J. OPERATOR THEORY
78:2(2017), 357–416
doi: 10.7900/jot.2016sep15.2128

STRUCTURE FOR REGULAR INCLUSIONS. I

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Dedicated to the memory of William B. Arveson

Communicated by Stefaan Vaes

ABSTRACT. We give general structure theory for pairs $(\mathcal{C}, \mathcal{D})$ of unital C^*-algebras where $\mathcal{D}$ is a regular and abelian C^*-subalgebra of $\mathcal{C}$.

When $\mathcal{D}$ is maximal abelian in $\mathcal{C}$, we prove existence and uniqueness of a completely positive unital map E of $\mathcal{C}$ into the injective envelope $I(\mathcal{D})$ of $\mathcal{D}$ such that $E|_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}$; E is a useful replacement for a conditional expectation when no expectation exists. When E is faithful, $(\mathcal{C}, \mathcal{D})$ has numerous desirable properties: e.g. the linear span of the normalizers has a unique minimal C^*-norm; $\mathcal{D}$ norms $\mathcal{C}$; and isometric isomorphisms of norm-closed subalgebras lying between $\mathcal{D}$ and $\mathcal{C}$ extend uniquely to their generated C^*-algebras.

KEYWORDS: *Inclusions of C^*-algebras, pseudo-expectation, regular homomorphism.*

MSC (2010): 46L05, 46L30, 47L30.

1. INTRODUCTION AND PRELIMINARIES

An *inclusion* is a pair $(\mathcal{C}, \mathcal{D})$ of unital C^*-algebras (with the same unit) where $\mathcal{D}$ is abelian and $\mathcal{D} \subseteq \mathcal{C}$. If the set of *normalizers*,

$$\mathcal{N}(\mathcal{C}, \mathcal{D}) := \{v \in \mathcal{C} : v\mathcal{D}v^* \cup v^*\mathcal{D}v \subseteq \mathcal{D}\}$$

has dense linear span in $\mathcal{C}$, $(\mathcal{C}, \mathcal{D})$ is a *regular inclusion*. Regular inclusions arise naturally in a wide variety of contexts. Here are several classes of examples:

(a) Any MASA $\mathcal{D}$ in a finite-dimensional C^*-algebra $\mathcal{C}$ yields a regular inclusion $(\mathcal{C}, \mathcal{D})$.

(b) If $\mathcal{M}$ is a von Neumann algebra and $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$, let $\mathcal{C}$ be the closed linear span of $\mathcal{N}(\mathcal{M}, \mathcal{D})$. Then $(\mathcal{C}, \mathcal{D})$ is a regular inclusion.

(c) Let X be a compact Hausdorff space, let Γ be a discrete group and $t \mapsto \alpha_t$ a homomorphism of Γ into the group of homeomorphisms of X. Let (π, U) be a covariant representation of the discrete dynamical system (X, Γ) and let $\mathcal{C}$ be the C^*-algebra generated by $\pi(C(X))$ and $\mathcal{D} = \pi(C(X))$. Then $(\mathcal{C}, \mathcal{D})$ is a regular inclusion.

(d) Let E be a row-finite directed graph with a finite vertex set, let $\mathcal{C} = C^*(E)$ and let $\mathcal{D}$ be the C^*-algebra generated by $\{S_\mu S_\mu^* : \mu \text{ is a finite path in } E\}$ (the notation is as in [31]). Then $(\mathcal{C}, \mathcal{D})$ is a regular inclusion.

The primary purposes of this paper are: (a) to give a number of structural results for regular inclusions; (b) to introduce the concept of *pseudo-expectation*, which is a technical tool useful when no conditional expectation is present; (c) to introduce a class of regular inclusions, the *virtual Cartan inclusions*, which provide a context for a number of results found in the literature; and (d) to give a setting in which we can establish the existence of maximal and minimal C^*-norms on the linear span of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ and explore some consequences of this.

Much of the work presented here was announced in our 2012 preprint, [28]. In some cases, the results of [28] have been extended or their proofs streamlined. We have also emphasized the role of Frolík's theorem and added a number of new results, particularly those related to minimal norms in Section 7 below. Due to length considerations, the results from [28] not appearing here will appear elsewhere.

In order to describe our results and put them into context, it will be helpful to list the principal types of inclusions we shall consider.

DEFINITION 1.1. The (not necessarily regular) inclusion $(\mathcal{C}, \mathcal{D})$ is a

MASA inclusion	if $\mathcal{D}$ is a MASA in $\mathcal{C}$;
EP-inclusion	if $\mathcal{D}$ has the extension property relative to $\mathcal{C}$, that is, every pure state σ on $\mathcal{D}$ has a unique extension to a state on $\mathcal{C}$;
virtual Cartan inclusion	if $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion such that the only closed, two-sided ideal J of $\mathcal{C}$ satisfying $J \cap \mathcal{D} = (0)$ is $J = (0)$;
Cartan inclusion	if $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion and there exists a faithful conditional expectation $E : \mathcal{C} \to \mathcal{D}$;
C^*-diagonal	if $(\mathcal{C}, \mathcal{D})$ is an EP-inclusion and also a Cartan inclusion.

In an important paper, Feldman and Moore [14] showed that when $\mathcal{D} \simeq L^\infty(X, \mu)$ is a Cartan MASA in a separably acting von Neumann algebra $\mathcal{M}$, there is a Borel equivalence relation $R \subseteq X \times X$ and a 2-cocycle c on R such that $\mathcal{M}$ is isomorphic to an algebra $M(R, c)$ consisting of certain measurable functions on R and $\mathcal{D}$ is isomorphic to the algebra $D(R, c)$ of functions supported on the diagonal $\{(x, x) : x \in R\}$ of R. The multiplication in $M(R, c)$ is essentially matrix multiplication twisted by the cocycle c. Feldman and Moore further show that the family of isomorphism classes of pairs $(\mathcal{M}, \mathcal{D})$ with $\mathcal{D}$ a Cartan MASA in the separably acting von Neumann algebra $\mathcal{M}$ is in bijective correspondence with the family of equivalence classes of pairs (R, c) where c is a 2-cocycle on the measured equivalence relation R.

The attractive character of the Feldman–Moore work led Kumjian to study a C^*-algebraic version of the Feldman–Moore formalism in [24]. In that article, Kumjian introduced the notion of a C^*-diagonal, as well as the notion of regularity reproduced above. (While the axioms for a C^*-diagonal in Definition 1.1 differ from those originally given by Kumjian, they are equivalent, aside from the fact that we do not require C^*-algebras to be separable, nor topological spaces to be second countable.) Roughly speaking, Kumjian showed there is a bijection between C^*-diagonals and certain families of twisted groupoids over a topological equivalence relation. This provided a satisfying parallel to the von Neumann algebraic context.

The requirement of the extension property in the axioms for a C^*-diagonal is at times too stringent, which is one of the advantages of Cartan inclusions (which need not have the extension property). For example, let $\mathcal{H} = \ell^2(\mathbb{N})$, with the usual orthonormal basis $\{e_n\}$, and let S be the unilateral shift, $Se_n = e_{n+1}$. Let $\mathcal{C} := C^*(S)$ be the Toeplitz algebra, and let $\mathcal{D} = C^*(\{S^n S^{*n} : n \geqslant 0\})$. Routine arguments show this is a Cartan inclusion, but the state $\rho_\infty(T) = \lim_{n\to\infty} \langle Te_n, e_n\rangle$ on $\mathcal{D}$ fails to have a unique extension to a state on $\mathcal{C}$.

Cartan inclusions were introduced by Renault in [32], in which he showed that if $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion (again with the separability and second countability hypotheses), then there is a satisfactory bijection between Cartan inclusions and certain twisted groupoids. In this paper, Renault makes a convincing case that Cartan inclusions are the appropriate analog of the Feldman–Moore setting in the C^*-context.

Crossed product constructions can be used to provide a supply of examples of C^*-diagonals and Cartan inclusions. When (X, Γ) is a discrete dynamical system with Γ acting freely on X, the inclusion

$$(C(X) \rtimes_{\mathrm{r}} \Gamma, C(X))$$

where $C(X) \rtimes_{\mathrm{r}} \Gamma$ is the reduced crossed product, is a C^*-diagonal. When the group Γ acts topologically freely on X, then $(C(X) \rtimes_{\mathrm{r}} \Gamma, C(X))$ is a Cartan inclusion, but is in general not a C^*-diagonal due to the fact that the extension property may fail.

Two notions closely related to our work appear in [25]. In that paper, Nagy and Reznikoff define the inclusion $(\mathcal{C}, \mathcal{D})$ to be a *pseudo-diagonal* if the set of pure states on $\mathcal{D}$ which have unique state extensions to $\mathcal{C}$ is weak*-dense in $\widehat{\mathcal{D}}$, and there exists a (necessarily unique) faithful conditional expectation $E : \mathcal{C} \to \mathcal{D}$. The second notion from [25] is that of an *abelian core*, which is a MASA inclusion $(\mathcal{C}, \mathcal{D})$ such that there exists a unique conditional expectation $E : \mathcal{C} \to \mathcal{D}$, E is faithful, and (0) is the only ideal of $\mathcal{C}$ which has trivial intersection with $\mathcal{D}$. Note that neither definition assumes regularity of $(\mathcal{C}, \mathcal{D})$.

A key technical tool for the results of Feldman–Moore, Kumjian, Nagy–Reznikoff, and Renault is the existence of a faithful conditional expectation. Unfortunately, conditional expectations do not always exist, even when $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, and the C^*-algebras involved are well-behaved. Here is a simple example, which is a special case of the far more general setting considered in Subsection 6.1.

EXAMPLE 1.2. Let X be a connected, compact Hausdorff space, and let $\alpha : X \to X$ be a homeomorphism such that α^2 is the identity map on X. Let F° be the interior of the set of fixed points for α; we assume that F° is neither empty nor all of X. (For a concrete example, take $X = \{z \in \mathbb{C} : |z| \leqslant 1 \text{ and } \operatorname{Re}(z)\operatorname{Im}(z) = 0\}$ and let $\alpha(z) = \overline{z}$.) Define $\theta : C(X) \to C(X)$ by $\theta(f) = f \circ \alpha^{-1}$, and set

$$\mathcal{C} := \left\{ \begin{pmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{pmatrix} : f_0, f_1 \in C(X) \right\} \quad \text{and} \quad \mathcal{D} := \left\{ \begin{pmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{pmatrix} : f_0 \in C(X) \right\}.$$

Then $\mathcal{C}$ is a C^*-subalgebra of $M_2(C(X))$, and $(\mathcal{C}, \mathcal{D})$ is a regular inclusion. (In fact, $\mathcal{C}$ is isomorphic to $C(X) \rtimes (\mathbb{Z}/2\mathbb{Z})$.)

A calculation shows that the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{C}$ is

$$\mathcal{D}^c = \left\{ \begin{pmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{pmatrix} \in \mathcal{C} : \operatorname{supp}(f_1) \subseteq F^\circ \right\}.$$

As $F^\circ \notin \{\emptyset, X\}$, we have $\mathcal{D} \subsetneq \mathcal{D}^c \subsetneq \mathcal{C}$, and another calculation shows $\mathcal{D}^c$ is abelian. Since $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathcal{N}(\mathcal{C}, \mathcal{D}^c)$, it follows that $(\mathcal{C}, \mathcal{D}^c)$ is a regular MASA inclusion.

Suppose $E : \mathcal{C} \to \mathcal{D}^c$ is a conditional expectation. Then for some $f_0, f_1 \in C(X)$ with $\operatorname{supp}(f_1) \subseteq F^\circ$, we have $E\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_0 & f_1 \\ \theta(f_1) & \theta(f_0) \end{smallmatrix}\right)$. Notice $\theta(f_1) = f_1$ and $\left(\begin{smallmatrix} 0 & f_1 \\ f_1 & 0 \end{smallmatrix}\right) \in \mathcal{D}^c$. We have

$$\begin{pmatrix} f_1^2 & f_1\theta(f_0) \\ f_1 f_0 & f_1^2 \end{pmatrix} = \begin{pmatrix} 0 & f_1 \\ f_1 & 0 \end{pmatrix} E \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = E\left(\begin{pmatrix} 0 & f_1 \\ f_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right) = \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix},$$

so $f_1^2 = f_1$. As X is connected, this yields $f_1 = 0$ or $f_1 = I$. But $\operatorname{supp}(f_1) \subseteq F^\circ \neq X$, so $f_1 = 0$. Thus $E(\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{smallmatrix}\right)$.

Now if $g_1 \in \mathcal{D}$ is such that $\operatorname{supp}(g_1) \subseteq F^\circ$, then $\theta(g_1) = g_1$, so $\left(\begin{smallmatrix} 0 & g_1 \\ g_1 & 0 \end{smallmatrix}\right) \in \mathcal{D}^c$. Thus,

$$\begin{aligned} \begin{pmatrix} 0 & g_1 \\ g_1 & 0 \end{pmatrix} &= E\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right) = \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} E \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} g_1 & 0 \\ 0 & g_1 \end{pmatrix} \begin{pmatrix} f_0 & 0 \\ 0 & \theta(f_0) \end{pmatrix} = \begin{pmatrix} g_1 f_0 & 0 \\ 0 & \theta(g_1 f_0) \end{pmatrix}. \end{aligned}$$

Hence $g_1 = 0$ for every such g_1. This implies that $F^\circ = \emptyset$, contrary to hypothesis. Hence no conditional expectation of $\mathcal{C}$ onto $\mathcal{D}^c$ exists.

While conditional expectations may fail to exist for a regular MASA inclusion, there is a map which often may be used as a replacement. Here is the relevant definition. (See page 365 below for a brief discussion of injective envelopes.)

DEFINITION 1.3. Let $(\mathcal{C},\mathcal{D})$ be an inclusion and let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$. A *pseudo-expectation for* ι is a unital completely positive map $E:\mathcal{C}\to I(\mathcal{D})$ such that $E|_{\mathcal{D}}=\iota$. When the context is clear, we sometimes drop the reference to ι and simply call E a pseudo-expectation. Denote the set of all pseudo-expectations for ι by $\operatorname{PsExp}(\mathcal{C},\mathcal{D},\iota)$ or more simply by $\operatorname{PsExp}(\mathcal{C},\mathcal{D})$.

The existence of pseudo-expectations follows immediately from the injectivity of $I(\mathcal{D})$. In general, $\operatorname{PsExp}(\mathcal{C},\mathcal{D})$ is a very large set. However, in Section 3 below, we show that for any regular MASA inclusion $(\mathcal{C},\mathcal{D})$, there is always a UNIQUE pseudo-expectation $E:\mathcal{C}\to I(\mathcal{D})$, see Theorem 3.5. Actually, we establish uniqueness of the pseudo-expectation for a larger class of inclusions, the *skeletal MASA inclusions*. The inclusion $(\mathcal{C},\mathcal{D})$ is a skeletal MASA inclusion if there is a $*$-monoid $\mathcal{M}\subseteq\mathcal{N}(\mathcal{C},\mathcal{D})$ such that $\overline{\operatorname{span}}\mathcal{M}=\mathcal{C}$ and $\mathcal{D}$ is a MASA in the linear span of $\mathcal{M}$. This generality is useful when considering certain common constructions. For example, if Γ is a discrete group which acts topologically freely on the compact Hausdorff space X, then the inclusion $(C(X)\rtimes_{\text{full}}\Gamma, C(X))$, is a skeletal MASA inclusion, but we do not know whether it is a regular MASA inclusion.

Let $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ be the family of all states on $\mathcal{C}$ whose restriction to $\mathcal{D}$ belongs to $\widehat{\mathcal{D}}$. When $(\mathcal{C},\mathcal{D})$ is a skeletal MASA inclusion, the family of states, $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D}):=\{\rho\circ E:\rho\in\widehat{I(\mathcal{D})}\}$ covers $\widehat{\mathcal{D}}$ in the sense that the restriction map, $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})\ni\rho\mapsto\rho|_{\mathcal{D}}\in\widehat{\mathcal{D}}$, is onto. Interestingly, $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$ is the unique minimal closed subset of $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ which covers $\widehat{\mathcal{D}}$, see Theorem 3.12. We also show that $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$ is closely related to the extension property. When $(\mathcal{C},\mathcal{D})$ is "countably generated", Theorem 3.12 also shows that $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$ is the closure of all states in $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ whose restrictions to $\mathcal{D}$ extend uniquely to $\mathcal{C}$.

For a skeletal MASA inclusion $(\mathcal{C},\mathcal{D})$, the intersection of the left kernels of the states in $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$ is the left kernel of the pseudo-expectation E, which we denote by $\mathcal{L}(\mathcal{C},\mathcal{D})$. Theorem 3.15 shows that $\mathcal{L}(\mathcal{C},\mathcal{D})$ is an ideal of $\mathcal{C}$, and moreover, is the unique ideal of $\mathcal{C}$ which is maximal with respect to the property of having trivial intersection with $\mathcal{D}$. The quotient of $\mathcal{C}$ by $\mathcal{L}(\mathcal{C},\mathcal{D})$ contains a canonical copy of $\mathcal{D}$, and Theorem 6.2 shows that when $\mathcal{D}^{\mathrm{c}}$ is the relative commutant of $\mathcal{D}$ in $\mathcal{C}/\mathcal{L}(\mathcal{C},\mathcal{D})$, the inclusion $(\mathcal{C}/\mathcal{L}(\mathcal{C},\mathcal{D}),\mathcal{D}^{\mathrm{c}})$ is a virtual Cartan inclusion. When the pseudo-expectation takes values in $\mathcal{D}$ (rather than $I(\mathcal{D})$), $\mathcal{D}^{\mathrm{c}}=\mathcal{D}$, and the ideal $\mathcal{L}(\mathcal{C},\mathcal{D})$ may be viewed as a measure of the failure of the inclusion to be Cartan in Renault's sense. Furthermore, it follows from Theorem 6.1 that a regular MASA inclusion is a virtual Cartan inclusion if and only if $\mathcal{L}(\mathcal{C},\mathcal{D})=(0)$, that is, when the pseudo-expectation is faithful.

As in Example 1.2 above, crossed products may be used to construct a variety of virtual Cartan inclusions. Theorem 6.14 shows that when $\mathcal{C}$ is the reduced crossed product of the abelian C^*-algebra $\mathcal{D}$ by a discrete group Γ, then, provided the relative commutant $\mathcal{D}^{\mathrm{c}}$ of $\mathcal{D}$ in $\mathcal{C}$ is abelian, $(\mathcal{C},\mathcal{D}^{\mathrm{c}})$ is a virtual Cartan inclusion. We characterize when $\mathcal{D}^{\mathrm{c}}$ is abelian in terms of the dynamics of the action of Γ on $\widehat{\mathcal{D}}$ in Theorem 6.11; this result shows that $\mathcal{D}^c$ is abelian precisely when the *germ isotropy* subgroup H^x of Γ is abelian for every $x \in \widehat{\mathcal{D}}$.

As mentioned earlier, virtual Cartan inclusions have a number of desirable properties. Here are some examples; throughout this list, $(\mathcal{C}, \mathcal{D})$ is a virtual Cartan inclusion.

(a) Theorem 8.2 shows that $\mathcal{D}$ norms $\mathcal{C}$ in the sense of Pop, Sinclair and Smith [30].

(b) Theorem 7.4 implies that the norm on $\mathcal{C}$ is the minimal C^*-norm on the linear span of a suitable subset (which we call a skeleton in Definition 1.7) of $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$.

(c) Theorem 7.4 also implies that if $\mathcal{M}$ is a skeleton for $(\mathcal{C}, \mathcal{D})$ and $\operatorname{span}(\mathcal{M})$ is completed with respect to any C^*-norm η to produce the C^*-algebra $\mathcal{C}_\eta$, then any ideal $J \subseteq \mathcal{C}_\eta$ with $J \cap \mathcal{D} = (0)$ lies in the kernel of the quotient mapping of $\mathcal{C}_\eta$ onto $\mathcal{C}$. These ideas have antecedents in the theory of crossed products. Indeed, a result of Archbold and Spielberg ([4], Theorem 1) shows that when Γ is a discrete group acting topologically freely on the compact Hausdorff space X and $J \subseteq C(X) \rtimes_{\text{full}} \Gamma$ is an ideal having trivial intersection with $C(X)$, then J is contained in the kernel of the canonical quotient map of $C(X) \rtimes_{\text{full}} \Gamma$ onto $C(X) \rtimes_{\text{red}} \Gamma$. Corollary 7.9 shows that the Archbold–Spielberg result fits into our context.

(d) Theorem 5.7 implies that there exists a C^*-diagonal $(\mathcal{C}_1, \mathcal{D}_1)$ and a one-to-one $*$-homomorphism $\alpha : \mathcal{C} \to \mathcal{C}_1$ such that $\alpha(\mathcal{N}(\mathcal{C}, \mathcal{D})) \subseteq \mathcal{N}(\mathcal{C}_1, \mathcal{D}_1)$. Thinking of $(\mathcal{C}_1, \mathcal{D}_1)$ as "enlarging" $(\mathcal{C}, \mathcal{D})$, this shows that in particular, one can enlarge $(\mathcal{C}, \mathcal{D})$ so that the extension property holds.

(e) It follows from Theorem 8.3 that if $\mathcal{A}$ is a norm-closed (not necessarily self-adjoint) subalgebra of $\mathcal{C}$ containing $\mathcal{D}$, then the C^*-envelope of $\mathcal{A}$ is the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$.

(f) If $(\mathcal{C}_i, \mathcal{D}_i)$ are virtual Cartan inclusions and $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$ are norm-closed subalgebras, Theorem 8.4 shows that any isometric isomorphism $\Theta : \mathcal{A}_1 \to \mathcal{A}_2$ uniquely extends to a $*$-isomorphism of the C^*-subalgebras of $\mathcal{C}_i$ generated by $\mathcal{A}_i$. Theorem 8.4 generalizes ([27], Theorem 2.16) to the context of virtual Cartan inclusions and allows for simplification of some arguments in the literature.

A corollary of Theorem 7.4 shows that when $\mathcal{M}$ is a skeleton for the virtual Cartan inclusion $(\mathcal{C}, \mathcal{D})$, there are minimal and maximal C^*-norms on $\operatorname{span}(\mathcal{M})$. These ideas lead to an interesting property for any regular EP-inclusion $(\mathcal{C}, \mathcal{D})$: Theorem 7.7 shows that if u is any C^*-semi-norm on the linear span of $\mathcal{N}(\mathcal{C}, \mathcal{D})$, the resulting inclusion obtained by completing the quotient of $\operatorname{span} \mathcal{N}(\mathcal{C}, \mathcal{D})$ by the null space of u is also a regular EP-inclusion.

Our work can be used to give an interpretation of certain uniqueness theorems for graph C^*-algebras. Using notation found in [31], if E is a row-finite directed graph (with finite vertex set), the set

$$\mathcal{M} := \{S_\mu : \mu \text{ is a finite path in } E\}$$

is a skeleton for $(C^*(E), \mathcal{D})$, where as above, $\mathcal{D}$ is the C^*-algebra generated by $\{S_\mu S_\mu^* : \mu \text{ is a finite path in } E\}$. The Gauge-invariant uniqueness theorem may then be interpreted as saying that $(C^*(E), \mathcal{D})$ is a virtual Cartan inclusion and that the minimal and maximal norms on $\operatorname{span}(\mathcal{M})$ coincide.

Theorem 5.7 does more than show that a virtual Cartan inclusion embeds into a C^*-diagonal: it characterizes which inclusions embed into a C^*-diagonal. To prove this result, we introduce a new family

$$\mathfrak{S}(\mathcal{C}, \mathcal{D}) \subseteq \operatorname{Mod}(\mathcal{C}, \mathcal{D}),$$

which we call *compatible states*. When $(\mathcal{C}, \mathcal{D})$ is a regular MASA inclusion, $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq \mathfrak{S}(\mathcal{C}, \mathcal{D})$. The intersection of the left kernels of the states in $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is an ideal of $\mathcal{C}$, $\operatorname{Rad}(\mathcal{C}, \mathcal{D})$. Theorem 5.7, shows that the regular inclusion $(\mathcal{C}, \mathcal{D})$ regularly embeds in a C^*-diagonal if and only if $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = (0)$.

Here are brief descriptions of the sections of the paper.

In Section 1 we give preliminary results regarding projective spaces and injective envelopes. Given an abelian C^*-algebra A, we also observe that the following Boolean algebras are isomorphic: the regular open subsets of $\widehat{\mathcal{A}}$; the regular ideals of $\mathcal{A}$; and the projection lattice of the injective envelope of $\mathcal{A}$.

In Section 2 we study dynamics of regular inclusions. It is well-known that there is a map $v \mapsto \beta_v$ from $\mathcal{N}(\mathcal{C}, \mathcal{D})$ into the inverse semigroup of partial homeomorphisms of $\widehat{\mathcal{D}}$. The key new idea we introduce here is the idea of a Frolík decomposition of $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. For each $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, we use Frolík's theorem to associate a set $\{K_i\}_{i=0}^4$ of five pairwise disjoint regular ideals in $\mathcal{D}$, whose span is an essential ideal. The ideal K_0 is associated with the regularization of the interior of the set of fixed points for β_v, K_4 is associated with the complement of $\operatorname{dom} \beta_v$, and for $i = 1, 2, 3$, K_i is associated with "free" parts of β_v.

In Section 3 we use Frolík decompositions to demonstrate uniqueness of the pseudo-expectation for a skeletal MASA inclusion $(\mathcal{C}, \mathcal{D})$; we show the left kernel of the pseudo-expectation is a two-sided ideal which is maximal with respect to having trivial intersection with $\mathcal{D}$; we identify the multiplicative domain for the pseudo-expectation; and give some results regarding the unique extension of states.

In Section 4 we introduce the notion of compatible states for any inclusion $(\mathcal{C}, \mathcal{D})$ and develop properties of compatible states we need for results which come later in the paper. While compatible states exist in abundance for any regular MASA inclusion, Theorem 4.13 implies that compatible states need not exist for a general regular inclusion.

In Section 5 we characterize when a regular inclusion embeds into a regular MASA inclusion and also into a C^*-diagonal. A key idea here is the notion of the radical of an inclusion.

Section 6 is devoted to virtual Cartan inclusions. After discussing some of their general properties, we show how to construct a virtual Cartan inclusion from any skeletal MASA inclusion. We characterize when the regular inclusions arising from reduced crossed products by discrete groups are virtual Cartan inclusions in dynamical terms, see Theorem 6.15. The embedding results of Section 5 are used in this analysis.

In Section 7 we apply some of our previous results to show that there exist minimal and maximal norms on span $\mathcal{M}$, where $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a skeleton such that $\mathcal{D}$ is a MASA in span $\mathcal{M}$, and we show how some of the results in the literature regarding crossed products follow from our setting.

In Section 8 we show that for any virtual Cartan inclusion $(\mathcal{C}, \mathcal{D})$, $\mathcal{D}$ norms $\mathcal{C}$, and we show how this result can be applied to the problem of extending isometric isomorphisms of nonselfadjoint algebras to their C^*-envelopes.

1.1. PRELIMINARIES. In this subsection, we collect a few preliminary facts. We continue with two sub-subsections, which contain a number of results to be used in the sequel.

STANDING ASSUMPTION. *All C^*-algebras will be unital, and if $\mathcal{D}$ is a sub-C^*-algebra of the C^*-algebra $\mathcal{C}$, we assume that the unit for $\mathcal{D}$ is the same as the unit for $\mathcal{C}$.*

NOTATION 1.4. The following notation will be used throughout the paper.

(i) Given a Banach space $\mathcal{X}$, we will use $\mathcal{X}^{\#}$ instead of the traditional $\mathcal{X}^*$ to denote the Banach space dual. Likewise if $\alpha : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map between Banach spaces, we use $\alpha^{\#}$ to denote the adjoint map, $\mathcal{Y}^{\#} \ni f \mapsto f \circ \alpha \in \mathcal{X}^{\#}$.

(ii) If X is a topological space and $E \subseteq X$, E° (respectively $\overline{E}$) denotes the interior (respectively closure) of E. Occasionally, we write $\operatorname{int}(E)$ (respectively $\operatorname{cl}(E)$) instead of E° (respectively $\overline{E}$). Also, for $f : X \to \mathbb{C}$, we write $\operatorname{supp} f$ for the set $\{x \in X : f(x) \neq 0\}$.

(iii) For any subset S of the Banach space $\mathcal{X}$, $\operatorname{span}(S)$ is the linear span of S; we will always write $\overline{\operatorname{span}}(S)$ when referring to the closed linear span.

(iv) When (x_λ) is a bounded increasing net in the self-adjoint part, $\mathcal{C}_{\text{s.a.}}$, of the C^*-algebra $\mathcal{C}$, $\sup\limits_{\mathcal{C}} x_\lambda$ means the least upper bound of (x_λ) in $\mathcal{C}_{\text{s.a.}}$. Note that writing $x = \sup\limits_{\mathcal{C}} x_\lambda$ implicitly asserts the supremum exists in $\mathcal{C}_{\text{s.a.}}$.

(v) Let $\mathcal{C}$ be a C^*-algebra, and let $\mathcal{S}(\mathcal{C})$ be the state space of $\mathcal{C}$. For $\rho \in \mathcal{S}(\mathcal{C})$ let

$$L_\rho = \{x \in \mathcal{C} : \rho(x^*x) = 0\}$$

be the left kernel of ρ, and let $(\pi_\rho, \mathcal{H}_\rho, \xi_\rho)$ be the GNS representation corresponding to ρ. We regard $\mathcal{C}/L_\rho$ as a dense subset of $\mathcal{H}_\rho$, and for $x \in \mathcal{C}$ will often write $x + L_\rho$ to denote the vector $\pi_\rho(x)\xi_\rho$. Denote the inner product on $\mathcal{H}_\rho$ by $\langle \cdot, \cdot \rangle_\rho$.

(vi) For any inclusion $(\mathcal{C}, \mathcal{D})$ we use $\mathcal{D}^{\mathrm{c}}$ to denote the relative commutant of $\mathcal{D}$ in $\mathcal{C}$, that is,

$$\mathcal{D}^{\mathrm{c}} := \{x \in \mathcal{C} : xd = dx \text{ for all } d \in \mathcal{D}\}.$$

LEMMA 1.5. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion, and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. The following statements hold:*

(i) *Let $d \in \mathcal{D}$. Then $v^*dv = dv^*v$ if and only if $vd = dv$.*

(ii) $\{d \in \mathcal{D} : vd = dv \in \mathcal{D}^c\} = \{d \in \mathcal{D} : v^*dhv = v^*vdh \text{ for all } h \in \mathcal{D}\}$.

Proof. (i) Let $d \in \mathcal{D}$. Suppose $v^*dv = dv^*v$. For every $n \in \mathbb{N}$,

$$0 = v^*dv - v^*vd = v^*(dv - vd) = vv^*(dv - vd) = (vv^*)^n(dv - vd).$$

Then for every polynomial p with $p(0) = 0$, $p(vv^*)(dv - vd) = 0$. Therefore, for every $n \in \mathbb{N}$,

$$0 = (vv^*)^{1/n}(dv - vd) = d(vv^*)^{1/n}v - (vv^*)^{1/n}vd.$$

Since $\lim_{n\to\infty}(vv^*)^{1/n}v = v$, we have $vd = dv$. As the reverse implication is obvious, part (i) holds.

