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ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS AND TOEPLITZ OPERATORS WITH MATRIX SYMBOL

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ABSTRACT. Truncated Toeplitz operators and their asymmetric versions are studied in the context of the Hardy space H_p of the half-plane for $1 < p < \infty$. The question of uniqueness of the symbol is solved via the characterization of the zero operator. It is shown that asymmetric truncated Toeplitz operators are equivalent after extension to 2×2 matricial Toeplitz operators, which allows one to deduce criteria for Fredholmness and invertibility. Shifted model spaces are presented in the context of invariant subspaces, allowing one to derive new Beurling–Lax theorems.

KEYWORDS: *Truncated Toeplitz operator, Toeplitz operator, model space, equivalence by extension, invariant subspace.*

MSC (2010): 47B35, 30H10.

INTRODUCTION

Certain classes of truncated Toeplitz operators (TTO), also known as skew Toeplitz operators, have been studied for many years [7], [8], [22], [30]. However, it is the paper of Sarason [27] that stimulated the most recent research in this area: see, for example [4], [10] and the recent survey [16]. Here we treat a more general class of operators, known as asymmetric truncated Toeplitz operators (ATTO), a natural generalisation of rectangular Toeplitz matrices. They appear in various contexts, such as in the study of finite-time convolution equations, signal processing, control theory, probability, approximation theory, and diffraction problems (see for instance [1], [2], [3], [4], [17], [18], [28]).

Motivated by these applications, where the natural variables are often time and frequency, we work mostly with the Hardy spaces $H_p^\pm$ of the upper and lower half-planes, for $1 < p < \infty$, recalling the decomposition $L_p(\mathbb{R}) = H_p^+ \oplus H_p^-$ (full definitions and notation will be given later). Many of our results may be rewritten for the disc, as we shall see later, although they may sometimes appear more complicated in this context. Most of the results we prove are new even for "standard" TTO in the Hardy spaces H_p.

For an inner function $\theta \in H_\infty^+$ the *model space* K_θ^p may be defined as

$$K_\theta = H_p^+ \cap \theta H_p^-. \tag{0.1}$$

We will omit the superscript p unless it is necessary for the sake of clarity. We then have

$$L_p(\mathbb{R}) = H_p^- \oplus K_\theta \oplus \theta H_p^+, \tag{0.2}$$

and we write P_θ to denote the associated projection $P_\theta : L_p(\mathbb{R}) \to K_\theta$.

Then for $g \in L_\infty(\mathbb{R})$ the standard TTO A_g^θ is defined as follows:

$$A_g^\theta : K_\theta \to K_\theta, \quad A_g^\theta = P_\theta(gI)_{|K_\theta} = P_\theta(gI)_{|P_\theta L_p}. \tag{0.3}$$

If α and θ are inner functions, we define the operator $A_g^{\alpha,\theta}$ as

$$A_g^{\alpha,\theta} := P_\alpha(gI)_{|K_\theta} = P_\alpha(gI)_{|P_\theta L_p}. \tag{0.4}$$

If α is an inner function that divides θ in H_∞^+ (we write this $\alpha \preceq \theta$), let $P_{\alpha,\theta}$ denote $P_\theta - P_\alpha$, a projection with range equal to the *shifted model space* $K_{\alpha,\theta} := \alpha K_{\overline{\alpha}\theta}$. Then we can define

$$B_g^{\alpha,\theta} := P_{\alpha,\theta}(gI)_{|K_\theta} = P_{\alpha,\theta}(gI)_{|P_\theta L_p}. \tag{0.5}$$

The operators $A_g^{\alpha,\theta}$ and $B_g^{\alpha,\theta}$ are particular cases of *general WH operators* (see [28]) in L_p, of the form

$$P_1 A_{|P_2 L_p}, \tag{0.6}$$

where P_1 and P_2 are projections and A is an operator in L_p. We say that $A_g^{\alpha,\theta}$ and $B_g^{\alpha,\theta}$ are *asymmetric truncated Toeplitz operators* (ATTO) in K_θ (that is, general WH operators where P_1 and P_2 are projections in K_θ and A is a Toeplitz operator).

In Section 1 we recall the definitions and basic properties of model spaces in an H_p context, while also introducing the notion of partial conjugation. Section 2 analyses an isometric isomorphism between L_p spaces on the disc and half-plane, which restricts to H_p^+ and indeed θH_p^+. For $p = 2$ it has further properties which aid in the study of ATTO. In Section 3 ATTO are treated in some detail, and we solve the question of uniqueness of symbol, via the characterization of the zero operator. In Section 4 we discuss the question which ATTO have finite rank. Next, in Section 5 it is shown that ATTO are equivalent by extension to Toeplitz operators with triangular 2×2 matrix symbol. This immediately enables one to obtain new results about ATTO (and even TTO) from known results about standard Toeplitz operators. In particular, we establish necessary and sufficient conditions for an ATTO to be Fredholm or invertible, and illustrate these results by describing the spectra of TTO in a particular class. Finally, Section 6 discusses kernels of ATTO and the link with invariant subspaces.

1. MODEL SPACES, PROJECTIONS AND TOEPLITZ OPERATORS

Recall that we write L_p for $L_p(\mathbb{R})$, H_p^+ and H_p^- for the Hardy spaces of the upper and lower half-planes $\mathbb{C}^+$ and $\mathbb{C}^-$ (here $1 \leqslant p \leqslant \infty$) and we denote by $P^\pm$ the Riesz projections $P^+ : L_p \to H_p^+$ and $P^- : L_p \to H_p^-$ for $1 < p < \infty$.

For θ an inner function (in H_∞^+), let $K_\theta = K_\theta^p$ denote the model space defined in (0.1), where we omit the superscript p unless it is necessary for clarity. If α and θ are inner functions, we say that αK_θ is a *shifted model space*. It is clear that $\alpha K_\theta \subset K_{\alpha\theta}$.

For any inner function θ, we have the decomposition (0.2), and

$$H_p^+ = K_\theta \oplus \theta H_p^+, \tag{1.1}$$

where the sum is orthogonal in the case $p = 2$. Let $P_\theta : L_p \to K_\theta$ be the projection from L_p onto K_θ defined by (0.2); we have

$$P_\theta = \theta P^- \overline{\theta} P^+ = P^+ \theta P^- \overline{\theta} I. \tag{1.2}$$

Let moreover Q_θ be the operator defined in L_p, $1 < p < \infty$, by

$$Q_\theta := P^+ - P_\theta, \tag{1.3}$$

and let us use the same notation P_θ, Q_θ for $P_{\theta|_{H_p^+}}, Q_{\theta|_{H_p^+}}$, respectively. For any $\varphi \in H_p^+$, we define

$$\varphi^\theta = P_\theta \varphi. \tag{1.4}$$

Now take $g \in L_\infty$. The *Toeplitz operator* with symbol g is the operator

$$T_g : H_p^+ \to H_p^+, \quad T_g = P^+ g I_+,$$

where I_+ denotes the identity operator in H_p^+. This definition can be generalised to the vectorial case straightforwardly, for a matricial symbol $g \in L_\infty^{n\times n}$.

If $\alpha, \theta \in H_\infty^+$ are inner functions, we say that $\alpha \preceq \theta$ if and only if there exists an inner function $\widetilde{\theta}$ such that $\theta = \alpha\widetilde{\theta}$, and $\alpha \prec \theta$ if and only if $\widetilde{\theta}$ is not constant. Of course $\alpha \preceq \theta \Rightarrow \overline{\alpha}\theta \preceq \theta$. We also have

$$\alpha \preceq \theta \Leftrightarrow K_\alpha \subset K_\theta \Leftrightarrow \ker P_\theta \subset \ker P_\alpha. \tag{1.5}$$

As a consequence of this we can also define, for $\alpha \preceq \theta$, a projection in L_p (or H_p^+) by

$$P_{\alpha,\theta} := P_\theta - P_\alpha, \tag{1.6}$$

and again we use the same notation for the operator defined by (1.6) in L_p and its restriction to H_p^+. We easily see that $P_{\alpha,\theta} = Q_\alpha P_\theta = P_\theta Q_\alpha = \alpha P_{\overline{\alpha}\theta} \overline{\alpha} I$, and it follows that the image of $P_{\alpha,\theta}$ is the shifted model space

$$K_{\alpha,\theta} := K_\theta \cap \alpha H_p^+ = \alpha K_{\overline{\alpha}\theta}. \tag{1.7}$$

Of course $K_{\alpha,\theta} = K_\theta$ if α is constant, and $K_{\alpha,\theta} = K_\theta \ominus K_\alpha$ if $p = 2$.

We introduce now a class of conjugate-linear operators in H_p^+ by generalising the notion of a conjugation in a complex Hilbert space $\mathcal{H}$ (i.e., an isometric conjugate-linear involution in $\mathcal{H}$).

DEFINITION 1.1. Let X, Y be closed subspaces of H_p^+ such that $x \perp y$ for all $x \in X \cap H_2^+, y \in Y \cap H_2^+$, and let $A = X \oplus Y$. We say that a conjugate-linear operator in H_p^+, $\mathcal{C}$, is a *partial conjugation* in A if and only if $\mathcal{C}_{|X}$ is an isometric involution on X and $\mathcal{C}_{|Y} = 0$. If $Y = \{0\}$ then $\mathcal{C}$ is a *conjugation* in A.

Let now $\mathcal{C}_\theta$ be the conjugate-linear operator defined in H_p^+, for each inner function θ, by

$$\mathcal{C}_\theta(\varphi_+) = \theta \overline{P_\theta \varphi_+}, \quad \varphi_+ \in H_p^+. \tag{1.8}$$

It is easy to see that $(\mathcal{C}_\theta)^2 = P_\theta$, $\mathcal{C}_\theta$ maps K_θ onto K_θ isometrically, and $\mathcal{C}_\theta(\theta H_p^+) = \{0\}$. Thus $\mathcal{C}_\theta$ is a partial conjugation in H_p^+ and, analogously $\mathcal{C}_\alpha$ is a partial conjugation in K_θ if $\alpha \preceq \theta$. Of course $\mathcal{C}_\alpha$ is a conjugation in K_α.

