde{L}^* D_{\widetilde{A}^*}F\widetilde{X} - D_{\widetilde{B}^*}H(r\widetilde{R}D_{\widetilde{B}}\widetilde{L}^* D_{\widetilde{A}^*})F\widetilde{X} + D_{\widetilde{B}^*}H(r\widetilde{R} - \widetilde{B}) \\
&= -[-\widetilde{B} + D_{\widetilde{B}^*}Hr\widetilde{R}D_{\widetilde{B}}]\widetilde{L}^* D_{\widetilde{A}^*}F\widetilde{X} + D_{\widetilde{B}^*}H(r\widetilde{R} - \widetilde{B}) \\
&= -[-\widetilde{B} + D_{\widetilde{B}^*}(I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1}r\widetilde{R}D_{\widetilde{B}}]\widetilde{L}^* D_{\widetilde{A}^*}F\widetilde{X} + D_{\widetilde{B}^*}H(r\widetilde{R} - \widetilde{B}) \\
&= -\Theta_B(rR)\widetilde{L}^* D_{\widetilde{A}^*}FD_{\widetilde{A}^*}\widetilde{L}D_{\widetilde{B}} + D_{\widetilde{B}^*}(I_{\Gamma\otimes\mathcal{H}_2} - r\widetilde{R}\widetilde{B}^*)^{-1}(r\widetilde{R} - \widetilde{B}) \\
&= -\Theta_B(rR)\widetilde{L}^* D_{\widetilde{A}^*}FD_{\widetilde{A}^*}\widetilde{L}D_{\widetilde{B}} + \Theta_B(rR)D_{\widetilde{B}} \quad \text{(by Lemma 2.1)} \\
&= \Theta_B(rR)[-\widetilde{L}^*(D_{\widetilde{A}^*}FD_{\widetilde{A}^*})\widetilde{L} + I_{\mathcal{D}_{\widetilde{B}}}]D_{\widetilde{B}} \\
&= \Theta_B(rR)[-\widetilde{L}^*(D_{\widetilde{A}^*}(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}D_{\widetilde{A}^*})\widetilde{L} + I_{\mathcal{D}_{\widetilde{B}}}]D_{\widetilde{B}} \\
&= \Theta_B(rR)[-\widetilde{L}^*(I_{\Gamma\otimes\mathcal{H}_1} + \Theta_A(rR)\widetilde{A}^*)\widetilde{L} + I_{\mathcal{D}_{\widetilde{B}}}]D_{\widetilde{B}} \quad \text{(by Lemma 2.1)} \\
&= \Theta_B(rR)(-\widetilde{L}^*\Theta_A(rR)\widetilde{A}^*\widetilde{L} + D^2_{\widetilde{L}})D_{\widetilde{B}}, \quad \text{and} \\
C_{21}(r) &= D_{\widetilde{L}^*}D_{\widetilde{A}^*}F(r\widetilde{R} - \widetilde{A}) \\
&= D_{\widetilde{L}^*}D_{\widetilde{A}^*}(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}(r\widetilde{R} - \widetilde{A}) = D_{\widetilde{L}^*}\Theta_A(rR)D_{\widetilde{A}},
\end{aligned}$$