Part (ii) follows directly from part (i). ∎

For any inclusion $(\mathcal{C}, \mathcal{D})$, the commutator of $\mathcal{D}$ in $\mathcal{C}$ will be denoted $[\mathcal{C}, \mathcal{D}]$; that is,

$$[\mathcal{C}, \mathcal{D}] := \overline{\operatorname{span}}\{cd - dc : c \in \mathcal{C},\, d \in \mathcal{D}\}.$$

The following observation of Kumjian follows from the fact that if $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ satisfies $v^2 = 0$, then $v(v^*v)^{1/n} - (v^*v)^{1/n}v = v(v^*v)^{1/n} \to v$.

LEMMA 1.6 (Kumjian, [24]). *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ be a free normalizer (i.e. $v^2 = 0$). Then $v \in [\mathcal{C}, \mathcal{D}]$.*

We have mentioned the notion of skeleton earlier. Before proceeding further, we give the formal definition.

DEFINITION 1.7. Given an inclusion $(\mathcal{C}, \mathcal{D})$, a *skeleton* for $(\mathcal{C}, \mathcal{D})$ is a $*$-semigroup $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that

(i) the linear span of $\mathcal{M}$ is dense in $\mathcal{C}$; and

(ii) $\mathcal{D} \subseteq \operatorname{span} \mathcal{M}$.

Note that when there exists a skeleton for an inclusion $(\mathcal{C}, \mathcal{D})$, then $(\mathcal{C}, \mathcal{D})$ is automatically a regular inclusion. Also, since $\mathcal{M}$ is a $*$-semigroup, $\operatorname{span} \mathcal{M}$ is a $*$-algebra. Evidently, $\mathcal{N}(\mathcal{C}, \mathcal{D})$ is an example of a skeleton for $(\mathcal{C}, \mathcal{D})$.

1.2. PROJECTIVE SPACES AND INJECTIVE ENVELOPES. In this subsection, we recall some facts about projective topological spaces, projective covers, and injective envelopes of C^*-algebras.

Following [18], given a compact Hausdorff space X, a pair (P, f) consisting of a compact Hausdorff space P and a continuous map $f : P \to X$ is called a *cover for X* (or simply a *cover*) if f is surjective. A cover (P, f) is *rigid* if the only continuous map $h : P \to P$ which satisfies $f \circ h = f$ is $h = \mathrm{id}_P$; the cover (P, f) is *essential* if whenever Y is a compact Hausdorff space, $h : Y \to P$ is continuous and satisfies $f \circ h$ is onto, then h is onto. Note that (P, f) is essential if and only if whenever $K \subseteq P$ is closed and $f(K) = X$, then $K = P$.

A compact Hausdorff space P is *projective* if whenever X and Y are compact Hausdorff spaces and $h : Y \to X$ and $f : P \to X$ are continuous maps with h

surjective, there exists a continuous map $g : P \to Y$ with $g \circ h = f$. A Hausdorff space which is extremally disconnected (i.e. the closure of every open set is open) and compact is *Stonean*. Gleason ([16], Theorem 2.5) proved that a compact Hausdorff space P is projective if and only if P is Stonean.

By Proposition 2.13 of [18], if (P, f) is a cover for X with P a projective space, then (P, f) is rigid if and only if (P, f) is essential. A *projective cover* for X is a rigid cover (P, f) for X such that P is projective. A projective cover for X always exists ([18], Theorem 2.16) and is unique in the sense that if (P_1, f_1) and (P_2, f_2) are projective covers for X, then there is a unique homeomorphism $h : P_1 \to P_2$ such that $f_1 = f_2 \circ h$.

Let $\mathfrak{O}$ be the category whose objects are operator systems and morphisms are completely positive (unital) maps. Recall that an operator system T is *injective* in $\mathfrak{O}$ if whenever R, S are operator systems with $S \subseteq R$ and $\phi : S \to T$ is a morphism, then there exists a morphism $\psi : R \to T$ such that $\psi|_S = \phi$. A C^*-algebra is *injective* if it is injective when viewed as an object in $\mathfrak{O}$.

Given a C^*-algebra $\mathcal{A}$, a pair $(\mathcal{B}, \sigma)$ consisting of a C^*-algebra $\mathcal{B}$ and a (unital) $*$-monomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ is called an *extension* of $\mathcal{A}$. Extensions $(\mathcal{B}_1, \sigma_1)$ and $(\mathcal{B}_2, \sigma_2)$ of $\mathcal{A}$ are *equivalent* if there exists a $*$-isomorphism $\tau : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\tau \circ \sigma_1 = \sigma_2$. The extension $(\mathcal{B}, \sigma)$ is *$\mathfrak{O}$-essential* if whenever $\mathcal{C}$ is a C^*-algebra and $\pi : \mathcal{B} \to \mathcal{C}$ is a completely positive unital map such that $\pi \circ \sigma$ is completely isometric on $\mathcal{A}$, then π is completely isometric on all of $\mathcal{B}$. Also, $(\mathcal{B}, \sigma)$ is *Hamana-regular* [21] if whenever $x \in \mathcal{B}$ is self-adjoint, x is the least upper bound of the set $\{\sigma(a) : a \in \mathcal{A}, a = a^* \text{ and } \sigma(a) \leqslant x\}$, where the least upper bound is taken in the self-adjoint part, $\mathcal{B}_{\text{s.a.}}$, of $\mathcal{B}$. When $\mathcal{B}$ is an injective C^*-algebra and the identity map on $\mathcal{B}$ is the unique completely positive linear map of $\mathcal{B}$ into itself which fixes $\sigma(\mathcal{A})$, then $(\mathcal{B}, \sigma)$ is an *injective envelope* of $\mathcal{A}$ (see [20]). Hamana shows that there is an injective object $I(\mathcal{A})$ in $\mathfrak{O}$ and a one-to-one completely positive $\iota : \mathcal{A} \to I(\mathcal{A})$ such that the extension $(I(\mathcal{A}), \iota)$ is rigid and $\mathfrak{O}$-essential. It follows from Theorem 3.1 of [10] that $I(\mathcal{A})$ is endowed with a product which makes it into a C^*-algebra (and ι a $*$-monomorphism). Thus, the pair $(I(\mathcal{A}), \iota)$ is an injective envelope for $\mathcal{A}$.

Like projective covers of compact Hausdorff spaces, injective envelopes of unital C^*-algebras have a uniqueness property. If $\mathcal{A}$ is a unital C^*-algebra, and $(\mathcal{B}_1, \sigma_1)$ and $(\mathcal{B}_2, \sigma_2)$ are injective envelopes for $\mathcal{A}$, then there exists a unique $*$-isomorphism $\theta : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\theta \circ \sigma_1 = \sigma_2$ ([20], Theorem 4.1).

Let $\mathfrak{C}$ be the category whose objects are unital abelian C^*-algebras and morphisms are $*$-homomorphisms. Then $\mathcal{A}$ is an injective object in $\mathfrak{C}$ if and only if $\mathcal{A}$ is an injective object in $\mathfrak{O}$, see Theorem 2.4 of [18]. (The statement of Theorem 2.4 in [18], mentions the category of operator systems without explicitly giving the morphisms, but the proof makes it clear that the authors mean the category of operator systems and unital, completely positive maps.) The concept of an injective envelope for an abelian unital C^*-algebras is dual to the concept of a projective cover of a compact Hausdorff space: if (P, f) is a cover for X, let $\iota : C(X) \to C(P)$

be the map $d \mapsto d \circ f$; then (P, f) is a projective cover if and only if $(C(P), \iota)$ is an injective envelope of $C(X)$ ([18], Corollary 2.18). These considerations lead to the following fact: in the category $\mathfrak{C}$, an extension $(\mathcal{B}, \sigma)$ of $\mathcal{A}$ is an injective envelope for $\mathcal{A}$ if and only if $\mathcal{B}$ is injective and $(\mathcal{B}, \sigma)$ is an *essential extension*, that is, whenever π is a $*$-homomorphism of $\mathcal{B}$ into the abelian C^*-algebra $\mathcal{C}$ such that $\pi \circ \sigma$ is faithful, then π is faithful.

Theorem 6.6 of [21] implies that if $\mathcal{A}$ is a unital, abelian C^*-algebra then an injective envelope $(I(\mathcal{A}), \iota)$ for $\mathcal{A}$ is Hamana-regular. When $x \in I(\mathcal{A})$ is positive, and $y \in \mathcal{A}_{\text{s.a.}}$ satisfies $\iota(y) \leqslant x$ then $\iota(y) \leqslant \iota(y_+) \leqslant x$, where $y_+ = 0 \vee y$. Thus,

$$x = \sup_{I(\mathcal{A})} \{\iota(a) : a \in \mathcal{A} \text{ and } 0 \leqslant \iota(a) \leqslant x\}.$$

It is important that $\mathcal{A}$ is abelian here, for Hamana has provided an example of a C^*-algebra $\mathcal{C}$, a Hamana-regular extension $(\mathcal{B}, \iota)$ of $\mathcal{C}$ with $\mathcal{B} \subseteq I(\mathcal{C})$, and a projection $0 \neq p \in \mathcal{B}$ such that $\{x \in \mathcal{C} : 0 \leqslant \iota(x) \leqslant p\} = \{0\}$, see Section 2 of [22].

Here is a description of a particular injective envelope of an abelian C^*-algebra. For details, see Theorem 1 of [17]. Let X be a compact Hausdorff space. Let $\mathfrak{B}(X)$ be the C^*-algebra of all bounded Borel complex-valued functions on X and let

$$\mathfrak{N} = \{f \in \mathfrak{B}(X) : f \text{ vanishes except on a set of first category}\}.$$

Then $\mathfrak{N}$ is an ideal in $\mathfrak{B}(X)$ and the quotient $\mathfrak{D}(X) := \mathfrak{B}(X)/\mathfrak{N}$ is called the *Dixmier algebra*. Define $j : C(X) \to \mathfrak{D}(X)$ by $j(f) = f + \mathfrak{N}$. Then $(\mathfrak{D}(X), j)$ is an injective envelope for $C(X)$. Because of this description, the injective envelope of $C(X)$ may be viewed as a topological analog of $L^\infty(X, \mu)$ where μ is a Borel measure on X with full support.

NOTATION 1.8. For a subset S of the abelian C^*-algebra $\mathcal{A}$, write

$$S^\perp := \{a \in \mathcal{A} : as = 0 \text{ for all } s \in \mathcal{S}\} \quad \text{and} \quad S^{\perp\perp} = (S^\perp)^\perp.$$

Notice that $S^\perp$ is an ideal in $\mathcal{A}$.

Let $\mathcal{A}$ be a unital abelian C^*-algebra and let $(I(\mathcal{A}), \iota)$ be an injective envelope for $\mathcal{A}$. For any ideal $\mathcal{J} \subseteq \mathcal{A}$, as $I(\mathcal{A})$ is an AW^*-algebra, there is a unique projection $Q \in I(\mathcal{A})$ such that $\iota(\mathcal{J})^{\perp\perp} = QI(\mathcal{A})$. We will sometimes call this projection the *support projection for $\mathcal{J}$ relative to ι*. When ι is understood, we will simply call Q the support projection of $\mathcal{J}$.

LEMMA 1.9. *Let $\mathcal{A}$ be an abelian C^*-algebra, let $\mathcal{J} \subseteq \mathcal{A}$ be a closed ideal, and let Q be the support projection for $\mathcal{J}$. If (u_λ) is an approximate unit for $\mathcal{J}$, then*

$$Q = \sup_{I(\mathcal{A})} \iota(u_\lambda).$$

For the proof, apply Lemma 1.1 of [23].

The following statement and its dual give further properties of Stonean spaces and injective abelian C^*-algebras.

LEMMA 1.10 ([36], Exercises 15G(1) and 19G(2)). *Suppose X is a Stonean space and $G \subseteq X$ is open. Then the Stone–Čech compactification βG of G is homeomorphic to $\overline{G}$.*

LEMMA 1.10A. *Suppose $\mathcal{A}$ is an abelian and injective C^*-algebra and $\mathcal{J} \subseteq \mathcal{A}$ is a closed ideal. Then the multiplier algebra $M(\mathcal{J})$ of $\mathcal{J}$ is $*$-isomorphic to $\mathcal{J}^{\perp\perp}$.*

The injective envelope has the following very useful property.

PROPOSITION 1.11. *Let $\mathcal{A}$ be a unital abelian C^*-algebra, and let $(I(\mathcal{A}), \iota)$ be an injective envelope for $\mathcal{A}$. For $k = 1, 2$, let $\mathcal{J}_k$ be closed ideals of $\mathcal{A}$ and let $Q_k \in I(\mathcal{A})$ be the support projection for $\mathcal{J}_k$. If $\alpha : \mathcal{J}_1 \to \mathcal{J}_2$ is a $*$-isomorphism, then there exists a unique $*$-isomorphism $\widetilde{\alpha} : Q_1 I(\mathcal{A}) \to Q_2 I(\mathcal{A})$ such that $\widetilde{\alpha} \circ \iota = \iota \circ \alpha$.*

Sketch of proof. For $k = 1, 2$, let $\mathcal{J}_k^+$ be the C^*-algebra generated by $\mathcal{J}_k \cup \{I\}$ and write $Q_k\iota$ for the map $x \mapsto Q_k\iota(x)$. Then $(Q_k I(\mathcal{A}), Q_k\iota)$ is an essential extension of $\mathcal{J}_k^+$. Since $Q_k I(\mathcal{A})$ is an injective algebra, $(Q_k I(\mathcal{A}), Q_k\iota)$ is an injective envelope for $\mathcal{J}_k^+$.

Clearly α extends to an isomorphism of $\mathcal{J}_1^+$ onto $\mathcal{J}_2^+$, so the result follows from the uniqueness property of injective envelopes. ∎

1.3. REGULAR IDEALS, REGULAR OPEN SETS, AND PROJECTIONS. We turn now to a brief discussion of the relationships between regular ideals in an abelian C^*-algebra $\mathcal{A}$, regular open sets in $\widehat{\mathcal{A}}$, and projections in an injective envelope for $\mathcal{A}$. These observations show that the Boolean algebras of regular ideals in $\mathcal{A}$ or the regular open sets of $\widehat{\mathcal{A}}$ in some sense determine $I(\mathcal{A})$. We leave many of the proofs to the reader.

Recall that an open subset G of the compact Hausdorff X space is a *regular open set* if G is the interior of its closure. It is well known (see [19]) that the family of regular open sets in X forms a complete Boolean algebra, $\mathrm{ROPEN}(X)$, under the operations,

$$G_1 \vee G_2 := \mathrm{int}(\mathrm{cl}(G_1 \cup G_2)), \quad G_1 \wedge G_2 := G_1 \cap G_2, \quad \text{and} \quad \neg G := \mathrm{int}(X \setminus G).$$

Note that if P is a Stonean space, $\mathrm{ROPEN}(P)$ is the collection of clopen subsets of P. As clopen subsets of P correspond to projections in $C(P)$ via the map

$$\mathrm{ROPEN}(P) \ni E \mapsto \chi_E \in \mathrm{PROJ}(C(P)),$$

we see that the Boolean algebras $\mathrm{ROPEN}(P)$ and the lattice of projections in $C(P)$, denoted $\mathrm{PROJ}(C(P))$, are isomorphic.

When (Y, g) is an essential cover for X, the map

$$\mathrm{ROPEN}(Y) \ni H \mapsto \mathrm{int}_X(g(\mathrm{cl}_Y(H))),$$

is a Boolean algebra isomorphism of $\mathrm{ROPEN}(Y)$ onto $\mathrm{ROPEN}(X)$.

An ideal $\mathcal{J}$ in the abelian C^*-algebra $\mathcal{A}$ is a *regular ideal* if $\mathcal{J}^{\perp\perp} = \mathcal{J}$. The regular ideals in $\mathcal{A}$ also form a complete Boolean algebra, denoted $\text{RIDEAL}(\mathcal{A})$, with the operations,

$$\mathcal{J}_1 \vee \mathcal{J}_2 := (\mathcal{J}_1 \cup \mathcal{J}_2)^{\perp\perp}, \quad \mathcal{J}_1 \wedge \mathcal{J}_2 := \mathcal{J}_1 \cap \mathcal{J}_2, \quad \text{and} \quad \neg\mathcal{J} := \mathcal{J}^{\perp}.$$

The map from $\text{RIDEAL}(\mathcal{A})$ to $\text{ROPEN}(\widehat{\mathcal{A}})$ given by

$$\mathcal{A} \supseteq \mathcal{J} \mapsto \{\rho \in \widehat{\mathcal{A}} : \rho|_{\mathcal{J}} \neq 0\},$$

is a Boolean algebra isomorphism.

Any injective abelian C^*-algebra is the closure of the linear span of its projections. So in some sense, the following fact shows how the injective envelope for an abelian C^*-algebra $\mathcal{A}$ may be obtained from very natural algebraic or topological objects associated to $\mathcal{A}$.

LEMMA 1.12. *Let $\mathcal{A}$ be an abelian C^*-algebra and let $(I(\mathcal{A}), \iota)$ be an injective envelope for $\mathcal{A}$. Then* $\text{RIDEAL}(\mathcal{A})$, $\text{ROPEN}(\widehat{\mathcal{A}})$ *and* $\text{PROJ}(I(A))$ *are isomorphic complete Boolean algebras (via the maps described above).*

2. DYNAMICS OF REGULAR INCLUSIONS

Given a regular inclusion $(\mathcal{C}, \mathcal{D})$, the $*$-semigroup $\mathcal{N}(\mathcal{C}, \mathcal{D})$ of normalizers acts via partial homeomorphisms on the maximal ideal space $\widehat{\mathcal{D}}$ of $\mathcal{D}$, and dually, acts on the family of closed ideals in $\mathcal{D}$. The purpose of this section is to discuss some of the features of these actions. A key tool is Frolík's theorem, which will allow us to decompose partial homeomorphisms and isomorphisms between ideals. We begin with some background facts regarding normalizers and a discussion of the partial action a normalizer determines. We then discuss regular homomorphisms, which are the correct morphisms for a category whose objects are regular inclusions, and give a useful example of a regular homomorphism. We utilize these ideas in the subsection to give a characterization of the extension property in terms of the dynamics associated with the action of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ on $\widehat{\mathcal{D}}$ (Theorem 2.20). Interesting consequences are a characterization of topological freeness for a discrete dynamical system (Proposition 2.17) and the fact that a regular inclusion $(\mathcal{C}, \mathcal{D})$ with $\mathcal{D}$ injective is an EP-inclusion if and only if it is a MASA inclusion, see Theorem 2.21.

2.1. NORMALIZERS, ISOMORPHISMS OF IDEALS, AND PARTIAL HOMEOMORPHISMS.

Recall that if $\mathcal{A}$ is an AW^*-algebra, then $\mathcal{A}$ has the *polar decomposition property*, that is, every $x \in \mathcal{A}$ factors as $x = u|x|$, where $u \in \mathcal{A}$ is a partial isometry such that

$$\mathcal{A}(I - uu^*) = \{a \in \mathcal{A} : ax = 0\} \quad \text{and} \quad (I - u^*u)\mathcal{A} = \{a \in \mathcal{A} : xa = 0\}$$

(use Corollary, p. 43; Theorem 1(iii), p. 129 and Proposition 2, p. 133 of [6]). The projections uu^* and u^*u are the smallest projections in $\mathcal{A}$ such that $uu^*x = x = xu^*u$. Thus, if $\mathcal{A}'$ is another AW^*-algebra with $\mathcal{A} \subseteq \mathcal{A}'$, it may be that the polar decomposition of x viewed as an element of $\mathcal{A}$ differs from the polar decomposition of x viewed as an element of $\mathcal{A}'$.

For an inclusion $(\mathcal{C}, \mathcal{D})$ it is not necessarily the case that $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ intertwines $\mathcal{D}$ in the sense that $v\mathcal{D} = \mathcal{D}v$. However, we now establish that v intertwines the (principal) ideals $\overline{vv^*\mathcal{D}}$ and $\overline{v^*v\mathcal{D}}$.

LEMMA 2.1. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $\mathcal{A}$ be an AW^*-algebra with $\mathcal{C} \subseteq \mathcal{A}$. Fix $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and let $v = |v^*|u$ be the factorization arising from the polar decomposition of v^* in $\mathcal{A}$. For $d \in \mathcal{D}$, $\|vv^*d\| = \|v^*dv\|$ and the map $vv^*d \mapsto v^*dv$ extends uniquely to a $*$-isomorphism θ_v of $\overline{vv^*\mathcal{D}}$ onto $\overline{v^*\mathcal{D}v} = \overline{v^*v\mathcal{D}}$. Furthermore, for every $h \in \overline{vv^*\mathcal{D}}$,*

$$v\theta_v(h) = hv \quad \textit{and} \quad \theta_v(h) = u^*hu. \tag{2.1}$$

Proof. Notice that for $d \in \mathcal{D}$,

$$dvv^* = uv^*dvu^*. \tag{2.2}$$

Therefore for $d \in \mathcal{D}$, $\|vv^*d\| = \|v^*dv\|$. For $d \in \mathcal{D}$, $v^*dv = \lim_{n\to\infty} (v^*v)^{1/n}v^*dv$, so $\overline{v^*\mathcal{D}v} \subseteq \overline{v^*v\mathcal{D}}$. If $\rho \in \widehat{\mathcal{D}}$ annihilates $v^*\mathcal{D}v$, then $\rho(v^*v) = 0$, so ρ annihilates $\overline{v^*v\mathcal{D}}$. Thus $\overline{v^*\mathcal{D}v} = \overline{v^*v\mathcal{D}}$.

As the map $vv^*d \mapsto v^*dv$ is a $*$-homomorphism, the existence and uniqueness of θ_v follows. The second equality in (2.1) follows by continuity and (2.2). The first equality in (2.1) is clear when $h \in vv^*\mathcal{D}$, and it follows for general $h \in \overline{vv^*\mathcal{D}}$ by continuity. ∎

For any topological space X, a *partial homeomorphism* is a homeomorphism $h : S \to R$, where S and R are open subsets of X. As usual, $\operatorname{dom}(h)$ and $\operatorname{ran}(h)$ will denote the domain and range of the partial homeomorphism h. We use $\operatorname{Inv}_{\mathcal{O}}(X)$ to denote the inverse semigroup of all partial homeomorphisms of X. When $\mathcal{S}$ is a $*$-semigroup, a semigroup homomorphism $\alpha : \mathcal{S} \to \operatorname{Inv}_{\mathcal{O}}(X)$ is a *$*$-homomorphism* if for every $s \in \mathcal{S}$, $\alpha(s^*) = \alpha(s)^{-1}$. A subset $\mathcal{G}$ of $\operatorname{Inv}_{\mathcal{O}}(X)$ which is closed under composition and inverses (i.e. a sub inverse semigroup) is sometimes called a *pseudo-group on X*.

Recall (see Proposition 6 of [24]) that a normalizer v determines a partial homeomorphism β_v with

$$\operatorname{dom}\beta_v = \{\sigma \in \widehat{\mathcal{D}} : \sigma(v^*v) > 0\} \quad \text{and} \quad \operatorname{range}\beta_v = \{\sigma \in \widehat{\mathcal{D}} : \sigma(vv^*) > 0\}$$

given by

$$\beta_v(\sigma)(d) = \frac{\sigma(v^*dv)}{\sigma(v^*v)} \quad (d \in \mathcal{D}). \tag{2.3}$$

We now observe that β_v is the restriction of $\theta_v^{\#}$ to the pure states of $\overline{v^*v\mathcal{D}}^{\#}$, that is, to $\operatorname{dom}\beta_v$. To be explicit, the relationship between β_v and θ_v is given in the following result, whose proof is immediate from the definitions.

LEMMA 2.2. *Let $\theta_v : \overline{vv^*\mathcal{D}} \to \overline{v^*v\mathcal{D}}$ be the isomorphism given in Lemma* 2.1. *Then for every $\sigma \in \operatorname{dom}\beta_v$ and $d \in \mathcal{D}$,*

$$\sigma(v^*v)[\beta_v(\sigma)(d)] = \sigma(\theta_v(vv^*d)).$$

Clearly $\mathcal{N}(\mathcal{C},\mathcal{D})$ is a $*$-semigroup under multiplication. Routine, but tedious, calculations show that the map $\mathcal{N}(\mathcal{C},\mathcal{D}) \ni v \mapsto \beta_v$ is a $*$-semigroup homomorphism $\beta : \mathcal{N}(\mathcal{C},\mathcal{D}) \to \operatorname{Inv}_{\mathcal{O}}(\widehat{\mathcal{D}})$. We record this fact as a proposition.

PROPOSITION 2.3 ([32], Lemma 4.10). *Suppose $(\mathcal{C},\mathcal{D})$ is an inclusion. Then the following statements hold:*

(i) *Suppose that $v, w \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $\rho \in \widehat{\mathcal{D}}$ satisfies $\rho(w^*v^*vw) \neq 0$. Then $\rho(w^*w) \neq 0$, and $\beta_{vw}(\rho) = \beta_v(\beta_w(\rho))$.*

(ii) *For every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, $\beta_{v^*} = (\beta_v)^{-1}$.*

DEFINITION 2.4. Let $(\mathcal{C},\mathcal{D})$ be an inclusion. A state ρ on $\mathcal{C}$ is *$\mathcal{D}$-modular* if for every $x \in \mathcal{C}$ and $d \in \mathcal{D}$,

$$\rho(dx) = \rho(d)\rho(x) = \rho(xd).$$

We let $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ be the collection of all $\mathcal{D}$-modular states on $\mathcal{C}$; equip $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ with the relative weak*-topology. Then $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ is closed and hence is compact. Using the Cauchy–Schwarz inequality, it is easy to see that

$$\operatorname{Mod}(\mathcal{C},\mathcal{D}) = \{\rho \in \mathcal{S}(\mathcal{C}) : \rho|_{\mathcal{D}} \in \widehat{\mathcal{D}}\}.$$

LEMMA 2.5. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion, and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$. If $\sigma \in \operatorname{dom}(\beta_v)$ and $\beta_v(\sigma) \neq \sigma$, then $\rho(v) = 0$ for every $\rho \in \{\tau \in \operatorname{Mod}(\mathcal{C},\mathcal{D}) : \tau|_{\mathcal{D}} = \sigma\}$.*

Proof. Suppose $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D})$ satisfies $\rho|_{\mathcal{D}} = \sigma$. Choose $d \in \mathcal{D}$ such that $\sigma(d) = 0$ and $\beta_v(\sigma)(d) = 1$. Then using Lemma 2.2 and the fact that $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D})$,

$$\begin{aligned}\sigma(v^*v)\rho(v) &= \rho(v)\sigma(\theta_v(vv^*d)) = \rho(v\theta_v(vv^*d)) = \rho((vv^*d)v)\\ &= \sigma(d)\sigma(vv^*)\rho(v) = 0.\end{aligned}$$

As $\sigma(v^*v) \neq 0$ by hypothesis, we are done. ∎

When $\rho \in \operatorname{Mod}(\mathcal{C},\mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ satisfies $\rho(v^*v) \neq 0$, the state $\beta'_v(\rho)$ on $\mathcal{C}$ given by

$$\beta'_v(\rho)(x) := \frac{\rho(v^*xv)}{\rho(v^*v)}$$

again belongs to $\operatorname{Mod}(\mathcal{C},\mathcal{D})$. When there is no danger of confusion with the notation in (2.3), we sometimes simplify notation and write $\beta_v(\rho)$ instead of $\beta'_v(\rho)$.

Thus $\mathcal{N}(\mathcal{C},\mathcal{D})$ also acts on $\mathrm{Mod}(\mathcal{C},\mathcal{D})$, and for every $\rho \in \mathrm{Mod}(\mathcal{C},\mathcal{D})$, we have $\beta'_v(\rho)|_{\mathcal{D}} = \beta_v(\rho|_{\mathcal{D}})$.

DEFINITION 2.6. A subset $F \subseteq \mathrm{Mod}(\mathcal{C},\mathcal{D})$ is $\mathcal{N}(\mathcal{C},\mathcal{D})$-*invariant* if for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $\rho \in F$ with $\rho(v^*v) \neq 0$, we have $\beta'_v(\rho) \in F$.

We record the following fact for use in the sequel.

PROPOSITION 2.7. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion and suppose $F \subseteq \mathrm{Mod}(\mathcal{C},\mathcal{D})$ is $\mathcal{N}(\mathcal{C},\mathcal{D})$-invariant. Then the set*

$$\mathcal{K}_F := \{x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in F\}$$

is a closed, two-sided ideal in $\mathcal{C}$. Moreover, if $\{\rho|_{\mathcal{D}} : \rho \in F\}$ is weak*-dense in $\widehat{\mathcal{D}}$, then $\mathcal{K}_F \cap \mathcal{D} = (0)$.

Proof. As $\mathcal{K}_F$ is the intersection of closed left-ideals, it remains only to prove that $\mathcal{K}_F$ is a right ideal. By regularity, it suffices to prove that if $x \in \mathcal{K}_F$ and $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, then $xv \in \mathcal{K}_F$. Let $\rho \in F$. If $\rho(v^*v) \neq 0$, then by hypothesis, we obtain $\rho(v^*x^*xv) = \beta_v(\rho)(x^*x)\rho(v^*v) = 0$. On the other hand, if $\rho(v^*v) = 0$, then $\rho(v^*x^*xv) \leqslant \|x^*x\|\rho(v^*v) = 0$. In either case, we find $\rho(v^*x^*xv) = 0$. As this holds for every $\rho \in \mathcal{F}$, we find $xv \in \mathcal{K}_F$, as desired. The final statement is obvious. ∎

DEFINITION 2.8. For $i = 1, 2$, let $(\mathcal{C}_i,\mathcal{D}_i)$ be inclusions. A $*$-homomorphism $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ is *regular* if $\alpha(\mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)) \subseteq \mathcal{N}(\mathcal{C}_2,\mathcal{D}_2)$ and $\alpha(I) = I$. We will sometimes use the notation $\alpha : (\mathcal{C}_1,\mathcal{D}_1) \to (\mathcal{C}_2,\mathcal{D}_2)$ to indicate that α is a regular $*$-homomorphism.

REMARK 2.9. Observe that if α is a regular homomorphism, then $\alpha(\mathcal{D}_1) \subseteq \mathcal{D}_2$. Indeed, for $d \in \mathcal{D}_1$ with $d \geqslant 0$, $d^{1/2} \in \mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)$, so $\alpha(d) = \alpha(d^{1/2})1\alpha(d^{1/2}) \in \mathcal{D}_2$. It follows that the dynamics of inclusions under regular homomorphisms are well-behaved in the sense that if $\alpha : (\mathcal{C}_1,\mathcal{D}_1) \to (\mathcal{C}_2,\mathcal{D}_2)$ is a regular homomorphism, then whenever $v \in \mathcal{N}(\mathcal{C}_1,\mathcal{D}_1) \setminus \ker\alpha$, the following diagram commutes:

$$\begin{array}{ccc}
\widehat{\mathcal{D}}_2 \supseteq \mathrm{dom}(\beta_{\alpha(v)}) & \xrightarrow{\beta_{\alpha(v)}} & \mathrm{range}(\beta_{\alpha(v)}) \subseteq \widehat{\mathcal{D}}_2 \\
{\scriptstyle \alpha^\#}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^\#} \\
\widehat{\mathcal{D}}_1 \supseteq \mathrm{dom}(\beta_v) & \xrightarrow{\beta_v} & \mathrm{range}(\beta_v) \subseteq \widehat{\mathcal{D}}_1.
\end{array}$$

Here is a very useful example of a regular $*$-monomorphism.