We will also use the following simple relations. Let r denote the function defined by

$$r(\xi) = \frac{\xi - \mathrm{i}}{\xi + \mathrm{i}} \tag{1.9}$$

for $\xi \in \mathbb{C}$ and let $\varphi_\pm \in H_p^\pm$. Then

$$P^+ r^{-1} \varphi_+ = r^{-1}\varphi_+ - 2\mathrm{i}\frac{\varphi_+(\mathrm{i})}{\xi - \mathrm{i}}, \quad P^- r \varphi_- = r\varphi_- + 2\mathrm{i}\frac{\varphi_-(-\mathrm{i})}{\xi + \mathrm{i}}. \tag{1.10}$$

Moreover, if θ is an inner function, taking into account that $\varphi_+ = \varphi_+^\theta + \theta\widetilde{\varphi}_+$ with $\widetilde{\varphi}_+ \in H_p^+$, we have

$$P_\theta h_+ \varphi_+ = P_\theta h_+ \varphi_+^\theta \tag{1.11}$$

whenever h_+ is such that $h_+\varphi_+ \in L_p$ and $h_+ Q_\theta \varphi_+ \in \theta H_p^+$ (in particular, if $h_+ \in H_\infty^+$), and

$$Q_\theta h_- \varphi_+^\theta = 0, \quad P_\theta h_- \varphi_+^\theta = P^+ h_- \varphi_+^\theta \tag{1.12}$$

whenever h_- is such that $h_-\varphi_+^\theta \in L_p$ and $h_-\overline{\theta}\varphi_+^\theta \in H_p^-$ (in particular, if $h_- \in H_\infty^-$). As a consequence of (1.11) and (1.12), we also have

$$\alpha \preceq \theta \Rightarrow P_\theta h_- \varphi_+^\alpha = P_\alpha h_- \varphi_+^\alpha, \quad P_\alpha h_+ \varphi_+^\theta = P_\alpha h_+ \varphi_+^\alpha. \tag{1.13}$$

2. EQUIVALENCE BETWEEN OPERATORS ON THE DISC AND HALF-PLANE

We now recall the details of the isometric isomorphism between the Hardy spaces H_p^+ on the upper half-plane $\mathbb{C}^+$ and $H_p(\mathbb{D})$ on the unit disc $\mathbb{D}$. It will be seen that this leads to an isometric bijective equivalence (i.e., an unitary equivalence in the case $p = 2$) between model spaces on the disc and half-plane; in the case $p = 2$ this leads to a unitary equivalence between (A)TTO on the disc and half-plane, enabling us to give an immediate translation of our results to the disc context. Our convention in this section is that lower case letters such as f denote functions on the disc, whereas capital letters denote functions on the half-plane.

Let $m : \mathbb{D} \to \mathbb{C}^+$ be the conformal bijection given by

$$m(z) = \mathrm{i}\Big(\frac{1-z}{1+z}\Big), \quad m^{-1}(\xi) = \frac{\mathrm{i}-\xi}{\mathrm{i}+\xi},$$

(other choices are possible) and $V : H_p(\mathbb{D}) \to H_p(\mathbb{C}^+)$ the isometric isomorphism given by

$$(Vf)(\xi) = \frac{1}{\pi^{1/p}} \frac{1}{(\mathrm{i}+\xi)^{2/p}} f(m^{-1}(\xi)), \quad (f \in H_p(\mathbb{D})), \tag{2.1}$$

(see, for example, [20], [23], [24]). The inverse mapping is given by

$$(V^{-1}F)(z) = \pi^{1/p}\Big(\frac{2\mathrm{i}}{1+z}\Big)^{2/p} F(m(z)), \quad (F \in H_p(\mathbb{C}^+)).$$

Now for $n \in \mathbb{Z}$ the function z^n is mapped by V to the function e_n given by

$$e_n^{(p)}(\xi) = \frac{1}{\pi^{1/p}} \frac{(\mathrm{i}-\xi)^n}{(\mathrm{i}+\xi)^{n+2/p}}.$$

The same formula (2.1) extends V to an isometric mapping from $L_p(\mathbb{T})$ onto $L_p(\mathbb{R})$, and for $p = 2$ it also maps $\overline{H_0^p(\mathbb{D})}$ into H_p^-.

Let θ be an inner function in $H^\infty(\mathbb{C}^+)$; then the function $\Theta := \theta \circ m^{-1}$ is an inner function in $H^\infty(\mathbb{D})$. Now for $f = \theta g$ with $g \in H^p(\mathbb{D})$ we have

$$(Vf)(\xi) = \Theta(\xi)(Vg)(\xi),$$

so V takes $\theta H^p(\mathbb{D})$ onto $\Theta H^p(\mathbb{C}^+)$. Letting q be the conjugate index to p, we also have that $(V^*)^{-1}$ maps $H_q(\mathbb{D})$ onto $H_q(\mathbb{C}^+)$ and takes its subspace K_θ to K_Θ.

The situation is better for $p = 2$, since V is unitary, and it maps $K_\theta = H_2(\mathbb{D}) \cap \theta\overline{H_0^2(\mathbb{D})}$ onto $K_\Theta = H_2^+ \cap \Theta H_2^-$; hence, the decomposition

$$L^2(\mathbb{T}) = \overline{H_0^2(\mathbb{D})} \oplus K_\theta \oplus \theta H^2(\mathbb{D})$$

is mapped by V term-wise into

$$L^2(\mathbb{R}) = H_2^- \oplus K_\Theta \oplus \Theta H_2^+.$$

This situation does not hold for $p \neq 2$.

Suppose now that $p = 2$ and $g \in L^\infty(\mathbb{D})$. We write $G := g \circ m^{-1}$ and $\mathcal{A} = \alpha \circ m^{-1}$. Then, we have the following commutative diagram, where $A_g^{\alpha,\theta}$ denotes an ATTO on the disc, as defined analogously to (0.3):

$$\begin{array}{ccc} K_\theta & \xrightarrow{A_g^{\alpha,\theta}} & K_\alpha \\ V\downarrow & & \downarrow V \\ K_\Theta & \xrightarrow{A_G^{\mathcal{A},\Theta}} & K_{\mathcal{A}} \end{array} \tag{2.2}$$

We see that this diagram commutes, since for $k \in K_\theta$ we have

$$V(gk)(\xi) = \frac{1}{\pi^{1/2}} \frac{1}{(\mathrm{i}+\xi)} g(m^{-1}(\xi))k(m^{-1}(\xi)) = G(\xi)(Vk)(\xi);$$

now, since $P_{\mathcal{A}}V = VP_\alpha$ we get

$$VP_\alpha(gk) = P_{\mathcal{A}}V(gk) = P_{\mathcal{A}}G(Vk),$$

so we have the required unitary equivalence between ATTO on the disc and half-plane.

3. ASYMMETRIC TRUNCATED TOEPLTZ OPERATORS

Let $g \in L_\infty$ and let $\alpha, \theta \in H_\infty^+$ be inner functions. As in Section 1, we define the *asymmetric truncated Toeplitz operators* (abbreviated to *ATTO*) $A_g^{\alpha,\theta}$ and, for $\alpha \preceq \theta$, $B_g^{\alpha,\theta}$ as follows:

$$A_g^{\alpha,\theta} = \alpha g P_\theta, \tag{3.1}$$

$$B_g^{\alpha,\theta} = P_{\alpha,\theta}\, g P_\theta, \tag{3.2}$$

where $A_g^{\alpha,\theta}$ and $B_g^{\alpha,\theta}$ can be seen as operators in H_p^+, or operators in K_θ if $\alpha \preceq \theta$, or as operators from K_θ into K_α and $K_{\alpha,\theta}$, respectively. We will assume the latter unless stated otherwise. If $\alpha = \theta$ then $A_g^{\alpha,\theta}$ is the *truncated Toeplitz operator* A_g^θ.

It is easy to see that $A_g^\theta = A_g^{\alpha,\theta} + B_g^{\alpha,\theta}$ and that an ATTO of the form (3.2) can be expressed in terms of ATTO of type (3.1), since we have

$$B_g^{\alpha,\theta} = P_{\alpha,\theta} T_{g|_{K_\theta}} = \alpha P_{\overline{\alpha}\theta} \overline{\alpha} T_{g|_{K_\theta}} = \alpha P_{\overline{\alpha}\theta} T_{\overline{\alpha}g|_{K_\theta}} = \alpha A_{\overline{\alpha}g}^{\overline{\alpha}\theta,\theta}. \tag{3.3}$$

We will therefore focus here on ATTO of type (3.1). Moreover, considering that

$$\left(A_g^{\alpha,\theta}\right)^* = A_{\overline{g}}^{\theta,\alpha},$$

we will assume in what follows that $\alpha \preceq \theta$.

We will use the following generalisation of the notion of a complex symmetric operator in a Hilbert space.

DEFINITION 3.1. Let A be a closed subspace of H_p^+. An operator $T : A \to H_p^+$ is a complex partially symmetric operator (respectively, a complex symmetric operator) if and only if there exists a partial conjugation (respectively, a conjugation) in A, $\mathcal{C}$, such that $\mathcal{C}T\mathcal{C} = \widetilde{T}$, where $\widetilde{T}$ coincides with T^* in $H_p^+ \cap H_q^+, 1/p + 1/q = 1$. In this case we say that T is *$\mathcal{PC}$-symmetric* (respectively, *$\mathcal{C}$-symmetric*).

PROPOSITION 3.2. *If $g \in L_\infty$, then*

$$\mathcal{C}_\alpha A_g^{\alpha,\theta} \mathcal{C}_\alpha = A_{\overline{g}}^\alpha.$$

Proof. Let $\varphi_+ \in K_\theta$. Then, for all $\varphi_+ \in H_p^+$,

$$\begin{aligned}\mathcal{C}_\alpha A_g^{\alpha,\theta}\mathcal{C}_\alpha\varphi_+ &= \alpha\overline{A_g^{\alpha,\theta}\mathcal{C}_\alpha\varphi_+} = \alpha\overline{P_\alpha g P_\theta \alpha\overline{P_\alpha\varphi_+}} = \alpha\overline{P_\alpha g\alpha\overline{P_\alpha\varphi_+}}\\ &= \alpha(\overline{\alpha}P^+\alpha P^-\overline{g\alpha}P_\alpha\varphi_+) = P^+\alpha P^-\overline{\alpha}(P^+ + P^-)\overline{g}P_\alpha\varphi_+\\ &= \alpha P^-\overline{\alpha}P^+\overline{g}P_\alpha\varphi_+ = P_\alpha\overline{g}P_\alpha\varphi_+. \quad ∎\end{aligned}$$

COROLLARY 3.3. *For $g \in L_\infty$, A_g^θ is $\mathcal{C}_\theta$-symmetric in K_θ and we have*

$$\mathcal{C}_\theta A_g^\theta = A_{\overline{g}}^\theta\mathcal{C}_\theta. \tag{3.4}$$

Let us consider now the case of analytic symbols $g_+ \in H_\infty^+$.