and finally

$$\begin{aligned}
C_{22}(r) &= -D_{\widetilde{L}^*}D_{\widetilde{A}^*}F\widetilde{X} = -D_{\widetilde{L}^*}D_{\widetilde{A}^*}(I_{\Gamma\otimes\mathcal{H}_1} - r\widetilde{R}\widetilde{A}^*)^{-1}D_{\widetilde{A}^*}\widetilde{L}D_{\widetilde{B}} \\
&= -D_{\widetilde{L}^*}[I_{\Gamma\otimes\mathcal{H}_1} + \Theta_A(rR)\widetilde{A}^*]\widetilde{L}D_{\widetilde{B}} \quad \text{(by Lemma 2.1)} \\
&= -D_{\widetilde{L}^*}\Theta_A(rR)\widetilde{A}^*\widetilde{L}D_{\widetilde{B}} - D_{\widetilde{L}^*}\widetilde{L}D_{\widetilde{B}} \\
&= -D_{\widetilde{L}^*}\Theta_A(rR)\widetilde{A}^*\widetilde{L}D_{\widetilde{B}} - \widetilde{L}D_{\widetilde{L}}D_{\widetilde{B}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(I_\Gamma \otimes \sigma_*)\Theta_T(rR)D_{\widetilde{T}} \\
&\quad = \begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR)D_{\widetilde{A}} & \Theta_B(rR)(-\widetilde{L}^*\Theta_A(rR)\widetilde{A}^*\widetilde{L} + D^2_{\widetilde{L}})D_{\widetilde{B}} \\ D_{\widetilde{L}^*}\Theta_A(rR)D_{\widetilde{A}} & -D_{\widetilde{L}^*}\Theta_A(rR)\widetilde{A}^*\widetilde{L}D_{\widetilde{B}} - \widetilde{L}D_{\widetilde{L}}D_{\widetilde{B}} \end{bmatrix} \\
&\quad = \begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR) & \Theta_B(rR)D_{\widetilde{L}} \\ D_{\widetilde{L}^*}\Theta_A(rR) & -\widetilde{L} \end{bmatrix} \begin{bmatrix} D_{\widetilde{A}} & -\widetilde{A}^*\widetilde{L}D_{\widetilde{B}} \\ 0 & D_{\widetilde{L}}D_{\widetilde{B}} \end{bmatrix} \\
&\quad = \begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR) & \Theta_B(rR)D_{\widetilde{L}} \\ D_{\widetilde{L}^*}\Theta_A(rR) & -\widetilde{L} \end{bmatrix} (I_\Gamma \otimes \sigma)D_{\widetilde{T}} \quad \text{(by Theorem 1.2)},
\end{aligned}$$

and we conclude that

$$(I_\Gamma \otimes \sigma_*)\Theta_T(rR) = \begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR) & \Theta_B(rR)D_{\widetilde{L}} \\ D_{\widetilde{L}^*}\Theta_A(rR) & -\widetilde{L} \end{bmatrix} (I_\Gamma \otimes \sigma).$$

We may rewrite this as

$$(I_\Gamma \otimes \sigma_*)\Theta_T(rR)(I_\Gamma \otimes \sigma^{-1}) = \begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR) & \Theta_B(rR)D_{\widetilde{L}} \\ D_{\widetilde{L}^*}\Theta_A(rR) & -\widetilde{L} \end{bmatrix}.$$

Finally, we observe that

$$\begin{aligned}\begin{bmatrix} \Theta_B(rR)\widetilde{L}^*\Theta_A(rR) & \Theta_B(rR)D_{\widetilde{L}} \\ D_{\widetilde{L}^*}\Theta_A(rR) & -\widetilde{L} \end{bmatrix} &= \begin{bmatrix} \Theta_B(rR) & 0 \\ 0 & I_{\mathcal{D}_{\widetilde{L}^*}} \end{bmatrix} \begin{bmatrix} \widetilde{L}^* & D_{\widetilde{L}} \\ D_{\widetilde{L}^*} & -\widetilde{L} \end{bmatrix} \begin{bmatrix} \Theta_A(rR) & 0 \\ 0 & I_{\mathcal{D}_{\widetilde{L}}} \end{bmatrix} \\ &= \begin{bmatrix} \Theta_B(rR) & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A(rR) & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_L} \end{bmatrix},\end{aligned}$$

so that the resulting formula is

$$\Theta_T(rR) = (I_\Gamma \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_B(rR) & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A(rR) & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_L} \end{bmatrix} (I_\Gamma \otimes \sigma).$$

The result follows by passing to the strong operator topology limit as $r \to 1$. ∎

In the following, we prove that the Julia–Halmos matrix factor J_L in the factorization of the above theorem is canonical. The proof is similar to the one for $n = 1$ case by Sz.-Nagy and Foias (see Theorem 3, page 209–212, [13]). We only sketch the main ideas and refer to [13] for full proof details.