LEMMA 2.10. *Suppose $(\mathcal{C},\mathcal{D})$ is an inclusion such that the relative commutant, $\mathcal{D}^c$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian. Then $(\mathcal{C},\mathcal{D}^c)$ is a* MASA *inclusion and $\mathcal{N}(\mathcal{C},\mathcal{D}) \subseteq \mathcal{N}(\mathcal{C},\mathcal{D}^c)$; in particular the identity map* $\mathrm{id} : (\mathcal{C},\mathcal{D}) \to (\mathcal{C},\mathcal{D}^c)$ *is a regular $*$-homomorphism.*

Proof. Since $\mathcal{D}$ and $\mathcal{D}^c$ are abelian, $(\mathcal{C},\mathcal{D}^c)$ is a MASA inclusion.

Now suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $x \in \mathcal{D}^{\mathrm{c}}$. Let $\theta_v : \overline{vv^*\mathcal{D}} \to \overline{v^*\mathcal{D}v}$ be the isomorphism from Lemma 2.1. Then for $d \in \mathcal{D}$,

$$\begin{aligned} vxv^*d &= \lim_{n\to\infty} vxv^*(vv^*)^{1/n}d = \lim_{n\to\infty} vx\theta_v((vv^*)^{1/n}d)v^* = \lim_{n\to\infty} v\theta_v((vv^*)^{1/n}d)xv^* \\ &= \lim_{n\to\infty}(vv^*)^{1/n}dvxv^* = dvxv^*. \end{aligned}$$

Therefore, $vxv^* \in \mathcal{D}^{\mathrm{c}}$. Similar considerations yield $v^*xv \in \mathcal{D}^{\mathrm{c}}$, so $v \in \mathcal{N}(\mathcal{C},\mathcal{D}^{\mathrm{c}})$. Thus, $(\mathcal{C},\mathcal{D}^{\mathrm{c}})$ is a MASA inclusion and the identity mapping id : $(\mathcal{C},\mathcal{D}) \to (\mathcal{C},\mathcal{D}^{\mathrm{c}})$ is a regular $*$-homomorphism. ∎

2.2. FROLÍK'S THEOREM AND DECOMPOSITIONS OF IDEALS. Recall that a Hausdorff topological space X is *extremally disconnected* if the closure of every open subset of X is open and that X is a *Stonean space* if X is compact, Hausdorff, and extremally disconnected.

The following topological proposition will be extremely useful in the sequel. The proof is a straightforward adaptation of the elegant proof by Arhangel′skii of Frolík's theorem ([15], Theorem 3.1) on fixed points of homeomorphisms of extremally disconnected spaces. We provide a sketch of the proof for the convenience of the reader.

PROPOSITION 2.11 (Frolík's theorem). *Let X be an extremally disconnected space, let V,W be clopen subsets of X, and suppose $h : V \to W$ is a homeomorphism of V onto W. Then the set of fixed points $F := \{x \in V : h(x) = x\}$ is a clopen subset of X. Moreover, there are three disjoint clopen subsets C_1, C_2, C_3 of X such that for $i = 1,2,3$, $h(C_i) \cap C_i = \emptyset = C_i \cap F$ and $V = F \cup C_1 \cup C_2 \cup C_3$.*

Proof. (see Theorem 1 of [5]). Call an open subset $A \subseteq V$ *h-simple* if $\emptyset = h(A) \cap A$. By the Hausdorff maximality theorem, there exists a maximal chain $\mathcal{G}$ of h-simple sets. Put $U = \bigcup \mathcal{G}$. Then U is also an h-simple subset of V, and since $\overline{U}$ is open, maximality shows that U is in fact clopen.

Next observe that $h(U) \cap V$ and $h^{-1}(U \cap W)$ are clopen h-simple sets, and put

$$M = U \cup (h(U) \cap V) \cup h^{-1}(U \cap W).$$

Since the intersection of F with any h-simple subset of V is empty, we have $M \cap F = \emptyset$. We shall show that $F = V \setminus M$.

Suppose to the contrary, that $x \in V \setminus M$ satisfies $h(x) \neq x$. Let H be an open subset of V such that $x \in H$ and $H \cap M$ and $h(H) \cap H$ are both empty. Then H is h-simple and

$$H \cap U = H \cap (h(U) \cap V) = H \cap h^{-1}(U \cap W) = \emptyset. \tag{2.4}$$

But (2.4) implies that $H \cup U$ is a h-simple set which properly contains U, contradicting the maximality of U. So $F = V \setminus M$.

Since both V and M are clopen, so is F. Finally, to complete the proof, take $C_1 := U$, $C_2 := h(U) \cap V$, and $C_3 := h^{-1}(U \cap W) \setminus (h(U) \cap V)$. ∎

Since the maximal ideal space of an abelian injective C^*-algebra is a Stonean space, the following is essentially a restatement of Frolík's theorem.

PROPOSITION 2.11A. *Let $\mathcal{D}$ be an abelian injective C^*-algebra, let $P, Q \in \mathcal{D}$ be projections and suppose $\alpha : P\mathcal{D} \to Q\mathcal{D}$ is a $*$-isomorphism. Then there exist subprojections S, R_1, R_2, R_3 of P such that: $P = S + \sum_{j=1}^{3} R_i$; $\alpha|_{S\mathcal{D}} = \mathrm{id}|_{S\mathcal{D}}$; and for $i = 1, 2, 3$, $R_i\alpha(R_i) = 0$.*

DEFINITION 2.12. Let X be a Stonean space, let V, W be clopen subsets of X and $h : V \to W$ a homeomorphism. A decomposition, $V = \bigcup_{i=0}^{3} C_i$, where $\{C_i\}_{i=0}^{3}$ is a pairwise disjoint family of clopen subsets of X such that $C_0 = \{x \in V : h(x) = x\}$ and $C_i \cap h(C_i) = \emptyset$ for $i = 1, 2, 3$ is called a *Frolík decomposition for h*.

Dually, if $\mathcal{D}$ is an injective abelian C^*-algebra, $P, Q \in \mathcal{D}$ are projections and $\alpha : P\mathcal{D} \to Q\mathcal{D}$ is a $*$-isomorphism, a set $\{R_i\}_{i=0}^{3}$ of projections in $\mathcal{D}$ such that $P = \sum_{i=0}^{3} R_i$, $\alpha|_{R_0\mathcal{D}} = \mathrm{id}|_{R_0\mathcal{D}}$, and $\alpha(R_i)R_i = 0$ for $i = 1, 2, 3$ is a *Frolík decomposition for α*.

Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Associated to $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a family of ideals which we will use to "approximately decompose" v. This will be the key idea used in the proof of the uniqueness of pseudo-expectations. Frolík's theorem applied to $\widetilde{\theta}_v$ gives this decomposition.

DEFINITION 2.13. Fix $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, let P and Q be the support projections in $I(\mathcal{D})$ for $\overline{vv^*\mathcal{D}}$ and $\overline{v^*v\mathcal{D}}$ respectively. Let $\widetilde{\theta}_v : P\mathcal{D}_1 \to Q\mathcal{D}_1$ be the isomorphism extending θ_v (see Lemma 2.1 and Proposition 1.11). Let $\{R_i\}_{i=0}^{3} \subseteq \mathrm{PROJ}(\mathcal{D}_1)$ be a Frolík decomposition for $\widetilde{\theta}_v$. For $i \in \{0, 1, 2, 3, 4\}$, let

$$K_i = \begin{cases} \iota^{-1}(R_i I(\mathcal{D})) & \text{for } 0 \leqslant i \leqslant 3, \\ \iota^{-1}(P^{\perp} I(\mathcal{D})) & \text{for } i = 4. \end{cases}$$

These are regular ideals in $\mathcal{D}$ with pairwise trivial intersections, and the ideal generated by the family $\{K_i\}_{i=0}^{4}$ is an essential ideal of $\mathcal{D}$. Furthermore, for $i \in \{1, 2, 3\}$,

$$K_i\theta_v(K_i) = (0)$$

because $R_i\widetilde{\theta}_v(R_i) = 0$. We shall call the family $\{K_i\}_{i=0}^{4}$ a *left Frolík family of ideals for v*. The proof of Frolík's theorem shows that K_i need not be uniquely determined for $i = 1, 2, 3$. However, since K_0 and K_4 are uniquely determined by v, so is $K_1 \vee K_2 \vee K_3 = \left(\operatorname{span} \bigcup_{i=1}^{3} K_i\right)^{\perp\perp}$. A *right Frolík family of ideals for v* is a left Frolík family of ideals for v^*. We shall call the the ideal K_0 the *fixed-point ideal* for v.

REMARK 2.14. Perhaps some explanation of the adjective "left" is appropriate. If one could multiply v by R_i on the left, then v would decompose from the left as $v = Pv = \sum_{i=0}^{3} R_i v$. Thus, a left Frolík decomposition for v may be thought of as an attempt to "approximate" v with sums of the form $\sum_{i=0}^{3} d_i v$, where $d_i \in K_i$ and $0 \leqslant d_i \leqslant I$.

The fixed point ideal for v is of particular interest and we describe it now.

LEMMA 2.15. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. Then*

$$K_0 = \{d \in (vv^*\mathcal{D})^{\perp\perp} : vd = dv \in \mathcal{D}^c\} = \{d \in (v^*v\mathcal{D})^{\perp\perp} : vd = dv \in \mathcal{D}^c\}.$$

Proof. Since $\widetilde{\theta}_v$ fixes each element of $I(\mathcal{D})R_0$ and ι is an isometry, for $d \in K_0$ and $h \in \mathcal{D}$, equation (2.1) of Lemma 2.1 yields,

$$vdh = v\theta_v(dh) = dhv.$$

Thus, $dv = vd \in \mathcal{D}^c$. Also, $K_0 \subseteq \{vv^*\}^{\perp\perp}$ because $\{vv^*\}^{\perp} = K_4$. (Here, $\{K_j\}_{j=0}^4$ is a left-Frolík family of ideals for v.) Thus, $K_0 \subseteq \{d \in \{vv^*\}^{\perp\perp} : dv = vd \in \mathcal{D}^c\}$.

Suppose now that $d \in L := \{d \in \{vv^*\}^{\perp\perp} : dv = vd \in \mathcal{D}^c\}$. Fix $i \in \{1, 2, 3\}$ and let $h \in K_i$. As L is an ideal in $\mathcal{D}$, $dh \in L$, so

$$\theta_v(h^2 d(vv^*)^2) = \theta_v(hdvv^*)\theta_v(hvv^*) = v^*hdv\theta_v(hvv^*) = v^*vdh\theta_v(hvv^*) = 0,$$

because $h\theta_v(hvv^*) \in K_i\theta_v(K_i) = 0$. Thus $h^2d(vv^*)^2 = 0$, so that $hd(vv^*) = 0$. Therefore, $hd \in \{vv^*\}^{\perp} \cap \{vv^*\}^{\perp\perp}$, so $hd = 0$. Hence $d \in K_i^{\perp}$. As this holds for $i = 1, 2, 3$ and $d \in K_4^{\perp}$, we find $d \in \left(\bigvee_{j=1}^{4} K_j\right)^{\perp} = K_0$. Thus, $K_0 = \{d \in \{vv^*\}^{\perp\perp} : dv = vd \in \mathcal{D}^c\}$. Replacing v with v^*, θ_v with θ_v^{-1}, and noting that R_0 is the same for both $\widetilde{\theta}_v$ and $\widetilde{\theta}_v^{-1}$, the previous argument also shows $K_0 = \{d \in \{v^*v\}^{\perp\perp} : dv = vd \in \mathcal{D}^c\}$. ∎

EXAMPLE 2.16. Here is an example showing the utility of Frolík decompositions applied to discrete dynamical systems. Proposition 2.17 below is very likely known, but we did not find a reference. We use the notation of Subsection 6.1. Let (X, Γ) be a discrete dynamical system and fix a projective cover (P, ϕ) for X. The rigidity of the projective cover implies that the action of Γ on X uniquely extends to an action of Γ on P, so that (P, Γ) is also a discrete dynamical system. Denoting by τ_s and $\widetilde{\tau}_s$ the homeomorphisms of X and P corresponding to $s \in \Gamma$, for every $s \in \Gamma$, $\tau_s \circ \phi = \phi \circ \widetilde{\tau}_s$.

PROPOSITION 2.17. *The action of Γ on X is topologically free if and only if the action of Γ on P is free.*

Sketch of Proof. Recall that ROPEN(P) is the family of clopen subsets of P. We will use the fact that the map ROPEN$(X) \ni G \mapsto \overline{\phi^{-1}(G)} \in$ ROPEN(P)

is a a Boolean algebra isomorphism with inverse $\text{ROPEN}(P) \ni F \mapsto \phi(F)^\circ \in \text{ROPEN}(X)$.

Suppose Γ acts topologically freely on X. Choose $e \neq s \in \Gamma$. Apply Frolík's theorem to $\widetilde{\tau}_s$ to obtain clopen subsets $C_i \subseteq P$ for $i = 0, \dots, 4$ where C_0 is the set of fixed points for $\widetilde{\tau}_s$. Then $G := (\phi(C_0))^\circ$ is a regular open set in X consisting of fixed points for τ_s. As Γ acts topologically freely, $G = \emptyset$. But $\overline{\phi^{-1}(G)} = C_0$, which shows $C_0 = \emptyset$. Hence Γ acts freely on P.

Suppose Γ acts freely on P and let $e \neq s \in \Gamma$. Let G be the interior of the set of fixed points for τ_s and let $H := \overline{\phi^{-1}(G)}$. Then $G \in \text{ROPEN}(X)$ and $H \in \text{ROPEN}(P)$. Moreover, $(H, \phi|_H)$ is a projective cover for $\overline{G}$. Also, $\widetilde{\tau}_v(\phi^{-1}(G)) \subseteq \phi^{-1}(G)$, from which it follows that $\widetilde{\tau}_s(H) \subseteq H$ and $(\phi|_H) \circ (\widetilde{\tau}_s|_H) = \phi|_H$. The rigidity property of projective covers shows that every element of H is a fixed point for $\widetilde{\tau}_s$. Thus $H = \emptyset$. As $H \supseteq \phi^{-1}(G)$, we obtain $G = \emptyset$, as desired. ∎

2.3. QUASI-FREENESS AND THE EXTENSION PROPERTY. By Proposition 2.3, the set $\mathcal{S} := \{\beta_v : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}$ is an inverse semigroup of partial homeomorphisms of $\widehat{\mathcal{D}}$. Recall that a group G of homeomorphisms of a space X acts freely if whenever $g \in G$ has a fixed point, then g is the identity. Paralleling the notion for groups, we make the following definition.

DEFINITION 2.18. Suppose that $\mathcal{S}$ is a $*$-semigroup, X is a compact Hausdorff space, and that $\alpha : \mathcal{S} \to \text{Inv}_{\mathcal{O}}(X)$ is a $*$-semigroup homomorphism. We say that $\mathcal{S}$ *acts quasi-freely* on X if whenever $s \in \mathcal{S}$, $\{x \in \text{dom}(\alpha(s)) : \alpha(s)(x) = x\}$ is an open set in X.

When Γ is a group acting quasi-freely on X, this says that for each $s \in \Gamma$, the set of fixed points of $\alpha(s)$ is a clopen set; in particular, when X is a connected set, the notions of free and quasi-free actions for a group (acting as homeomorphisms) on X coincide.

In some circumstance, quasi-freeness is automatic. Using Frolík's theorem, we now show that any action of a $*$-semigroup on a Stonean space is quasi-free.

THEOREM 2.19. *Suppose that X is a Stonean space, $\mathcal{S}$ is a $*$-semigroup, and $\alpha : \mathcal{S} \to \text{Inv}_{\mathcal{O}}(X)$ is a $*$-semigroup homomorphism. Then $\mathcal{S}$ acts quasi-freely on X.*

Proof. Fix $s \in \mathcal{S}$, and consider the open sets $G := \text{dom}(\alpha(s))$ and $H := \text{ran}(\alpha(s))$. Since $\alpha(s)$ is a homeomorphism of G onto H, Lemma 1.10 and general properties of the Stone–Čech compactification show that $\alpha(s)$ extends to a homeomorphism h of $\overline{G}$ onto $\overline{H}$.

Let $F \subseteq \overline{G}$ be the set of fixed points for h; Proposition 2.11 shows that F is clopen in X. Therefore,

$$\{x \in \text{dom}(\alpha(s)) : \alpha(s)(x) = x\} = F \cap \text{dom}(\alpha(s))$$

is open in X. Thus $\mathcal{S}$ acts quasi-freely on X. ∎

Quasi-freeness is intimately related to the extension property; the following result shows the relationship and generalizes ([32], Proposition 5.11).

THEOREM 2.20. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion and let $\mathcal{M}$ be a skeleton for $(\mathcal{C}, \mathcal{D})$. Then the following statements are equivalent:*

(i) *$\mathcal{D}$ has the extension property in $\mathcal{C}$;*

(ii) *$\mathcal{D}$ is a* MASA *in* span$(\mathcal{M})$ *and the action $v \mapsto \beta_v$ of the $*$-semigroup $\mathcal{M}$ is a quasi-free action on $\widehat{\mathcal{D}}$;*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathcal{D}$ has the extension property. Then Corollary 2.7 of [3] shows that $\mathcal{D}$ is a MASA in $\mathcal{C}$, so is in particular a MASA in span $\mathcal{M}$. Moreover, the extension property ensures there exists a conditional expectation $E : \mathcal{C} \to \mathcal{D}$. We now show quasi-freeness of the action $v \mapsto \beta_v$. Suppose that $v \in \mathcal{M}$, and $\sigma \in \widehat{\mathcal{D}}$ satisfies $\sigma(v^*v) > 0$ and $\beta_v(\sigma) = \sigma$. A calculation (or Proposition 3.12 of [12]) shows $v^*E(v) \in \mathcal{D}$. Also, if G is the unitary group of $\mathcal{D}$, we have for $g \in G$,

$$\begin{aligned}\sigma(v^*gvg^{-1}) &= \sigma(v^*gv)\sigma(g^{-1}) = \beta_v(\sigma)(g)\sigma(v^*v)\sigma(g^{-1}) \\ &= \sigma(g)\sigma(v^*v)\sigma(g^{-1}) = \sigma(v^*v).\end{aligned}$$

The extension property and Theorem 3.7 of [3] show that $E(v) \in \overline{co}\{gvg^{-1} : g \in G\}$, so that

$$\sigma(v^*E(v)) = \sigma(v^*v),$$

whence $\sigma(v^*E(v)) = \sigma(v^*v) \neq 0$.

Hence there exists an open set $U \subseteq \widehat{\mathcal{D}}$ so that $\sigma \in U$ and $\tau(v^*E(v)) \neq 0$ for every $\tau \in U$. Since $v^*E(v) \in \mathcal{D}$, we have $\tau = \beta_{v^*E(v)}(\tau) = \beta_{v^*}(\beta_{E(v)}(\tau)) = \beta_{v^*}(\tau)$ for every $\tau \in U$. But $\beta_v^{-1} = \beta_{v^*}$, so $\beta_v(\tau) = \tau$ for $\tau \in U$. Thus $\{\sigma \in \widehat{\mathcal{D}} : \sigma(v^*v) > 0 \text{ and } \beta_v(\sigma) = \sigma\}$ is open in $\widehat{\mathcal{D}}$, so the semigroup $\mathcal{M}$ acts quasi-freely on $\widehat{\mathcal{D}}$.

Now suppose (ii) holds. For $i = 1, 2$, suppose ρ_i are states on $\mathcal{C}$ such that $\sigma := \rho_i|_{\mathcal{D}} \in \widehat{\mathcal{D}}$. Since span $\mathcal{M}$ is dense in $\mathcal{C}$, to show that $\rho_1 = \rho_2$, it suffices to show that for every $v \in \mathcal{M}$, $\rho_1(v) = \rho_2(v)$.

So fix $v \in \mathcal{M}$. If $\sigma \notin \operatorname{dom} \beta_v$, then $\sigma(v^*v) = 0$ and the Cauchy–Schwarz inequality gives $\rho_1(v) = \rho_2(v) = 0$.

Next, suppose $\sigma \in \operatorname{dom} \beta_v$ and $\beta_v(\sigma) \neq \sigma$. Lemma 2.5 shows that $\rho_1(v) = \rho_2(v) = 0$.

Finally, suppose $\sigma \in \operatorname{dom} \beta_v$ and $\beta_v(\sigma) = \sigma$. By hypothesis, the set $F := \{\tau \in \widehat{\mathcal{D}} : \beta_v(\tau) = \tau\}$ is an open subset of $\widehat{\mathcal{D}}$. Let $h \in \mathcal{D}$ be such that $\sigma(h) = 1$ and $\operatorname{supp} \widehat{h} \subseteq F$. Notice that $F \subseteq \operatorname{dom}(\beta_v) \cap \operatorname{range}(\beta_v)$, so h belongs to the ideal, $\overline{vv^*\mathcal{D}} \cap \overline{v^*v\mathcal{D}}$. By Lemma 2.2, $\theta_v(dh) = dh$ for every $d \in \mathcal{D}$. Hence for $d \in \mathcal{D}$

$$(dh)v = v\theta_v(dh) = v(dh).$$

It follows that $vh = hv \in \mathcal{D}^c$. As $vh \in \operatorname{span} \mathcal{M}$ and $\mathcal{D}$ is a MASA in $\operatorname{span} \mathcal{M}$, we have $vh = hv \in \mathcal{D}$. Then

$$\rho_1(v) = \rho_1(v)\sigma(h) = \rho_1(vh) = \sigma(vh) = \rho_2(vh) = \rho_2(v)\sigma(h) = \rho_2(v).$$

This exhausts all cases, so we obtain $\rho_1(v) = \rho_2(v)$ for every $v \in \mathcal{M}$. Hence $\rho_1 = \rho_2$, as desired. ∎

As an immediate corollary of our work, we have the following theorem.

THEOREM 2.21. *Suppose $(\mathcal{C}, \mathcal{D})$ is a regular inclusion, with $\mathcal{D}$ an injective C^*-algebra. Then $(\mathcal{C}, \mathcal{D})$ has the extension property if and only if $\mathcal{D}$ is a* MASA *in $\mathcal{C}$.*

Proof. Suppose first that $\mathcal{D}$ is a MASA. By results of Dixmier and Gonshor [11], [17], $\mathcal{D}$ is injective if and only if $\widehat{\mathcal{D}}$ is a Stonean space. Now combine Theorem 2.19 with Theorem 2.20.

The converse follows from Theorem 3.4 of [2]. ∎

EXAMPLE 2.22. Suppose that $\mathcal{M}$ is a von Neumann algebra and $\mathcal{D} \subseteq \mathcal{M}$ is a MASA. Let $\mathcal{C}$ be the norm-closure of $\mathcal{N}(\mathcal{M}, \mathcal{D})$. Then $(\mathcal{C}, \mathcal{D})$ has the extension property. Note that in particular, when $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$ in the sense of Feldman and Moore [14], then $(\mathcal{C}, \mathcal{D})$ is a C^*-diagonal.

REMARK 2.23. It is not possible to remove the regularity hypothesis in Theorem 2.21. Indeed, Corollary 4.7 of [1] shows that when $\mathcal{C}$ is the hyperfinite II_1 factor, and $\mathcal{D} \subseteq \mathcal{C}$ is any MASA, then $(\mathcal{C}, \mathcal{D})$ fails to have the extension property.

3. PSEUDO-EXPECTATIONS FOR SKELETAL MASA INCLUSIONS

Throughout this section, $(\mathcal{C}, \mathcal{D})$ is an inclusion and $(I(\mathcal{D}), \iota)$ is an injective envelope for $\mathcal{D}$. Recall that a pseudo-expectation for $(\mathcal{C}, \mathcal{D})$ is a unital completely positive map $E : \mathcal{C} \to I(\mathcal{D})$ such that $\iota = E|_{\mathcal{D}}$. By Corollary 3.19 of [26], a pseudo-expectation E for $(\mathcal{C}, \mathcal{D})$ preserves the $\mathcal{D}$-bimodule structure in the sense that for every $d_1, d_2 \in \mathcal{D}$ and $x \in \mathcal{C}$,

$$E(d_1 x d_2) = \iota(d_1)E(x)\iota(d_2). \tag{3.1}$$

Our main purpose in this section is to prove that any regular MASA inclusion has a unique pseudo-expectation. We actually prove this in a somewhat more general context, namely when $\mathcal{D}$ is maximal abelian in the linear span of a skeleton. As noted in the introduction, the reason for this generality is to allow us to apply our results to settings such as the full crossed product of $C(X)$ by a discrete group acting topologically freely on X.

Here is the main idea for establishing uniqueness of the pseudo-expectation. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, let $\{K_i\}_{i=0}^4$ be a left Frolík family of ideals for v (Definition 2.13) and let $R_i \in I(\mathcal{D})$ be the support projection for K_i. If one could multiply v by R_i, then any pseudo-expectation $E : \mathcal{C} \to I(\mathcal{D})$ would satisfy

$$E(v) = \sum_{i=0}^{4} E(R_i v) = \sum_{i=0}^{4} R_i E(v) R_i = \sum_{i=0}^{4} E(R_i v \widetilde{\theta}_v(R_i) R_i) = E(R_0 v).$$

Since the multiplications need not be defined, we replace R_i with elements from K_i instead. As $K_0 v$ is a sufficiently rich subset of $\mathcal{D}$, this will give uniqueness. We now give the definition of the class of inclusions for which we establish a unique pseudo-expectation.

DEFINITION 3.1. The inclusion $(\mathcal{C}, \mathcal{D})$ is a *skeletal MASA inclusion* if there exists a skeleton $\mathcal{M}$ for $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D}$ is a MASA in the linear span of $\mathcal{M}$. A skeleton $\mathcal{M}$ for $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{D}$ is a MASA in span $\mathcal{M}$ will be called a *MASA skeleton for* $(\mathcal{C}, \mathcal{D})$.

QUESTION 3.2. *We do not know whether Definition* 3.1 *is independent of the choice of skeleton for* $(\mathcal{C}, \mathcal{D})$. *In particular, is it the case that* $\mathcal{D}$ *is a* MASA *in* span $\mathcal{M}$ *if and only if* $\mathcal{D}$ *is a* MASA *in* span $\mathcal{N}(\mathcal{C}, \mathcal{D})$*?*

We require two preparatory lemmas. First, Hamana regularity yields the following lemma.

LEMMA 3.3. *Let* $\mathcal{D}$ *be an abelian* C^**-algebra and let* $(I(\mathcal{D}), \iota)$ *be an injective envelope for* $\mathcal{D}$. *Let* $x \in I(\mathcal{D})$. *If* $\{d \in \mathcal{D} : x\iota(d) = 0\}$ *is an essential ideal in* $\mathcal{D}$, *then* $x = 0$.

Proof. Let $J := \overline{xI(\mathcal{D})}$ be the closed ideal of $I(\mathcal{D})$ generated by x. Then $\iota^{-1}(J)$ is an ideal of $\mathcal{D}$ which has trivial intersection with $\{d \in \mathcal{D} : x\iota(d) = 0\}$. Hence $\iota^{-1}(J) = (0)$. By Hamana regularity, $x^*x = 0$, so $x = 0$. ∎

Second, we need the following algebraic fact.

PROPOSITION 3.4. *Let* $(\mathcal{C}, \mathcal{D})$ *be a skeletal* MASA *inclusion, let* $\mathcal{M}$ *be a* MASA *skeleton for* $(\mathcal{C}, \mathcal{D})$, *and let* $(I(\mathcal{D}), \iota)$ *be an injective envelope for* $\mathcal{D}$. *Let* $\mathcal{S} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D}) \cap$ span $\mathcal{M}$ *be such that* $\mathcal{D} \subseteq \mathcal{S}$ *and* $d_1 v d_2 \in \mathcal{S}$ *whenever* $v \in \mathcal{S}$ *and* $d_1, d_2 \in \mathcal{D}$. *For* $i = 1, 2$, *suppose* $\Delta_i : \mathcal{S} \to I(\mathcal{D})$ *is a map with the following property: for every* $h, k \in \mathcal{D}$ *and* $v \in \mathcal{S}$,

$$\Delta_i(hvk) = \iota(h)\Delta_i(v)\iota(k).$$

If $\Delta_1(I) = \Delta_2(I)$, *then* $\Delta_1 = \Delta_2$.

Proof. Choose $v \in \mathcal{S}$ and let $\{K_i\}_{i=0}^4$ be a left Frolík family of ideals for v. We claim that

$$J := \{d \in \mathcal{D} : (\Delta_1(v) - \Delta_2(v))\iota(d) = 0\}$$

is an essential ideal of $\mathcal{D}$. To see this, we will show that $K_i \subseteq J$ for $0 \leqslant i \leqslant 4$.

If $d \in K_0$, Lemma 2.15 shows that $dv = vd \in \mathcal{D}^c$. As $dv \in$ span $\mathcal{M}$, we have $dv \in \mathcal{D}$ because $\mathcal{D}$ is a MASA in span $\mathcal{M}$. Hence

$$\Delta_1(v)\iota(d) = \Delta_1(vd) = \Delta_1(I)\iota(vd) = \Delta_2(I)\iota(vd) = \Delta_2(vd) = \Delta_2(v)\iota(d).$$

Thus, $K_0 \subseteq J$. Next, if $d \in K_4$, then $dvv^* = 0$, so that $dv = 0$. Thus, $\iota(d)\Delta_1(v) = \iota(d)\Delta_2(v) = 0$, so $K_4 \subseteq J$. Finally, suppose that $i \in \{1,2,3\}$ and $d \in K_i$. Let (u_λ) be an approximate unit for K_i. Then (using Lemma 2.1),

$$\iota(d)\Delta_1(v) = \lim \iota(d)\Delta_1(u_\lambda v) = \lim \iota(d)\Delta_1(v\theta_v(u_\lambda)) = \lim \iota(d)\Delta_1(v)\iota(\theta_v(u_\lambda)) = 0,$$

because $\theta_v(K_i)K_i = (0)$. Likewise $\iota(d)\Delta_2(v) = 0$, so $K_i \subseteq J$. Therefore, J is an essential ideal of $\mathcal{D}$.

An application of Lemma 3.3 completes the proof. ∎

With this preparation, we now prove that every regular skeletal MASA inclusion has a unique pseudo-expectation.

THEOREM 3.5. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion, and let $\mathcal{M}$ be a* MASA *skeleton for $(\mathcal{C}, \mathcal{D})$. Then there exists a unique pseudo-expectation $E : \mathcal{C} \to I(\mathcal{D})$ for ι. Furthermore, suppose $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and let $\{K_i\}_{i=0}^4$ be a right Frolík family for v. Then*

(i) $E(vh) = \iota(vh)$ *for every* $h \in K_0$;

(ii) $E(vh) = 0$ *for every* $h \in \overline{\operatorname{span}} \bigcup_{i=1}^{4} K_i$; *and*

(iii) $|E(v)|^2 = R_0\iota(v^*v)$, *where $R_0 \in I(\mathcal{D})$ is the support projection for K_0.*

Proof. Suppose for $i = 1, 2$ that E_i are pseudo-expectations for ι. Equation 3.1 and Proposition 3.4 applied with $\mathcal{S} = \mathcal{N}(\mathcal{C}, \mathcal{D}) \cap \operatorname{span} \mathcal{M}$ show that $E_1|_{\mathcal{M}} = E_2|_{\mathcal{M}}$. But span $\mathcal{M}$ is dense in $\mathcal{C}$, so $E_1 = E_2$.