PROPOSITION 3.4. (i) *If $g_+ \in H_\infty^+$ and α, θ are inner functions with $\alpha \preceq \theta$, then*

$$A_{g_+}^{\alpha,\theta}\varphi_+ = A_{g_+}^\alpha\varphi_+, \quad A_{\overline{g}_+}^{\theta,\alpha}\varphi_+ = A_{\overline{g}_+}^\alpha\varphi_+,$$

for all $\varphi_+ \in H_p^+$.

(ii) *If $\alpha \preceq \beta$ and $\beta \preceq \theta$, then $A_{g_+}^{\alpha,\beta}A_{f_+}^{\beta,\theta} = A_{g_+f_+}^{\alpha,\theta}$.*

Proof. (i) follows from (1.13).

(ii) $A_{g_+}^{\alpha,\beta}A_{f_+}^{\beta,\theta} = P_\alpha g_+ P_\beta f_+ P_\theta = P_\alpha g_+(P^+ - Q_\beta)f_+P_\theta = P_\alpha g_+f_+P_\theta = A_{g_+f_+}^{\alpha,\theta}$. ∎

As an immediate consequence we have, for $g_+ \in H_\infty^+, n \in \mathbb{N}$,

$$(A_{g_+}^\theta)^n = A_{g_+^n}^\theta. \tag{3.5}$$

From Propositions 3.2 and 3.4 we also have the following.

PROPOSITION 3.5. *If $g_+ \in H_\infty^+$, then $A_{g_+}^{\alpha,\theta}$ and $A_{\overline{g}_+}^{\theta,\alpha}$ are $\mathcal{PC}_\alpha$-symmetric and*

$$\mathcal{C}_\alpha A_{g_+}^{\alpha,\theta} = A_{\overline{g}_+}^\alpha\mathcal{C}_\alpha = A_{\overline{g}_+}^{\theta,\alpha}\mathcal{C}_\alpha.$$

Proof. By Proposition 3.2 we have $\mathcal{C}_\alpha A_{g_+}^{\alpha,\theta} = A_{\overline{g}_+}^\alpha\mathcal{C}_\alpha$ and, by Proposition 3.4(i), $A_{\overline{g}_+}^{\theta,\alpha} = A_{\overline{g}_+}^\alpha$. ∎

Let us now consider the functions k_w^θ and $\widetilde{k}_w^\theta$ defined, for each $w \in \mathbb{C}^+$, by

$$k_w^\theta(\xi) := \frac{1 - \overline{\theta(w)}\theta(\xi)}{\xi - \overline{w}}, \tag{3.6}$$

$$\widetilde{k}_w^\theta(\xi) := \frac{\theta(\xi) - \theta(w)}{\xi - w}, \tag{3.7}$$

which will play an important role in this section. We have $k_w^\theta, \widetilde{k}_w^\theta \in K_\theta$, with

$$k_w^\theta = P_\theta\frac{1}{\xi - \overline{w}}, \quad \widetilde{k}_w^\theta = P_\theta\frac{\theta}{\xi - w} = \mathcal{C}_\theta k_w^\theta. \tag{3.8}$$

If $\alpha \preceq \theta$, the functions $k_w^\alpha, \widetilde{k}_w^\alpha$ are related to $k_w^\theta, \widetilde{k}_w^\theta$, respectively, by

$$P_\alpha k_w^\theta = k_w^\alpha, \quad P_\alpha\widetilde{k}_w^\theta = (\overline{\alpha}\theta)(w)\widetilde{k}_w^\alpha. \tag{3.9}$$

PROPOSITION 3.6. *k_{i}^θ is a cyclic vector for A_r^θ and $\widetilde{k}_{\mathrm{i}}^\theta$ is a cyclic vector for $A_{r^{-1}}^\theta$.*

Proof. By (3.5) and (1.11),

$$(A_r^\theta)^n k_{\mathrm{i}}^\theta = A_{r^n}^\theta k_{\mathrm{i}}^\theta = P_\theta r^n P_\theta \frac{1}{\xi+\mathrm{i}} = P_\theta\Big(r^n \frac{1}{\xi+\mathrm{i}}\Big),$$

so $\{(A_r^\theta)^n k_{\mathrm{i}}^\theta : n \in \mathbb{N}\}$ is dense in K_θ. On the other hand, since $T_\theta T_{\overline{\theta}} T_\theta = T_\theta$, we have

$$(A_{r^{-1}}^\theta)^n \widetilde{k}_{\mathrm{i}}^\theta = A_{r^{-n}}^\theta \mathcal{C}_\theta k_{\mathrm{i}}^\theta = \mathcal{C}_\theta A_{r^n}^\theta k_{\mathrm{i}}^\theta$$

and, since $\mathcal{C}_\theta$ is an isometry in K_θ, it follows that $\widetilde{k}_{\mathrm{i}}^\theta$ is a cyclic vector for $A_{r^{-1}}^\theta$. ∎

PROPOSITION 3.7. *The operators P_α–$A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha}$ and P_α–$A_{r^{-1}}^{\theta,\alpha} A_r^{\alpha,\theta}$ on H_p^+ are rank -one operators, with range equal to* span$\{k_{\mathrm{i}}^\alpha\}$*and* span$\{\widetilde{k}_{\mathrm{i}}^\alpha\}$*, respectively, and we have*

$$(P_\alpha - A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha})\varphi_+ = 2\mathrm{i}\varphi_+^\alpha(\mathrm{i}) k_{\mathrm{i}}^\alpha, \tag{3.10}$$

$$(P_\alpha - A_{r^{-1}}^{\theta,\alpha} A_r^{\alpha,\theta})\varphi_+ = -2\mathrm{i}\varphi_-^\alpha(-\mathrm{i}) \widetilde{k}_{\mathrm{i}}^\alpha, \tag{3.11}$$

where $\varphi_-^\alpha = \overline{\alpha}\varphi_+^\alpha = \overline{\mathcal{C}_\alpha \varphi_+^\alpha}$.

Proof. We have

$$A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha}\varphi_+ = P_\alpha r P_\theta r^{-1}\varphi_+^\alpha = P_\alpha r P^+ r^{-1}\varphi_+^\alpha = \varphi_+^\alpha - 2\mathrm{i}\varphi_+^\alpha(\mathrm{i}) P_\alpha \frac{1}{\xi+\mathrm{i}} = \varphi_+^\alpha - 2\mathrm{i}\varphi_+^\alpha(\mathrm{i}) k_{\mathrm{i}}^\alpha,$$

where we used (1.10), and (3.10) follows from this equality. On the other hand, by Proposition 3.4, Proposition 3.5, (3.8) and (3.10),

$$\begin{aligned} A_{r^{-1}}^{\theta,\alpha} A_r^{\alpha,\theta}\varphi_+ &= A_{r^{-1}}^\alpha (\mathcal{C}_\alpha)^2 A_r^\alpha \varphi_+ = \mathcal{C}_\alpha A_r^\alpha A_{r^{-1}}^\alpha \mathcal{C}_\alpha \varphi_+ \\ &= \mathcal{C}_\alpha A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} \mathcal{C}_\alpha\varphi_+ = -2\mathrm{i}\alpha\overline{(\mathcal{C}_\alpha\varphi_+)_{(\mathrm{i})}}\,\overline{k}_{\mathrm{i}}^\alpha = -2\mathrm{i}\varphi_-^\alpha(-\mathrm{i})\widetilde{k}_{\mathrm{i}}^\alpha. \end{aligned}$$ ∎

In particular, for $\alpha = \theta$, we have the defect operators ([27]) $I_{K_\theta} - A_r^\theta A_{r^{-1}}^\theta$ and $I_{K_\theta} - A_{r^{-1}}^\theta A_r^\theta$ in K_θ, where I_{K_θ} denotes the identity operator in K_θ, with

$$(I_{K_\theta} - A_r^\theta A_{r^{-1}}^\theta)\varphi_+^\theta = 2\mathrm{i}\varphi_+^\theta(\mathrm{i}) k_{\mathrm{i}}^\theta \tag{3.12}$$

$$(I_{K_\theta} - A_{r^{-1}}^\theta A_r^\theta)\varphi_+^\theta = -2\mathrm{i}\varphi_-^\theta(-\mathrm{i}) \widetilde{k}_{\mathrm{i}}^\theta. \tag{3.13}$$

Next we address the question when an ATTO is zero, which is equivalent to obtaining conditions for two ATTO to be equal. For this purpose, it will be useful to note that a symbol $g \in L_\infty$ admits the following decompositions:

$$g = G_+ + G_-, \quad \text{with } G_\pm = (\xi+\mathrm{i})P^\pm \frac{g}{\xi+\mathrm{i}}, \tag{3.14}$$

$$g = g_+ + g_-, \quad \text{with } g_\pm = (\xi-\mathrm{i})P^\pm \frac{g}{\xi-\mathrm{i}}, \tag{3.15}$$

$$g = \gamma_+ + \gamma_- + C, \quad \text{with } \gamma_\pm = (\xi\pm\mathrm{i})P^\pm \frac{g}{\xi\pm\mathrm{i}},\ C \in \mathbb{C}. \tag{3.16}$$

The third decomposition can easily be related to any of the other two; for instance,

$$G_+ = \gamma_+, G_- = \gamma_- + C, \quad \text{with } C = -2\mathrm{i}P^-\Big(\frac{g}{\xi - \mathrm{i}}\Big)(-\mathrm{i}).$$

It is clear that an ATTO does not have a unique symbol, since we can have $A_g^{\alpha,\theta} = 0$ with $g \neq 0$. In fact, using the previous results and defining $\mathcal{H}_p^\pm := \lambda_\pm H_p^\pm$ where $\lambda_\pm(\xi) = \xi \pm \mathrm{i}$, we have the following.