THEOREM 2.3. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{F}$ and $\mathcal{F}_*$ be Hilbert spaces and $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ be row contractions on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $w \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{F}, \mathcal{D}_B \oplus \mathcal{F}_*)$ be a unitary operator and*

$$\Theta = \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{F}_*} \end{bmatrix} (I_\Gamma \otimes w) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma\otimes\mathcal{F}} \end{bmatrix} : \Gamma \otimes (\mathcal{D}_A \oplus \mathcal{F}) \to \Gamma \otimes (\mathcal{D}_{B^*} \oplus \mathcal{F}_*)$$

be a purely contractive multi-analytic operator. Then Θ and Θ_T coincide where

$$T = \begin{bmatrix} A & D_{A^*}(P_{\mathcal{D}_{A^*}} w^*|_{\mathcal{D}_B})D_B \\ 0 & B \end{bmatrix} : (\textstyle\bigoplus_1^n \mathcal{H}_1) \oplus (\bigoplus_1^n \mathcal{H}_2) \to \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Proof. Let $w^* = \begin{bmatrix} L & M \\ N & K \end{bmatrix}$ where $L = P_{\mathcal{D}_{A^*}} w^*|_{\mathcal{D}_B} \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$, $M \in \mathcal{B}(\mathcal{F}_*, \mathcal{D}_{A^*})$, $N \in \mathcal{B}(\mathcal{D}_B, \mathcal{F})$ and $K \in \mathcal{B}(\mathcal{F}_*, \mathcal{F})$ are contractions. Define $\mathcal{F}' := \mathcal{F} \ominus N\mathcal{D}_B$ and $\mathcal{F}'_* := \mathcal{F}_* \ominus M^*\mathcal{D}_{A^*}$. Following the same line of argument as in the proof of the first part of Theorem 3 in [13] we have

$$w\mathcal{F}' = \mathcal{F}'_*.$$

In particular, for each $f' \in \mathcal{F}' (\subset \mathcal{F})$ we have $wf' \in \mathcal{F}'_* (\subset \mathcal{F}_*)$ and

$$\begin{aligned}\Theta(e_\emptyset \otimes f') &= \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{F}_*} \end{bmatrix} (I_\Gamma \otimes w) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma\otimes\mathcal{F}} \end{bmatrix} (e_\emptyset \otimes (0 \oplus f')) \\ &= \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{F}_*} \end{bmatrix} (e_\emptyset \otimes (0 \oplus wf')) = 0 \oplus (e_\emptyset \otimes wf').\end{aligned}$$

Then $\|P_{e_\emptyset \otimes (\mathcal{D}_{B^*} \oplus \mathcal{F}_*)} \Theta(e_\emptyset \otimes f')\|^2 = \|e_\emptyset \otimes wf'\|^2 = \|f'\|^2$. Since Θ is purely contractive, $f' = 0$, that is, $\mathcal{F}' = \{0\}$ and hence $\mathcal{F}'_* = \{0\}$. Hence $\overline{N\mathcal{D}_B} = \mathcal{F}$ and $\overline{M^*\mathcal{D}_{A^*}} = \mathcal{F}_*$. Consequently, $U \in \mathcal{B}(\mathcal{F}, \mathcal{D}_L)$ and $V \in \mathcal{B}(\mathcal{F}_*, \mathcal{D}_{L^*})$ defined by

$$U(Nx) = D_L x \quad \text{and} \quad V(M^* y) = D_{L^*} y \quad (x \in \mathcal{D}_B, y \in \mathcal{D}_{A^*}),$$