Part (i) is evident. For part (ii), let $1 \leqslant i \leqslant 4$. Given $d \in K_i$ write $h = h_1h_2$ for $h_1, h_2 \in K_i$. Then (as in the proof of Proposition 3.4) $E(vh) = E(v)\iota(h_1h_1) = E(h_1vh_2) = 0$, and part (ii) follows.

For part (iii), let $H := \{d \in \mathcal{D} : (E(v^*)E(v) - R_0\iota(v^*v))\iota(d) = 0\}$. By parts (i) and (ii), $K_i \subseteq H$ for $0 \leqslant i \leqslant 4$. Therefore, H is an essential ideal in $\mathcal{D}$. Part (iii) now follows from Lemma 3.3. ∎

More can be said when $(I(\mathcal{D}), \iota)$ is chosen to be the Dixmier algebra of $\widehat{\mathcal{D}}$. The following corollary, whose proof we leave to the reader, gives a formula for $E(v)$ in this case.

COROLLARY 3.6. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion. Suppose $I(\mathcal{D}) = \mathfrak{B}(\widehat{\mathcal{D}})/\mathfrak{N}$ is the Dixmier algebra for $\widehat{\mathcal{D}}$, and for $d \in \mathcal{D}$, $\iota(d) = \widehat{d} + \mathfrak{N}$. Let $E : \mathcal{C} \to I(\mathcal{D})$ be the pseudo-expectation, fix $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and let K_0 be the fixed-point ideal for v. Define a function $\widehat{v}$ on $\widehat{\mathcal{D}}$ by*

$$\widehat{v}(\sigma) := \begin{cases} 0 & \text{if } \sigma|_{K_0} \equiv 0, \\ \sigma(vd)/\sigma(d) & \text{if } d \in K_0 \text{ and } \sigma(d) \neq 0. \end{cases}$$

Then $\widehat{v}$ is a well-defined, bounded Borel function on $\widehat{\mathcal{D}}$ and $E(v) = \widehat{v} + \mathfrak{N}$.

The hypothesis in Theorem 3.5 that $(\mathcal{C}, \mathcal{D})$ is a skeletal MASA inclusion can be weakened, as we now observe. The hypothesis in Corollary 3.7 that $\mathcal{D}^c \supseteq \mathcal{D}$ is an essential extension is needed: consider the inclusion $(C(X), \mathbb{C})$.

COROLLARY 3.7. *Suppose $(\mathcal{C},\mathcal{D})$ is a regular inclusion such that $\mathcal{D}^c$ is abelian and $(\mathcal{D}^c,\subseteq)$ is an essential extension of $\mathcal{D}$. Then there exists a unique pseudo-expectation $E:\mathcal{C}\to I(\mathcal{D})$ for ι.*

Proof. While Corollary 3.21 of [29] can be used to establish this result, the following argument is a bit more self-contained. Recall that $I(\mathcal{D})$ is an injective object viewed in either of the categories $\mathfrak{C}$ or $\mathfrak{O}$. Injectivity of $I(\mathcal{D})$ ensures that ι extends to a $*$-homomorphism $\widetilde{\iota}:\mathcal{D}^c\to I(\mathcal{D})$. Since $\mathcal{D}^c$ is an essential extension of $\mathcal{D}$, $\widetilde{\iota}$ is one-to-one. The rigidity property of $(I(\mathcal{D}),\iota)$ (in the category $\mathfrak{O}$) ensures that $\widetilde{\iota}$ is the unique unital completely positive extension of ι to $\mathcal{D}^c$. Moreover, $(I(\mathcal{D}),\widetilde{\iota})$ is an essential extension of $\mathcal{D}^c$ and hence is an injective envelope for $\mathcal{D}^c$. Theorem 3.5 gives a unique pseudo-expectation $E:\mathcal{C}\to I(\mathcal{D})$ for $\widetilde{\iota}$. Suppose now that $E_1:\mathcal{C}\to I(\mathcal{D})$ is a pseudo-expectation for ι. The restriction of E_1 to $\mathcal{D}^c$ extends ι, so $E_1|_{\mathcal{D}^c}=\widetilde{\iota}$. Thus, E_1 is also a pseudo-expectation for $\widetilde{\iota}$, whence $E=E_1$. ∎

REMARK 3.8. Suppose that $(\mathcal{C},\mathcal{D})$ is a skeletal MASA inclusion and there exists a conditional expectation $\mathbb{E}:\mathcal{C}\to\mathcal{D}$. Notice that in this context, $\mathbb{E}$ is unique. Indeed, if $(I(\mathcal{D}),\iota)$ is an injective envelope for $\mathcal{D}$, then $E:=\iota\circ\mathbb{E}$ is the pseudo-expectation for $(\mathcal{C},\mathcal{D})$. Uniqueness of E together with the fact that ι is one-to-one gives uniqueness of $\mathbb{E}$. When this occurs, we will identify $\mathbb{E}$ with E and will simply say that the conditional expectation is the pseudo-expectation.

Here is a result "dual" to Theorem 3.5. Notice that in the context of Theorem 3.5, when $\rho\in\widehat{I(\mathcal{D})}$, $E^\#(\rho)=\rho\circ E\in\operatorname{Mod}(\mathcal{C},\mathcal{D})$.

THEOREM 3.9. *Let $(\mathcal{C},\mathcal{D})$ be a skeletal* MASA *inclusion, let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$, and let E be the pseudo-expectation for ι. The map $E^\#:\widehat{I(\mathcal{D})}\to$* $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ *is the unique continuous map of $\widehat{I(\mathcal{D})}$ into* $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ *such that for every $\rho\in\widehat{I(\mathcal{D})}$, $E^\#(\rho)|_{\mathcal{D}}=\rho\circ\iota$.*

Proof. Clearly $E^\#$ has the stated property, so we need only prove uniqueness.

Suppose that ℓ is a continuous map of $\widehat{I(\mathcal{D})}$ into $\operatorname{Mod}(\mathcal{C},\mathcal{D})$ such that for every $\rho\in\widehat{I(\mathcal{D})}$, $\ell(\rho)|_{\mathcal{D}}=\rho\circ\iota$. For $x\in\mathcal{C}$, define a function

$$\phi_x:\widehat{I(\mathcal{D})}\to\mathbb{C}\quad\text{by } \phi_x(\rho)=\ell(\rho)(x).$$

Since ℓ is continuous, ϕ_x is continuous. Hence there exists a unique element $E_1(x)\in I(\mathcal{D})$ such that ϕ_x is the Gelfand transform of $E_1(x)$. Using the fact that $\operatorname{Mod}(\mathcal{C},\mathcal{D})\subseteq\mathcal{S}(\mathcal{C})$, we find E_1 is linear, bounded, unital and positive. Since $I(\mathcal{D})$ is abelian, E_1 is completely positive. For $d\in\mathcal{D}$ we have $\rho(E_1(d))=\ell(\rho)(d)=\rho(\iota(d))$. Therefore, $E_1|_{\mathcal{D}}=\iota$, so E_1 is a pseudo-expectation for ι. By Theorem 3.5, $E_1=E$, hence $\ell=E^\#$. ∎

We now present an interesting result concerning uniqueness of extensions of pure states on $\mathcal{D}$ to $\mathcal{C}$. This result generalizes a result found in [32], however, the proof is rather different. Notice that Theorem 3.10 holds when $\mathcal{C}$ is separable or when there is a countable subset $X \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ is the C^*-algebra generated by $\mathcal{D}$ and X. We shall use Theorem 3.10 in the proof of Theorem 8.2.

THEOREM 3.10. *Suppose $(\mathcal{C}, \mathcal{D})$ is a skeletal* MASA *inclusion and that $N \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a countable set such that the norm-closed $\mathcal{D}$-bimodule generated by N is $\mathcal{C}$. Let*

$$\mathfrak{U} := \{\sigma \in \widehat{\mathcal{D}} : \sigma \text{ has a unique state extension to } \mathcal{C}\}.$$

Then $\mathfrak{U}$ is dense in $\widehat{\mathcal{D}}$.

Proof. For each $v \in N$, let $H_v := \overline{\operatorname{span}} \bigcup_{j=0}^{4} K_j$, where $\{K_j\}_{j=0}^{4}$ is a right Frolík family for v, and let $X_v := \{\sigma \in \widehat{\mathcal{D}} : \sigma|_{H_v} \neq 0\}$. Clearly X_v is open in $\widehat{\mathcal{D}}$ and since H_v is an essential ideal in $\mathcal{D}$, X_v is dense in $\widehat{\mathcal{D}}$. Baire's theorem shows that

$$P := \bigcap_{v \in N} X_v$$

is dense in $\widehat{\mathcal{D}}$.

Let $\sigma \in P$ and suppose for $i = 1, 2$, ρ_i are states on $\mathcal{C}$ such that $\rho_i|_{\mathcal{D}} = \sigma$. The Cauchy–Schwarz inequality shows that $\rho_i : \mathcal{C} \to \mathbb{C}$ are $\mathcal{D}$-modular maps.

Fix $v \in N$. Since $\sigma \in X_v$ and the ideals in a right Frolík family $\{K_j\}$ for v have pairwise trivial intersection, we may find $j \in \{0, \ldots, 4\}$ and $h \in K_j$ such that $\sigma(h) = 1$. Then (as in the proof of Theorem 3.5(ii)),

$$\rho_1(v) = \rho_1(v)\sigma(h) = \rho_1(vh) = \rho_2(vh) = \rho_2(v)\sigma(h) = \rho_2(v).$$

Since N generates $\mathcal{C}$ as a $\mathcal{D}$-bimodule and ρ_i are $\mathcal{D}$-modular, we see that $\rho_1 = \rho_2$. Hence $P \subseteq \mathfrak{U}$, and the proof is complete. ∎

DEFINITION 3.11. Let $(\mathcal{C}, \mathcal{D})$ be a skeletal MASA inclusion, let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$, and let E be the (unique) pseudo-expectation for ι. Define

$$\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) := \{\rho \circ E : \rho \in \widehat{I(\mathcal{D})}\}.$$

We shall call states belonging to $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ *strongly compatible* states. Then $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ is a closed subset of $\operatorname{Mod}(\mathcal{C}, \mathcal{D})$. Observe that $\widehat{\mathcal{D}} = \{\tau|_{\mathcal{D}} : \tau \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D})\}$; this is because $\widehat{\mathcal{D}} = \{\rho \circ \iota : \rho \in \widehat{I(\mathcal{D})}\}$.

Let $r : \operatorname{Mod}(\mathcal{C}, \mathcal{D}) \to \widehat{\mathcal{D}}$ be the restriction map, $r(\rho) = \rho|_{\mathcal{D}}$. We now show that $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ is the unique minimal closed subset of $\operatorname{Mod}(\mathcal{C}, \mathcal{D})$ for which r is onto. In a certain sense, this allows us to determine $\mathfrak{S}_s(\mathcal{C}, \mathcal{D})$ without the use of the pseudo-expectation.

THEOREM 3.12. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion and suppose F is a closed subset of* $\operatorname{Mod}(\mathcal{C}, \mathcal{D})$ *which satisfies $r(F) = \widehat{\mathcal{D}}$. Then $\mathfrak{S}_s(\mathcal{C}, \mathcal{D}) \subseteq F$.*

Suppose further that there exists a countable subset $N \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ such that the norm-closed $\mathcal{D}$-bimodule generated by N is $\mathcal{C}$ and set

$$\mathfrak{U}(\mathcal{C},\mathcal{D}) := \{\rho \in \mathrm{Mod}(\mathcal{C},\mathcal{D}) : \rho|_{\mathcal{D}} \textit{ has a unique state extension to } \mathcal{C}\}.$$

Then

$$\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D}) = \overline{\mathfrak{U}(\mathcal{C},\mathcal{D})}^{\mathrm{w}^*}.$$

Proof. Since F is closed and $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ is compact, F is compact and Hausdorff. As $\widehat{I(\mathcal{D})}$ is projective (in the category of compact Hausdorff spaces and continuous maps) and r maps F onto $\widehat{\mathcal{D}}$, there exists a continuous map $\ell : \widehat{I(\mathcal{D})} \to F$ such that $\iota^{\#} = r \circ \ell$. Let $\epsilon : F \to \mathrm{Mod}(\mathcal{C},\mathcal{D})$ be the inclusion map. Then $\ell' := \epsilon \circ \ell$ is a continuous map of $\widehat{I(\mathcal{D})}$ into $\mathrm{Mod}(\mathcal{C},\mathcal{D})$ such that $r \circ \ell' = \iota^{\#}$. Theorem 3.9 shows that $\ell' = E^{\#}$. Therefore, the range of $E^{\#}$ is contained in F, that is, $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D}) \subseteq F$.

Suppose now that there is a countable subset $N \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ which generates $\mathcal{C}$ as a $\mathcal{D}$-bimodule. Theorem 3.10 implies that $\widehat{\mathcal{D}} = r(\overline{\mathfrak{U}(\mathcal{C},\mathcal{D})})$, so we have $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D}) \subseteq \overline{\mathfrak{U}(\mathcal{C},\mathcal{D})}$. To complete the proof, observe that $\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$ is closed and $\mathfrak{U}(\mathcal{C},\mathcal{D}) \subseteq \mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$. ∎

3.1. THE LEFT KERNEL AND MULTIPLICATIVE DOMAIN OF THE PSEUDO-EXPECTATION. We now show the left kernel of the pseudo-expectation on a skeletal MASA inclusion is the unique ideal which is maximal with respect to being disjoint from $\mathcal{D}$. We also identify the multiplicative domain of the pseudo-expectation.

NOTATION 3.13. For a skeletal MASA inclusion $(\mathcal{C},\mathcal{D})$, let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$, and let $E : \mathcal{C} \to I(\mathcal{D})$ be the pseudo-expectation. Denote by $\mathcal{L}(\mathcal{C},\mathcal{D})$ the left kernel of E, that is,

$$\mathcal{L}(\mathcal{C},\mathcal{D}) := \{x \in \mathcal{C} : E(x^*x) = 0\}.$$

When there is no danger of confusion, we will sometimes write $\mathcal{L}$ instead of $\mathcal{L}(\mathcal{C},\mathcal{D})$. Recall that if $(\mathcal{D}_1,\alpha)$ is another injective envelope for $\mathcal{D}$, then there exists a unique $*$-isomorphism $\theta : I(\mathcal{D}) \to \mathcal{D}_1$ such that $\theta \circ \iota = \alpha$. Thus, $\mathcal{L}(\mathcal{C},\mathcal{D})$ is intrinsic to the inclusion $(\mathcal{C},\mathcal{D})$ and does not depend upon the choice of injective envelope for $\mathcal{D}$.

It would be extremely convenient if for $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $x \in \mathcal{C}$, we had $E(v^*xv) = v^*E(x)v$. In general, the multiplications on right side of this formula are not defined, as $E(x)$ need not belong to $\mathcal{C}$. However, if E happens to be a conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$, then $v^*E(x)v = \theta_v(vv^*E(x)) = \theta_v(E(vv^*x))$. This suggests the formula in the following proposition, which provides the key step in showing $\mathcal{L}(\mathcal{C},\mathcal{D})$ is an ideal in $\mathcal{C}$.

PROPOSITION 3.14. *Let $(\mathcal{C},\mathcal{D})$ be a skeletal* MASA *inclusion, let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$ and let $E : \mathcal{C} \to I(\mathcal{D})$ be the pseudo-expectation. Then for*

$v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $x \in \mathcal{C}$,

$$E(v^*xv) = \widetilde{\theta}_v(E(vv^*x)). \tag{3.2}$$

Proof. Let $\mathcal{M}$ be a MASA skeleton for $(\mathcal{C},\mathcal{D})$ and fix $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$.

Since $\operatorname{span}(\mathcal{M})$ is dense in $\mathcal{C}$, to obtain (3.2), it suffices to show that for each $w \in \mathcal{M}$,

$$E(v^*wv) = \widetilde{\theta}_v(E(vv^*w)). \tag{3.3}$$

To do this, by Lemma 3.3, it suffices to show that the ideal

$$H := \{d \in \mathcal{D} : (E(v^*wv) - \widetilde{\theta}_v(E(vv^*w)))\iota(d) = 0\}$$

is an essential ideal of $\mathcal{D}$.

So let $w \in \mathcal{M}$. Let $\{K_i\}_{i=0}^4$ be a left Frolík family of ideals for w, let $J = \overline{vv^*\mathcal{D}}$ and let P be the support projection in $I(\mathcal{D})$ for J. Let

$$A := \theta_v(J)^\perp \cup \bigcup_{i=0}^4 \theta_v(J \cap K_i).$$

Then $A^\perp = \{0\}$, so A generates an essential ideal of $\mathcal{D}$. Thus if we show that $A \subseteq H$, it will follow that H is an essential ideal and this is what we shall do.

Notice that $\theta_v(J)^\perp = (\overline{v^*v\mathcal{D}})^\perp$. Therefore, if $d \in \theta_v(J)^\perp$, then $vd = 0$. Hence

$$\begin{aligned}(E(v^*wv) - \widetilde{\theta}_v(E(vv^*w)))\iota(d) &= 0 - \widetilde{\theta}_v(\iota(vv^*))\widetilde{\theta}_v(PE(w))\iota(d)\\ &= -\iota(v^*v)\widetilde{\theta}_v(PE(w))\iota(d) = 0.\end{aligned}$$

This shows $\theta_v(J)^\perp \subseteq H$.

Let $0 \leqslant i \leqslant 4$ and choose $d \in J \cap K_i$. Factor $d = hk$ (e.g. Cohen's factorization theorem), where $h,k \in J \cap K_i$. Using equation (2.1) of Lemma 2.1,

$$\begin{aligned}E(v^*wv)\iota(\theta_v(d)) &= \iota(\theta_v(h))E(v^*wv)\iota(\theta_v(k)) = E(\theta_v(h)v^*wv\theta_v(k))\\ &= E(v^*hwkv).\end{aligned} \tag{3.4}$$

Similarly (using the fact that $\iota \circ \theta_v = \widetilde{\theta}_v \circ \iota$),

$$\widetilde{\theta}_v(E(vv^*w))\iota(\theta_v(d)) = \iota(\theta_v(h))\widetilde{\theta}_v(E(vv^*w))\iota(\theta_v(k)) = \widetilde{\theta}_v(E(vv^*hwk)). \tag{3.5}$$

Recalling that $K_iwK_i = (0)$ for $1 \leqslant i \leqslant 4$, we see that equations (3.4) and (3.5) give $\theta_v(J \cap K_i) \subseteq H$ for $1 \leqslant i \leqslant 4$.

Finally, we examine the case $i = 0$. Lemma 2.15 and the fact that $\mathcal{D}$ is a MASA in span $\mathcal{M}$ gives $kw \in \mathcal{D}$. Therefore, $wd = dw = h(kw) \in J \cap K_0$. Thus,

$$\begin{aligned}E(v^*wv)\iota(\theta_v(d)) &= E(v^*wdv) = \iota(v^*wdv)\\ &= \iota(\theta_v(vv^*wd)) = \widetilde{\theta}_v(E(vv^*wd)) = \widetilde{\theta}_v(E(vv^*w))\iota(\theta_v(d)).\end{aligned}$$

Thus, $\theta_v(J \cap K_0) \subseteq H$ as well.

We conclude that $A \subseteq H$, which completes the proof. ∎

The following theorem is one of our main structural results.

THEOREM 3.15. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion. Then $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is an ideal of $\mathcal{C}$ such that $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \mathcal{D} = (0)$.*

Moreover, if $\mathcal{K} \subseteq \mathcal{C}$ is an ideal such that $\mathcal{K} \cap \mathcal{D} = (0)$, then $\mathcal{K} \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D})$.

Proof. As usual, let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Proposition 3.14 shows that if $x \in \mathcal{L}(\mathcal{C}, \mathcal{D})$ and $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, then

$$E(v^*x^*xv) = \widetilde{\theta}_v(E(vv^*x^*x)) = \widetilde{\theta}_v(\iota(vv^*)E(x^*x)) = 0,$$

so $xv \in \mathcal{L}(\mathcal{C}, \mathcal{D})$. Regularity of $(\mathcal{C}, \mathcal{D})$ now shows that if $y \in \mathcal{C}$ and $x \in \mathcal{L}(\mathcal{C}, \mathcal{D})$, then $xy \in \mathcal{L}(\mathcal{C}, \mathcal{D})$, so that $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is a right ideal in $\mathcal{C}$. As it is clear that $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is a left ideal, $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is an ideal in $\mathcal{C}$. That $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \mathcal{D} = (0)$ is obvious.

Suppose now that $\mathcal{K} \subseteq \mathcal{C}$ is an ideal with $\mathcal{K} \cap \mathcal{D} = (0)$ and let $q : \mathcal{C} \to \mathcal{C}/\mathcal{K}$ be the quotient map. Then $q|_{\mathcal{D}}$ is faithful, and we let $\kappa := \iota \circ (q|_{\mathcal{D}})^{-1}$. Then $(I(\mathcal{D}), \kappa)$ is an injective envelope for $q(\mathcal{D})$. Hence there exists a unital completely positive map $\Phi : \mathcal{C}/\mathcal{K} \to I(\mathcal{D})$ such that $\Phi(d + \mathcal{K}) = \kappa(d + \mathcal{K})$ for every $d + \mathcal{K} \in q(\mathcal{D})$. The uniqueness of the pseudo-expectation on $\mathcal{C}$ shows that $E = \Phi \circ q$. Therefore, for $x \in \mathcal{K}$, $E(x^*x) = \Phi(q(x^*x)) = 0$, so $x \in \mathcal{L}(\mathcal{C}, \mathcal{D})$. ∎

The ideal $\mathcal{L}(\mathcal{C}, \mathcal{D})$ behaves well with respect to certain $*$-homomorphisms.

PROPOSITION 3.16. *Suppose for $i = 1, 2$ that $(\mathcal{C}_i, \mathcal{D}_i)$ are skeletal* MASA *inclusions, and $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ is a $*$-homomorphism (not necessarily regular) such that $\alpha|_{\mathcal{D}_1}$ is one-to-one and $\alpha(\mathcal{D}_1) \subseteq \mathcal{D}_2$. Then*

$$\{x \in \mathcal{C}_1 : \alpha(x) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)\} \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1), \tag{3.6}$$

with equality holding in (3.6) *if $(\mathcal{D}_2, \alpha|_{\mathcal{D}_1})$ is an essential extension of $\mathcal{D}_1$.*

Proof. The set $\{x \in \mathcal{C}_1 : \alpha(x) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)\}$ is an ideal of $\mathcal{C}_1$ whose intersection with $\mathcal{D}_1$ is trivial, hence Theorem 3.15 gives

$$\{x \in \mathcal{C}_1 : \alpha(x) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)\} \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1).$$

Now suppose that $(\mathcal{D}_2, \alpha|_{\mathcal{D}_1})$ is an essential extension of $\mathcal{D}_1$. Let $(I(\mathcal{D}_2), \iota_2)$ be an injective envelope for $\mathcal{D}_2$. Then $(I(\mathcal{D}_2), \iota_2 \circ \alpha|_{\mathcal{D}_1})$ is an essential extension for $\mathcal{D}_1$, so that $(I(\mathcal{D}_2), \iota_2 \circ \alpha|_{\mathcal{D}_1})$ is an injective envelope for $\mathcal{D}_1$.

Let $E_2 : \mathcal{C}_2 \to I(\mathcal{D}_2)$ be the pseudo-expectation for ι_2. Then $E_2 \circ \alpha|_{\mathcal{D}_1} = \iota_2 \circ \alpha|_{\mathcal{D}_1}$ so $E_2 \circ \alpha : \mathcal{C}_1 \to I(\mathcal{D}_2)$ is the pseudo-expectation for $\iota_2 \circ \alpha|_{\mathcal{D}_1}$. Thus if $x \in \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1)$, then $E_2(\alpha(x^*x)) = 0$, so $\alpha(x) \in \mathcal{L}(\mathcal{C}_2, \mathcal{D}_2)$. ∎

We conclude this section by identifying the multiplicative domain of the pseudo-expectation. Recall that the multiplicative domain for E is the set

$$\{x \in \mathcal{C} : \text{for every } y \in \mathcal{C},\ E(yx) = E(y)E(x) = E(xy)\}.$$

PROPOSITION 3.17. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion, let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$, let $E : \mathcal{C} \to I(\mathcal{D})$ be the pseudo-expectation and set*

$$\mathcal{A} := \{x \in \mathcal{C} : xd - dx \in \mathcal{L}(\mathcal{C}, \mathcal{D}) \text{ for all } d \in \mathcal{D}\}.$$

Then $\mathcal{A}$ is the multiplicative domain for E. Moreover, the following statements hold:

(i) $\mathcal{A}\cap\mathcal{N}(\mathcal{C},\mathcal{D}) \subseteq \mathcal{D}^c$ *and if* $\mathcal{M}$ *is a* MASA *skeleton for* $(\mathcal{C},\mathcal{D})$*, then* $\mathcal{A}\cap\mathcal{M} \subseteq \mathcal{D}$.
(ii) *If there is a conditional expectation of* $\mathcal{C}$ *onto* $\mathcal{D}$*, then* $\mathcal{A} = \mathcal{D} + \mathcal{L}(\mathcal{C},\mathcal{D})$.

Proof. Fix $x \in \mathcal{A}$. For $i = 1, 2$, define $\Delta_i : \mathcal{C} \to I(\mathcal{D})$ by

$$\Delta_1(y) := E(yx) \quad \text{and} \quad \Delta_2(y) := E(y)E(x).$$

Obviously, $\Delta_1(I) = \Delta_2(I) = E(x)$. For $d_1, d_2 \in \mathcal{D}$ and $y \in C$,

$$\begin{aligned}\Delta_1(d_1 y d_2) &= E(d_1 y d_2 x) = E(d_1 y x d_2) + E(d_1 y(d_2 x - x d_2)) = E(d_1 y x d_2)\\ &= \iota(d_1)E(yx)\iota(d_2) = \iota(d_1)\Delta_1(y)\iota(d_2).\end{aligned}$$

Also, $\Delta_2(d_1 y d_2) = \iota(d_1)\Delta_2(y)\iota(d_2)$.

By Proposition 3.4, $\Delta_1|_{\mathcal{N}(\mathcal{C},\mathcal{D})} = \Delta_2|_{\mathcal{N}(\mathcal{C},\mathcal{D})}$, that is, for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$,

$$E(vx) = E(v)E(x).$$

Similarly, $E(xv) = E(x)E(v)$ for every $v \in \mathcal{N}(C,\mathcal{D})$. By regularity of $(\mathcal{C},\mathcal{D})$ and continuity of E, we obtain $E(yx) = E(y)E(x) = E(xy)$ for every $y \in \mathcal{C}$. Thus $\mathcal{A}$ is contained in the multiplicative domain of E.

Conversely, suppose x belongs to the multiplicative domain of E. For $d \in \mathcal{D}$,

$$E((xd - dx)^*(xd - dx)) = E(d^*x^*xd) - E(x^*d^*xd) - E(d^*x^*dx) + E(x^*d^*dx) = 0,$$

so $x \in \mathcal{A}$. Hence $\mathcal{A}$ is the multiplicative domain for E.

For part (i), suppose $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ and $dv - vd \in \mathcal{L}$ for every $d \in \mathcal{D}$. As $\mathcal{L}$ is an ideal, we obtain $v^*dv - v^*vd \in \mathcal{L}\cap\mathcal{D} = (0)$, so $v^*dv - v^*vd = 0$ for all $d \in \mathcal{D}$. By Lemma 1.5, $v \in \mathcal{D}^c$. If $\mathcal{M}$ is a MASA skeleton for $(\mathcal{C},\mathcal{D})$ and $v \in \mathcal{M}\cap\mathcal{A}$, then $v \in \mathcal{D}$ because $\mathcal{D}$ is a MASA in span $\mathcal{M}$. This gives part (i).

Finally, suppose there is a conditional expectation $\mathcal{C}$ onto $\mathcal{D}$. By Remark 3.8, this map may be identified with the pseudo-expectation. Let $x \in \mathcal{A}$. Since $\mathcal{A}$ is the multiplicative domain for E,

$$E((x^* - E(x^*))(x - E(x))) = E(x^*x - E(x)^*x - x^*E(x) + E(x^*)E(x)) = 0.$$

Therefore $x - E(x) \in \mathcal{L}$, so that

$$x = E(x) + (x - E(x)) \in \mathcal{D} + \mathcal{L}.$$

Thus, $\mathcal{A} \subseteq \mathcal{D} + \mathcal{L} \subseteq \mathcal{A}$. ∎

4. COMPATIBLE STATES

Since the extension property does not always hold for an inclusion $(\mathcal{C},\mathcal{D})$, we identify a useful class of states in $\mathrm{Mod}(\mathcal{C},\mathcal{D})$, which we call $\mathcal{D}$-compatible states. We will use these states in the proofs of the embedding theorems found in Section 5 and also in the proof of Theorem 8.2.

To motivate the definition, observe that when $(\mathcal{C},\mathcal{D})$ is a regular EP-inclusion, the only way to extend a pure state $\sigma \in \widehat{\mathcal{D}}$ to $\mathcal{C}$ is via composition with the expectation: $\rho := \sigma \circ E$. Then the GNS-representation $(\pi_\rho, \mathcal{H}_\rho)$ arising from ρ has the property that for any $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$ either $I + L_\rho$ and $v + L_\rho$ are orthogonal in the Hilbert space $\mathcal{H}_\rho$, or one is a scalar multiple of the other, according to whether or not the Gelfand transform of $E(v)$ is zero in a neighborhood of σ. Furthermore, the techniques used in the proof of Proposition 5.4 of [12] show that $\pi_\rho(\mathcal{D})''$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$ and also that for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$. The intersection $\mathcal{J}$ of the kernels of such representations is the left kernel of the expectation E, $\mathcal{D} \cap \mathcal{J} = (0)$, and the quotient of $(\mathcal{C},\mathcal{D})$ by $\mathcal{J}$ yields a C^*-diagonal with the same coordinate system as $(\mathcal{C},\mathcal{D})$, see Theorem 4.8 of [12].

We shall define the set of compatible states to be those states ρ on $\mathcal{C}$ for which the vectors $\{v + L_\rho : v \in \mathcal{N}(\mathcal{C},\mathcal{D})\}$ form an orthogonal set of vectors in $\mathcal{H}_\rho$. These states have many of the properties listed in the previous paragraph, but have the advantage of not needing the extension property or a conditional expectation for their definition. Here is the formal definition.