THEOREM 3.8. *$A_g^{\alpha,\theta} = 0$ if and only if $g = \overline{\theta}\widetilde{g}_- + \alpha\widetilde{g}_+$ with $\widetilde{g}_\pm \in \mathcal{H}_p^\pm$.*

Proof. First we prove that $A_g^{\alpha,\theta} = 0$ if $g = \overline{\theta}\widetilde{g}_- + \alpha\widetilde{g}_+$. For $z_+ \in \mathbb{C}^+$, let $k_{z_+}^\theta := (1 - \overline{\theta(z_+)}\theta)/(\xi - \overline{z}_+) = P_\theta(1/(\xi - \overline{z}_+))$; then

$$\begin{aligned} A_g^{\alpha,\theta}k_{z_+}^\theta &= P_\alpha\Big[g\frac{1 - \overline{\theta(z_+)}\theta}{\xi - \overline{z}_+}\Big] = P_\alpha\Big[(\overline{\theta}\widetilde{g}_- + \alpha\widetilde{g}_+)\frac{1 - \overline{\theta(z_+)}\theta}{\xi - \overline{z}_+}\Big] \\ &= P_\alpha\Big[\widetilde{g}_-\frac{\overline{\theta} - \overline{\theta(z_+)}}{\xi - \overline{z}_+}\Big] + P_\alpha\Big[\alpha\widetilde{g}_+\frac{1 - \overline{\theta(z_+)}\theta}{\xi - \overline{z}_+}\Big] = 0 \end{aligned}$$

since $\widetilde{g}_-(\overline{\theta} - \overline{\theta(z_+)})/(\xi - \overline{z}_+) \in H_p^-$ and $\alpha\widetilde{g}_+(1 - \overline{\theta(z_+)}\theta)/(\xi - \overline{z}_+) \in \alpha H_p^+$. The converse will be proved in several steps. Assuming that $A_g^{\alpha,\theta} = 0$, we show that:

(i) $A_{G_+}^{\alpha,\theta}A_r^{\alpha,\theta}A_{r^{-1}}^{\theta,\alpha}k_\mathrm{i}^\alpha = A_r^{\alpha,\theta}A_{r^{-1}}^{\theta,\alpha}A_{G_+}^{\alpha,\theta}k_\mathrm{i}^\alpha$;
(ii) $\gamma_+ = \alpha f_+ + C_1$ for some $f_+ \in \mathcal{H}_p^+$ and some $C_1 \in \mathbb{C}$;
(iii) $\gamma_- = \overline{\theta}f_- + C_2$ for some $f_- \in \mathcal{H}_p^-$ and some $C_2 \in \mathbb{C}$;
(vi) $C_1 + C_2 + C = 0$, where C is the constant in (3.16);

so that $g = \alpha f_+ + \overline{\theta}f_-$ with $f_\pm \in \mathcal{H}_p^\pm$.

(i) Let $G_\pm$ be defined as in (3.14). We have, from (3.10),

$$A_{G_+}^{\alpha,\theta}A_r^{\alpha,\theta}A_{r^{-1}}^{\theta,\alpha}k_\mathrm{i}^\alpha = (1 - 2\mathrm{i}k_\mathrm{i}^\alpha(\mathrm{i}))P_\alpha G_+k_\mathrm{i}^\alpha.$$

Now, if $A_g^{\alpha,\theta} = 0$ then $A_{G_++G_-}^{\alpha,\theta} = 0$ and

$$A_{G_+}^{\alpha,\theta}\varphi_+ = -A_{G_-}^{\alpha,\theta}\varphi_+ \tag{3.17}$$

for all φ_+ such that $G_\pm\varphi_+^\theta \in H_p^+$ (where we define $A_{G_\pm}^{\alpha,\theta}\varphi_+ = P_\alpha G_\pm\varphi_+^\theta$). Also note that

$$P_\alpha G_-k_\mathrm{i}^\alpha = P^+G_-k_\mathrm{i}^\alpha. \tag{3.18}$$

Using (3.17), (3.18), (1.12), (1.13), and taking into account that

$$P_\theta r^{-1}k_\mathrm{i}^\alpha = P_\alpha r^{-1}k_\mathrm{i}^\alpha = P^+r^{-1}k_\mathrm{i}^\alpha = r^{-1}k_\mathrm{i}^\alpha - 2\mathrm{i}\frac{k_\mathrm{i}^\alpha(\mathrm{i})}{\xi - \mathrm{i}},$$

we have

$$\begin{aligned} A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha} &= -A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} A_{G_-}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha} = -A_r^{\alpha,\theta} (P_\alpha r^{-1} G_- k_{\mathrm{i}}^{\alpha}) \\ &= -A_r^{\alpha,\theta} \Big[P_\alpha G_- (r^{-1} k_{\mathrm{i}}^{\alpha} - 2\mathrm{i} \frac{k_{\mathrm{i}}^{\alpha}(\mathrm{i})}{\xi - \mathrm{i}}) \Big] = -A_r^{\alpha} A_{G_-}^{\alpha,\theta} r^{-1} k_{\mathrm{i}}^{\alpha} \\ &= A_r^{\alpha} A_{G_+}^{\alpha,\theta} r^{-1} k_{\mathrm{i}}^{\alpha} = P_\alpha r P_\alpha G_+ P_\theta r^{-1} k_{\mathrm{i}}^{\alpha} = P_\alpha r G_+ P_\theta r^{-1} k_{\mathrm{i}}^{\alpha} \\ &= P_\alpha r G_+ P_\alpha r^{-1} k_{\mathrm{i}}^{\alpha} = P_\alpha G_+ k_{\mathrm{i}}^{\alpha} - 2\mathrm{i} k_{\mathrm{i}}^{\alpha}(\mathrm{i}) P_\alpha G_+ P_\alpha \frac{1}{\xi + \mathrm{i}} \\ &= (1 - 2\mathrm{i} k_{\mathrm{i}}^{\alpha}(\mathrm{i})) P_\alpha G_+ k_{\mathrm{i}}^{\alpha}. \end{aligned}$$

Thus, $A_{G_+}^{\alpha,\theta} A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} k_{\mathrm{i}}^{\alpha} = A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha}$.

(ii) From (i) we get

$$(A_{G_+}^{\alpha,\theta} - A_{G_+}^{\alpha,\theta} A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha}) k_{\mathrm{i}}^{\alpha} = (A_{G_+}^{\alpha,\theta} - A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha} A_{G_+}^{\alpha,\theta}) k_{\mathrm{i}}^{\alpha}$$

and thus

$$A_{G_+}^{\alpha,\theta} (P_\alpha - A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha}) k_{\mathrm{i}}^{\alpha} = (P_\alpha - A_r^{\alpha,\theta} A_{r^{-1}}^{\theta,\alpha}) A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha}$$

which, by Proposition 3.7, is equivalent to

$$A_{G_+}^{\alpha,\theta} 2\mathrm{i} k_{\mathrm{i}}^{\alpha}(\mathrm{i}) k_{\mathrm{i}}^{\alpha} = 2\mathrm{i} (A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha})(\mathrm{i}) k_{\mathrm{i}}^{\alpha}.$$

Therefore,

$$A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha} = C_1 k_{\mathrm{i}}^{\alpha} \quad \text{where } C_1 \in \mathbb{C} \setminus \{0\}, \quad \text{and}$$

$$A_{G_+}^{\alpha,\theta} k_{\mathrm{i}}^{\alpha} = C_1 k_{\mathrm{i}}^{\alpha} \Leftrightarrow P_\alpha (G_+ - C_1) k_{\mathrm{i}}^{\alpha} = 0 \Leftrightarrow P_\alpha \frac{G_+ - C_1}{\xi + \mathrm{i}} = 0 \Leftrightarrow \frac{G_+ - C_1}{\xi + \mathrm{i}} \in \alpha H_p^+.$$

Since $G_+ = \gamma_+$, we have $\gamma_+ = \alpha f_+ + C_1$ with $f_+ \in \mathcal{H}_p^+$ and $C_1 \in \mathbb{C}$.

(iii) Since $\overline{g} = (\overline{g})_+ + (\overline{g})_-$, where

$$(\overline{g})_\pm = (\xi + \mathrm{i}) P^\pm \frac{\overline{g}}{\xi + \mathrm{i}},$$

so that $(\overline{g})_+ = \overline{\gamma}_-$, to study the condition on γ_- we use the equivalence $A_g^{\alpha,\theta} = 0 \Leftrightarrow A_{\overline{g}}^{\theta,\alpha} = 0 \Leftrightarrow P_\theta \overline{g} P_\alpha = 0$, where the equality on the right-hand side means that

$$P_\alpha \overline{g} P_\alpha = 0 \quad \text{and} \quad P_{\alpha,\theta} \overline{g} P_\alpha = 0. \tag{3.19}$$

From the first equality in (3.19) and from (ii) we conclude that, for some constant $C_2 \in \mathbb{C}$,

$$\frac{(\overline{g})_+ - C_2}{\xi + \mathrm{i}} \in \alpha H_p^+. \tag{3.20}$$

On the other hand we have, from the second equality in (3.19),

$$P_{\alpha,\theta} (\overline{g})_+ k_{\mathrm{i}}^{\alpha} = -P_{\alpha,\theta} (\overline{g})_- k_{\mathrm{i}}^{\alpha} = -\alpha P_{\overline{\alpha}\theta} \overline{\alpha} (I - P^-) (\overline{g})_- k_{\mathrm{i}}^{\alpha} = 0. \tag{3.21}$$

Since we also have $P_{\alpha,\theta}C_2k_{\mathrm{i}}^{\alpha} = 0$, taking this and (3.21) into account we get

$$0 = P_{\alpha,\theta}((\overline{g})_+ - C_2)k_{\mathrm{i}}^{\alpha} = P_{\alpha,\theta}\Big(\frac{(\overline{g})_+ - C_2}{\xi + \mathrm{i}}(1 - \overline{\alpha(\mathrm{i})}\alpha)\Big)$$

which, by (3.20), implies that

$$0 = P_{\theta}(f + (1 - \overline{\alpha(\mathrm{i})}\alpha)) \quad \text{with } f_+ = \frac{(\overline{g})_+ - C_2}{\xi + \mathrm{i}}.$$

Now,

$$P_{\theta}[f_+(1 - \overline{\alpha(\mathrm{i})}\alpha)] = 0 \Rightarrow P_{\theta}f_+ = 0,$$

because $P_{\theta}[f_+(1 - \overline{\alpha(\mathrm{i})}\alpha)] = 0$ implies that $f_+(1 - \overline{\alpha(\mathrm{i})}\alpha) = \theta\widetilde{f}_+$, with $\widetilde{f}_+ \in H_p^+$ and, if $\widetilde{f}_+^{\mathrm{i}}, \widetilde{f}_+^{\mathrm{o}}$ are the inner and outer factors of $\widetilde{f}_+$, respectively, that is equivalent to having $f_+^{\mathrm{i}} f_+^{\mathrm{o}}(1 - \overline{\alpha(\mathrm{i})}\alpha) = \theta\widetilde{f}_+$. Since $1 - \overline{\alpha(\mathrm{i})}\alpha$ is an outer function in H_∞^+, we conclude that θ divides f_+^{i} and thus $P_\theta f_+ = 0$. Thus $f_+ \in \theta H_p^+$ and we conclude that $\gamma_- = \overline{(\overline{g})_+} = \overline{\theta} f_- + C_2$ with $f_- \in \mathcal{H}_p^-$.