are unitary operators. Also

$$N^* = D_L|_{\mathcal{D}_L} U.$$

Then

$$w = \begin{bmatrix} L^* & N^* \\ M^* & K^* \end{bmatrix} = \begin{bmatrix} L^* & D_L|_{\mathcal{D}_L} U \\ M^* & V^* K_1 U \end{bmatrix} = v^* J u,$$

where $K_1 = VK^*U^* \in \mathcal{B}(\mathcal{D}_L, \mathcal{D}_{L^*})$, $u = \begin{bmatrix} I_{\mathcal{D}_{A^*}} & 0 \\ 0 & U \end{bmatrix}$, $v = \begin{bmatrix} I_{\mathcal{D}_B} & 0 \\ 0 & V \end{bmatrix}$ and $J = \begin{bmatrix} L^* & D_L|_{\mathcal{D}_L} \\ D_{L^*} & K_1 \end{bmatrix}$.

Since $J \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{D}_L, \mathcal{D}_B \oplus \mathcal{D}_{L^*})$ is a unitary operator we have (see page 211 in [13]) $K_1 = -L|_{\mathcal{D}_L}$. Now for $u' := \begin{bmatrix} I_{\mathcal{D}_A} & 0 \\ 0 & U \end{bmatrix}$ and $v' := \begin{bmatrix} I_{\mathcal{D}_{B^*}} & 0 \\ 0 & V \end{bmatrix}$, we have

$$(I_\Gamma \otimes v') \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{F}_*} \end{bmatrix} = \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes v), \quad \text{and}$$
$$(I_\Gamma \otimes u) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{F}} \end{bmatrix} = \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_L} \end{bmatrix} (I_\Gamma \otimes u').$$

This implies that

$$\begin{aligned}(I_\Gamma \otimes v')\Theta(I_\Gamma \otimes u'^*) &= (I_\Gamma \otimes v') \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{F}_*} \end{bmatrix} (I_\Gamma \otimes w) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{F}} \end{bmatrix} (I_\Gamma \otimes u'^*) \\ &= \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes v)(I_\Gamma \otimes w)(I_\Gamma \otimes u^*) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_L} \end{bmatrix} \\ &= \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_L} \end{bmatrix}.\end{aligned}$$

Since $L = P_{\mathcal{D}_{A^*}} w^*|_{\mathcal{D}_B}$ is a contraction, Theorem 1.1 shows that the n-tuple T defined by $T = \begin{bmatrix} A & D_{A^*} L D_B \\ 0 & B \end{bmatrix}$ is a row contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and Theorem 2.2 implies that

$$(I_\Gamma \otimes \sigma_*)\Theta_T(I_\Gamma \otimes \sigma^{-1}) = \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma \otimes \mathcal{D}_L} \end{bmatrix}$$

where $\sigma : \mathcal{D}_T \to \mathcal{D}_A \oplus \mathcal{D}_L$ and $\sigma_* : \mathcal{D}_{T^*} \to \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*}$ are unitary operators as in Theorem 1.2. Therefore, $(I_\Gamma \otimes v')\Theta(I_\Gamma \otimes u'^*) = (I_\Gamma \otimes \sigma_*)\Theta_T(I_\Gamma \otimes \sigma^{-1})$, that is, Θ_T coincides with Θ. ∎

3. FACTORIZATIONS OF CHARACTERISTIC FUNCTIONS OF CONSTRAINED ROW CONTRACTIONS

The main objective of this section is to study factorizations of characteristic functions of row contractions in noncommutative varieties. The notion of a noncommutative variety was introduced by G. Popescu in [11].