DEFINITION 4.1. Let $(\mathcal{C},\mathcal{D})$ be an inclusion. A state ρ on $\mathcal{C}$ is called $\mathcal{D}$-*compatible* if for every $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$,

$$|\rho(v)|^2 \in \{0, \rho(v^*v)\}.$$

When the context is clear, we will simply use the term *compatible state* instead of $\mathcal{D}$-compatible state. We will use $\mathfrak{S}(\mathcal{C},\mathcal{D})$ to denote the set of all $\mathcal{D}$-compatible states on $\mathcal{C}$. Topologize $\mathfrak{S}(\mathcal{C},\mathcal{D})$ with the relative weak*-topology.

NOTATION 4.2. For $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, let

$$\Delta_\rho := \{v \in \mathcal{N}(\mathcal{C},\mathcal{D}) : \rho(v) \neq 0\} \quad \text{and} \quad \Lambda_\rho := \{v \in \mathcal{N}(\mathcal{C},\mathcal{D}) : \rho(v^*v) > 0\}.$$

Also, define a relation $\sim_\rho$ on Λ_ρ by $(v,w) \in \sim_\rho$ if and only if $v^*w \in \Delta_\rho$. (We shall prove that $\sim_\rho$ is an equivalence relation momentarily, and then will simply write $v \sim_\rho w$.)

REMARK 4.3. (i) When $(\mathcal{C},\mathcal{D})$ is a MASA inclusion, Proposition 4.6 below shows that compatible states exist in abundance. For general (non-MASA) inclusions, it is possible that $\mathfrak{S}(\mathcal{C},\mathcal{D}) = \emptyset$, see Theorem 4.13.

(ii) As $|\rho(x)|^2 \leqslant \rho(x^*x)$ for any state $\rho \in \mathcal{C}^\#$ and any $x \in \mathcal{C}$, we see that $\mathcal{D}$-compatible states satisfy an extremal property relative to the normalizers for $\mathcal{D}$, and one might expect an inclusion relationship between compatible states and pure states. However, there is not. Indeed, Example 7.17 of [28] gives an example of a Cartan inclusion $(\mathcal{C},\mathcal{D})$ and element of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ which is not a pure state on $\mathcal{C}$, while Example 7.16 of [28] gives an example of a Cartan inclusion $(\mathcal{C},\mathcal{D})$ and a pure state ρ on $\mathcal{C}$ such that $\rho \in \mathrm{Mod}(\mathcal{C},\mathcal{D})$, yet $\rho \notin \mathfrak{S}(\mathcal{C},\mathcal{D})$. As we shall see momentarily, $\mathfrak{S}(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Mod}(\mathcal{C},\mathcal{D})$. Thus no such inclusion relationship exists.

(iii) The following simple observation will be useful during the sequel: for $i = 1,2$, let $(\mathcal{C}_i,\mathcal{D}_i)$ be inclusions and suppose that $\alpha : \mathcal{C}_1 \to \mathcal{C}_2$ is a regular and unital

$*$-homomorphism. Then

$$\alpha^{\#}(\mathfrak{S}(\mathcal{C}_2,\mathcal{D}_2)) \subseteq \mathfrak{S}(\mathcal{C}_1,\mathcal{D}_1). \tag{4.1}$$

Here are some properties of elements of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ which hold for any inclusion.

PROPOSITION 4.4. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion and let $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$. The following statements hold:*

(i) *Suppose $v \in \Delta_\rho$. Then for every $x \in \mathcal{C}$,*

$$\rho(vx) = \rho(v)\rho(x) = \rho(xv) \quad \textit{and} \quad \rho(v^*xv) = \rho(v^*v)\rho(x).$$

(ii) *The restriction of ρ to $\mathcal{D}$ is a multiplicative linear functional on $\mathcal{D}$.*

(iii) *If $v_1, v_2 \in \Lambda_\rho$ and $(v_1,v_2) \in \sim_\rho$, then*

$$|\rho(v_1^*v_2)|^2 = \rho(v_1^*v_1)\rho(v_2^*v_2).$$

Moreover, $\sim_\rho$ is an equivalence relation on Λ_ρ.

(iv) *$\mathfrak{S}(\mathcal{C},\mathcal{D})$ is an $\mathcal{N}(\mathcal{C},\mathcal{D})$-invariant subset of* $\operatorname{Mod}(\mathcal{C},\mathcal{D})$.

(v) *If $v \in \Lambda_\rho$, then $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$; in particular, for every $d \in \mathcal{D}$,*

$$\pi_\rho(d)(v+L_\rho) = \frac{\rho(v^*dv)}{\rho(v^*v)}v + L_\rho.$$

(vi) *The set $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is weak* closed in $\mathcal{C}^{\#}$ and the restriction map, $\mathfrak{S}(\mathcal{C},\mathcal{D}) \ni \rho \mapsto \rho|_{\mathcal{D}}$, is a weak*-weak* continuous map from $\mathfrak{S}(\mathcal{C},\mathcal{D})$ into $\widehat{\mathcal{D}}$.*

Proof. (i) Since $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, a calculation yields $v - \rho(v)I \in L_\rho$. But L_ρ is a left ideal and $L_\rho \subseteq \ker\rho$. So for $x \in \mathcal{C}$, we have $\rho(x(v-\rho(v)I)) = 0$. So $\rho(xv) = \rho(x)\rho(v)$. As $\rho(v^*) = \overline{\rho(v)} \neq 0$, a similar argument shows that $0 = \rho((v-\rho(v)I)x)$. The equality $\rho(v^*xv) = \rho(v^*v)\rho(x)$ now follows from the fact that Δ_ρ is closed under the adjoint operation. So part (i) holds.

(ii) Since $\mathcal{D} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$, this follows from part (i) and continuity of ρ.

(iii) Let $\sigma = \rho|_{\mathcal{D}}$ and for $i = 1,2$ put $\sigma_i = \beta_{v_i}(\sigma)$. Then $\sigma_1 = \sigma_2$ by statement (i) and Proposition 2.3. Therefore, since $\rho(v_1^*v_2) \neq 0$, we have

$$|\rho(v_1^*v_2)|^2 = \rho(v_2^*v_1v_1^*v_2) = \sigma_2(v_1v_1^*)\sigma(v_2^*v_2) = \sigma_1(v_1v_1^*)\sigma(v_2^*v_2) = \rho(v_1^*v_1)\rho(v_2^*v_2).$$

Clearly the relation $\sim_\rho$ is reflexive and symmetric on Λ_ρ. For $i = 1,2,3$, suppose $v_i \in \Lambda_\rho$, $(v_1,v_2) \in \sim_\rho$ and $(v_2,v_3) \in \sim_\rho$. The equality verified in the previous paragraph shows that in $\mathcal{H}_\rho$, $|\langle v_1 + L_\rho, v_2 + L_\rho\rangle_\rho|^2 = \|v_1 + L_\rho\|_\rho^2\,\|v_2 + L_\rho\|_\rho^2$. Hence there exists a non-zero scalar t such that $tv_1 + L_\rho = v_2 + L_\rho$. Similarly, there exists a non-zero scalar s such that $v_2 + L_\rho = sv_3 + L_\rho$. So $\{v_1 + L_\rho, v_3 + L_\rho\}$ is a linearly dependent set of non-zero vectors in $\mathcal{H}_\rho$. Thus $\rho(v_1^*v_3) \neq 0$, whence $(v_1,v_3) \sim_\rho$.

(iv) Let $v \in \Lambda_\rho$. For $w \in \mathcal{N}(\mathcal{C},\mathcal{D})$, we claim $|\rho(w^*v)|^2 \in \{0, \rho(w^*w)\rho(v^*v)\}$. If $\rho(w^*v) \neq 0$, then as $|\rho(w^*v)|^2 \leqslant \rho(w^*w)\rho(v^*v)$, we find that $w \in \Lambda_\rho$ and

$w \sim_\rho v$, so the claim holds by statement (iii). Hence

$$|\beta_v(\rho)(w)|^2 = \frac{|\rho(v^*(wv))|^2}{\rho(v^*v)^2} \in \left\{0, \frac{\rho(v^*v)\rho(v^*w^*wv)}{\rho(v^*v)^2}\right\} = \{0, \beta_v(\rho)(w^*w)\},$$

so $\beta_v(\rho) \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

(v) Suppose $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and that $\rho(v^*v) \neq 0$. For $d \in \mathcal{D}$, let $\sigma_1(d) = \rho(v^*dv)/\rho(v^*v)$. Then $\sigma_1 \in \widehat{\mathcal{D}}$, and for $d \in \mathcal{D}$, we have

$$\begin{aligned}\|(\pi_\rho(d) - \sigma_1(d)I)v + L_\rho\|_\rho^2 &= \rho(v^*(d - \sigma_1(d)I)^*(d - \sigma_1(d)I)v) \\ &= \sigma_1((d - \sigma_1(d)I)^*(d - \sigma_1(d)I))\rho(v^*v) = 0.\end{aligned}$$

We conclude that $\pi_\rho(d)v + L_\rho = \sigma_1(d)v + L_\rho$, so $v + L_\rho$ is an eigenvector for $\pi_\rho(\mathcal{D})$ and statement (v) holds.

(vi) Suppose $(\rho_\lambda)_{\lambda \in \Lambda}$ is a net in $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that ρ_λ converges weak* to $\rho \in \mathcal{C}^\#$. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$. If $\rho(v) \neq 0$, then for large enough λ, $\rho_\lambda(v) \neq 0$. Hence $|\rho(v)|^2 = \lim_\lambda |\rho_\lambda(v)|^2 = \lim_\lambda \rho_\lambda(v^*v) = \rho(v^*v)$. It follows that $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, so $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ is weak* closed. The continuity of the restriction mapping is obvious. ∎

REMARK 4.5. Statement (i) says that if $v \in \Delta_\rho$, then $v \in \mathfrak{M}_\rho$, where

$$\mathfrak{M}_\rho = \{x \in \mathcal{C} : \rho(xy) = \rho(yx) = \rho(x)\rho(y)\ \forall\, y \in \mathcal{C}\},$$

see [2]. Also, if $\mathcal{B}$ is the closed linear span of Δ_ρ, then $\mathcal{B}$ is a C^*-algebra because Δ_ρ is closed under multiplication. Clearly $\mathcal{D} \subseteq \mathcal{B}$, so that $(\mathcal{B}, \mathcal{D})$ is an inclusion enjoying the properties of regularity or MASA inclusion when $(\mathcal{C}, \mathcal{D})$ has the same properties.

PROPOSITION 4.6. *Let $(\mathcal{C}, \mathcal{D})$ be an inclusion (not assumed regular) such that $\mathcal{D}$ is a* MASA *in* span $\mathcal{N}(\mathcal{C}, \mathcal{D})$ *and let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$.*

(i) *If $E : \mathcal{C} \to I(\mathcal{D})$ is a pseudo-expectation, then*

$$\{\sigma \circ E : \sigma \in \widehat{I(\mathcal{D})}\} \subseteq \mathfrak{S}(\mathcal{C}, \mathcal{D}).$$

(ii) *When $E : \mathcal{C} \to \mathcal{D}$ is a conditional expectation, $E^\#|_{\widehat{\mathcal{D}}}$ is a continuous one-to-one map of $\widehat{\mathcal{D}}$ into $\mathfrak{S}(\mathcal{C}, \mathcal{D})$. If in addition, $(\mathcal{C}, \mathcal{D})$ has the extension property, $E^\#|_{\widehat{\mathcal{D}}}$ is a homeomorphism of $\widehat{\mathcal{D}}$ onto $\mathfrak{S}(\mathcal{C}, \mathcal{D})$.*

Proof. (i) Let $\mathcal{C}_1 := \overline{\text{span}}(\mathcal{N}(\mathcal{C}, \mathcal{D}))$. Then $(\mathcal{C}_1, \mathcal{D})$ is a skeletal MASA inclusion and $\mathcal{N}(\mathcal{C}, \mathcal{D}) = \mathcal{N}(\mathcal{C}_1, \mathcal{D})$. By Theorem 3.5, $E|_{\mathcal{C}_1}$ is the unique pseudo-expectation for $(\mathcal{C}_1, \mathcal{D})$. Let $\sigma \in \widehat{I(\mathcal{D})}$ and put $\rho = \sigma \circ E$. Suppose $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ is such that $\rho(v) \neq 0$. Then $0 \neq |\sigma(E(v))|^2$. Using Theorem 3.5(iii), $|\rho(v)|^2 = \sigma(|E(v)|^2) = \sigma(\iota(v^*v)) = \sigma(E(v^*v)) = \rho(v^*v)$. Thus, $\sigma \circ E \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.

(ii) Since E is onto, $E^\#$ is one-to-one and continuous, and by part (i), the image of $\widehat{\mathcal{D}}$ under $E^\#$ is contained in $\mathfrak{S}(\mathcal{C}, \mathcal{D})$.

Now suppose that $(\mathcal{C}, \mathcal{D})$ has the extension property. If $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$, put $\sigma = \rho|_{\mathcal{D}}$. Then $\sigma \in \widehat{\mathcal{D}}$. By the extension property, we have $\rho = \sigma \circ E$, so $\rho = E^\#(\sigma)$, whence $E^{\#}|_{\widehat{\mathcal{D}}}$ is onto. So $E^{\#}$ is a continuous bijection of $\widehat{\mathcal{D}}$ onto $\mathfrak{S}(\mathcal{C},\mathcal{D})$. Since $\widehat{\mathcal{D}}$ and $\mathfrak{S}(\mathcal{C},\mathcal{D})$ are both compact and Hausdorff, $E^{\#}|_{\widehat{\mathcal{D}}}$ is a homeomorphism. ∎

Our next goal is to show that the construction of elements of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ found in Proposition 4.6(i), can be modified to show that every $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$ arises from a suitable representation π of $\mathcal{C}$ together with a conditional expectation of $\mathcal{B}(\mathcal{H}_\pi)$ onto $\pi(\mathcal{D})''$, see Theorem 4.9. To achieve this, it is necessary to consider representations arising from states in $\mathfrak{S}(\mathcal{C},\mathcal{D})$. We begin with a simple lemma concerning states on regular inclusions, whose proof we leave to the reader.

LEMMA 4.7. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion and suppose that ρ is a state on $\mathcal{C}$. Then*

$$\operatorname{span}\{v + L_\rho : v \in \mathcal{N}(\mathcal{C},\mathcal{D}),\ \rho(v^*v) > 0\}$$

is norm-dense in $\mathcal{H}_\rho$.

PROPOSITION 4.8. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion, let $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, and let $T \subseteq \Lambda_\rho$ be chosen so that for every $v \in T$, $\rho(v^*v) = 1$ and T contains exactly one element from each $\sim_\rho$ equivalence class. Then the following statements hold:*

(i) *$\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$.*

(ii) *For $v \in T$, let $\mathcal{K}_v := \{\xi \in \mathcal{H}_\rho : \pi_\rho(d)\xi = \rho(v^*dv)\xi$ for all $d \in \mathcal{D}\}$ and let $\sigma = \rho|_{\mathcal{D}}$. Then $\mathcal{K}_v = \overline{\operatorname{span}}\{w + L_\rho : w \in T$ and $\beta_w(\sigma) = \beta_v(\sigma)\}$.*

(iii) *For $v \in T$, let P_v be the orthogonal projection of $\mathcal{H}_\rho$ onto $\mathcal{K}_v$. Then P_v is a minimal projection in $\pi_\rho(\mathcal{D})''$ and $\bigvee_{v\in T} P_v = I$.*

(iv) *$\pi_\rho(\mathcal{D})''$ is an abelian and atomic von Neumann algebra.*

Proof. (i) If $v, w \in T$ are distinct, then $\rho(v^*w) = 0$, so that $\{v + L_\rho : v \in T\}$ is an orthonormal set. Part (iii) of Proposition 4.4 and the Cauchy–Schwarz inequality show that if $v \in T$, and $w \in \Lambda_\rho$ is such that $v \sim_\rho w$, then $w + L_\rho \in \operatorname{span}\{v + L_\rho\}$. This, together with Lemma 4.7, shows that

$$\overline{\operatorname{span}}\{v+L_\rho : v\in T\} = \overline{\operatorname{span}}\{v+L_\rho : v\in\Lambda_\rho\} = \overline{\operatorname{span}}\{v+L_\rho : v\in\mathcal{N}(\mathcal{C},\mathcal{D})\} = \mathcal{H}_\rho.$$

Thus $\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$.

(ii) If $\xi \in \overline{\operatorname{span}}\{w + L_\rho : w \in T$ and $\beta_w(\sigma) = \beta_v(\sigma)\}$, part (v) of Proposition 4.4 implies that $\xi \in \mathcal{K}_v$. For the opposite inclusion, suppose $\xi \in \mathcal{K}_v$. Then for $w \in T$ and $d \in \mathcal{D}$ we have

$$\beta_v(\sigma)(d)\langle \xi, w+L_\rho\rangle = \langle \pi_\rho(d)\xi, w+L_\rho\rangle = \langle \xi, \pi_\rho(d^*)(w+L_\rho)\rangle = \beta_w(\rho)(d)\langle \xi, w+L_\rho\rangle.$$

Hence if $\langle \xi, w + L_\rho\rangle \neq 0$, then $\beta_v(\sigma) = \beta_w(\sigma)$. This yields

$$\xi \in \overline{\operatorname{span}}\{w + L_\rho : w \in T \text{ and } \beta_w(\sigma) = \beta_v(\sigma)\}.$$

(iii) First note that for $v \in T$, $v + L_\rho \in \mathcal{K}_v$; thus, since $\{v + L_\rho : v \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$, we obtain $\bigvee_{v\in T} P_v = I$.

Let $X \in \pi_\rho(\mathcal{D})'$ and $\xi \in \mathcal{K}_v$. Then for $d \in \mathcal{D}$,

$$\pi_\rho(d)X\xi = X\pi_\rho(d)\xi = \rho(v^*dv)X\xi.$$

Therefore $X\xi \in \mathcal{K}_v$, showing that $\mathcal{K}_v$ is an invariant subspace for X. As this holds for every $X \in \pi_\rho(\mathcal{D})'$, we conclude that $P_v \in \pi_\rho(\mathcal{D})''$.

Let $v \in T$ and suppose that $Q \in \pi_\rho(\mathcal{D})''$ is a projection with $0 \leqslant Q \leqslant P_v$. For all $d \in \mathcal{D}$ we have

$$\pi_\rho(d)P_v = \beta_v(\sigma)(d)P_v = \langle \pi_\rho(d)(v + L_\rho), v + L_\rho \rangle P_v.$$

The Kaplansky density theorem shows that for every $X \in \pi_\rho(\mathcal{D})''$ we have $XP_v = \langle X(v + L_\rho), v + L_\rho \rangle P_v \in \mathbb{C}P_v$. Since Q commutes with P_v, QP_v is a projection; hence $QP_v \in \{0, P_v\}$, so P_v is a minimal projection in $\pi_\rho(\mathcal{D})''$.

(iv) This follows from statement (iii). ∎

The following result shows that elements of $\mathfrak{S}(\mathcal{C}, \mathcal{D})$ arise from regular representations π of $(\mathcal{C}, \mathcal{D})$, which can be taken so that $\pi(\mathcal{D})''$ is atomic. For vectors h_1, h_2 in a Hilbert space $\mathcal{H}$ we use the notation $h_1 h_2^*$ for the rank-one operator $h \mapsto \langle h, h_2 \rangle h_1$.

THEOREM 4.9. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion. The following statements hold:*

(i) *Let $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ and let*

$$\mathcal{A}_\rho := \{(v + L_\rho)(v + L_\rho)^* : v \in \mathcal{N}(\mathcal{C}, \mathcal{D})\}'' \subseteq \mathcal{B}(\mathcal{H}_\rho).$$

Then $\mathcal{A}_\rho$ is an atomic MASA *in $\mathcal{B}(\mathcal{H}_\rho)$ and $\pi_\rho : (\mathcal{C}, \mathcal{D}) \to (\mathcal{B}(\mathcal{H}_\rho), \mathcal{A}_\rho)$ is a regular $*$-homomorphism.*

(ii) *Conversely, suppose $\pi : \mathcal{C} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism with $\pi(\mathcal{D})''$ a (not necessarily atomic)* MASA *in $\mathcal{B}(\mathcal{H})$, and let $E : \mathcal{B}(\mathcal{H}) \to \pi(\mathcal{D})''$ be any conditional expectation. Then for any pure state σ of $\pi(\mathcal{D})''$, $\sigma \circ E \circ \pi \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$.*

Proof. For the first statement, choose T as in the statement of Proposition 4.8. For $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, we have $v + L_\rho = 0$ if $v \notin \Lambda_\rho$. When $v \in \Lambda_\rho$, there exists $w \in T$ such that $v \sim_\rho w$, so $(v + L_\rho)(v + L_\rho)^* \in \mathbb{C}(w + L_\rho)(w + L_\rho)^*$. Since $\mathcal{B} := \{w + L_\rho : w \in T\}$ is an orthonormal basis for $\mathcal{H}_\rho$, we see that $\mathcal{A}_\rho$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$.

We now show that π_ρ is a regular homomorphism. Let $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$ and let $w \in T$. Then $\pi_\rho(v)(w + L_\rho)(w + L_\rho)^* \pi_\rho(v)^* = (vw + L_\rho)(vw + L_\rho)^* \in \mathcal{A}_\rho$. As $\operatorname{span}\{(w + L_\rho)(w + L_\rho)^* : w \in T\}$ is weakly dense in $\mathcal{A}_\rho$, we conclude that $\pi_\rho(v)\mathcal{A}_\rho\pi_\rho(v)^* \subseteq \mathcal{A}_\rho$. Similarly $\pi_\rho(v)^*\mathcal{A}_\rho\pi_\rho(v) \subseteq \mathcal{A}_\rho$. Thus π_ρ is a regular $*$-homomorphism.

For the second statement, Proposition 4.6 shows that if $\sigma \in \widehat{\pi_\rho(\mathcal{D})''}$, then $\sigma \circ E \in \mathfrak{S}(\mathcal{B}(\mathcal{H}), \pi(\mathcal{D})'')$. Remark 4.3(iii) completes the proof. ∎

REMARK 4.10. We have $\pi_\rho(\mathcal{D})'' \subseteq \mathcal{A}_\rho$ always, but in general they can be very different. In fact, there exist a Cartan pair $(\mathcal{C}, \mathcal{D})$ and $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$ such that $\pi_\rho(\mathcal{D})'' = \mathbb{C}I$, while $\mathcal{A}_\rho$ is a MASA.

The following proposition characterizes when $\pi_\rho(\mathcal{D})''$ and $\mathcal{A}_\rho$ coincide. We first make a definition.

DEFINITION 4.11. Let $(\mathcal{C}, \mathcal{D})$ be an inclusion and let $f \in \operatorname{Mod}(\mathcal{C}, \mathcal{D})$. The *$\mathcal{D}$-stabilizer* of f is the set,

$$\mathcal{D}\text{-stab}(f) := \{v \in \mathcal{N}(\mathcal{C}, \mathcal{D}) : \text{for all } d \in \mathcal{D}, f(v^*dv) = f(d)\}.$$

If for every $v \in \mathcal{D}$-stab(f) and $x \in \mathcal{C}$, we have $f(x) = f(v^*xv)$, then we call f a *$\mathcal{D}$-rigid state*.

PROPOSITION 4.12. *Let $(\mathcal{C}, \mathcal{D})$ be a regular inclusion, and suppose that $\rho \in \mathfrak{S}(\mathcal{C}, \mathcal{D})$. The following statements are equivalent:*
(i) $\pi_\rho(\mathcal{D})''$ *is a* MASA *in* $\mathcal{B}(\mathcal{H}_\rho)$.
(ii) *If $v \in \mathcal{D}$-stab(ρ), then $\rho(v) \neq 0$.*
(iii) ρ *is a pure and $\mathcal{D}$-rigid state.*

Proof. Throughout the proof, we let $\sigma = \rho|_{\mathcal{D}}$, which by Proposition 4.4(ii), belongs to $\widehat{\mathcal{D}}$.

Suppose $\pi_\rho(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\rho)$ and let $v \in \mathcal{D}$-stab(ρ), so $v \in \Lambda_\rho$ and $\beta_v(\sigma) = \sigma$. Then, using the notation of Proposition 4.8, we find that $P_I(v + L_\rho) = v + L_\rho$. Since $\pi_\rho(\mathcal{D})''$ is a MASA, P_I is the orthogonal projection onto $\mathbb{C}(I + L_\rho)$. We conclude that $v + L_\rho$ is a non-zero scalar multiple of $I + L_\rho$. Hence $0 \neq \langle v + L_\rho, I + L_\rho\rangle = \rho(v)$. Thus $v \in \Delta_\rho$, so statement (i) implies statement (ii).

Now suppose statement (ii) holds. We first prove that ρ is pure. So suppose that $t \in [0, 1]$ and that for $i = 1, 2$, ρ_i are states on $\mathcal{C}$ and $\rho = t\rho_1 + (1 - t)\rho_2$. As $\rho|_{\mathcal{D}}$ is a pure state on $\mathcal{D}$, we have $\rho_i|_{\mathcal{D}} = \sigma$. We claim that for every $v \in \mathcal{N}(\mathcal{C}, \mathcal{D})$, $\rho_1(v) = \rho_2(v) = \rho(v)$. By Lemma 2.5, we need only prove this for $v \in \Lambda_\rho$ such that $\beta_v(\sigma) = \sigma$ and $\rho(v^*v) = 1$. So suppose v has this property. By the hypothesis in statement (ii), $\rho(v) \neq 0$. Clearly $|\rho_i(v)| \leqslant \rho_i(v^*v)^{1/2} = \rho(v^*v)^{1/2} = |\rho(v)|$. Thus we have

$$t\frac{\rho_1(v)}{\rho(v)} + (1 - t)\frac{\rho_2(v)}{\rho(v)} = 1,$$

which expresses 1 as a convex combination of elements of the closed unit disk. Hence $\rho_i(v) = \rho(v)$, establishing the claim. By regularity, we conclude that $\rho_1 = \rho_2 = \rho$, so ρ is a pure state.

Next, if $v \in \Lambda_\rho$ and $\beta_v(\sigma) = \sigma$, then by hypothesis, $\rho(v) \neq 0$. So the final part of statement (iii) follows from part (i) of Proposition 4.4. Thus statement (ii) implies statement (iii).

Finally, suppose that statement (iii) holds. Let $v, w \in \Lambda_\rho$ be such that $\beta_v(\sigma) = \beta_w(\sigma)$. We shall show that $\{v + L_\rho, w + L_\rho\}$ is a linearly dependent set, showing that $\mathcal{K}_v$ is one-dimensional. We have $\beta_{w^*v}(\sigma) = \sigma = \beta_{v^*w}(\sigma)$, so

$$\rho(v^*ww^*v)^{-1/2}w^*v \in \mathcal{D}\text{-stab}(\rho).$$

By hypothesis, $\rho(x) = \rho(v^*wxw^*v)(\rho(v^*ww^*v))^{-1}$ for every $x \in \mathcal{C}$. Thus if $\eta = \rho(v^*ww^*v)^{-1/2}w^*v + L_\rho$, we have $\langle \pi_\rho(x)\eta, \eta\rangle = \langle \pi_\rho(x)(I + L_\rho), I + L_\rho\rangle$ for every $x \in \mathcal{C}$. Since ρ is pure, $\pi_\rho(\mathcal{C})'' = \mathcal{B}(\mathcal{H}_\rho)$, so that for every $X \in \mathcal{B}(\mathcal{H}_\rho)$ we obtain

$$\langle X\eta, \eta\rangle = \langle X(I + L_\rho), I + L_\rho\rangle.$$

Hence $\{\eta, I+L_\rho\}$ is a linearly dependent set. Thus, $\{w^*v+L_\rho, I+L_\rho\}$ is linearly dependent. Since both vectors in this set are non-zero, we find

$$0 \neq \langle w^*v+L_\rho, I+L_\rho\rangle = \rho(w^*v).$$

Applying part (iii) of Proposition 4.4 and the Cauchy–Schwarz inequality, we obtain $\{v+L_\rho, w+L_\rho\}$ is linearly dependent, as desired.

As $\mathcal{K}_v$ is one-dimensional, Proposition 4.8 implies $\pi_\rho(\mathcal{D})''$ is a MASA. ∎

We turn now to a result which shows that there are inclusions with few compatible states. In fact, some inclusions have no compatible states. This result applies when the relative commutant of $\mathcal{D}$ in $\mathcal{C}$ is all of $\mathcal{C}$, e.g. $(\mathcal{C}, \mathbb{C}I)$. The result shows that when $\mathcal{N}(\mathcal{C},\mathcal{D})$ is too large, it may happen that $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is empty. For example, when $\mathcal{C}$ is a unital simple C^*-algebra with $\dim(\mathcal{C})>1$, then $\mathfrak{S}(\mathcal{C},\mathbb{C}I)=\emptyset$.

THEOREM 4.13. *Let $(\mathcal{C},\mathcal{D})$ be an inclusion and let $\mathcal{U}(\mathcal{C})$ be the unitary group of $\mathcal{C}$. Assume that $\mathcal{U}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$. Then $(\mathcal{C},\mathcal{D})$ is regular and $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is the set of all multiplicative linear functionals on $\mathcal{C}$.*

Proof. Since $\operatorname{span}(\mathcal{U}(\mathcal{C})) = \mathcal{C}$, $(\mathcal{C},\mathcal{D})$ is a regular inclusion. As every multiplicative linear functional on $\mathcal{C}$ is a compatible state, we need only prove that every element of $\mathfrak{S}(\mathcal{C},\mathcal{D})$ is a multiplicative linear functional.

Fix $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$. Then for every unitary $U \in \mathcal{C}$ we have $\rho(U) \in \{0\} \cup \mathbb{T}$. Let π be a universal representation of $\mathcal{C}$, and identify $\mathcal{C}^{\#\#}$ with the von Neumann algebra $\pi(\mathcal{C})''$. Also, regard $\mathcal{C}$ as a subalgebra of $\mathcal{C}^{\#\#}$. Let $\rho^{\#\#}$ denote the normal state on $\mathcal{C}^{\#\#}$ obtained from ρ. By II.4.11 of [34], every unitary in $\mathcal{C}^{\#\#}$ is the strong* limit of a net of unitaries in $\mathcal{C}$. Since $\rho^{\#\#}$ is normal, $\rho^{\#\#}(W) \in \{0\} \cup \mathbb{T}$ for every unitary $W \in \mathcal{C}^{\#\#}$.

Let P be a projection in $\mathcal{C}^{\#\#}$. We shall show that $\rho^{\#\#}(P) \in \{0,1\}$. We argue by contradiction. Suppose that $0 < \rho^{\#\#}(P) < 1$. Then

$$0 < |\rho^{\#\#}(P) + \mathrm{i}\rho^{\#\#}(I-P)| < 1.$$

Put $W = P + \mathrm{i}(I-P)$. Then W is a unitary belonging to $\mathcal{C}^{\#\#}$, and therefore we may find a net U_α of unitaries in $\mathcal{C}$ so that U_α converges strong* to W. But then $|\rho(U_\alpha)| \to |\rho^{\#\#}(W)| \in (0,1)$. This implies that there exists a unitary $U \in \mathcal{C}$ such that $|\rho(U)| \in (0,1)$, which is a contradiction. Therefore $\rho^{\#\#}(P) \in \{0,1\}$ for every projection $P \in \mathcal{C}^{\#\#}$.