(iv) It follows from (ii), (iii) and (3.16) that $g = \alpha f_+ + \overline{\theta} f_- + B$ where B is a constant. Since $A_g^{\alpha,\theta} = 0$, it follows from the first part of the proof that we must then have $A_B^{\alpha,\theta} = 0$, which implies that $B = 0$. ∎

For $p = 2$ we may use the unitary equivalence derived earlier to obtain a generalisation of Sarason's result for TTO in [27], which, it seems, cannot be proved directly using his techniques. It seems natural to conjecture that an analogous result holds in the disc for all $1 < p < \infty$, although no direct translation of the half-plane result seems to be possible for $p \neq 2$.

COROLLARY 3.9. *In the case of $p = 2$ and for Hardy spaces on $\mathbb{D}$, the asymmetric truncated Toeplitz operator $A_g^{\alpha,\theta}$ is zero if and only if $g \in \alpha H^2(\mathbb{D}) + \overline{\theta H^2(\mathbb{D})}$.*

Proof. Note that $g \in \alpha H^2(\mathbb{D})$ if and only if $g \circ m^{-1} \in (\alpha \circ m^{-1})\lambda_+ H_2^+$ and $g \in \overline{\theta H^2(\mathbb{D})}$ if and only if $g \circ m^{-1} \in (\theta \circ m^{-1})\lambda_- H_2^-$. Now the result follows directly from Theorem 3.8 using the equivalence given in (2.2). ∎

4. FINITE RANK ASYMMETRIC TRUNCATED TOEPLITZ OPERATORS

In this section we assume again that α, θ are inner functions with $\alpha \preceq \theta$. It is clear from any of the decompositions (3.14)–(3.16) of $g \in L_\infty$ that we can represent g in the form

$$g = a_+\overline{\theta} + a_-\alpha \tag{4.1}$$

with $a_\pm \in \mathcal{H}_p^\pm$. If $a_\pm \in \mathbb{C}$, then by Theorem 3.8 we have $A_g^{\alpha,\theta} = 0$. It now seems natural to consider symbols of the form

$$g = \frac{\alpha}{\xi - z_+} \quad \text{and} \quad g = \frac{\overline{\theta}}{\xi - \overline{z}_+} \quad (z_+ \in \mathbb{C}^+) \tag{4.2}$$

as being the simplest corresponding to a non-zero ATTO $A_g^{\alpha,\theta}$.

Some other symbols seem equally simple. Let θ have a non-tangential limit $\theta(\xi_0)$ at $\xi_0 \in \mathbb{R}$ and suppose, in addition, that the functions

$$\frac{\alpha(\xi) - \alpha(\xi_0)}{\xi - \xi_0} \quad \text{and} \quad \frac{\theta(\xi) - \theta(\xi_0)}{\xi - \xi_0} \quad \text{lie in } L_\infty, \tag{4.3}$$

in which case the functions in (4.3) lie in K_α and K_θ respectively, and hence in H_p^+. We can then consider bounded symbols of the form (4.1) with

$$a_- = \frac{\overline{\theta} - \overline{\theta(\xi_0)}}{\xi - \xi_0}, \quad a_+ = \frac{\alpha(\xi_0) - \alpha}{\xi - \xi_0},$$

i.e.,

$$g = \frac{\alpha(\xi_0)\overline{\theta} - \overline{\theta(\xi_0)}\alpha}{\xi - \xi_0}. \tag{4.4}$$

Analogously, if θ admits a non-tangential limit $\theta(\infty)$ at ∞, i.e., the inner function $\theta(\mathrm{i}(1+z)/(1-z))$ in the unit disc has a non-tangential limit $\theta(\infty)$ at 1, and in addition the functions

$$\xi[\alpha(\xi) - \alpha(\infty)] \quad \text{and} \quad \xi[\theta(\xi) - \theta(\infty)] \quad \text{lie in } L_\infty, \tag{4.5}$$

then we can consider bounded symbols of the form

$$g = \xi[\alpha(\infty)\overline{\theta} - \overline{\theta(\infty)}\alpha]. \tag{4.6}$$

We remark that, if (4.5) holds, then

$$\widetilde{k}_\infty^\alpha := \alpha - \alpha(\infty) \in K_\alpha \quad \text{and} \quad \widetilde{k}_\infty^\theta := \theta - \theta(\infty) \in K_\theta.$$

THEOREM 4.1. *The asymmetric truncated Toeplitz operators $A_g^{\alpha,\theta}$ with g of the form* (4.2), (4.4) *and* (4.6), *are rank-one operators.*

Proof. Suppose that $g = \alpha/(\xi - z_+)$ with $z_+ \in \mathbb{C}^+$. Then for any $w \in \mathbb{C}^+$ and k_w^θ given by (3.6), we have

$$\begin{aligned} A_g^{\alpha,\theta} k_w^\theta &= \alpha P^- \overline{\alpha} P^+ \frac{\alpha}{\xi - z_+} k_w^\theta = \alpha P^- \overline{\alpha} \Big(\frac{\alpha k_w^\theta - \alpha(z_+) k_w^\theta(z_+)}{\xi - z_+} \Big) \\ &= k_w^\theta(z_+) \frac{\alpha - \alpha(z_+)}{\xi - z_+} = k_w^\theta(z_+) \widetilde{k}_{z_+}^\alpha, \end{aligned}$$

where $\widetilde{k}_{z_+}^\alpha$ is defined in (3.7). Analogously, if $g = \overline{\theta}/(\xi - \overline{z}_+)$ with $z_+ \in \mathbb{C}^+$, then $A_g^{\alpha,\theta} k_w^\theta = -(\overline{\theta} k_w^\theta)(z_+) k_{z_+}^\theta$ for all $w \in \mathbb{C}^+$. Suppose now that g takes the form (4.4).

Then, taking into account the fact that, for all $w \in \mathbb{C}^+$

$$\frac{k_w^\theta - k_w^\theta(\xi_0)}{\xi - \xi_0} = \Big(C_1 + C_2 \frac{\theta - \theta(\xi_0)}{\xi - \xi_0}\Big)\frac{1}{\xi - \overline{w}} \in H_p^+,$$

where

$$C_1 = \frac{1 + \overline{\theta(w)}\theta(\xi_0)}{\xi_0 - \overline{w}} \quad \text{and} \quad C_2 = -\frac{\xi_0\overline{\theta(w)}}{\xi_0 - \overline{w}}, \quad \text{and}$$

$$\frac{\overline{\theta}k_w^\theta - (\overline{\theta}k_w^\theta)(\xi_0)}{\xi - \xi_0} = \Big(\widetilde{C}_1 + \widetilde{C}_2 \frac{\overline{\theta} - \overline{\theta(\xi_0)}}{\xi - \xi_0}\Big)\frac{1}{\xi - \overline{w}} \in H_p^-,$$

where

$$\widetilde{C}_1 = \frac{\overline{\theta(w)} - \overline{\theta(\xi_0)}}{\xi_0 - w} \quad \text{and} \quad \widetilde{C}_2 = \frac{\xi_0 - \overline{w}}{\xi_0 - w},$$

we have

$$\begin{aligned}
A_g^{\alpha,\theta}k_w^\theta &= P_\alpha \frac{\alpha(\xi_0)\overline{\theta} - \overline{\theta(\xi_0)}\alpha}{\xi - \xi_0}k_w^\theta \\
&= P_\alpha\Big[\alpha(\xi_0)\underbrace{\frac{\overline{\theta}k_w^\theta - (\overline{\theta}k_w^\theta)(\xi_0)}{\xi - \xi_0}}_{\in H_p^-} - \overline{\theta}(\xi_0)\alpha \underbrace{\frac{k_w^\theta - k_w^\theta(\xi_0)}{\xi - \xi_0}}_{\in H_p^+} - (\overline{\theta}k_w^\theta)(\xi_0)\underbrace{\frac{\alpha - \alpha(\xi_0)}{\xi - \xi_0}}_{\in K_\alpha}\Big] \\
&= -(\overline{\theta}k_w^\theta)(\xi_0)\widetilde{k}_{\xi_0}^\alpha,
\end{aligned}$$

where $\widetilde{k}_{\xi_0}^\alpha := (\alpha - \alpha(\xi_0))/(\xi - \xi_0)$.

Let now g take the form (4.6). Then, for all $w \in \mathbb{C}^+$, we have

$$\begin{aligned}
A_g^{\alpha,\theta}k_w^\theta &= P_\alpha\Big[\xi[\alpha(\infty)\overline{\theta} - \overline{\theta(\infty)}\alpha]\frac{1 - \overline{\theta(w)}\theta}{\xi - \overline{w}}\Big] \\
&= P_\alpha\Big[\alpha(\infty)\frac{\xi(\overline{\theta} - \overline{\theta(w)})}{\xi - \overline{w}} + \frac{\overline{\theta(w)} - \overline{\theta(\infty)}}{\xi - \overline{w}}\xi[\alpha - \alpha(\infty)] + \overline{\theta(\infty)\theta(w)}\alpha \underbrace{\frac{\xi(\theta - \theta(\infty))}{\xi - \overline{w}}}_{\in H_p^+}\Big] \\
&= \alpha P^-\overline{\alpha}P^+\Big[\alpha(\infty)\frac{\xi(\overline{\theta} - \overline{\theta(\infty)})}{\xi - \overline{w}}\Big] + \alpha P^-\overline{\alpha}\frac{\xi[\alpha - \alpha(\infty)]}{\xi - \overline{w}}(\overline{\theta(w)} - \overline{\theta(\infty)}) \\
&= \alpha(\infty)\alpha P^-\overline{\alpha}\Big(\overline{w}\frac{\overline{\theta(w)} - \overline{\theta(\infty)}}{\xi - \overline{w}}\Big) \\
&\quad + (\overline{\theta(w)} - \overline{\theta(\infty)})\alpha(\infty)\alpha\Big[\frac{\xi(\overline{\alpha(\infty)} - \overline{\alpha}) - \overline{w}(\overline{\alpha(\infty)} - \overline{\alpha(w)})}{\xi - \overline{w}}\Big] \\
&= \alpha(\infty)(\overline{\theta(w)} - \overline{\theta(\infty)})\Big[\overline{w}\frac{1 - \overline{\alpha(w)}\alpha}{\xi - \overline{w}} + \frac{\alpha\xi(\overline{\alpha(\infty)} - \overline{\alpha}) - \alpha\overline{w}\overline{\alpha(\infty)} + \alpha\overline{w}\overline{\alpha(w)}}{\xi - \overline{w}}\Big] \\
&= \alpha(\infty)(\overline{\theta(w)} - \overline{\theta(\infty)})(-1 + \alpha\overline{\alpha(\infty)}) \\
&= (\overline{\theta(w)} - \overline{\theta(\infty)})(\alpha - \alpha(\infty)) = (\overline{\theta(w)} - \overline{\theta(\infty)})k_w^\alpha.
\end{aligned}$$