We first recollect some basic definitions, notations, and results that will be used subsequently. For details, we refer to [11], [12] and references therein. Let $\mathcal{P}_J \subset \mathcal{F}_n^\infty$ be a family of noncommutative polynomials and J be the WOT-closed two sided ideal of $\mathcal{F}_n^\infty$ generated by $\mathcal{P}_J$. In what follows, we always assume that $J \neq \mathcal{F}_n^\infty$. Then

$$\mathcal{M}_J := \overline{\text{span}}\{\phi \otimes \psi : \phi \in J, \psi \in \Gamma\} \quad \text{and} \quad \mathcal{N}_J := \Gamma \ominus \mathcal{M}_J,$$

are proper joint $(L_1, \ldots, L_n)$ and $(L_1^*, \ldots, L_n^*)$ invariant subspaces of Γ, respectively. Define constrained left creation operators and constrained right creation operators on $\mathcal{N}_J$ by

$$V_j := P_{\mathcal{N}_J} L_j|_{\mathcal{N}_J} \quad \text{and} \quad W_j := P_{\mathcal{N}_J} R_j|_{\mathcal{N}_J} \quad (j = 1, \ldots, n),$$

respectively.

Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces and $M \in \mathcal{B}(\mathcal{N}_J \otimes \mathcal{E}, \mathcal{N}_J \otimes \mathcal{E}_*)$. Then M is said to be a *constrained multi-analytic operator* if

$$M(V_j \otimes I_{\mathcal{E}}) = (V_j \otimes I_{\mathcal{E}_*})M \quad (j = 1, \ldots, n).$$

We say that $M \in \mathcal{B}(\mathcal{N}_J \otimes \mathcal{E}, \mathcal{N}_J \otimes \mathcal{E}_*)$ is *purely contractive* if M is a contraction and $e_\emptyset \in \mathcal{N}_J$ and

$$\|P_{e_\emptyset \otimes \mathcal{E}_*} M(e_\emptyset \otimes \eta)\| < \|\eta\| \quad (\eta \neq 0, \eta \in \mathcal{E}).$$

It has been shown by Popescu [11] that the set of all constrained multi-analytic operators in $\mathcal{B}(\mathcal{N}_J \otimes \mathcal{E}, \mathcal{N}_J \otimes \mathcal{E}_*)$ coincides with

$$\mathcal{W}(W_1, \ldots, W_n) \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*) = P_{\mathcal{N}_J \otimes \mathcal{E}_*}[\mathcal{R}_n^\infty \overline{\otimes} \mathcal{B}(\mathcal{E}, \mathcal{E}_*)]|_{\mathcal{N}_J \otimes \mathcal{E}}.$$

where $\mathcal{W}(W_1, \ldots, W_n)$ is the WOT-closed algebra generated by $\{I, W_1, \ldots, W_n\}$ and $\mathcal{R}_n^\infty = U^* \mathcal{F}_n^\infty U$ and U is the flipping operator.

A row contraction $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$ is said to be a *J-constrained row contraction*, or simply a *constrained row contraction* if J is clear from the context, if

$$p(T_1, \ldots, T_n) = 0 \quad (p \in \mathcal{P}_J).$$

The *constrained characteristic function* (see Popescu [11]) of a constrained row contraction $T = (T_1, \ldots, T_n)$ on $\mathcal{H}$ is the constrained multi-analytic operator $\Theta_{J,T} : \mathcal{N}_J \otimes \mathcal{D}_T \to \mathcal{N}_J \otimes \mathcal{D}_{T^*}$ defined by

$$\Theta_{J,T} = P_{\mathcal{N}_J \otimes \mathcal{D}_{T^*}} \Theta_T|_{\mathcal{N}_J \otimes \mathcal{D}_T}.$$

Since $\mathcal{N}_J$ is a joint $(R_1^* \otimes I_{\mathcal{D}_{T^*}}, \ldots, R_n^* \otimes I_{\mathcal{D}_{T^*}})$ invariant subspace and

$$W_j = P_{\mathcal{N}_J} R_j|_{\mathcal{N}_J}, \quad j = 1, \ldots, n,$$

it follows that (see [11])

$$\Theta_T^*(\mathcal{N}_J \otimes \mathcal{D}_{T^*}) \subset \mathcal{N}_J \otimes \mathcal{D}_T \quad \text{and} \quad \Theta_T(\mathcal{M}_J \otimes \mathcal{D}_T) \subset \mathcal{M}_J \otimes \mathcal{D}_{T^*}. \tag{3.1}$$

From here onwards to maintain simplicity of notations, we often omit the subscript J.