Now let $P, Q \in \mathcal{C}^{\#\#}$ be projections. We claim that

$$\rho^{\#\#}(PQ) = \rho^{\#\#}(P)\rho^{\#\#}(Q).$$

By the Cauchy–Schwarz inequality, $|\rho^{\#\#}(PQ)| \leqslant \rho^{\#\#}(P)\rho^{\#\#}(Q)$, so that $\rho^{\#\#}(PQ) = 0$ if $0 \in \{\rho^{\#\#}(P), \rho^{\#\#}(Q)\}$. Suppose then that $\rho^{\#\#}(P) = \rho^{\#\#}(Q) = 1$. Since $2P-I$ and $2Q-I$ are unitaries in $\mathcal{C}^{\#\#}$, we may find nets of unitaries u_α and v_α in $\mathcal{C}$ so that u_α and v_α converge $*$-strongly to $2P-I$ and $2Q-I$ respectively. Both $\rho(u_\alpha)$ and $\rho(v_\alpha)$ are eventually non-zero because

$$\lim \rho(u_\alpha) = \rho^{\#\#}(2P-I) = 1 = \rho^{\#\#}(2Q-I) = \lim \rho(v_\alpha).$$

As multiplication on bounded subsets of $\mathcal{C}^{\#\#}$ is jointly continuous in the strong* topology, $u_\alpha v_\alpha$ converges strongly to $(2P-I)(2Q-I)$. By Proposition 4.4(i),

$$\rho^{\#\#}((2P-I)(2Q-I)) = \lim \rho(u_\alpha v_\alpha) = \lim \rho(u_\alpha)\rho(v_\alpha) = \rho^{\#\#}(2P-I)\rho^{\#\#}(2Q-I) = 1.$$

On the other hand, a calculation shows that

$$\rho^{\#\#}((2P-I)(2Q-I)) = 4\rho^{\#\#}(PQ) - 3.$$

Combining these equalities gives $\rho^{\#\#}(PQ) = 1$, as desired. The claim follows.

Let $X = \sum_{j=1}^{n} \lambda_j P_j$ and $Y = \sum_{j=1}^{n} \mu_j Q_j$ be linear combinations of projections $\{P_j\}_{j=1}^n$ and $\{Q_j\}_{j=1}^n$ in $\mathcal{C}^{\#\#}$. It follows from the previous paragraph that $\rho^{\#\#}(XY) = \rho^{\#\#}(X)\rho^{\#\#}(Y)$. Since any von Neumann algebra is the norm closure of the span of its projections, $\rho^{\#\#}$ is multiplicative on $\mathcal{C}^{\#\#}$. It then follows that ρ is multiplicative on $\mathcal{C}$. ∎

5. EMBEDDING THEOREMS

Our purpose in this section is to prove a pair of embedding theorems. We characterize when a regular inclusion regularly embeds into a MASA inclusion, see Theorem 5.4. Also, we characterize when a regular inclusion regularly embeds into a C^*-diagonal, Theorem 5.7.

Proposition 4.6 shows compatible states exist in abundance for any MASA inclusion; in fact, if $r : \mathfrak{S}_s(\mathcal{C},\mathcal{D}) \to \widehat{\mathcal{D}}$ is the map $r(\rho) := \rho|_{\mathcal{D}}$, then $(\mathfrak{S}_s(\mathcal{C},\mathcal{D}), r)$ is an essential cover of $\widehat{\mathcal{D}}$. Moreover, for any regular MASA inclusion $(\mathcal{C},\mathcal{D})$, $\mathcal{L}(\mathcal{C},\mathcal{D})$ is the intersection of the left kernels of the elements of $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$, that is,

$$\mathcal{L}(\mathcal{C},\mathcal{D}) = \{x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in \mathfrak{S}_s(\mathcal{C},\mathcal{D})\}. \tag{5.1}$$

We wish to replace the family $\mathfrak{S}_s(\mathcal{C},\mathcal{D})$ with $\mathfrak{S}(\mathcal{C},\mathcal{D})$ in this formula. For regular inclusions, this produces an ideal which we show can be used to determine whether the inclusion embeds into a C^*-diagonal. Here is the general definition.

DEFINITION 5.1. For an inclusion $(\mathcal{C},\mathcal{D})$, the *$\mathcal{D}$-radical of* $(\mathcal{C},\mathcal{D})$ is the set

$$\mathrm{Rad}(\mathcal{C},\mathcal{D}) := \{x \in \mathcal{C} : \pi_\rho(x) = 0 \text{ for all } \rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})\},$$

provided $\mathfrak{S}(\mathcal{C},\mathcal{D}) \neq \emptyset$; otherwise define $\mathrm{Rad}(\mathcal{C},\mathcal{D}) = \mathcal{C}$.

When $(\mathcal{C},\mathcal{D})$ is a regular inclusion, we have the following description of $\mathrm{Rad}(\mathcal{C},\mathcal{D})$.

PROPOSITION 5.2. *Suppose that $(\mathcal{C},\mathcal{D})$ is a regular inclusion. Then*

$$\mathrm{Rad}(\mathcal{C},\mathcal{D}) = \{x \in \mathcal{C} : \rho(x^*x) = 0 \textit{ for all } \rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})\}.$$

Proof. Let $J := \{x \in \mathcal{C} : \rho(x^*x) = 0 \text{ for all } \rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})\}$. If $x \in \mathrm{Rad}(\mathcal{C},\mathcal{D})$ and $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, then $\rho(x^*x) = \|\pi_\rho(x)(I+L_\rho)\|^2 = 0$, so $\mathrm{Rad}(\mathcal{C},\mathcal{D}) \subseteq J$. For the opposite inclusion, let $x \in J$. Part (iv) of Proposition 4.4 and Corollary 2.7 show that J is a closed, two-sided ideal of $\mathcal{C}$. Hence for every $c \in \mathcal{C}$ and $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$ we have $\rho(c^*x^*xc) = 0$, which means that $\pi_\rho(x) = 0$ for every ρ. So $x \in \mathrm{Rad}(\mathcal{C},\mathcal{D})$, showing $\mathrm{Rad}(\mathcal{C},\mathcal{D}) = J$. ∎

PROPOSITION 5.3. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion, and let $(\mathcal{C}_1,\mathcal{D}_1)$ be a regular inclusion such that $\mathcal{D}_1$ is maximal abelian in the linear span of $\mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)$. Suppose $\alpha : \mathcal{C} \to \mathcal{C}_1$ is a regular (unital) $*$-homomorphism. The following statements hold:*

(i) $\mathrm{Rad}(\mathcal{C},\mathcal{D}) \subseteq \alpha^{-1}(\mathcal{L}(\mathcal{C}_1,\mathcal{D}_1))$*; and*

(ii) $\alpha(\mathcal{D}^c) \subseteq \mathcal{D}_1$.

Proof. (i) By Remark 4.3(iii), $\alpha^\#$ carries $\mathfrak{S}(\mathcal{C}_1,\mathcal{D}_1)$ into $\mathfrak{S}(\mathcal{C},\mathcal{D})$, and hence $\alpha^\#(\mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1)) \subseteq \mathfrak{S}(\mathcal{C},\mathcal{D})$. Thus if $x \in \mathrm{Rad}(\mathcal{C},\mathcal{D})$, then for every $\rho \in \mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1)$, $\rho(\alpha(x^*x)) = 0$, so $\alpha(x) \in \mathcal{L}(\mathcal{C}_1,\mathcal{D}_1)$.

(ii) Let $(I(\mathcal{D}_1),\iota_1)$ be an injective envelope for $\mathcal{D}_1$ and let $E_1 : \mathcal{C}_1 \to I(\mathcal{C}_1)$ be the pseudo-expectation for ι_1.

Observe that $(\mathcal{D}^c,\mathcal{D})$ is a regular inclusion, and $\mathcal{N}(\mathcal{D}^c,\mathcal{D}) \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$. Let $\rho \in \mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1)$. As α is a regular homomorphism and $\mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1) \subseteq \mathfrak{S}(\mathcal{C}_1,\mathcal{D}_1)$, $\rho\circ\alpha \in \mathfrak{S}(\mathcal{C},\mathcal{D})$. Hence $\rho\circ\alpha|_{\mathcal{D}^c} \in \mathfrak{S}(\mathcal{D}^c,\mathcal{D})$. Theorem 4.13 implies $\rho\circ\alpha|_{\mathcal{D}^c}$ is a multiplicative linear functional on $\mathcal{D}^c$. By the definition of $\mathfrak{S}_s(\mathcal{C}_1,\mathcal{D}_1)$, we see that for every $\tau \in \widehat{I(\mathcal{D}_1)}$, $\tau\circ E_1\circ\alpha|_{\mathcal{D}^c}$ is a multiplicative linear functional on $\mathcal{D}^c$. We conclude $E_1\circ\alpha|_{\mathcal{D}^c}$ is a $*$-homomorphism of $\mathcal{D}^c$ into $I(\mathcal{D}_1)$.

Let $x \in \mathcal{D}^c$ be a unitary element. Then $x \in \mathcal{N}(\mathcal{C},\mathcal{D})$, so $\alpha(x) \in \mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)$. Since $E_1\circ\alpha$ is multiplicative on $\mathcal{D}^c$, $E_1(\alpha(x))$ is a unitary element of $I(\mathcal{D}_1)$. Hence the fixed point ideal (i.e. K_0) for $\alpha(x)$ is an essential ideal of $\mathcal{D}_1$. But the fixed point ideal is also a regular ideal in $\mathcal{D}_1$, so $K_0 = \mathcal{D}_1$. By Lemma 2.15, $\alpha(x)$ commutes with $\mathcal{D}_1$. As $\mathcal{D}_1$ is a MASA in $\operatorname{span}\mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)$, $\alpha(x) \in \mathcal{D}_1$. As every element of $\mathcal{D}^c$ is a linear combination of four unitary elements of $\mathcal{D}^c$, part (ii) follows. ∎

We now give the characterization of when a regular inclusion regularly embeds into a MASA inclusion.

THEOREM 5.4. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion. The following are equivalent:*

(i) *There exists a regular* MASA*-inclusion $(\mathcal{C}_1,\mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : \mathcal{C} \to \mathcal{C}_1$.*

(ii) *There exists a regular inclusion $(\mathcal{C}_1,\mathcal{D}_1)$ such that $\mathcal{D}_1$ is maximal abelian in the linear span of $\mathcal{N}(\mathcal{C}_1,\mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : \mathcal{C} \to \mathcal{C}_1$.*

(iii) *The relative commutant of $\mathcal{D}$ in $\mathcal{C}$ is abelian.*

Proof. That (i) implies (ii) is trivial. Suppose (ii) holds. Proposition 5.3(ii) shows that $\alpha(\mathcal{D}^c) \subseteq \mathcal{D}_1$. As α is one-to-one, $\mathcal{D}^c$ is abelian. The implication (iii) $\Rightarrow$ (i) follows from Lemma 2.10. ∎

The following is immediate, but interesting as it shows the role of $\mathcal{N}(\mathcal{C},\mathcal{D})$ in determining whether $\mathcal{D}^c$ is abelian.

COROLLARY 5.5. *Suppose $(\mathcal{C},\mathcal{D})$ is a regular inclusion such that $\mathcal{D}$ is maximal abelian in* $\operatorname{span}\mathcal{N}(\mathcal{C},\mathcal{D})$. *Then $\mathcal{D}^c$ is abelian.*

We now turn to the problem of regularly embedding a regular inclusion into a C^*-diagonal. The following result shows that one can construct a C^*-diagonal and a regular $*$-homomorphism such that the inclusion in Proposition 5.3(i) is an equality.

THEOREM 5.6. *For a regular inclusion $(\mathcal{C},\mathcal{D})$, there exists a C^*-diagonal $(\mathcal{C}_1,\mathcal{D}_1)$ and a regular $*$-homomorphism $\alpha : (\mathcal{C},\mathcal{D}) \to (\mathcal{C}_1,\mathcal{D}_1)$ with* $\ker\alpha = \operatorname{Rad}(\mathcal{C},\mathcal{D})$.

Proof. For each $\rho \in \mathfrak{S}(\mathcal{C},\mathcal{D})$, let $(\pi_\rho,\mathcal{H}_\rho)$ be the GNS representation of $\mathcal{C}$ arising from ρ. Let $\mathcal{H} := \bigoplus_{\rho\in\mathfrak{S}(\mathcal{C},\mathcal{D})} \mathcal{H}_\rho$ and let $\mathcal{D}_1 = \bigoplus_{\rho\in\mathfrak{S}(\mathcal{C},\mathcal{D})} \mathcal{A}_\rho$, where $\mathcal{A}_\rho$ is as in the statement of Theorem 4.9. As $\mathcal{A}_\rho$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\rho)$, we see that $\mathcal{D}_1$ is an atomic MASA in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{C}_1 = \overline{\operatorname{span}}\mathcal{N}(\mathcal{B}(\mathcal{H}),\mathcal{D}_1)$. By Theorem 2.21 and the fact that the expectation onto an atomic MASA in $\mathcal{B}(\mathcal{H})$ is faithful, $(\mathcal{C}_1,\mathcal{D}_1)$ is a C^*-diagonal.

For each $v \in \mathcal{N}(\mathcal{C},\mathcal{D})$, the regularity of π_ρ (Theorem 4.9) yields

$$\bigoplus_{\rho\in\mathfrak{S}(\mathcal{C},\mathcal{D})} \pi_\rho(v) \in \mathcal{N}(\mathcal{B}(\mathcal{H}),\mathcal{D}_1).$$

Hence for each $x \in \mathcal{C}$, $\bigoplus_{\rho\in\mathfrak{S}(\mathcal{C},\mathcal{D})} \pi_\rho(x) \in \mathcal{C}_1$. Thus if $\alpha : \mathcal{C} \to \mathcal{C}_1$ is given by $\alpha(x) = \bigoplus_{\rho\in\mathfrak{S}(\mathcal{C},\mathcal{D})} \pi_\rho(x)$, then α is a regular $*$-homomorphism. By construction, $\ker\alpha = \operatorname{Rad}(\mathcal{C},\mathcal{D})$. ∎

We are now in a position to characterize when a regular inclusion embeds into a C^*-diagonal.

THEOREM 5.7. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion. There exists a C^*-diagonal $(\mathcal{C}_1,\mathcal{D}_1)$ and a regular $*$-monomorphism $\alpha : (\mathcal{C},\mathcal{D}) \to (\mathcal{C}_1,\mathcal{D}_1)$ if and only if* $\operatorname{Rad}(\mathcal{C},\mathcal{D})$ *is* (0).

Proof. Suppose that $(\mathcal{C}_1,\mathcal{D}_1)$ is a C^*-diagonal and $\alpha : (\mathcal{C},\mathcal{D}) \to (\mathcal{C}_1,\mathcal{D}_1)$ is a regular $*$-monomorphism. Since $\mathcal{L}(\mathcal{C}_1,\mathcal{D}_1) = (0)$, Proposition 5.3 gives $\operatorname{Rad}(\mathcal{C},\mathcal{D}) \subseteq \ker\alpha = (0)$.

The converse follows from Theorem 5.6. ∎

We close this section with two questions.

QUESTION 5.8. *Let $(\mathcal{C},\mathcal{D})$ be a regular inclusion such that $\mathcal{D}^c$ is abelian. Lemma* 2.10 *gives $\mathcal{N}(\mathcal{C},\mathcal{D}) \subseteq \mathcal{N}(\mathcal{C},\mathcal{D}^c)$, so that $\mathfrak{S}(\mathcal{C},\mathcal{D}^c) \subseteq \mathfrak{S}(\mathcal{C},\mathcal{D})$. Therefore,* $\operatorname{Rad}(\mathcal{C},\mathcal{D})$

$\subseteq \mathrm{Rad}(\mathcal{C}, \mathcal{D}^c)$. *As* $(\mathcal{C}, \mathcal{D}^c)$ *is a regular* MASA *inclusion,*

$$\mathrm{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \mathrm{Rad}(\mathcal{C}, \mathcal{D}^c) \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D}^c). \tag{5.2}$$

Must equality hold throughout in (5.2)*? The case of primary interest to us is when* $(\mathcal{C}, \mathcal{D})$ *is a skeletal* MASA *inclusion, in which case the question reduces to determining whether* $\mathrm{Rad}(\mathcal{C}, \mathcal{D}) = \mathcal{L}(\mathcal{C}, \mathcal{D})$.

QUESTION 5.9. *When does a regular inclusion regularly embed into a regular* EP-*inclusion?*

As any EP-inclusion is a MASA inclusion, we obviously must have $\mathcal{D}^c$ is abelian for such an embedding to exist. Thus the question really boils down to asking when a regular MASA inclusion embeds into an EP-extension.

6. VIRTUAL CARTAN INCLUSIONS

If $(\mathcal{C}, \mathcal{D})$ is a Cartan inclusion, the conditional expectation $E : \mathcal{C} \to \mathcal{D}$ is the pseudo-expectation, and $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, by the hypothesis that E is faithful. In Definition 1.1, we defined a virtual Cartan inclusion to be a regular MASA inclusion $(\mathcal{C}, \mathcal{D})$ such that (0) is the only ideal of $\mathcal{C}$ having trivial intersection with $\mathcal{D}$. By Theorem 3.15, $(\mathcal{C}, \mathcal{D})$ is a virtual Cartan inclusion if and only if the pseudo-expectation is faithful, and this is the reason for the name.

Our purpose in this section is to develop some properties of virtual Cartan inclusions and to produce a large class of virtual Cartan inclusions using reduced crossed products. We first observe that virtual Cartan inclusions have very nice mapping properties. Our proof that (i) implies (ii) in the following is an adaptation of an observation by Breuillard, Kalantar, Kennedy and Ozawa, which is to be included in a future version of [8].

THEOREM 6.1. *Let* $(\mathcal{C}, \mathcal{D})$ *be a regular* MASA *inclusion. The following statements are equivalent:*

(i) $(\mathcal{C}, \mathcal{D})$ *is a virtual Cartan inclusion.*

(ii) *If* $\mathcal{S}$ *is an operator system and* $\phi : \mathcal{C} \to \mathcal{S}$ *is a unital completely contractive map such that* $\phi|_{\mathcal{D}}$ *is completely isometric, then* ϕ *is faithful, (i.e. if* $x \in \mathcal{C}$ *and* $\phi(x^*x) = 0$*, then* $x = 0$*).*

(iii) *If* $\mathcal{A}$ *is a* C^**-algebra and* $\alpha : \mathcal{C} \to \mathcal{A}$ *is a* $*$*-homomorphism such that* $\alpha|_{\mathcal{D}}$ *is one-to-one, then* α *is one-to-one on* $\mathcal{C}$.

Proof. (i) $\Rightarrow$ (ii) Suppose $(\mathcal{C}, \mathcal{D})$ is a virtual Cartan inclusion, $\mathcal{S}$ is an operator system, and $\phi : \mathcal{C} \to \mathcal{S}$ is a unital completely contractive map such that $\phi|_{\mathcal{D}}$ is completely isometric. Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Since $I(\mathcal{D})$ is an injective object in $\mathfrak{O}$, and $\phi|_{\mathcal{D}}$ is completely isometric, there exists a unital completely contractive map $\psi : \mathcal{S} \to I(\mathcal{D})$ such that $\psi \circ (\phi|_{\mathcal{D}}) = \iota$. Then $\psi \circ \phi : \mathcal{C} \to I(\mathcal{D})$ is a pseudo-expectation. By the uniqueness of the pseudo-expectation,

$E = \psi \circ \phi$. Hence if $\phi(x^*x) = 0$, then $x \in \mathcal{L}(\mathcal{C}, \mathcal{D})$, and therefore $x = 0$. Hence ϕ is faithful.

(ii) $\Rightarrow$ (iii) A $*$-homomorphism is completely contractive and is completely isometric if and only if it is one-to-one.

(iii) $\Rightarrow$ (i) Let $\alpha : \mathcal{C} \to \mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D})$ be the quotient map. Since $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \mathcal{D} = (0)$ (Theorem 3.15), $\alpha|_{\mathcal{D}}$ is one-to-one. Hence α is one-to-one on $\mathcal{C}$, so $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, and $(\mathcal{C}, \mathcal{D})$ is a virtual Cartan inclusion. ∎

Recall that an inclusion $(\mathcal{C}, \mathcal{D})$ is *C^*-essential* if the following property holds: if $J \subseteq \mathcal{C}$ is a ideal such that $J \cap \mathcal{D} = (0)$, then $J = (0)$. Thus, the equivalence of (i) and (iii) in the previous theorem may be stated as saying that virtual Cartan inclusions are those regular MASA inclusions which are C^*-essential.

Our next goal is to show how to construct a virtual Cartan inclusion from a skeletal MASA inclusion $(\mathcal{C}, \mathcal{D})$ by examining the inclusion obtained from the quotient of $\mathcal{C}$ by $\mathcal{L}(\mathcal{C}, \mathcal{D})$. Let $q : \mathcal{C} \to \mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D})$ be the quotient map; we will sometimes write $q(\mathcal{C})$ instead of $\mathcal{C}/\mathcal{L}(\mathcal{C}, \mathcal{D})$. Then $q(\mathcal{D})$ is isomorphic to $\mathcal{D}$ because $\mathcal{L}(\mathcal{C}, \mathcal{D})$ has trivial intersection with $\mathcal{D}$. Because $(\mathcal{C}, \mathcal{D})$ is regular, so is $(q(\mathcal{C}), q(\mathcal{D}))$. Since $\mathcal{L}(\mathcal{C}, \mathcal{D})$ is the largest ideal in $\mathcal{C}$ disjoint from $\mathcal{D}$, it is natural to conjecture that $(q(\mathcal{C}), q(\mathcal{D}))$ is a virtual Cartan inclusion. However, while we shall observe in a moment that $q(\mathcal{D})^c$ is abelian, it seems unlikely that in general, $q(\mathcal{D})$ is a MASA in $q(\mathcal{C})$. Nevertheless, $q(\mathcal{D})$ is "nearly" a MASA in the sense that $q(\mathcal{D})^c$ is an essential extension of $q(\mathcal{D})$, and the inclusion $(q(\mathcal{C}), q(\mathcal{D})^c)$ is a virtual Cartan inclusion.

THEOREM 6.2. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion. Then the relative commutant, $q(\mathcal{D})^c$, of $q(\mathcal{D})$ in $q(\mathcal{C})$ is abelian. Moreover, the following statements hold:*

(i) *$(q(\mathcal{C}), q(\mathcal{D})^c)$ is a virtual Cartan inclusion.*

(ii) *$(q(\mathcal{C}), q(\mathcal{D}))$ is a C^*-essential inclusion.*

(iii) *$(q(\mathcal{D})^c, \subseteq)$ is an essential extension of $q(\mathcal{D})$.*

(iv) *If there exists a conditional expectation $E : \mathcal{C} \to \mathcal{D}$, then $(q(\mathcal{C}), q(\mathcal{D}))$ is a Cartan inclusion.*

Proof. Throughout the proof, we abbreviate $\mathcal{L}(\mathcal{C}, \mathcal{D})$ by $\mathcal{L}$. Observe first that the inverse image under q of $q(\mathcal{D})^c$ is

$$q^{-1}(q(\mathcal{D})^c) = \{x \in \mathcal{C} : xd - dx \in \mathcal{L} \text{ for all } d \in \mathcal{D}\}.$$

Write $\mathcal{A} := q^{-1}(q(\mathcal{D})^c)$. Proposition 3.17 shows that $\mathcal{A}$ is the multiplicative domain for E. Hence $E|_{\mathcal{A}}$ is a homomorphism of $\mathcal{A}$ into $I(\mathcal{D})$ and $\ker E|_{\mathcal{A}} = \mathcal{L}$. Since $E(\mathcal{A})$ is abelian, $\ker E|_{\mathcal{A}}$ contains the commutator ideal of $\mathcal{A}$. Thus, $q(\mathcal{A}) = q(\mathcal{D})^c$ is abelian.

Since $(q(\mathcal{C}), q(\mathcal{D}))$ is a regular inclusion, Lemma 2.10 shows $(q(\mathcal{C}), q(\mathcal{D})^c)$ is a regular MASA inclusion.

Factor E as $E = \widetilde{E} \circ q$, where $\widetilde{E} : q(\mathcal{C}) \to I(\mathcal{D})$ is given by $\widetilde{E}(x + \mathcal{L}) = E(x)$. Since $E|_{\mathcal{A}}$ is a homomorphism, $\widetilde{E}|_{q(\mathcal{D})^c}$ is a $*$-monomorphism of $q(\mathcal{D})^c$ into $I(\mathcal{D})$.

Hence $(I(\mathcal{D}), \widetilde{E}|_{q(\mathcal{D})^c})$ is an extension of $q(\mathcal{D})^c$. Also, $\iota \circ (q|_{\mathcal{D}})^{-1} = \widetilde{E}|_{q(\mathcal{D})}$. As $q|_{\mathcal{D}}$ is an isomorphism of $\mathcal{D}$ onto $q(D)$, uniqueness of injective envelopes shows $(I(\mathcal{D}), \widetilde{E}|_{q(\mathcal{D})})$ is an injective envelope for $q(\mathcal{D})$. In particular, $(I(\mathcal{D}), \widetilde{E}|_{q(\mathcal{D})})$ is an essential extension for $q(\mathcal{D})$, whence $(I(\mathcal{D}), \widetilde{E}|_{q(\mathcal{D})^c})$ is also an essential extension for $q(\mathcal{D})^c$. Therefore, $(I(\mathcal{D}), \widetilde{E}|_{q(\mathcal{D})^c})$ is an injective envelope for $q(\mathcal{D})^c$.

It is now clear that $\widetilde{E}$ is a pseudo-expectation for both $(q(\mathcal{C}), q(\mathcal{D})^c)$ and $(q(\mathcal{C}), q(\mathcal{D}))$. But $\widetilde{E}$ is faithful on $q(\mathcal{C})$ by construction, so $(q(\mathcal{C}), q(\mathcal{D})^c)$ is a virtual Cartan inclusion.

Part (ii) follows from faithfulness of $\widetilde{E}$ and Theorem 3.15.

For part (iii), we use ideas from the proof of Corollary 3.22 in [29]. Notice first that $\widetilde{E}|_{q(\mathcal{D})^c}$ is the unique $*$-homomorphism of $q(\mathcal{D})^c$ into $I(\mathcal{D})$ which extends $\widetilde{E}|_{q(\mathcal{D})}$. Indeed, if ψ is another such map, use injectivity to extend ψ to a completely contractive unital map $\Psi : q(\mathcal{C}) \to I(\mathcal{D})$. Then $\Psi = \widetilde{E}$ by uniqueness of the pseudo-expectation. Restricting Ψ to $q(\mathcal{D})^c$ gives $\psi = \widetilde{E}|_{q(\mathcal{D})^c}$, as desired.

Now if $J \subseteq q(\mathcal{D})^c$ is an ideal such that $J \cap q(\mathcal{D}) = (0)$, let $\gamma : q(\mathcal{D})^c \to q(\mathcal{D})^c/J$ be the quotient map. Since $J \cap q(\mathcal{D}) = (0)$, $\gamma|_{q(\mathcal{D})}$ is faithful. Use injectivity again to produce a homomorphism $\theta : q(\mathcal{D})^c/J \to I(\mathcal{D})$ such that $\theta \circ \gamma|_{q(\mathcal{D})} = \widetilde{E}|_{q(\mathcal{D})}$. The observation in the preceding paragraph shows that $\widetilde{E}|_{q(\mathcal{D})^c} = \theta \circ \gamma$. But $\widetilde{E}$ is faithful, therefore so is γ. Thus statement (iii) holds.

Finally, suppose that $E : \mathcal{C} \to \mathcal{D}$ is a conditional expectation. Proposition 3.17 then shows that $q(\mathcal{D})^c = q(\mathcal{D})$, so $(q(\mathcal{C}), q(\mathcal{D}))$ is a Cartan inclusion (with conditional expectation $x + \mathcal{L} \mapsto E(x) + \mathcal{L})$). ∎

The following shows that a skeletal MASA inclusion $(\mathcal{C}, \mathcal{D})$ with faithful pseudo-expectation is very nearly a virtual Cartan inclusion.

COROLLARY 6.3. *Suppose* $(\mathcal{C}, \mathcal{D})$ *is a skeletal* MASA *inclusion. If* $\mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, *then* $\mathcal{D}^c$ *is abelian,* $(\mathcal{D}^c, \subseteq)$ *is an essential extension of* $\mathcal{D}$, *and* $(\mathcal{C}, \mathcal{D}^c)$ *is a virtual Cartan inclusion.*

REMARK 6.4. Corollary 6.3 also follows from work in [29]. Indeed, if $(\mathcal{C}, \mathcal{D})$ is a skeletal MASA inclusion with faithful pseudo-expectation E, Corollary 3.14 of [29] shows that $\mathcal{D}^c$ is abelian, and Corollary 3.22 of [29] shows $(\mathcal{D}^c, \subseteq)$ is an essential extension of $\mathcal{D}$.

6.1. AN EXAMPLE: REDUCED CROSSED PRODUCTS BY DISCRETE GROUPS.

In this subsection, we consider the regular inclusion $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C} = \mathcal{D} \rtimes_{\mathrm{r}} \Gamma$ is the reduced crossed product of the unital abelian C^*-algebra $\mathcal{D} = C(X)$ by a discrete group Γ of homeomorphisms of X. We will use the reduced crossed product construction to produce a large class of examples of virtual Cartan inclusions.

The main results of this subsection are: Theorem 6.11, which characterizes when the relative commutant $\mathcal{D}^c$ of $\mathcal{D}$ in $\mathcal{D} \rtimes_{\mathrm{r}} \Gamma$ is abelian in terms of the associated dynamical system; Theorem 6.14, which shows that when $\mathcal{D}^c$ is abelian,

$(\mathcal{D} \rtimes_{\mathrm{r}} \Gamma, \mathcal{D}^{\mathrm{c}})$ is a virtual Cartan inclusion; and a summary result, Theorem 6.15 which gives a number of characterizations for when $(\mathcal{D} \rtimes_{\mathrm{r}} \Gamma, \mathcal{D})$ regularly embeds into a C^*-diagonal. By choosing the space X and group Γ appropriately, the methods in this section can be used to produce an example of a virtual Cartan inclusion $(\mathcal{C}, \mathcal{D})$ where $\mathcal{C}$ is not nuclear, see Theorem 4.4.3 or Theorem 5.1.6 of [9]. Some of the results presented in this subsection complement results from [33].

We begin by establishing some notation. This is standard material, but we include it because there are a number of variations in the literature.

Throughout, let X be a compact Hausdorff space, let Γ be a discrete group with unit element e acting on X as homeomorphisms of X. Thus there is a homomorphism Ξ of Γ into the group of homeomorphisms of X, and for $(s,x) \in \Gamma \times X$, we will write sx instead of $\Xi(s)(x)$. We will refer to the pair (X, Γ) as a *discrete dynamical system.* For $s \in \Gamma$, let $\alpha_s \in \mathrm{Aut}(C(X))$ be given by

$$(\alpha_s(f))(x) = f(s^{-1}x), \quad f \in C(X),\, x \in X.$$

If Y is any set, and $z \in Y$, we use δ_z to denote the characteristic function of the singleton set $\{z\}$.