Since the span of $\{k_w^\theta : w \in \mathbb{C}^+\}$ is dense in K_θ, we have proved the result. ∎

One can show analogously that if

(i) $g = \alpha/(\xi - z_+)^n$ or $g = \overline{\theta}/(\xi - \overline{z}_+)^n$, with $n \in \mathbb{N}$, or

(ii) $\theta, \theta', \ldots, \theta^{(n-1)}$ have non-tangential limits at $\xi_0 \in \mathbb{R}$, while the functions a_+ and a_- are given by

$$a_+(\xi) = \frac{\alpha(\xi) - \sum_{j=0}^{n-1} \alpha^{(j)}(\xi_0)(\xi-\xi_0)^j/j!}{(\xi - \xi_0)^n} \quad \text{and} \quad \overline{a_-(\xi)} = \frac{\theta(\xi) - \sum_{j=0}^{n-1} \theta^{(j)}(\xi_0)(\xi-\xi_0)^j/j!}{(\xi - \xi_0)^n}$$

lie in L_∞, and $g = a_+\overline{\theta} + a_-\alpha$, or

(iii) $\theta, \theta', \ldots, \theta^{(n-1)}$ have non-tangential limits at ∞, while the functions a_+ and a_- satisfying

$$a_+(\xi) = \xi^n \left[a(\xi) - \sum_{j=0}^{n-1} a^{(j)}(\infty)\xi^{-j}/j! \right] \quad \text{and} \quad \overline{a_-(\xi)} = \xi^n \left[\theta(\xi) - \sum_{j=0}^{n-1} \theta^{(j)}(\infty)\xi^{-j}/j! \right]$$

lie in L_∞, and $g = a_+\overline{\theta} + a_-\alpha$, then $A_g^{\alpha,\theta}$ is a finite-rank operator.

Finite-rank truncated Toeplitz operators ($\alpha = \theta$) were completely characterized by Sarason [27] and Bessonov [9] in the setting of the disk, for $p = 2$. Whether, in our case, every rank-one ATTO with symbol in L_∞ is of the form considered in Theorem 4.1, or every finite-rank ATTO with symbol in L_∞ is a linear combination of those given above is an open question, whose study necessarily involves a characterization of ATTO with L_p symbols, which is beyond the scope of the present paper.

5. EQUIVALENCE AFTER EXTENSION OF ATTO AND TOEPLITZ OPERATORS WITH TRIANGULAR MATRIX SYMBOLS

In this section we show that asymmetric truncated Toeplitz operators are equivalent after extension to Toeplitz operators with triangular symbols of a certain form.

Recall that here, as in the previous sections, by an operator we mean a bounded linear operator acting between complex Banach spaces.

DEFINITION 5.1 ([5], [19], [29]). The operators $T : X \to \widetilde{X}$ and $S : Y \to \widetilde{Y}$ are said to be *(algebraically and topologically) equivalent* if and only if $T = ESF$ where E, F are invertible operators. More generally, T and S are *equivalent after extension* if and only if there exist (possibly trivial) Banach spaces X_0, Y_0, called *extension spaces*, and invertible bounded linear operators $E : \widetilde{Y} \oplus Y_0 \to \widetilde{X} \oplus X_0$ and $F : X \oplus X_0 \to Y \oplus Y_0$, such that

$$\begin{pmatrix} T & 0 \\ 0 & I_{X_0} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Y_0} \end{pmatrix} F. \tag{5.1}$$

In this case we say that $T \overset{*}{\sim} S$.

The relation $\overset{*}{\sim}$ is an equivalence relation. Operators that are equivalent after extension have many features in common. In particular, using the notation $X \simeq Y$ to say that two Banach spaces X and Y are isomorphic, i.e., that there exists an invertible operator from X onto Y, and the notation $\operatorname{Im} A$ to denote the range of an operator A, we have the following.

THEOREM 5.2 ([5]). *Let $T : X \to \widetilde{X}$, $S : Y \to \widetilde{Y}$ be operators and assume that $T \overset{*}{\sim} S$. Then*

(i) $\ker T \simeq \ker S$;

(ii) $\operatorname{Im} T$ *is closed if and only if* $\operatorname{Im} S$ *is closed and, in that case,* $\widetilde{X}/\operatorname{Im} T \simeq \widetilde{Y}/\operatorname{Im} S$;

(iii) *if one of the operators T, S is generalised (left, right) invertible, then the other is generalised (left, right) invertible too;*

(iv) *T is Fredholm if and only if S is Fredholm and in that case* $\dim\ker T = \dim\ker S$, $\operatorname{codim}\operatorname{Im} T = \operatorname{codim}\operatorname{Im} S$.

More properties can be found in [5], [29], for instance.

Now let us consider the operator $A_g^{\alpha,\theta} : K_\theta \to K_\alpha$ and the operator

$$P_\alpha g P_\theta + Q_\theta : H_p^+ \to K_\alpha \oplus \theta H_p^+. \tag{5.2}$$

It is easy to see that

$$A_g^{\alpha,\theta} \overset{*}{\sim} P_\alpha g P_\theta + Q_\theta \tag{5.3}$$

because

$$\begin{pmatrix} A_g^{\alpha,\theta} & 0 \\ 0 & I_{\theta H_p^+} \end{pmatrix} = E_1 \begin{pmatrix} P_\alpha g P_\theta + Q_\theta & 0 \\ 0 & I_{\{0\}} \end{pmatrix} F_1 \tag{5.4}$$

where

$$F_1 : K_\theta \oplus \theta H_p^+ \to H_p^+ \oplus \{0\}, \tag{5.5}$$

$$E_1 : (K_\alpha \oplus \theta H_p^+) \oplus \{0\} \to K_\alpha \oplus \theta H_p^+, \tag{5.6}$$

are invertible operators (defined in an obvious way). On the other hand, it is clear that

$$P_\alpha g P_\theta + Q_\theta \overset{*}{\sim} \begin{pmatrix} P_\alpha g P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} \tag{5.7}$$

where the operator on the right-hand side is defined from $(H_p^+)^2$ into $(K_\alpha \oplus \theta H_p^+) \times H_p^+$. Now, from (5.2) we have

$$P_\alpha g P_\theta + Q_\theta = (P^+ - P_\alpha T_g Q_\theta)(P_\alpha T_g + Q_\theta) \tag{5.8}$$

where we have the following.

LEMMA 5.3. *The following operator is invertible:*

$$P^+ - P_\alpha T_g Q_\theta : K_\alpha \oplus \theta H_p^+ \to K_\alpha \oplus \theta H_p^+. \tag{5.9}$$

Proof. First we prove that $P^+ \pm P_\alpha T_g Q_\theta$ maps $K_\alpha \oplus \theta H_p^+$ into $K_\alpha \oplus \theta H_p^+$. Indeed, let $\varphi_\alpha \in K_\alpha$, $\varphi_+ \in H_p^+$; then

$$(P^+ \pm P_\alpha T_g Q_\theta)(\varphi_\alpha + \theta\varphi_+) = \varphi_\alpha + \theta\varphi_+ \pm P_\alpha T_g(\theta\varphi_+)$$

because $Q_\theta \varphi_\alpha = 0$. For the same reason ($Q_\theta P_\alpha = 0$), we have

$$(P^+ \pm P_\alpha T_g Q_\theta)(P^+ \mp P_\alpha T_g Q_\theta) = P^+ \mp P_\alpha T_g Q_\theta \pm P_\alpha T_g Q_\theta = P^+$$

and therefore the operator (5.9) is invertible, with inverse

$$P^+ + P_\alpha T_g Q_\theta : K_\alpha \oplus \theta H_p^+ \to K_\alpha \oplus \theta H_p^+. \quad ∎$$

Thus, with

$$T = \begin{pmatrix} P^+ - P_\alpha T_g Q_\theta & 0 \\ 0 & P^+ \end{pmatrix},$$

we can write

$$\begin{aligned}\begin{pmatrix} P_\alpha g P_\theta + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} &= T \begin{pmatrix} P_\alpha T_g + Q_\theta & 0 \\ 0 & P^+ \end{pmatrix} \\ &= T \begin{pmatrix} T_\theta & P_\alpha \\ -P^+ & T_{\overline{\alpha}} \end{pmatrix} \begin{pmatrix} T_{\overline{\theta}} & 0 \\ T_g - Q_\alpha(T_g - T_{\alpha\overline{\theta}}) & T_\alpha \end{pmatrix} \\ &= T \begin{pmatrix} T_\theta & P_\alpha \\ -P^+ & T_{\overline{\alpha}} \end{pmatrix} \begin{pmatrix} T_{\overline{\theta}} & 0 \\ T_g & T_\alpha \end{pmatrix} \begin{pmatrix} P^+ & 0 \\ -T_{\overline{\alpha}}(T_g - T_{\alpha\overline{\theta}}) & P^+ \end{pmatrix}. \end{aligned} \tag{5.10}$$

On the right-hand side of the last equality,

(i) the first factor, T, is invertible in $(K_\alpha \oplus \theta H_p^+) \times H_p^+$ by Lemma 5.3;

(ii) the second factor is invertible as an operator from $(H_p^+)^2$ into $(K_\alpha \oplus \theta H_p^+) \times H_p^+$ by Lemma 5.4 below;

(iii) the last factor is invertible in $(H_p^+)^2$ by Lemma 5.5 below.

LEMMA 5.4. *The operator $T_1 : (H_p^+)^2 \to (K_\alpha \oplus \theta H_p^+) \times H_p^+$ defined by the following equation is invertible:*

$$T_1(\varphi_{1+}, \varphi_{2+}) = \begin{pmatrix} T_\theta & P_\alpha \\ -P^+ & T_{\overline{\alpha}} \end{pmatrix} \begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix}. \tag{5.11}$$

Proof. Given any $(\psi_{1+}, \psi_{2+}) \in (K_\alpha \oplus \theta H_p^+) \times H_p^+$, it follows from (5.11) that

$$T_1(\varphi_{1+}, \varphi_{2+}) = (\psi_{1+}, \psi_{2+}) \tag{5.12}$$

$$\Leftrightarrow \begin{cases} \theta\varphi_{1+} + P_\alpha \varphi_{2+} = \psi_{1+}, \\ -\varphi_{1+} + T_{\overline{\alpha}}\varphi_{2+} = \psi_{2+}. \end{cases} \tag{5.13}$$

The first equation in (5.13) implies that

$$\theta\varphi_{1+} = Q_\theta \psi_{1+}, \quad P_\alpha \varphi_{2+} = P_\alpha \psi_{1+} \tag{5.14}$$

and from the second equation in (5.13) we have

$$\varphi_{1+} + \psi_{2+} = \overline{\alpha} Q_\alpha \varphi_{2+}; \tag{5.15}$$

therefore

$$Q_\alpha \varphi_{2+} = \alpha\varphi_{1+} + \alpha\psi_{2+} = \alpha\overline{\theta} Q_\theta \psi_{1+} + \alpha\psi_{2+}. \tag{5.16}$$

From (5.14) and (5.16) we see that (5.12) implies that

$$\varphi_{1+} = \overline{\theta} Q_\theta \psi_{1+}, \quad \varphi_{2+} = (P_\alpha + \alpha\overline{\theta} Q_\theta)\psi_{1+} + T_\alpha \psi_{2+}. \tag{5.17}$$

It follows that T_1 is injective (replacing ψ_{1+} and ψ_{2+} by 0) and surjective (since for any $\psi_{1+} \in K_\alpha \oplus \theta H_p^+$ and any $\psi_{2+} \in H_p^+$ there exist $\varphi_{1+}, \varphi_{2+} \in H_p^+$, given by (5.17), such that (5.12) holds.