Now we are ready to prove a factorization of constrained characteristic functions corresponding to upper triangular constrained row contractions.

THEOREM 3.1. *Let* $T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix}$ *be a constrained row contraction on* $\mathcal{H}_1 \oplus \mathcal{H}_2$ *where* $A = (A_1, \dots, A_n)$ *on* $\mathcal{H}_1$ *and* $B = (B_1, \dots, B_n)$ *on* $\mathcal{H}_2$ *are row contractions and* $L \in \mathcal{B}(\mathcal{D}_B, \mathcal{D}_{A^*})$ *is a contraction. Then* A *and* B *are also constrained row contractions and*

$$\Theta_{J,T} = (I_{\mathcal{N}} \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_{J,B} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_{\mathcal{N}} \otimes J_L) \begin{bmatrix} \Theta_{J,A} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_L} \end{bmatrix} (I_{\mathcal{N}} \otimes \sigma)$$

where $\sigma \in \mathcal{B}(\mathcal{D}_T, \mathcal{D}_A \oplus \mathcal{D}_L)$ *and* $\sigma_* \in \mathcal{B}(\mathcal{D}_{T^*}, \mathcal{D}_{B^*} \oplus \mathcal{D}_{L^*})$ *are unitary operators as in Theorem* 1.2.

Proof. It is straightforward to verify that A and B are constrained row contractions. For the remaining part, first we observe that

$$\Theta_{J,T} = P_{\mathcal{N}\otimes\mathcal{D}_{T^*}} (I_\Gamma \otimes \sigma_*^{-1}) \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L^*}} \end{bmatrix} (I_\Gamma \otimes J_L) \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_L} \end{bmatrix} (I_\Gamma \otimes \sigma)|_{\mathcal{N}\otimes\mathcal{D}_T}.$$

Since $P_{\mathcal{N}\otimes\mathcal{D}_{T^*}} (I_\Gamma \otimes \sigma_*^{-1}) = (I_{\mathcal{N}} \otimes \sigma_*^{-1}) P_{\mathcal{N}\otimes(\mathcal{D}_{B^*}\oplus\mathcal{D}_{L^*})}$ and

$$P_{\mathcal{N}\otimes(\mathcal{D}_{B^*}\oplus\mathcal{D}_{L^*})} \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_{L^*}} \end{bmatrix} = P_{\mathcal{N}\otimes(\mathcal{D}_{B^*}\oplus\mathcal{D}_{L^*})} \begin{bmatrix} \Theta_B & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_{L^*}} \end{bmatrix} P_{\mathcal{N}\otimes(\mathcal{D}_B\oplus\mathcal{D}_{L^*})},$$

and $P_{\mathcal{N}\otimes(\mathcal{D}_B\oplus\mathcal{D}_{L^*})}(I_\Gamma \otimes J_L) = I_{\mathcal{N}} \otimes J_L = (I_{\mathcal{N}} \otimes J_L) P_{\mathcal{N}\otimes(\mathcal{D}_{A^*}\oplus\mathcal{D}_L)}$, and

$$P_{\mathcal{N}\otimes(\mathcal{D}_{A^*}\oplus\mathcal{D}_L)} \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\Gamma\otimes\mathcal{D}_L} \end{bmatrix} = P_{\mathcal{N}\otimes(\mathcal{D}_{A^*}\oplus\mathcal{D}_L)} \begin{bmatrix} \Theta_A & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{D}_L} \end{bmatrix} P_{\mathcal{N}\otimes(\mathcal{D}_A\oplus\mathcal{D}_L)},$$

we have the required equality. ∎

We now state a similar result to Theorem 2.3 for constrained row contractions. We omit the proof, which uses similar techniques to the proof of Theorem 2.3 (and Theorem 3 in [13]).