Let $\mathcal{D} = C(X)$, and let $C_{\mathrm{c}}(\Gamma, \mathcal{D})$ be the set of all functions $a : \Gamma \to \mathcal{D}$ such that $\{s \in \Gamma : a(s) \neq 0\}$ is a finite set. We will sometimes write $a(s,x)$ for the value of $a(s)$ at $x \in X$ instead of $a(s)(x)$. Then $C_{\mathrm{c}}(\Gamma, \mathcal{D})$ is a $*$-algebra under the usual twisted convolution product and adjoint operation: for $a, b \in C_{\mathrm{c}}(\Gamma, \mathcal{D})$,

$$(ab)(t) = \sum_{r \in \Gamma} a(r)\alpha_r(b(r^{-1}t)) \quad \text{and} \quad (a^*)(t) = \alpha_t(a(t^{-1}))^*. \tag{6.1}$$

Let $\mathcal{C} = C(X) \rtimes_{\mathrm{r}} \Gamma$ be the reduced crossed product of $C(X)$ by Γ.

The group Γ is naturally embedded into $\mathcal{C}$ via $s \mapsto w_s$, where w_s is the element of $C_{\mathrm{c}}(\Gamma, \mathcal{D})$ given by $w_s(t) = \begin{cases} 0 & \text{if } t \neq s, \\ I & \text{if } t = s. \end{cases}$ Also, $\mathcal{D}$ is embedded into $C_{\mathrm{c}}(\Gamma, \mathcal{D})$ via the map $d \mapsto dw_e$ and we identify $\mathcal{D}$ with its image under this map. Now $w_s d w_{s^{-1}} = \alpha_s(d)$ and $\mathrm{span}\{dw_s : d \in \mathcal{D}, s \in \Gamma\}$ is norm dense in $\mathcal{C}$, so $\{w_s : s \in \Gamma\} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$. Thus $(\mathcal{C}, \mathcal{D})$ is a regular inclusion.

It is well known (for example, see the discussion of crossed products in [9]) that the map $\mathbb{E} : C_{\mathrm{c}}(\Gamma, \mathcal{D}) \to \mathcal{D}$ given by $\mathbb{E}(a) = a(e)$ extends to a faithful conditional expectation $\mathbb{E}$ of $\mathcal{C}$ onto $\mathcal{D}$. Likewise, the maps $\mathbb{E}_s : C_{\mathrm{c}}(\Gamma, \mathcal{D}) \to \mathcal{D}$ given by $\mathbb{E}_s(a) = a(s)$ extend to norm-one linear mappings $\mathbb{E}_s$ of $\mathcal{C}$ onto $\mathcal{D}$. Notice that for $a \in \mathcal{C}$ and $s \in \Gamma$,

$$\mathbb{E}_s(a) = \mathbb{E}(aw_{s^{-1}}).$$

The maps $\mathbb{E}_s$ allow a useful "Fourier series" viewpoint for elements of $\mathcal{C}$: $a \sim \sum_{s \in \Gamma} \mathbb{E}_s(a) w_s$.

The following is well-known. We provide a proof for convenience of the reader.

PROPOSITION 6.5. *If $a \in \mathcal{C}$ and $\mathbb{E}_s(a) = 0$ for every $s \in \Gamma$, then $a = 0$.*

Proof. For $a \in C_c(\Gamma, \mathcal{D})$ and $s, t \in \Gamma$, a calculation shows that

$$\mathbb{E}_s(w_t a w_{t^{-1}}) = \alpha_t(\mathbb{E}_{t^{-1}st}(a)); \tag{6.2}$$

a continuity argument then shows that (6.2) actually holds for every $a \in \mathcal{C}$.

Let $J = \{a \in \mathcal{C} : \mathbb{E}_t(a) = 0 \ \forall t \in \Gamma\}$. Clearly J is closed. Then (6.2) shows that if $a \in J$ and $s \in \Gamma$, then $w_s a w_{s^{-1}} \in J$. Easy calculations now show that if $d \in \mathcal{D}$, $s \in \Gamma$ and $a \in J$, then $\{da, ad, w_s a, a w_s\} \subseteq J$, and by taking linear combinations and closures, we find that J is a closed two-sided ideal of $\mathcal{C}$. Thus, if $a \in J$, $a^*a \in J$, so that $\mathbb{E}_e(a^*a) = \mathbb{E}(a^*a) = 0$. Hence $a = 0$ by faithfulness of $\mathbb{E}$. This shows that $J = (0)$, completing the proof. ∎

DEFINITION 6.6. We make the following definitions:

(i) For $s \in \Gamma$, let $F_s = \{x \in X : sx = x\}$ be the set of fixed points of s.

(ii) For $s \in \Gamma$, let $\mathfrak{F}_s = \{f \in \mathcal{D} : \operatorname{supp}(f) \subseteq F_s^\circ\}$. Thus $\{\mathfrak{F}_s : s \in \Gamma\}$ is a family of closed ideals in $\mathcal{D}$.

(iii) For $x \in X$, let $\Gamma^x := \{s \in \Gamma : sx = x\}$ be the isotropy group at x.

(iv) For $x \in X$, let $H^x := \{s \in \Gamma : x \in (F_s)^\circ\}$. We will call H^x the *germ isotropy group at x.*

REMARK 6.7. We chose the terminology "germ isotropy" because $s \in H^x$ if and only if the homeomorphisms s and $\mathrm{id}|_X$ agree in a neighborhood of x, that is, they have the same germ. It is easy to see that H^x is a group; in fact, H^x is a normal subgroup of Γ^x. To see that H^x a normal subgroup in Γ^x, fix $x \in X$ and let $s \in H^x$. Then there exists an open neighborhood V of x such that $V \subseteq F_s$. Let $t \in \Gamma^x$ and put $W = t^{-1}V$. Since $tx = x$, $x \in W$. For $y \in W$, $ty \in V$, so $sty = ty$. Hence $t^{-1}sty = y$. Therefore, $W \subseteq F_{t^{-1}st}$. As W is open and $x \in W$, we see that x belongs to the interior of $F_{t^{-1}st}$, so $t^{-1}st \in H^x$ as desired.

Simple examples show the inclusion of H^x in Γ^x can be proper.

We record a description of the relative commutant of $\mathcal{D}$ in $\mathcal{C}$.

PROPOSITION 6.8. *We have*

$$\begin{aligned}\mathcal{D}^c &= \{a \in \mathcal{C} : \alpha_s(d)\mathbb{E}_s(a) = d\mathbb{E}_s(a) \text{ for all } d \in \mathcal{D} \text{ and all } s \in \Gamma\}\\ &= \{a \in \mathcal{C} : \mathbb{E}_s(a) \in \mathfrak{F}_s \text{ for all } s \in \Gamma\}.\end{aligned}$$

Proof. A computation shows that for $a \in \mathcal{C}$, $d \in \mathcal{D}$ and $s \in \Gamma$,

$$\mathbb{E}_s(da - ad) = (d - \alpha_s(d))\mathbb{E}_s(a). \tag{6.3}$$

Thus if $a \in \mathcal{D}^c$, we obtain $\alpha_s(d)\mathbb{E}_s(a) = d\mathbb{E}_s(a)$ for every $d \in \mathcal{D}$ and $s \in \Gamma$. Conversely, if $\mathbb{E}_s(a)\alpha_s(d) = d\mathbb{E}_s(a)$ for every $d \in \mathcal{D}$ and $s \in \Gamma$, Proposition 6.5 gives $a \in \mathcal{D}^c$.

For the second equality, suppose that $a \in \mathcal{C}$ and $\mathbb{E}_s(a) \in \mathfrak{F}_s$ for every $s \in \Gamma$. Since $\mathbb{E}_s(a)$ is supported in F_s°, an examination of (6.3) shows that $\mathbb{E}_s(da - ad) = 0$ for every $d \in \mathcal{D}$. By Proposition 6.5 again, $a \in \mathcal{D}^c$. For the reverse inclusion, suppose that $a \in \mathcal{D}^c$. Then for $d \in \mathcal{D}$ and $s \in \Gamma$, $0 = (d - \alpha_s(d))\mathbb{E}_s(a)$. Thus if $x \in X$ and $\mathbb{E}_s(a)(x) \neq 0$, we have $d(x) - d(s^{-1}x) = 0$ for every $d \in \mathcal{D}$. It follows that the support of $\mathbb{E}_s(a)$ is contained in F_s. But $\operatorname{supp}(\mathbb{E}_s(a))$ is open, so the reverse inclusion holds. ∎

We now describe a representation useful for establishing certain formulae.

THE VERY DISCRETE REPRESENTATION. Let $\mathcal{H} = \ell^2(\Gamma \times X)$. Then $\{\delta_{(t,y)} : (t,y) \in \Gamma \times X\}$ is an orthonormal basis for $\mathcal{H}$. For $f \in C(X)$, $s \in \Gamma$, and $\xi \in \mathcal{H}$, define representations π of $C(X)$ and U of Γ on $\mathcal{H}$ by

$$(\pi(f)\xi)(t,y) = f(ty)\xi(t,y) \quad \text{and} \quad (U_s\xi)(t,y) = \xi(s^{-1}t,y).$$

In particular,

$$\pi(f)\delta_{(t,y)} = f(ty)\delta_{(t,y)} \quad \text{and} \quad U_s\delta_{(t,y)} = \delta_{(st,y)}.$$

The C^*-algebra generated by the images of π and U is isometrically isomorphic to the reduced crossed product of $C(X)$ by Γ (see pages 117–118 of [9]), and hence determines a faithful representation $\theta : \mathcal{C} \to \mathcal{B}(\mathcal{H})$.

A computation shows that for $a \in \mathcal{C}$, $t, r \in \Gamma$ and $x, y \in X$,

$$\langle \theta(a)\delta_{(t,y)}, \delta_{(r,x)} \rangle = \begin{cases} 0 & \text{if } x \neq y, \\ \mathbb{E}_{rt^{-1}}(a)(ry) & \text{if } x = y. \end{cases}$$

Also for $a \in \mathcal{C}$, $t \in \Gamma$ and $y \in X$, we have

$$\theta(a)\delta_{(t,y)} = \sum_{s \in \Gamma} \mathbb{E}_s(a)(sty)\delta_{(st,y)}. \tag{6.4}$$

We now define some notation. Let $\lambda : \Gamma \to \mathcal{B}(\ell^2(\Gamma))$ be the left regular representation, and for $x \in X$, regard $\ell^2(H^x)$ as a subspace of $\ell^2(\Gamma)$. Then $C^*_r(H^x)$ is the C^*-algebra generated by $\{\lambda_s|_{\ell^2(H^x)} : s \in H^x\}$. Define $V_x : \ell^2(H^x) \to \mathcal{H}$ by

$$(V_x\eta)(s,y) = \begin{cases} 0 & \text{if } (s,y) \notin H^x \times \{x\}, \\ \eta(s) & \text{if } (s,y) \in H^x \times \{x\}. \end{cases}$$

Then for $r \in H^x$, we have $V_x\delta_r = \delta_{(r,x)}$, so V_x is an isometry.

PROPOSITION 6.9. *For $x \in X$ and $a \in \mathcal{C}$, define $\Phi_x(a) := V_x^*\theta(a)V_x$. Then Φ_x is a completely positive unital mapping of $\mathcal{C}$ onto $C^*_r(H^x)$ and $\Phi_x|_{\mathcal{D}^c}$ is a $*$-epimorphism of $\mathcal{D}^c$ onto $C^*_r(H^x)$.*

Proof. Clearly Φ_x is completely positive and unital. For $d \in \mathcal{D}$, $r \in \Gamma$ and $s, t \in H^x$ we have

$$\begin{aligned} \langle \Phi_x(dw_r)\delta_s, \delta_t \rangle &= \langle V_x^*\theta(dw_r)V_x\delta_s, \delta_t \rangle = \langle \pi(d)U_r\delta_{(s,x)}, \delta_{(t,x)} \rangle \\ &= \langle \pi(d)\delta_{(rs,x)}, \delta_{(t,x)} \rangle = d(rsx)\langle \delta_{(rs,x)}, \delta_{(t,x)} \rangle = d(rx)\langle \delta_{rs}, \delta_t \rangle \\ &= d(rx)\langle \lambda_r\delta_s, \delta_t \rangle. \end{aligned} \tag{6.5}$$

Hence for every $d \in \mathcal{D}$ and $r \in \Gamma$,

$$\Phi_x(dw_r) = \begin{cases} 0 & \text{if } r \notin H^x, \\ d(x)\lambda_r|_{\ell^2(H^x)} & \text{if } r \in H^x. \end{cases} \tag{6.6}$$

Therefore Φ_x maps a set of generators for $\mathcal{C}$ into $C^*_r(H^x)$, giving $\Phi_x(\mathcal{C}) \subseteq C^*_r(H^x)$.

To show that $\Phi_x|_{\mathcal{D}^c}$ is a $*$-homomorphism, it suffices to prove that the range of V_x is an invariant subspace for $\theta(\mathcal{D}^c)$. Note that $\operatorname{range}(V_x) = \overline{\operatorname{span}}\{\delta_{(t,x)} : t \in H^x\}$. Let $a \in \mathcal{D}^c$ and fix $t \in H^x$. We claim that if $s \in \Gamma$, $d \in \mathfrak{F}_s$ and $stx \in \operatorname{supp}(d)$, then $s \in H^x$. Indeed, suppose that $stx \in \operatorname{supp}(d)$. As $t \in H^x$, $stx = sx$. So $sx \in F_s^\circ = F_{s^{-1}}^\circ$, which yields $x \in F_s^\circ$. Thus $s \in H^x$, so the claim holds.

Next by (6.4) and Proposition 6.8, for $t \in H^x$, we have

$$\theta(a)\delta_{(t,x)} = \sum_{s\in\Gamma} \mathbb{E}_s(a)(stx)\delta_{(st,x)} = \sum_{s\in H^x} \mathbb{E}_s(a)(stx)\delta_{(st,x)} \in \operatorname{range}(V_x),$$

as desired. It follows that $\Phi_x|_{\mathcal{D}^c}$ is a $*$-homomorphism.

It remains to show $\Phi_x(\mathcal{D}^c) = C^*_r(H^x)$. If $s \in H^x$, let $d \in \mathfrak{F}_s$ be such that $d(x) = 1$, and put $a = dw_s$. Then (6.6) shows that $\Phi_x(a) = \lambda_s|_{\ell^2(H^x)}$. By Proposition 6.8, $a \in \mathcal{D}^c$, and hence $\Phi_x(\mathcal{D}^c)$ is dense in $C^*_r(H^x)$. Since $\Phi_x|_{\mathcal{D}^c}$ is a homomorphism, it has closed range. Therefore $\Phi_x(\mathcal{D}^c) = C^*_r(H^x)$. ∎

Let

$$\bigoplus_{x\in X} C^*_r(H^x) := \Big\{ f \in \prod_{x\in X} C^*_r(H^x) : \sup_{x\in X} \|f(x)\| < \infty \Big\}$$

and for $f \in \bigoplus_{x\in X} C^*_r(H^x)$, define $\|f\| = \sup_{x\in X} \|f(x)\|$. Then with product, addition, scalar multiplication and involution defined point-wise, $\bigoplus_{x\in X} C^*_r(H^x)$ is a C^*-algebra.

COROLLARY 6.10. *The map $\Phi : \mathcal{C} \to \bigoplus_{x\in X} C^*_r(H^x)$ given by $\Phi(a)(x) = \Phi_x(a)$ is a faithful completely positive unital mapping such that $\Phi|_{\mathcal{D}^c}$ is a $*$-monomorphism.*

Proof. It follows from the definition of Φ_x that Φ is unital and completely positive. Proposition 6.9 shows that $\Phi|_{\mathcal{D}^c}$ is a $*$-homomorphism; it remains to check that Φ is faithful.

For $x \in X$, let tr_x be the the trace on $C^*_r(H^x)$. For $d \in \mathcal{D}$ and $s \in \Gamma$ equation (6.6) gives,

$$\operatorname{tr}_x(\Phi_x(dw_s)) = \left\{\begin{matrix} 0 & \text{if } s \neq e, \\ d(x) & \text{if } s = e, \end{matrix}\right\} = \mathbb{E}(dw_s)(x).$$

This formula extends by linearity and continuity, so that for $a \in \mathcal{C}$, $\operatorname{tr}_x(\Phi_x(a)) = \mathbb{E}(a)(x)$. So if $a \geqslant 0$ belongs to $\mathcal{C}$ and $\Phi(a) = 0$, then $\mathbb{E}(a) = 0$, so $a = 0$. Thus, Φ is faithful. ∎

THEOREM 6.11. *The relative commutant, $\mathcal{D}^c$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian if and only if H^x is an abelian group for every $x \in X$.*

Proof. Corollary 6.10 shows that if H^x is abelian for every $x \in X$, then $\mathcal{D}^c$ is abelian.

For the converse, we prove the contrapositive. Suppose that H^x is non-abelian for some $x \in X$. Fix $s, t \in H^x$ so that $st \neq ts$. Then $x \in (F_s)^\circ \cap (F_t)^\circ$, so we may find $d \in \mathcal{D}$ so that $d(x) = 1$ and $\overline{\operatorname{supp}}(d) \subseteq (F_s)^\circ \cap (F_t)^\circ$. Then for $h \in \mathcal{D}$ and $z \in X$ we have (by examining the cases $z \in F_s$ and $z \notin F_s$),

$$(\alpha_s(h)(z) - h(z))d(z) = (h(s^{-1}z) - h(z))d(z) = 0.$$

Proposition 6.8 shows that $dw_s \in \mathcal{D}^c$. Likewise, $dw_t \in \mathcal{D}^c$.

Then $dw_s dw_t = d\alpha_s(d)w_{st}$. Note that by choice of d, $s\,\overline{\operatorname{supp}}(d) = \overline{\operatorname{supp}}(d)$. For $z \in X$,

$$\alpha_s(d)(z) = d(s^{-1}z) = \begin{cases} 0 & \text{if } z \notin \operatorname{supp}(d), \\ d(z) & \text{if } z \in \operatorname{supp}(d). \end{cases}$$

Thus, $\alpha_s(d) = d$, and likewise, $\alpha_t(d) = d$. Therefore,

$$(dw_s)(dw_t) = d^2 w_{st} \neq d^2 w_{ts} = (dw_t)(dw_s),$$

so $\mathcal{D}^c$ is not abelian. ∎

PROPOSITION 6.12. *Let (X, Γ) be a discrete dynamical system such that for each $x \in X$, the germ isotropy group H^x is abelian. Let $\Gamma_1 \subseteq \Gamma$ be a subgroup of Γ, set*

$$\mathcal{C}_1 := \mathcal{D} \rtimes_{\mathrm{r}} \Gamma_1, \quad \textit{and let} \quad \mathcal{D}_1 = \{x \in \mathcal{C}_1 : xd = dx \textit{ for all } d \in \mathcal{D}\}.$$

Then $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA *inclusion and $\mathcal{C}_1 \cap \mathcal{L}(\mathcal{C}, \mathcal{D}^c) \subseteq \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1)$.*

Proof. As $\mathcal{D}_1 \subseteq \mathcal{D}^c$, $\mathcal{D}_1$ is abelian, and as $\mathcal{D}_1$ is the relative commutant of $\mathcal{D}$ in $\mathcal{C}_1$, $(\mathcal{C}_1, \mathcal{D}_1)$ is a regular MASA inclusion. Let $\epsilon : \mathcal{C}_1 \to \mathcal{C}$ be the inclusion map. Notice that each map in the diagram,

$$(\mathcal{C}_1, \mathcal{D}_1) \xrightarrow{\epsilon} (\mathcal{C}, \mathcal{D}_1) \xrightarrow{\mathrm{id}} (\mathcal{C}, \mathcal{D}^c)$$

is a regular map. The first is clearly regular, while the regularity of the second follows from the fact that the relative commutant of $\mathcal{D}_1$ in $\mathcal{C}$ is $\mathcal{D}^c$ and an application of Lemma 2.10. Therefore, $\epsilon : (\mathcal{C}_1, \mathcal{D}_1) \to (\mathcal{C}, \mathcal{D}^c)$ is a regular $*$-monomorphism. An application of Proposition 3.16 completes the proof. ∎

NOTATION 6.13. When G is an abelian group with dual group $\widehat{G}$, we use the notation $\langle g, \gamma \rangle$ to denote the value of $\gamma \in \widehat{G}$ at $g \in G$. Also, we will identify $C^*(G)$ with $C(\widehat{G})$; lastly, for $\gamma \in \widehat{G}$ and $a \in C^*(G)$, we will write $\gamma(a)$ instead of $\widehat{a}(\gamma)$.

THEOREM 6.14. *Suppose that (X, Γ) is a discrete dynamical system such that for each $x \in X$, the germ isotropy group H^x is abelian. Then $(\mathcal{C}, \mathcal{D}^c)$ is a virtual Cartan inclusion.*

Proof. By Theorem 6.11, $(\mathcal{C}, \mathcal{D}^c)$ is a regular MASA inclusion. We must show $\mathcal{L}(\mathcal{C}, \mathcal{D}^c) = (0)$.

First assume that Γ is a countable discrete group. Let

$$P = \prod_{x \in X} \widehat{H}^x$$

be the Cartesian product of the dual groups. Denote by $p(x)$ the "x-th component" of $p \in P$. For $(x, p) \in X \times P$, define a state $\rho_{(x,p)}$ on $\mathcal{C}$ by

$$\rho_{(x,p)}(a) = p(x)(\Phi_x(a)) \quad \text{(here } a \in \mathcal{C}\text{)}, \quad \text{and let} \quad A := \{\rho_{(x,p)} : (x, p) \in X \times P\}.$$

Corollary 6.10 shows that the restriction of $\rho_{(x,p)}$ to $\mathcal{D}^c$ is a multiplicative linear functional, so in particular, $A \subseteq \operatorname{Mod}(\mathcal{C}, \mathcal{D}^c)$.

For each $s \in \Gamma$, let

$$X_s := (X \setminus F_s) \cup F_s^\circ.$$

Then X_s is a dense, open subset of X. Set

$$Y := \bigcap_{s \in \Gamma} X_s \quad \text{and} \quad B := \{\rho_{(y,p)} : (y, p) \in Y \times P\}.$$

Our goal is to show that

$$\overline{B} \subseteq \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c). \tag{6.7}$$

Fix $(y, p) \in Y \times P$, and suppose that $\tau \in \operatorname{Mod}(\mathcal{C}, \mathcal{D}^c)$ satisfies $\rho_{(y,p)}|_{\mathcal{D}^c} = \tau|_{\mathcal{D}^c}$. We claim that $\rho_{(y,p)} = \tau$. To see this, it suffices to show that for each $s \in \Gamma$, $\rho_{(y,p)}(w_s) = \tau(w_s)$. Given $s \in \Gamma$, if $sy \neq y$, we may choose $d \in \mathcal{D}$ so that $d(sy) = 1$ and $d(y) = 0$. Using (6.6),

$$\rho_{(y,p)}(d) = p(y)(\Phi_y(d)) = 0 \;\text{ and }\; \rho_{(y,p)}(w_s^* d w_s) = \rho_{(y,p)}(\alpha_{s^{-1}}(d)) = p(y)(d(sy)I) = 1.$$

Then

$$\rho_{(y,p)}(w_s) = \rho_{(y,p)}(w_s)\rho_{(y,p)}(w_s^* d w_s) = \rho_{(y,p)}(w_s(w_s^* d w_s)) = \rho_{(y,p)}(d)\rho_{(y,p)}(w_s) = 0.$$

Likewise, $\tau(w_s) = 0$, so $\tau(w_s) = \rho_{(y,p)}(w_s) = 0$ when $y \notin F_s$.

On the other hand, if $sy = y$, then as $y \in X_s$, we have $y \in F_s^\circ$, so $s \in H^y$. Choose $d \in \mathcal{D}$ so that $\widehat{d}(y) = 1$ and $\operatorname{supp} \widehat{d} \subseteq F_s^\circ$. Then $dw_s \in \mathcal{D}^c$, so that

$$\rho_{(p,y)}(w_s) = \rho_{(y,p)}(dw_s) = \tau(dw_s) = \tau(w_s).$$

Therefore, $\rho_{(p,y)} = \tau$.

Let $\mathfrak{U}(\mathcal{C}, \mathcal{D}^c) = \{\tau \in \operatorname{Mod}(\mathcal{C}, \mathcal{D}^c) : \tau|_{\mathcal{D}^c} \text{ extends uniquely to } \mathcal{C}\}$. The previous paragraph shows that $B \subseteq \mathfrak{U}(\mathcal{C}, \mathcal{D}^c)$. By Theorem 3.12, $\overline{B} \subseteq \overline{\mathfrak{U}(\mathcal{C}, \mathcal{D}^c)} = \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c)$, so (6.7) holds.

Suppose now that $a \in \mathcal{L}(\mathcal{C}, \mathcal{D}^c)$. Then for every $\rho \in \mathfrak{S}_s(\mathcal{C}, \mathcal{D}^c)$, we have $\rho(a^*a) = 0$. In particular, for each $(y, p) \in Y \times P$,

$$0 = \rho_{(y,p)}(a^*a) = p(y)(\Phi_y(a^*a)).$$

Now $\widehat{H}^y = \{p(y) : p \in P\}$, so holding y fixed and varying p, yields $\Phi_y(a^*a) = 0$. Hence, we have $\mathbb{E}_e(a^*a)(y) = \mathbb{E}(a^*a)(y) = 0$ for every $y \in Y$. By Baire's theorem, Y is dense in X, so that $\mathbb{E}(a^*a) = 0$. Since $\mathbb{E}$ is faithful, $a = 0$. This gives the theorem in the case when Γ is countable.

We turn now to the general case. Let Γ be any discrete group and suppose $a \in \mathcal{L}(\mathcal{C}, \mathcal{D}^{\mathrm{c}})$. Then there exists a countable subgroup $\Gamma_1 \subseteq \Gamma$ such that $a \in \mathcal{D} \rtimes_{\mathrm{r}} \Gamma_1$. Put $\mathcal{C}_1 = \mathcal{D} \rtimes_{\mathrm{r}} \Gamma_1$ and let $\mathcal{D}_1 = \{x \in \mathcal{C}_1 : dx = xd \text{ for all } d \in \mathcal{D}\}$ be the relative commutant of $\mathcal{D}$ in $\mathcal{C}_1$. By Proposition 6.12, we have $a \in \mathcal{L}(\mathcal{C}_1, \mathcal{D}_1) = (0)$. This completes the proof. ∎

We summarize the results of this subsection into a main theorem.

THEOREM 6.15. *Let X be a compact Hausdorff space and let Γ be a discrete group acting as homeomorphisms on X. Let $\mathcal{C} = C(X) \rtimes_{\mathrm{r}} \Gamma$ and $\mathcal{D} = C(X)$. The following statements are equivalent:*

(i) *For every $x \in X$, the germ isotropy group H^x is abelian.*

(ii) *The relative commutant, $\mathcal{D}^{\mathrm{c}}$, of $\mathcal{D}$ in $\mathcal{C}$ is abelian.*

(iii) *$(\mathcal{C}, \mathcal{D}^{\mathrm{c}})$ is a virtual Cartan inclusion.*

(iv) $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) = (0)$.

(v) *$(\mathcal{C}, \mathcal{D})$ regularly embeds into a C^*-diagonal.*

Proof. (i) $\Leftrightarrow$ (ii) This is Theorem 6.11.

(ii) $\Rightarrow$ (iii) Apply Theorem 6.14.

(iii) $\Rightarrow$ (iv) If $(\mathcal{C}, \mathcal{D}^{\mathrm{c}})$ is a virtual Cartan inclusion, then in particular, $(\mathcal{C}, \mathcal{D}^{\mathrm{c}})$ is a regular MASA inclusion. Combine Lemma 2.10 and Proposition 5.3 to obtain $\operatorname{Rad}(\mathcal{C}, \mathcal{D}) \subseteq \mathcal{L}(\mathcal{C}, \mathcal{D}^{\mathrm{c}}) = (0)$.

(iv) $\Leftrightarrow$ (v) Use Theorem 5.7.

(v) $\Rightarrow$ (ii) This follows from Theorem 5.4. ∎

7. MINIMAL NORMS

In this section, we show that under suitable hypotheses on the regular inclusion $(\mathcal{C}, \mathcal{D})$, the span of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ has unique minimal and maximal C^*-norms. Actually, for certain applications, it is more convenient to work with a skeleton $\mathcal{M}$ for $(\mathcal{C}, \mathcal{D})$ rather than the full monoid $\mathcal{N}(\mathcal{C}, \mathcal{D})$. In general, minimal norms on span $\mathcal{M}$ do not exist, as the following simple example shows.

EXAMPLE 7.1. Consider the inclusion $(C(\mathbb{T}), \mathbb{C}I)$. If $\phi_n(z) = z^n$, then $\mathcal{M} := \{\phi_n : n \in \mathbb{Z}\}$ is a skeleton for this inclusion. Note that $\mathcal{A} := \operatorname{span} \mathcal{M}$ is just the set of all trigonometric polynomials. Suppose η is a C^*-norm on $\mathcal{A}$ and let $\mathcal{C}_\eta$ be the completion of $\mathcal{A}$ with respect to η. Then there exists a $*$-homomorphism $\pi : C(\mathbb{T}) \twoheadrightarrow \mathcal{C}_\eta$. Let $U = \pi(\phi_1)$. Since η is a norm on $\mathcal{A}$, the spectrum of U, $\sigma(U)$, is an infinite subset of $\mathbb{T}$. Choose an infinite compact set $K \subsetneq \sigma(U)$. For

$p = \sum_{n=-N}^{N} a_n\phi_n \in \mathcal{A}$, put $\|p\|_K := \sup_{z\in K}\left|\sum_{n=-N}^{N} a_n z^n\right|$. Since K is an infinite set, this defines a C^*-norm on $\mathcal{A}$. Also, $\|p\|_K \leqslant \sup_{z\in\sigma(U)}\left|\sum_{n=-N}^{N} a_n z^n\right| = \eta(p)$ and if $w \in \sigma(U)\setminus K$, then $\|\phi_1 + w\|_K < \eta(\phi_1 + w) = 2$. This shows there is no minimal C^*-norm on $\mathcal{A}$.

Despite this example, we shall see that in certain interesting cases, $\operatorname{span}(\mathcal{M})$ does have a unique minimal norm. We begin with two lemmas.