Moreover (5.17) yields an expression for the inverse operator:

$$T_1^{-1} : (K_\alpha \oplus \theta H_p^+) \times H_p^+ \to (H_p^+)^2,$$

$$T_1^{-1} \begin{pmatrix} \psi_{1+} \\ \psi_{2+} \end{pmatrix} = \begin{pmatrix} T_{\overline{\theta}} & 0 \\ P_\alpha + \alpha\overline{\theta} Q_\theta & T_\alpha \end{pmatrix} \begin{pmatrix} \psi_{1+} \\ \psi_{2+} \end{pmatrix}. \quad ∎ \tag{5.18}$$

LEMMA 5.5. *The operator*

$$T_2 : (H_p^+)^2 \to (H_p^+)^2, \quad T_2 = \begin{pmatrix} P^+ & 0 \\ -T_{\overline{\alpha}}(T_g - T_{\alpha\overline{\theta}}) & P^+ \end{pmatrix} \tag{5.19}$$

is invertible, with inverse given by

$$T_2^{-1} = \begin{pmatrix} P^+ & 0 \\ T_{\overline{\alpha}}(T_g - T_{\alpha\overline{\theta}}) & P^+ \end{pmatrix}. \tag{5.20}$$

Proof. This follows from the fact that T_2 is of the form

$$\begin{pmatrix} P^+ & 0 \\ A & P^+ \end{pmatrix}$$

where A is an operator in H_p^+ which commutes with P^+. ∎

From (5.3), (5.7), (5.10) and Lemmas 5.3, 5.4 and 5.5 we now conclude the following.

THEOREM 5.6. $A_g^{\alpha,\theta} \overset{*}{\sim} T_G$ *where* $G = \begin{pmatrix} \overline{\theta} & 0 \\ g & \alpha \end{pmatrix}$.

As an immediate consequence of Theorem 5.6, one may study properties of ATTO (or TTO), such as Fredholmness and invertibility, using known results for Toeplitz operators with matricial symbols and vice-versa. For the simplest inner functions, such as $\theta(z) = z^n$ on $\mathbb{T}$ and $\theta(\xi) = e^{i\mu\xi}$ on $\mathbb{R}$, old results linking the invertibility of $A_g^{\theta,\theta}$ and T_G may be found in [11], for example. However, we are now able to consider all the properties listed in Theorem 5.2. It is well known, for instance, that T_G is Fredholm if and only if G admits a Wiener–Hopf (or generalized) p-factorization ([11], [12], [21])

$$G = G_- D G_+^{-1} \tag{5.21}$$

where, taking $\lambda_\pm(\xi) = \xi \pm \mathrm{i}$ and $1/p' = 1 - 1/p$, we have

$$D = \operatorname{diag}\left\{\left(\frac{\lambda_-}{\lambda_+}\right)^{-k}, \left(\frac{\lambda_-}{\lambda_+}\right)^{k}\right\} \quad \text{with } k \in \mathbb{Z}, \tag{5.22}$$

$$\lambda_\pm^{-1} G_\pm \in (H_p^\pm)^{2\times 2}, \quad \lambda_\pm^{-1} G_\pm^{-1} \in (H_{p'}^\pm)^{2\times 2}, \tag{5.23}$$

$$\begin{array}{c} G_+ P^+ G_-^{-1} I \text{ is defined in a dense subset of } (L_p(\mathbb{R}))^2 \\ \text{and admits a bounded extension to } L_p(\mathbb{R})^2. \end{array} \tag{5.24}$$

Moreover, T_G is invertible if and only if $k=0$ in (5.22). We have thus the following.

COROLLARY 5.7. *The operator $A_g^{\alpha,\theta}$ is Fredholm in K_θ^p if and only if the matrix symbol G admits a Wiener–Hopf p-factorization, and it is invertible if and only if $k = 0$ in* (5.22).

As an illustration, we consider the following class of TTO. Let $\theta(\xi) = \mathrm{e}^{\mathrm{i}\xi}$, $e_\lambda(\xi) = \mathrm{e}^{\mathrm{i}\lambda\xi}$ for $\lambda \in \mathbb{R}$, and

$$g_\lambda = b e_{-\beta} - \lambda + \sum_{k=1}^{n} (a_k e_{k\alpha})$$

where $\alpha, \beta \in (0,1)$, $\alpha + \beta > 1$, $\alpha/\beta \notin \mathbb{Q}$, $b, \lambda, a_k \in \mathbb{C}$ for $k = 1, \dots, n$, and $n = [1/\alpha]$ is the integer part of $1/\alpha$. For $p = 2$ this can be seen as corresponding, via the Fourier transform, to a finite interval delay equation, involving shifts in opposite directions in the time domain.

By Theorem 5.6, $A_{g_\lambda}^\theta$ is invertible, or Fredholm, if and only if the same holds for T_{G_λ} with

$$G_\lambda = \begin{pmatrix} e_{-1} & 0 \\ g_\lambda & e_1 \end{pmatrix}.$$

For $\lambda \neq 0$, T_{G_λ} is invertible by Theorem 5.1 in [13]. For $\lambda = 0$ we have $G_\lambda H_+ = H_-$ with $H_\pm \in (H_\infty^\pm)^2$ given by

$$H_+ = \left(e_\beta, -e_{\alpha+\beta-1} \textstyle\sum_{k=1}^{n} (a_k e_{(k-1)\alpha})\right), \quad H_- = (e_{\beta-1}, b),$$

and by Theorem 5.3 in [12] it follows that $\dim\ker T_{G_0} = \infty$, so that T_{G_0} (and, consequently, $A_{g_0}^\theta$) is not Fredholm. Since $A_{g_\lambda}^\theta = A_{g_0-\lambda}^\theta$, we conclude that

$$\sigma_{ess}(A_{g_0}^\theta) = \sigma(A_{g_0}^\theta) = \sigma_p(A_{g_0}^\theta) = \{0\}.$$

6. KERNELS OF ATTO WITH ANALYTIC SYMBOLS AND INVARIANT SUBSPACES

TTO have generated much interest, and so have T-kernels (kernels of Toeplitz operators) — see, for example [15], [26] and the references therein. We are therefore led to consider kernels of ATTO. If we do so, we immediately see that, given an inner function θ and any inner function α such that $\alpha \preceq \theta$, we have

$$\ker A_g^{\theta} \subset \ker A_g^{\alpha,\theta} \tag{6.1}$$

(see Figure 1).

More precisely,

$$\ker A_g^{\theta} = \ker A_g^{\alpha,\theta} \cap \ker B_g^{\alpha,\theta} \tag{6.2}$$

where all the spaces involved are kernels of ATTO of different kinds (considering that the TTO A_g^{θ} is a particular case of an ATTO).

Since, according to (6.2), $\ker A_g^{\alpha,\theta}$ is "bigger" than $\ker A_g^{\theta}$, it is natural to think that it may be simpler to characterize. Thus, determining the former can be seen as a first step towards determining the latter; the elements $\varphi_+ \in \ker A_g^{\theta}$ may then be singled out by adding the condition

$$B_g^{\alpha,\theta}\varphi_+ = 0.$$

This line of reasoning was used in [14] to study Toeplitz operators with 2×2 triangular matrix symbols with almost periodic entries.

By Theorems 5.6 and 5.2, $\ker A_g^{\alpha,\theta} \simeq \ker T_G$ where $g \in L_\infty$ and

$$G = \begin{pmatrix} \overline{\theta} & 0 \\ g & \alpha \end{pmatrix}. \tag{6.3}$$

Denoting by P_j the projection defined by

$$P_j(\varphi_1, \varphi_2) = \varphi_j \quad (j = 1, 2),$$

we have $\ker T_G \simeq P_1(\ker T_G)$. Indeed, $\varphi_+ = (\varphi_{1+}, \varphi_{2+}) \in \ker T_G$ if and only if we have

$$G\varphi_+ = \varphi_- \quad \text{with } \varphi_- \in (H_p^-)^2,$$

which is equivalent to

$$\overline{\theta}\varphi_{1+} = \varphi_{1-} \quad \text{and} \quad g\varphi_{1+} + \alpha\varphi_{2+} = \varphi_{2-}, \tag{6.4}$$

and it is clear from (6.4) that φ_{1+} uniquely defines $\varphi_{1-}, \varphi_{2+}$ and φ_{2-}, since we have

$$\varphi_{1-} = \overline{\theta}\varphi_{1+}, \quad \varphi_{2-} = P^-(g\varphi_{1+}) = 0, \quad \text{and} \quad \varphi_{2+} = -\overline{\alpha}(g\varphi_{1+}).$$

It is also easy to see that

$$\varphi_{1+} \in \ker A_{g_+}^{\alpha,\theta} \Leftrightarrow \varphi_{1+} \in P_1(\ker T_G), \tag{6.5}$$

i.e., the elements of $\ker A_{g_+}^{\alpha,\theta}$ are the first components of the elements of $\ker T_G$, where G is given by (6.3).

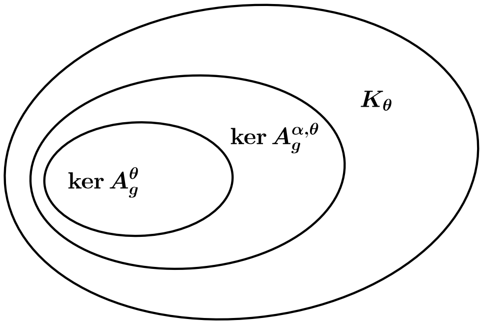

FIGURE 1.

Let us now consider asymmetric truncated Toeplitz operators with symbols in H_∞^+, of the form $A_{g_+}^{\alpha,\theta}$, where α and θ are inner functions such that $\alpha \preceq \theta$ and $g_+ \in H_\infty^+$.