THEOREM 3.2. *Let* $\mathcal{H}_1, \mathcal{H}_2, \mathcal{F}$ *and* $\mathcal{F}_*$ *be Hilbert spaces and* $A = (A_1, \dots, A_n)$ *and* $B = (B_1, \dots, B_n)$ *be constrained row contractions on* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively, and* $e_\emptyset \in \mathcal{N}$. *Let* $w \in \mathcal{B}(\mathcal{D}_{A^*} \oplus \mathcal{F}, \mathcal{D}_B \oplus \mathcal{F}_*)$ *be a unitary operator and* $T = \begin{bmatrix} A & D_{A^*}(P_{\mathcal{D}_{A^*}} w^*|_{\mathcal{D}_B})D_B \\ 0 & B \end{bmatrix}$ *be a constrained row contraction and*

$$\Theta = \begin{bmatrix} \Theta_{J,B} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{F}_*} \end{bmatrix} (I_{\mathcal{N}} \otimes w) \begin{bmatrix} \Theta_{J,A} & 0 \\ 0 & I_{\mathcal{N}\otimes\mathcal{F}} \end{bmatrix}$$

be a purely contractive constrained multi-analytic operator. Then Θ coincides with $\Theta_{J,T}$.

A particularly important example of noncommutative variety is the one given by $\mathcal{P}_{J_c} = \{L_iL_j - L_jL_i : i,j = 1,\dots,n\}$. In this case $\mathcal{N}_{J_c} = \Gamma_s$ is the symmetric Fock space, $V_j = P_{\Gamma_s}L_j|_{\Gamma_s}, j = 1,\dots,n$, are the creation operators on Γ_s (see [3], [11]). Moreover, one can identify $(V_1,\dots,V_n)$ on Γ_s with the multiplication operator tuple $(M_{z_1},\dots,M_{z_n})$ on the Drury–Arveson space H^2_n (see [1]). Recall that the Drury–Arveson space is a reproducing kernel Hilbert space with kernel function $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ defined by

$$k(z,w) = (1 - \langle z,w\rangle_{\mathbb{C}^n})^{-1} \quad (z,w \in \mathbb{B}^n).$$

Under this identification, the set of constrained multi-analytic operators $P_{\Gamma_s}\mathcal{F}^\infty_n|_{\Gamma_s}$ corresponds to the multiplier algebra of H^2_n.

Note too that a row contraction $T = (T_1,\dots,T_n)$ on $\mathcal{H}$ is a constrained row contraction if and only if T is a commuting row contraction, that is, $T_iT_j = T_jT_i$, $i,j = 1,\dots,n$. In this case, we can identify the constrained characteristic function $\Theta_{J_c,T} = P_{\mathcal{N}_{J_c}\otimes\mathcal{D}_{T^*}}\Theta_T|_{\mathcal{N}_{J_c}\otimes\mathcal{D}_T}$ with the bounded operator-valued analytic function θ_T on $\mathbb{B}^n$ defined by (see [2], [3], and [11])

$$\theta_T(z) = -T + D_{T^*}(I_{\mathcal{H}} - ZT^*)^{-1}ZD_T \quad (z \in \mathbb{B}^n),$$

where $Z = (z_1I_{\mathcal{H}},\dots,z_nI_{\mathcal{H}}), z \in \mathbb{B}^n$.

In this setting, Theorems 3.1 and 3.2 can be stated as follows.