LEMMA 7.2. *Let $(\mathcal{C},\mathcal{D})$ be a skeletal* MASA *inclusion, let $(I(\mathcal{D}),\iota)$ be an injective envelope for $\mathcal{D}$ and let E be the pseudo-expectation. Assume $I(\mathcal{D}) \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and let π_E be the Stinespring representation for E. Then for every $x \in \mathcal{C}$,*

$$\sup\{\|\pi_\rho(x)\| : \rho \in \mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})\} = \|\pi_E(x)\| = \operatorname{dist}(x,\mathcal{L}(\mathcal{C},\mathcal{D})).$$

Proof. We claim that

$$\mathcal{L}(\mathcal{C},\mathcal{D}) = \ker\pi_E = \bigcap_{\rho\in\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})} \ker\pi_\rho.$$

Indeed, if $x \in \mathcal{L}(\mathcal{C},\mathcal{D})$, $z \in \mathcal{C}$ and $\xi \in \mathcal{H}$, then

$$\|\pi_E(x)(z\otimes\xi)\|^2 = \langle xz\otimes\xi, xz\otimes\xi\rangle = \langle E(z^*x^*xz)\xi,\xi\rangle = 0,$$

as $\mathcal{L}(\mathcal{C},\mathcal{D})$ is an ideal. Thus $x \in \ker\pi_E$. Suppose $x \in \ker\pi_E$ and $\rho \in \mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$. Then for every $z \in \mathcal{C}$, $0 \leqslant \rho(z^*x^*xz) \leqslant \|E(z^*x^*xz)\| = 0$. Therefore $x \in \ker\pi_\rho$. Finally, suppose for every $\rho \in \mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})$, $\pi_\rho(x) = 0$. Let $z \in \mathcal{C}$. Then for every $\sigma \in \widehat{I(\mathcal{D})}$, $\sigma(E(z^*x^*xz)) = 0$, so $E(z^*x^*xz) = 0$. Taking $z = I$, we obtain $x \in \mathcal{L}(\mathcal{C},\mathcal{D})$.

Now let $\mathcal{K} = \bigoplus_{\rho\in\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})} \mathcal{H}_\rho$ and let $\tau = \bigoplus_{\rho\in\mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})} \pi_\rho$ be the direct sum of the representations. Then $\mathcal{C}/\ker\tau = \mathcal{C}/\mathcal{L}(\mathcal{C},\mathcal{D}) = \mathcal{C}/\ker\pi_E$, so that for every $x \in \mathcal{C}$,

$$\operatorname{dist}(x,\mathcal{L}(\mathcal{C},\mathcal{D})) = \|\pi_E(x)\| = \sup\{\|\pi_\rho(x)\| : \rho \in \mathfrak{S}_{\mathrm{s}}(\mathcal{C},\mathcal{D})\}. \quad ∎$$

LEMMA 7.3. *Suppose $(\mathcal{C},\mathcal{D})$ is a skeletal* MASA *inclusion with $\mathcal{D}$ injective and let $\mathcal{S}$ be the linear span of $\mathcal{N}(\mathcal{C},\mathcal{D})$ (with no closure taken). Then*

$$\mathcal{L}(\mathcal{C},\mathcal{D}) \cap \mathcal{S} = (0).$$

Proof. Let E be the pseudo-expectation for $(\mathcal{C},\mathcal{D})$; as $\mathcal{D}$ is injective, E is the unique conditional expectation of $\mathcal{C}$ onto $\mathcal{D}$.

Fix $x \in \mathcal{S} \cap \mathcal{L}(\mathcal{C},\mathcal{D})$ and find a finite set $F \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ so that $x = \sum_{v\in F} v$. We shall show that for every $\rho \in \widehat{\mathcal{D}}$, there exists a projection $R_\rho \in \mathcal{D}$ such that $\rho(R_\rho) = 1$ and $xR_\rho = 0$.

Towards this end, let $\rho \in \widehat{\mathcal{D}}$. For every $(v,w) \in F\times F$, let $\{K_i(w^*v)\}_{i=0}^4$ be a right Frolík family of ideals for w^*v, and let $\{P_i(w^*v)\}_{i=0}^4 \subseteq \mathcal{D}$ be their corresponding support projections. As $K_i(w^*v)$ is a regular ideal in $\mathcal{D}$, and $\mathcal{D}$ is injective, $K_i(w^*v) = P_i(w^*v)\mathcal{D}$. As $\bigvee_{j=0}^{4} K_j(w^*v) = \mathcal{D}$

$$I = \sum_{i=0}^{4} P_i(w^*v) \quad \text{and (using Corollary 3.6)} \quad E(w^*v) = w^*vP_0(w^*v).$$

Therefore, ρ is non-zero on exactly one of the projections in the set $\{P_i(w^*v) : 0 \leqslant i \leqslant 4\}$; let R_{ρ,w^*v} be this projection. Put

$$R_\rho = \prod_{(v,w)\in F\times F} R_{\rho,w^*v}.$$

By construction, $\rho(R_\rho) = 1$, and, as

$$R_{\rho,w^*v}w^*vR_{\rho,w^*v} = \begin{cases} 0 & \text{if } R_{\rho,w^*v} \in \{P_i(w^*v) : i = 1, \dots, 4\}, \\ E(w^*v) & \text{if } R_{\rho,w^*v} = P_0(w^*v), \end{cases}$$

we obtain

$$R_\rho w^*vR_\rho = E(w^*v)R_\rho.$$

Since $x \in \mathcal{L}(\mathcal{C}, \mathcal{D})$,

$$R_\rho x^*xR_\rho = \sum_{v,w\in F} R_\rho w^*vR_\rho = \sum_{v,w\in F} E(w^*v)R_\rho = E(x^*x)R_\rho = 0.$$

This gives $xR_\rho = 0$ as desired.

To complete the proof, observe that the support of $\widehat{R}_\rho$ is a clopen subset of $\widehat{\mathcal{D}}$, so compactness of $\widehat{\mathcal{D}}$ ensures that we can find $n \in \mathbb{N}$ and $\rho_1, \dots, \rho_n \in \widehat{\mathcal{D}}$ so that $d := \sum_{j=1}^{n} R_{\rho_j}$ is an invertible element of $\mathcal{D}$. Then $xd = 0$, so that $x = 0$. ∎

THEOREM 7.4. *Suppose $(\mathcal{C}, \mathcal{D})$ is a skeletal* MASA *inclusion, $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ is a* MASA *skeleton for $(\mathcal{C}, \mathcal{D})$, and $\mathcal{A} = \operatorname{span} \mathcal{M}$. For $x \in \mathcal{A}$, let $\eta_0(x) = \operatorname{dist}(x, \mathcal{L}(\mathcal{C}, \mathcal{D}))$. Let η be a C^*-norm on $\mathcal{A}$ and denote by $\mathcal{C}_\eta$ the completion of $\mathcal{A}$ with respect to η. Then the following statements hold:*

(i) *There exists a unique pseudo-expectation $E_\eta : \mathcal{C}_\eta \to I(\mathcal{D})$ and $E_\eta|_{\mathcal{A}} = E|_{\mathcal{A}}$.*

(ii) *For each $x \in \mathcal{A}$,*

$$\eta_0(x) \leqslant \eta(x).$$

(iii) *If $\mathcal{D}$ is injective, then η_0 is a C^*-norm on $\mathcal{A}$.*

(iv) *If $(\mathcal{C}, \mathcal{D})$ is a regular* EP-*inclusion, then $(\mathcal{C}_\eta, \mathcal{D})$ is a regular* EP-*inclusion.*

Proof. Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Both $(\mathcal{C}_\eta, \mathcal{D})$ and $(\mathcal{C}, \mathcal{D})$ are skeletal MASA inclusions and $\mathcal{M}$ is a MASA skeleton for both. By Theorem 3.5, there exist unique pseudo-expectations $E : \mathcal{C} \to I(\mathcal{D})$ and $E_\eta : \mathcal{C}_\eta \to I(\mathcal{D})$. Proposition 3.4 shows that $E|_{\mathcal{M}} = E_\eta|_{\mathcal{M}}$. Hence $E|_{\mathcal{A}} = E_\eta|_{\mathcal{A}}$ and part (i) follows.

As $\mathcal{D} \subseteq \mathcal{M}$, $(\mathcal{C}_\eta, \mathcal{D})$ is a regular inclusion. For $\sigma \in \widehat{I(\mathcal{D})}$, let π_σ and $\pi_{\sigma,\eta}$ be the cyclic representations of $\mathcal{C}$ and $\mathcal{C}_\eta$ obtained from the states $\sigma \circ E$ and $\sigma \circ E_\eta$ respectively. Then for $x \in \mathcal{A}$,

$$\begin{aligned}\|\pi_\sigma(x)\|^2 &= \sup\{\sigma(E(y^*x^*xy)) : y \in \mathcal{A} \text{ and } \sigma(E(y^*y)) = 1\}\\ &= \sup\{\sigma(E_\eta(y^*x^*xy)) : y \in \mathcal{A} \text{ and } \sigma(E_\eta(y^*y)) = 1\} = \|\pi_{\sigma,\eta}(x)\|^2.\end{aligned}$$

Applying Lemma 7.2, for every $x \in \mathcal{A}$,

$$\eta_0(x) = \sup\{\|\pi_\sigma(x)\| : \sigma \in \widehat{I(\mathcal{D})}\} = \sup\{\|\pi_{\sigma,\eta}(x)\| : \sigma \in \widehat{I(\mathcal{D})}\} \leqslant \eta(x).$$

Thus statement (ii) holds.

Part (iii) follows from Lemma 7.3 and an application of Theorem 2.20 gives part (iv). ∎

COROLLARY 7.5. *Let $(\mathcal{C}, \mathcal{D})$ be a skeletal* MASA *inclusion, $\mathcal{M}$ a* MASA *skeleton for $(\mathcal{C}, \mathcal{D})$ and $\mathcal{A} = \operatorname{span} \mathcal{M}$. If $\mathcal{A} \cap \mathcal{L}(\mathcal{C}, \mathcal{D}) = (0)$, then there are maximal and minimal C^*-norms on $\mathcal{A}$.*

Proof. Theorem 7.4 shows that η_0 is the smallest C^*-norm on $\mathcal{A}$. Let η be a C^*-norm on $\mathcal{A}$. Recall that $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$, so that $v^*v \in \mathcal{D}$ for $v \in \mathcal{M}$. Thus, for $v \in \mathcal{M}$, we have $\eta(v)^2 = \|v^*v\|_\mathcal{C}$. Given $x \in \mathcal{A}$, we may find $N \in \mathbb{N}$, $m_i \in \mathcal{M}$ and scalars c_i $(1 \leqslant i \leqslant N)$ such that $x = \sum_{j=1}^{N} c_j m_j$. Therefore, $\eta(x)$ is dominated by $\sum |c_j| \|m_j\|_\mathcal{C}$. Hence

$$\|x\|_{\max} := \sup\{\eta(x) : \eta \text{ is a } C^*\text{-norm on } \mathcal{A}\}$$

is finite for each $x \in \mathcal{A}$. As $\|\cdot\|_{\max}$ is a C^*-norm which dominates any other C^*-norm, the proof is complete. ∎

QUESTION 7.6. *Is the hypothesis in Corollary* 7.5 *that $\mathcal{L}(\mathcal{C}, \mathcal{D}) \cap \operatorname{span} \mathcal{M} = (0)$ automatically satisfied? Lemma* 7.3 *shows this is the case when $(\mathcal{C}, \mathcal{D})$ is a skeletal* MASA *inclusion with $\mathcal{D}$ injective.*

Suppose $(\mathcal{C}, \mathcal{D})$ is an EP-inclusion, let $J \subseteq \mathcal{C}$ be an ideal, and let $q : \mathcal{C} \to \mathcal{C}/J$ be the quotient map. Then Lemma 3.1 of [3] shows $(q(\mathcal{C}), q(\mathcal{D}))$ is an EP-inclusion, that is, smaller norms (or semi-norms) on $\operatorname{span}(\mathcal{N}(\mathcal{C}, \mathcal{D}))$ yield EP-inclusions. It seems interesting that when $(\mathcal{C}, \mathcal{D})$ is a regular EP-inclusion, Theorem 7.4(iii) and Corollary 7.5 combined show that all semi-norms also produce regular EP-inclusions. The following corollary of our work extends Proposition 4 of [35] from the setting of represented free transformation groups to general regular EP-inclusions.

THEOREM 7.7. *Suppose $(\mathcal{C}, \mathcal{D})$ is a regular* EP-*inclusion and let u be a C^*-semi-norm on $\mathcal{A} := \operatorname{span} \mathcal{N}(\mathcal{C}, \mathcal{D})$. Let $N := \{x \in \mathcal{A} : u(x) = 0\}$ and let $(\mathcal{C}_u, \mathcal{D}_u)$ be the inclusion obtained by completing $\mathcal{A}/N$ and $\mathcal{D}/(N \cap \mathcal{D})$ with respect to u. Then $(\mathcal{C}_u, \mathcal{D}_u)$ is a regular* EP-*inclusion.*

Proof. As in the proof of Corollary 7.5, there exists a largest C^*-norm η on $\operatorname{span}(\mathcal{N}(\mathcal{C},\mathcal{D}))$. By Theorem 7.4(iv), $(\mathcal{C}_\eta,\mathcal{D})$ is a regular EP-inclusion. Also, there is a regular $*$-epimorphism $q : \mathcal{C}_\eta \to \mathcal{C}_u$. Now apply Lemma 3.1 of [3] to $J = \ker q$. ∎

Many constructions of C^*-algebras utilize a combinatorial object (e.g. a directed graph) to obtain a $*$-algebra, which is then appropriately completed to produce the C^*-algebra in question. One interpretation of Theorem 7.4 is that when this occurs, the minimal norm is uniquely determined by the combinatorial object. Here is an example of this. We utilize terminology, notation and results from Section 6. (See Definition 6.6 for the definition of the germ isotropy group.) Given a (discrete) group Γ and a C^*-algebra $\mathcal{D}$, $w_s : \Gamma \to \mathcal{D}$ is the function $t \mapsto \delta_{st} I$, where δ_{st} is the Kronecker δ function.

COROLLARY 7.8. *Let (X,Γ) be a discrete dynamical system such that for every $x \in X$, the germ isotropy group, H^x is abelian. Let $\mathcal{C} = C(X) \rtimes_{\mathrm{r}} \Gamma$, $\mathcal{D} = C(X)$ and let $\mathcal{D}^{\mathrm{c}}$ be the relative commutant of $\mathcal{D}$ in $\mathcal{C}$. Let $\mathcal{M} = \{dw_t : d \in \mathcal{D}^{\mathrm{c}}$ and $t \in \Gamma\}$, and let $\mathcal{A} = \operatorname{span}\mathcal{M}$. Then the norm on $\mathcal{A}$ obtained by restricting the norm on $\mathcal{C}$ to $\mathcal{A}$ is the unique minimal C^*-norm on $\mathcal{A}$.*

Proof. By Theorem 6.15, $\mathcal{D}^{\mathrm{c}}$ is abelian and $\mathcal{L}(\mathcal{C},\mathcal{D}^{\mathrm{c}}) = (0)$. By Lemma 2.10, $\mathcal{M} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D}^{\mathrm{c}})$. An application of Theorem 7.4 completes the proof. ∎

Here is an alternate proof of a result of Archbold and Spielberg regarding certain crossed products.

COROLLARY 7.9 ([4], Theorem 1). *Suppose (X,Γ) is a discrete dynamical system such that Γ acts topologically freely on X. Let $\pi : C(X) \rtimes_{\mathrm{full}} \Gamma \to C(X) \rtimes_{\mathrm{r}} \Gamma$ be the canonical quotient map. If $J \subseteq C(X) \rtimes_{\mathrm{full}} \Gamma$ is an ideal such that $J \cap C(X) = (0)$, then $J \subseteq \ker\pi$.*

Proof. Let $\mathcal{C} := C(X) \rtimes_{\mathrm{r}} \Gamma$, $\mathcal{D} := C(X)$ and $\mathcal{M} = \{dw_t : d \in C(X)$ and $t \in \Gamma\}$. Since Γ acts topologically freely, $(\mathcal{C},\mathcal{D})$ is a Cartan inclusion, $\mathcal{M}$ is a MASA skeleton for $(\mathcal{C},\mathcal{D})$ and $\operatorname{span}\mathcal{M} = C_{\mathrm{c}}(\Gamma,\mathcal{D})$. Let η be the full crossed product norm on $C_{\mathrm{c}}(\Gamma,\mathcal{D})$, so that $\mathcal{C}_\eta$ is the full crossed product. Note that $(\mathcal{C}_\eta,\mathcal{D})$ is a skeletal MASA inclusion and, by Theorem 7.4, $\ker\pi = \mathcal{L}(\mathcal{C}_\eta,\mathcal{D})$. Theorem 3.15 shows that $J \subseteq \ker\pi$. ∎

8. NORMING ALGEBRAS

Here is an application of our work to norming algebras. We begin with a definition. Recall that $\mathcal{N}(\mathcal{C},\mathcal{D})$ is a closed $*$-semigroup containing $\mathcal{D}$.

DEFINITION 8.1. A $*$-subsemigroup $\mathcal{F} \subseteq \mathcal{N}(\mathcal{C},\mathcal{D})$ with $\mathcal{D} \subseteq \mathcal{F}$ is *countably generated over $\mathcal{D}$* if there exists a countable set $F \subseteq \mathcal{F}$ so that the smallest

$*$-subsemigroup of $\mathcal{N}(\mathcal{C},\mathcal{D})$ containing $F\cup\mathcal{D}$ is $\mathcal{F}$. The set F will be called a *generating set* for $\mathcal{F}$.

We will say that the inclusion $(\mathcal{C},\mathcal{D})$ is *countably regular* if there exists a $*$-subsemigroup $\mathcal{F}\subseteq\mathcal{N}(\mathcal{C},\mathcal{D})$ such that $\mathcal{F}$ is countably generated over $\mathcal{D}$ and $\mathcal{C}=\overline{\operatorname{span}}(\mathcal{F})$.

The following result generalizes Lemma 2.15 of [27] and gives a large class of norming algebras. In particular, notice that the result holds for Cartan inclusions.

THEOREM 8.2. *Suppose $(\mathcal{C},\mathcal{D})$ is a virtual Cartan inclusion. Then $\mathcal{D}$ norms $\mathcal{C}$.*

Proof. Let $\mathcal{F}\subseteq\mathcal{N}(\mathcal{C},\mathcal{D})$ be a $*$-subsemigroup which is countably generated over $\mathcal{D}$ by the (countable) set F. Let $\mathcal{C}_{\mathcal{F}}\subseteq\mathcal{C}$ be the C^*-subalgebra generated by $\mathcal{F}$. (Notice that $\mathcal{C}_{\mathcal{F}}$ is simply the closed linear span of $\mathcal{F}$.) Then $(\mathcal{C}_{\mathcal{F}},\mathcal{D})$ is a countably regular MASA inclusion.

We will show that $\mathcal{D}$ norms $\mathcal{C}_{\mathcal{F}}$. Let

$$Y:=\{\sigma\in\widehat{\mathcal{D}}:\sigma\text{ has a unique state extention to }\mathcal{C}_{\mathcal{F}}\}.$$

Theorem 3.10 shows that Y is dense in $\widehat{\mathcal{D}}$. For each element $\sigma\in Y$, let σ' denote the unique extension of σ to all of $\mathcal{C}_{\mathcal{F}}$. Notice that if $\rho\in\widehat{I(\mathcal{D})}$ and $\rho\circ\iota=\sigma$, then $\sigma'=\rho\circ E$ because $\sigma'|_{\mathcal{D}}=\sigma=\rho\circ E|_{\mathcal{D}}$.

For $\sigma\in Y$, let π_σ be the GNS-representation for σ'. Proposition 4.12 shows that $\pi_\sigma(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\sigma)$.

Define an equivalence relation R on Y by $\sigma_1\sim\sigma_2$ if and only if there exists $v\in\mathcal{F}$ such that $\sigma_2=\beta_v(\sigma_1)$. (Since $\mathcal{F}$ is a $*$-semigroup, this is an equivalence relation.)

We claim that if π_{σ_1} is unitarily equivalent to π_{σ_2}, then $\sigma_1\sim\sigma_2$. To see this, we use a modification of the argument in Lemma 5.8 of [12]. Let $U\in\mathcal{B}(\mathcal{H}_{\sigma_2},\mathcal{H}_{\sigma_1})$ be a unitary operator such that

$$U^*\pi_{\sigma_1}U=\pi_{\sigma_2}.$$

Let L_{σ_i} be the left kernel of σ_i'. Since π_{σ_i} is irreducible, $\mathcal{C}/L_{\sigma_i}=\mathcal{H}_{\sigma_i}$. Hence we may find $X\in\mathcal{C}$ such that $U(I+L_{\sigma_2})=X+L_{\sigma_1}$. Then for every $x\in\mathcal{C}$,

$$\sigma_2'(x)=\langle\pi_{\sigma_2}(x)(I+L_{\sigma_2}),(I+L_{\sigma_2})\rangle=\sigma_1'(X^*xX).$$

Fix $\rho_i\in\widehat{I(\mathcal{D})}$ such that $\rho_i\circ\iota=\sigma_i$.

The map $\mathcal{C}\ni x\mapsto\sigma_1'(X^*x)$ is a non-zero linear bounded linear functional on $\mathcal{C}$. Since $\operatorname{span}(\mathcal{F})$ is dense in $\mathcal{C}$, there exists $v\in\mathcal{F}$ so that $\sigma_1'(X^*v)\neq 0$. The Cauchy–Schwarz inequality for completely positive maps shows that for any $d\in\mathcal{D}$,

$$\begin{aligned}|\sigma_1'(X^*vd)|^2&=\rho_1(E(X^*vd)E(d^*v^*X))\leqslant\rho_1(E(X^*vdd^*v^*X))\\&=\sigma_1'(X^*vdd^*v^*X)=\sigma_2(vdd^*v^*).\end{aligned}$$

When $d \in \mathcal{D}$ and $\sigma_1(d) \neq 0$, we have $\sigma_1'(X^*vd) = \sigma_1'(X^*v)\sigma_1(d) \neq 0$. Therefore, when $d \in \mathcal{D}$ satisfies $\sigma_1(d) \neq 0$,

$$0 < \sigma_2(vdd^*v^*). \tag{8.1}$$

In particular, $\sigma_2(vv^*) \neq 0$. For any $d \in \mathcal{D}$, we have

$$\beta_{v^*}(\sigma_2)(d) = \frac{\sigma_2(vdv^*)}{\sigma_2(vv^*)}.$$

If $\beta_{v^*}(\sigma_2) \neq \sigma_1$, then there exists $d \in \mathcal{D}$ with $\sigma_1(dd^*) \neq 0$ and $\beta_{v^*}(\sigma_2)(dd^*) = 0$. But this is impossible by (8.1). So $\beta_{v^*}(\sigma_2) = \sigma_1$. Hence $\sigma_1 \sim \sigma_2$ as claimed.

Thus, if $\sigma_1 \not\sim \sigma_2$, then π_{σ_1} and π_{σ_2} are disjoint representations (as they are both irreducible).

Let $\mathcal{Y} \subseteq Y$ be chosen so that $\mathcal{Y}$ contains exactly one element from each equivalence class of Y. Put

$$\pi = \bigoplus_{\sigma \in \mathcal{Y}} \pi_\sigma.$$

Then

$$\begin{aligned} \ker \pi &= \bigcap_{\sigma \in \mathcal{Y}} \ker \pi_\sigma = \bigcap_{\sigma \in Y} \ker \pi_\sigma \\ &= \{x \in \mathcal{C}_{\mathcal{F}} : \sigma'(z^*x^*xz) = 0 \text{ for all } \sigma \in Y \text{ and all } z \in \mathcal{C}_{\mathcal{F}}\}. \end{aligned} \tag{8.2}$$

We next prove that

$$\mathcal{L}(\mathcal{C}_{\mathcal{F}}, \mathcal{D}) \supseteq \ker \pi. \tag{8.3}$$

Suppose to obtain a contradiction, that $x \in \ker \pi$ and that $E(x^*x)$ is a non-zero element of $I(\mathcal{D})$. By the Hamana-regularity of the extension $(I(\mathcal{D}), \iota)$, there exists $d \in \mathcal{D}$ such that $0 \leqslant d \leqslant E(x^*x)$ and $d \neq 0$. Since Y is dense in $\widehat{\mathcal{D}}$, there exists $\sigma \in Y$ such that $\sigma(d) \neq 0$. Choose $\rho \in \widehat{I(\mathcal{D})}$ such that $\sigma' = \rho \circ E$. Then

$$0 \neq \sigma(d) = \rho(E(d)) \leqslant \rho(E(x^*x)) = \sigma'(x^*x),$$

contradicting (8.2). Hence (8.3) holds.

Since $\mathcal{L}(\mathcal{C}, \mathcal{D}) \supseteq \mathcal{L}(\mathcal{C}_{\mathcal{F}}, \mathcal{D}) \supseteq \ker \pi$, we see that π is a faithful representation of $\mathcal{C}_{\mathcal{F}}$.

Since the representations in the definition of π are disjoint and each $\pi_\sigma(\mathcal{D})''$ is a MASA in $\mathcal{B}(\mathcal{H}_\sigma)$, $\pi(\mathcal{D})''$ is an atomic MASA in $\mathcal{B}(\mathcal{H}_\pi)$. Therefore, $\pi(\mathcal{D})''$ is locally cyclic (see p. 173 of [30]) for $\mathcal{B}(\mathcal{H}_\pi)$. By Theorem 2.7 and Lemma 2.3 of [30] $\pi(\mathcal{D})$ norms $\mathcal{B}(\mathcal{H}_\pi)$. But then $\pi(\mathcal{D})$ norms $\pi(\mathcal{C}_{\mathcal{F}})$. Since π is faithful, $\mathcal{D}$ norms $\mathcal{C}_{\mathcal{F}}$.

Finally, suppose that $k \in \mathbb{N}$ and that $x = (x_{ij}) \in M_n(\mathcal{C})$. For each $n \in \mathbb{N}$ and $i, j \in \{1, \ldots, k\}$, we may find a finite set $F_{n,i,j} \subseteq \mathcal{N}(\mathcal{C}, \mathcal{D})$ so that

$$\Big\| x_{ij} - \sum_{v \in F_{n,i,j}} v \Big\| < \frac{1}{n}.$$

Let

$$F = \bigcup \{F_{n,i,j} : n \in \mathbb{N}, i, j \in \{1, \dots, k\}\}.$$

Then F is countable. Let $\mathcal{F}$ be the closed $*$-subsemigroup of $\mathcal{N}(\mathcal{C}, \mathcal{D})$ generated by F and $\mathcal{D}$. Then for $i, j \in \{1, \dots, k\}$, $x_{ij} \in \mathcal{C}_{\mathcal{F}}$. Since $\mathcal{D}$ norms $\mathcal{C}_{\mathcal{F}}$, we conclude that

$$\begin{aligned}\|x\|_{M_k(\mathcal{C})} &= \|x\|_{M_k(\mathcal{C}_{\mathcal{F}})} \\ &= \sup\{\|RxC\| : R \in M_{1,n}(\mathcal{D}), C \in M_{n,1}(\mathcal{D}), \|R\| \leqslant 1, \|C\| \leqslant 1\}.\end{aligned}$$

Hence $\mathcal{D}$ norms $\mathcal{C}$. ∎

For any norm closed subalgebra $\mathcal{A}$ of the C^*-algebra $\mathcal{C}$, let $C^*(\mathcal{A})$ be the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$, and let $C^*_{\mathrm{e}}(\mathcal{A})$ be the C^*-envelope of $\mathcal{A}$. (There are a number of references which discuss C^*-envelopes; see [7], [13], [26].)

The following result significantly generalizes Theorem 4.21 of [12]. Theorem 8.3 was observed by Vrej Zarikian, who has kindly consented to its inclusion here.

THEOREM 8.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be C^*-algebras, with $\mathcal{D} \subseteq \mathcal{C}$ ($\mathcal{D}$ is not assumed abelian). Let $(I(\mathcal{D}), \iota)$ be an injective envelope for $\mathcal{D}$. Suppose there exists a unique unital completely positive map $\Phi : \mathcal{C} \to I(\mathcal{D})$ such that $\Phi|_{\mathcal{D}} = \iota$, and assume also that Φ is faithful. Let $\mathcal{A}$ be a norm-closed (not necessarily self-adjoint) subalgebra of $\mathcal{C}$ such that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{C}$. Then the C^*-subalgebra of $\mathcal{C}$ generated by $\mathcal{A}$ is the C^*-envelope of $\mathcal{A}$.*

Proof. Let $\theta : \mathcal{A} \to C^*_{\mathrm{e}}(\mathcal{A})$ be a unital completely isometric (unital) homomorphism such that the C^*-algebra generated by the image of θ is $C^*_{\mathrm{e}}(\mathcal{A})$. Then there exists a unique $*$-epimorphism $q : C^*(\mathcal{A}) \to C^*_{\mathrm{e}}(\mathcal{A})$ such that $q|_{\mathcal{A}} = \theta$. Our task is to show that q is one-to-one.

Since $I(\mathcal{D})$ is injective in the category of operator systems and completely contractive maps, there exists a unital completely contractive map $\Phi_{\mathrm{e}} : C^*_{\mathrm{e}}(\mathcal{A}) \to I(\mathcal{D})$ such that $\Phi_{\mathrm{e}} \circ \theta|_{\mathcal{D}} = \iota$. Also, there exists a unital completely contractive map $\Delta : \mathcal{C} \to I(\mathcal{D})$ so that $\Delta|_{C^*(\mathcal{A})} = \Phi_{\mathrm{e}} \circ q$. Then for $d \in \mathcal{D}$, we have $\theta(d) = q(d)$, so $\iota(d) = \Phi_{\mathrm{e}}(\theta(d)) = \Phi_{\mathrm{e}}(q(d)) = \Delta(d)$. The uniqueness of Φ gives $\Delta = \Phi$. Then if $x \in C^*(\mathcal{A})$ and $q(x) = 0$, we have

$$\Phi(x^*x) = \Delta(x^*x) = \Phi_{\mathrm{e}}(q(x^*x)) = 0,$$

so $x^*x = 0$ by the faithfulness of Φ. Thus q is one-to-one, and the proof is complete. ∎

We now obtain the following generalization of Theorem 2.16 in [27]. The outline of the proof is the same as the proof of Theorem 2.16 in [27], however the details in obtaining norming subalgebras are different.

THEOREM 8.4. *For $i = 1, 2$, suppose that $(\mathcal{C}_i, \mathcal{D}_i)$ are virtual Cartan inclusions and that $\mathcal{A}_i \subseteq \mathcal{C}_i$ are norm closed subalgebras such that $\mathcal{D}_i \subseteq \mathcal{A}_i \subseteq \mathcal{C}_i$. Let $C^*(\mathcal{A}_i)$ be* *the C^*-subalgebra of $\mathcal{C}_i$ generated by $\mathcal{A}_i$. If $u : \mathcal{A}_1 \to \mathcal{A}_2$ is an isometric isomorphism, then u extends uniquely to a $*$-isomorphism of $C^*(\mathcal{A}_1)$ onto $C^*(\mathcal{A}_2)$.*

Proof. Theorem 8.2 implies that $\mathcal{D}_i$ norms $\mathcal{C}_i$. Taken together, the uniqueness of the pseudo-expectations for $(\mathcal{C}_i, \mathcal{D}_i)$ from Theorem 3.5 and Theorem 8.3 imply that $C^*(\mathcal{A}_i)$ is the C^*-envelope of $\mathcal{A}_i$. Finally, an application of Corollary 1.5 in [27] completes the proof. ∎

Acknowledgements. The author is grateful for the support of the University of Nebraska's NSF ADVANCE grant #0811250 in the completion of this paper. This work was also partially supported by a grant from the Simons Foundation (#316952 to David Pitts).

The author thanks Ken Davidson, Allan Donsig, William Grilliette, Vern Paulsen, and especially Vrej Zarikian for numerous helpful conversations and suggestions.

Finally, the author is indebted to William Arveson, who greatly influenced the author and also the field of operator algebras. His passing saddens us all.

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Received September 15, 2016; revised October 16, 2016.