In what follows recall that $K_{\alpha,\theta} = \alpha K_{\overline{\alpha}\theta}$, the shifted model space that is the image of the projection $P_{\alpha,\theta} = P_\theta - P_\alpha$, and that

$$K_\theta = K_\alpha \oplus K_{\alpha,\theta},$$

where the sum is orthogonal if $p = 2$. The next theorem shows that shifted model spaces are the kernels of ATTO with analytic symbols. First, however, we prove an auxiliary result.

LEMMA 6.1. *Given $g_+ \in H_\infty^+ \setminus \{0\}$ and an inner function θ,*

$$g_+\varphi_+ \in \theta H_p^+ \Leftrightarrow \varphi_+ \in \theta\overline{\beta}H_p^+$$

with $\beta = GCD(g_+^{\mathrm{i}}, \theta)$, where g_+^{i} is the inner factor of the inner-outer factorization $g_+ = g_+^{\mathrm{i}} g_+^{\mathrm{o}}$.

Proof. Let $g_+\varphi_+ = \theta\psi_+$ with $\psi_+ \in H_p^+$. Using the superscripts i and o to denote the inner and outer factors respectively, we have

$$g_+^{\mathrm{i}} g_+^{\mathrm{o}} \varphi_+^{\mathrm{i}} \varphi_+^{\mathrm{o}} = -\theta\psi_+^{\mathrm{i}}\psi_+^{\mathrm{o}},$$

so that $g_+^{\mathrm{i}}\varphi_+^{\mathrm{i}} = C\theta\psi_+^{\mathrm{i}}$ for some $C \in \mathbb{C}$ with $|C| = 1$. Dividing both sides of this equation by $\beta = \mathrm{GCD}(\theta, g_+^{\mathrm{i}})$ we obtain

$$\frac{g_+^{\mathrm{i}}}{\beta}\varphi_+^{\mathrm{i}} = C\frac{\theta}{\beta}\psi_+^{\mathrm{i}}$$

and since g_+^{i}/β and θ/β are relatively prime, it follows that θ/β divides φ_+^{i}; thus $\varphi_+ \in \theta\overline{\beta}H_p^+$. Conversely, if $\varphi_+ = \theta\overline{\beta}\psi_+$ with $\psi_+ \in H_p^+$, then $g_+\varphi_+ = (g_+^{\mathrm{i}}\overline{\beta})g_+^{\mathrm{o}}\theta\psi_+ \in \theta H_p^+$. ∎

THEOREM 6.2. *Let α and θ be inner functions with $\alpha \preceq \theta$, and suppose that $g_+ \in H_\infty^+ \setminus \{0\}$. Then* $\ker A_{g_+}^{\alpha,\theta} = K_{\gamma,\theta}$, *with $\gamma = \alpha/\beta$ where, denoting by g_+^{i} the inner factor in an inner-outer factorization of g_+, we have $\beta = \mathrm{GCD}(\alpha, g_+^{\mathrm{i}})$.*

Proof. We have $\varphi_{1+} \in \ker A_{g_+}^{\alpha,\theta}$ if and only if

$$\begin{pmatrix} \overline{\theta} & 0 \\ g_+ & \alpha \end{pmatrix} \begin{pmatrix} \varphi_{1+} \\ \varphi_{2+} \end{pmatrix} = \begin{pmatrix} \varphi_{1-} \\ \varphi_{2-} \end{pmatrix}, \tag{6.6}$$

where as usual $\varphi_j^\pm \in H_p^\pm$ for $j = 1, 2$. Thus $g_+\varphi_{1+} + \alpha\varphi_{2+} = \varphi_{2-} = 0$, and therefore $g_+\varphi_{1+} = -\alpha\varphi_{2+}$. By Lemma 6.1, we have $\varphi_{1+} \in \gamma H_p^+$ and thus $\varphi_{1+} \in \gamma H_p^+ \cap K_\theta = K_{\gamma,\theta}$.

Conversely, if $\varphi_{1+} \in K_{\gamma,\theta} \subset \gamma H_p^+$, then by Lemma 6.1 we have $g_+\varphi_{1+} \in \alpha H_p^+$, so that we can write $g_+\varphi_{1+} + \alpha\varphi_{2+} = \varphi_{2-}$ with $\varphi_{2+} \in H_p^+$ and $\varphi_{2-} = 0$. Hence (6.6) is satisfied and $\varphi_{1+} \in \ker A_{g_+}^{\alpha,\theta}$. ∎

COROLLARY 6.3. *Let α and θ be inner functions with $\alpha \preceq \theta$. Then $K_{\alpha,\theta} = \ker A_1^{\alpha,\theta}$ and $K_\theta = \ker A_\alpha^{\alpha,\theta}$.*

COROLLARY 6.4. *With the same assumptions as in Theorem* 6.2, *if $p = 2$ we have*

$$\ker A_{g_+}^{\alpha,\theta} = K_\theta \ominus K_\gamma = \gamma H_2^+ \ominus \theta H_2^+.$$

This holds, in particular for the TTO $A_{g_+}^\theta$, where $\alpha = \theta$, in which case we have ([22])

$$\ker A_{g_+}^\theta = \frac{\theta}{\beta} H_2^+ \ominus \theta H_2^+.$$

Moreover, for all $p \in (1, \infty)$, we have the following.

COROLLARY 6.5. *With the same assumptions as in Theorem* 6.2 *we have the following:*

(i) $A_{g_+}^{\alpha,\theta} = 0$ *if and only if $g_+ \in \alpha H_\infty^+$;*

(ii) $A_{g_+}^{\alpha,\theta}$ *is injective if and only if $\alpha = \theta$ and β is a constant;*

(iii) $\dim\ker A_{g_+}^{\alpha,\theta} < \infty$ *if and only if $\overline{\alpha}\theta$ and β are finite Blaschke products and, in that case,* $\dim\ker A_{g_+}^{\alpha,\theta} = n_1 + n_2$ *where n_1 and n_2 are the number of zeroes of $\overline{\alpha}\theta$ and β, respectively.*

(iv) *for $\alpha = \theta$,* $\dim\ker A_{g_+}^\theta < \infty$ *if and only if β is a finite Blaschke product and, in that case,* $\dim\ker A_{g_+}^\theta$ *is equal to the number of common zeroes of g_+^{i} and θ.*

As an immediate consequence we see that, in the particular case of the truncated shift with symbol r given by (1.9), we have $\ker A_r^\theta = \{0\}$ if $\theta(\mathrm{i}) \neq 0$, and $\ker A_r^\theta = (\theta/r)K_r = \mathrm{span}\{\theta/(\xi - \mathrm{i})\}$ if $\theta(\mathrm{i}) = 0$.

Shifted model spaces are also associated with ATTO in a different way: they are the (closed) invariant subspaces of the truncated shift A_r^θ.

THEOREM 6.6. *The lattice* $\mathrm{Lat}(A_r^\theta)$ *consists of the spaces $K_{\alpha,\theta}$, where $\alpha \preceq \theta$.*

Proof. For $\alpha \preceq \theta$ and $\beta = \theta\overline{\alpha}$, we have $K_{\alpha,\theta} = \alpha K_\beta$; let k^+ be any function in K_β. Then $k^+ = P_\beta \varphi_+$ for some $\varphi_+ \in H_p^+$ and

$$P_\theta r(\alpha k^+) = P_\theta r(\alpha P_\beta \varphi_+) = P_\theta r P_\theta \alpha \varphi_+ = P_\theta r \alpha \varphi_+ = \alpha P_\beta (r\varphi_+) \in \alpha K_\beta.$$

Thus every space $K_{\alpha,\theta}$ is invariant for A_r^θ. To show the converse, we begin with the observation that for the Hardy space $H_p(\mathbb{D})$ of the unit disc, we have a version of Beurling's theorem for each $1 < p < \infty$; namely that the nontrivial invariant subspaces of the shift T_z are all of the form αH_p for some inner function α. See, for example, Corollary C.2.1.20 of [23]. By means of the standard isometric isomorphism between $H_p(\mathbb{D})$ and H_p^+ given in (2.1) we see that the same result holds for the shift T_r on H_p^+.

Next, using the duality between H_p^+ and H_q^+ (up to isomorphism), we see that the T_r^*-invariant subspaces in H_q^+ are the annihilators of the invariant subspaces for T_r, i.e., the model spaces

$$K_\alpha^q = \left\{ f \in H_q^+ : \int_{\mathbb{R}} f\overline{g} = 0 \;\forall g \in \alpha H^p \right\} = \alpha H_q^- \cap H_q^+.$$

Now if A_r^θ is a restricted shift on H_p^+, then its Banach space adjoint is the restriction of T_r^* to its invariant subspace K_θ^q, so that its adjoint has invariant subspaces K_α^q where $\alpha \preceq \theta$.

Using duality once more we conclude that the invariant subspaces of A_r^θ take the form

$$\left\{ f \in K_\theta^p : \int_{\mathbb{R}} f\overline{g} = 0 \;\forall g \in K_\alpha^q \right\} = K_\theta^p \cap \alpha H^p = K_{\alpha,\theta},$$

where $\alpha \preceq \theta$. ∎

COROLLARY 6.7. $\mathrm{Lat}(A_r^\theta) = \{\ker A_{g_+}^{\alpha,\theta} : \alpha \preceq \theta, g_+ \in H_\infty^+\}$.

We may now prove a theorem of Lax–Beurling flavour for the "truncated shift" semigroup on K_θ given by $T(t) = A_{e_t}^\theta$, $(t \geqslant 0)$, where $e_t \in H_\infty^+$ is the inner function given by $e_t(\xi) = \mathrm{e}^{\mathrm{i}t\xi}$.

THEOREM 6.8. *The common invariant subspaces of the semigroup $(T(t))_{t\geqslant 0}$ are the shifted model spaces $K_{\alpha,\theta}$, where $\alpha \preceq \theta$.*

Proof. It is easy to see that these subspaces are all invariant under the semigroup, since if α divides a function $f \in K_\theta$ then it also divides $T(t)f$.

The converse is proved as in Theorem 3.1.5 of [25], the standard Lax –Beurling theorem. By writing

$$\frac{1}{\xi + \mathrm{i}} = \frac{1}{\mathrm{i}} \int_0^\infty \mathrm{e}^{-t} \mathrm{e}^{\mathrm{i}t\xi} \mathrm{d}t,$$

and approximating the integral by Riemann sums, we see that the ATTO operator with symbol $1/(\xi + \mathrm{i})$ is the strong limit of a sequence of finite linear combinations of the ATTO with symbols e_t. Hence any closed subspace invariant under the semigroup is also invariant under A_r^θ, and thus is a shifted model space, as required. ∎

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