THEOREM 3.3. *Let $T = \begin{bmatrix} A & D_{A^*}LD_B \\ 0 & B \end{bmatrix}$ be a commuting row contraction on $\mathcal{H}_1 \oplus \mathcal{H}_2$ where A and B are commuting row contractions on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $L \in \mathcal{B}(\mathcal{D}_B,\mathcal{D}_{A^*})$ is a contraction. Then θ_T coincides with*

$$\begin{bmatrix} \theta_B & 0 \\ 0 & I_{H^2_n\otimes\mathcal{D}_{L^*}} \end{bmatrix}(I_{H^2_n}\otimes J_L)\begin{bmatrix} \theta_A & 0 \\ 0 & I_{H^2_n\otimes\mathcal{D}_L} \end{bmatrix}.$$

Moreover, if $\widehat{T} = \begin{bmatrix} A & D_{A^*}(P_{\mathcal{D}_{A^*}}w^*|_{\mathcal{D}_B})D_B \\ 0 & B \end{bmatrix}$ is a commuting row contraction for some unitary operator $w \in \mathcal{B}(\mathcal{D}_{A^*}\oplus\mathcal{F},\mathcal{D}_B\oplus\mathcal{F}_*)$ and Hilbert spaces $\mathcal{F}$ and $\mathcal{F}_*$, and if

$$\theta = \begin{bmatrix} \theta_B & 0 \\ 0 & I_{H^2_n\otimes\mathcal{F}_*} \end{bmatrix}(I_{H^2_n}\otimes w)\begin{bmatrix} \theta_A & 0 \\ 0 & I_{H^2_n\otimes\mathcal{F}} \end{bmatrix}$$

is a purely contractive multiplier then θ coincides with $\theta_{\widehat{T}}$.

Now let $\mathcal{H}_1$ be a closed subspace of a Hilbert space $\mathcal{H}$ and $T = (T_1,\dots,T_n)$ be an n-tuple on $\mathcal{H}$. Let $\mathcal{H}_1$ be a joint T invariant subspace of $\mathcal{H}$ (that is, $T_i\mathcal{H}_1 \subseteq \mathcal{H}_1$ for all $i = 1,\dots,n$) and $\mathcal{H}_2 = \mathcal{H}\ominus\mathcal{H}_1$. Then we can represent, with respect to $\mathcal{H} = \mathcal{H}_1\oplus\mathcal{H}_2$, T_j as an upper triangular operator matrix

$$T_j = \begin{bmatrix} A_j & X_j \\ 0 & B_j \end{bmatrix},$$

where $A_j = T_j|_{\mathcal{H}_1} \in \mathcal{B}(\mathcal{H}_1)$, $B_j = P_{\mathcal{H}_2} T_j|_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_2)$ and $X_j = P_{\mathcal{H}_1} T_j|_{\mathcal{H}_2} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$, $j = 1, \dots, n$. In other words

$$T = \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} : (\bigoplus_1^n \mathcal{H}_1) \oplus (\bigoplus_1^n \mathcal{H}_2) \to \mathcal{H}_1 \oplus \mathcal{H}_2, \tag{3.2}$$

where $A = (A_1, \dots, A_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_1, \mathcal{H}_1)$, $B = (B_1, \dots, B_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_2)$ and $X = (X_1, \dots, X_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_1)$.

Conversely, let T be a row operator on $\mathcal{H}$ and $\mathcal{H}_1$ and $\mathcal{H}_2$ be closed subspaces of $\mathcal{H}$. If T admits an upper triangular representation as in (3.2) for some row operators

$$A = (A_1, \dots, A_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_1, \mathcal{H}_1), \quad B = (B_1, \dots, B_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_2) \quad \text{and}$$
$$X = (X_1, \dots, X_n) \in \mathcal{B}(\bigoplus_1^n \mathcal{H}_2, \mathcal{H}_1),$$

then $\mathcal{H}_1$ is a joint T-invariant subspace of $\mathcal{H}$. In other words, T has a non-trivial joint invariant subspace if and only if T admits an upper triangular representation as in (3.2). This is also equivalent to the regular factorizations of the characteristic function Θ_T in terms of Θ_A and Θ_B (see Sz.-Nagy and Foias [14] for $n = 1$ case and Popescu [10] for general case). It is not known, in the general case, how one relates regular factorizations of characteristic functions and the one obtained in this paper. We do not know the answer even if $n = 1$.

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