BOUNDARY REPRESENTATIONS OF OPERATOR SPACES AND COMPACT RECTANGULAR MATRIX CONVEX SETS

ADAM H. FULLER, MICHAEL HARTZ, and MARTINO LUPINI

Communicated by Stefaan Vaes

ABSTRACT. We initiate the study of matrix convexity for operator spaces. We define the notion of compact rectangular matrix convex set, and prove the natural analogs of the Krein–Milman and the bipolar theorems in this context. We deduce a canonical correspondence between compact rectangular matrix convex sets and operator spaces. We also introduce the notion of boundary representation for an operator space, and prove the natural analog of Arveson's conjecture: every operator space is completely normed by its boundary representations. This yields a canonical construction of the triple envelope of an operator space.

KEYWORDS: *Operator space, operator system, boundary representation, compact matrix convex set, matrix-gauged space.*

MSC (2010): Primary 46L07, 47L25; Secondary 46E22, 47L07.

INTRODUCTION

It is hard to overstate the importance of the theory of convexity in analysis. This is all the more true in the study of operator systems, which can be seen as the noncommutative analog of compact convex sets. Indeed, given any operator system S, the space of matrix-valued unital completely positive maps on S is endowed with a natural notion of convex combinations with matrix coefficients (matrix convex combination), and a topology which is compact as long as one restricts the target to a fixed matrix algebra. The *compact matrix convex sets* that arise in this way have been initially studied by Effros and Wittstock in [19], [48]. The program of developing the theory of compact matrix convex sets as the noncommutative analog of compact convex sets has been proposed by Effros in [19]. This program has been pursued in [21], [46], where matricial generalizations of the classical Krein–Milman and bipolar theorems are proved. Compact matrix convex sets and the corresponding notion of matrix extreme points have been subsequently studied in a number of papers. This line of research

has recently found outstanding applications. These include the matrix convexity proof of Arveson's conjecture on boundary representations due to Davidson and Kennedy [16] building on previous work of Farenick [23], [24], and the work of Helton, Klep, and McCullogh in free real algebraic geometry [30], [31].

The main goal of this paper is to provide the nonselfadjoint analog of the results above in the setting of operator spaces. Precisely, we introduce the notion of compact *rectangular* matrix convex set, which is the natural analog of the notion of compact matrix convex set where convex combinations with *rectangular* matrices are considered. We then prove generalizations of the Krein–Milman and bipolar theorems in the setting of compact rectangular matrix convex sets. We then deduce that compact rectangular matrix convex sets are in canonical functorial one-to-one correspondence with operator spaces. It follows from this that any operator space is completely normed by the matrix-valued completely contractive maps that are rectangular matrix extreme points.

We also introduce the notion of boundary representation for operator spaces. The natural operator space analog of Arveson's conjecture is then established: any operator space is completely normed by its boundary representations. This gives an explicit description of the triple envelope of an operator space in terms of boundary representations. We also obtain in this setting an analog of Arveson's boundary theorem. As an application, we compute boundary representations for multiplier spaces associated with pairs of reproducing kernel Hilbert spaces.

The results of this paper can be seen as the beginning of a convexity theory approach to the study of operator spaces. Convexity theory has played a crucial role in the setting of Banach spaces, such as in the groundbreaking work of Alfsen and Effros on M-ideals in Banach spaces [2], [3] or the work of Lazar and Lindenstrauss on L^1-predual spaces [35], [36]. This work can be seen as a first step towards establishing noncommutative analogs of the results of Alfsen–Effros and Lazar–Lindenstrauss mentioned above. We will see below that the crucial notion of collinearity for bounded linear functionals on a Banach space, which is of key importance for the work of Alfsen and Effros on facial cones and M-ideals, has a natural interpretation in the setting of rectangular matrix convexity.

The results of the present paper have already found application in [37]. The fact established here that an operator space is completely normed by its matrix-valued completely contractive maps that are rectangular matrix extreme is used there to prove that the noncommutative Gurarij space introduced by Oikhberg in [38] is the unique separable nuclear operator space with the property that the canonical map from the maximal TRO to the triple envelope is injective.

This paper is divided into three sections, besides the introduction. In Section 1 we introduce the notion of boundary representation for operator spaces, and prove that any operator space is completely normed by its boundary representations. The boundary theorem for operator spaces and applications to multiplier spaces for pairs of reproducing kernel Hilbert spaces are also considered in this section. In Section 2 we introduce the notion of compact rectangular matrix

convex set and rectangular matrix extreme point. We prove the Krein–Milman and bipolar theorem for compact rectangular matrix convex sets, and deduce the correspondence between compact rectangular matrix convex sets and operator spaces. Finally in Section 3 we consider the notion of matrix-gauged space. Such a concept has recently been introduced by Russell in [41] in order to provide an abstract characterization of selfadjoint subspaces of C^*-algebras (selfadjoint operator spaces), and to capture the injectivity property of $B(H)$ in such a category. We prove in Section 3 that the construction of the injective envelope and the C^*-envelope of an operator system can be naturally generalized to the setting of matrix-gauged spaces. A geometric approach to the study of matrix-gauged spaces and selfadjoint operator spaces is also possible, as we show that matrix-gauged spaces are in functorial one-to-one correspondence with compact matrix convex sets with a distinguished matrix extreme point.

1. BOUNDARY REPRESENTATIONS AND THE SHILOV BOUNDARY OF AN OPERATOR SPACE

1.1. NOTATION AND PRELIMINARIES. Recall that a ternary ring of operators (TRO) T is a subspace of the C^*-algebra $B(H)$ of bounded linear operators on a Hilbert space H that is closed under the triple product $(x, y, z) \mapsto xy^*z$. An important example of a TRO is the space $B(H, K)$, where H, K are Hilbert spaces. A TRO has a canonical operator space structure coming from the inclusion $T \subset B(H)$, which does not depend on the concrete representation of T as a ternary ring of operators on H. A triple morphism between TROs is a linear map that preserves the triple product. Any TRO can be seen as the 1-2 corner of a canonical C^*-algebra $\mathcal{L}(T)$ called its *linking algebra*. A triple morphism between TROs can be seen as the 1-2 corner of a $*$-homomorphism between the corresponding linking algebras [29]; see also Corollary 8.3.5 of [9].

   The notions of (nondegenerate, irreducible, faithful) representations admit natural generalizations from C^*-algebras to TROs. A representation of a TRO T is a triple morphism $\theta : T \to B(H, K)$ for some Hilbert spaces H, K. A linear map $\psi : T \to B(H, K)$ is *nondegenerate* if, whenever p, q are projections in $B(H)$ and $B(K)$, respectively, such that $q\theta(x) = \theta(x)p = 0$ for every $x \in T$, one has $p = 0$ and $q = 0$. Similarly a representation θ of T is *irreducible* if , whenever p, q are projections in $B(H)$ and $B(K)$, respectively, such that $q\theta(x)p + (1 - q)\theta(x)(1 - p) = \theta(x)$ for every $x \in T$ (equivalently, $q\theta(x) = \theta(x)p$ for every $x \in T$), one has $p = 1$ and $q = 1$, or $p = 0$ and $q = 0$. Finally, θ is called *faithful* if it is injective or, equivalently, completely isometric. Various characterizations of nondegenerate and irreducible representations are obtained in Lemma 3.1.4 and Lemma 3.1.5 of [10]. A concrete TRO $T \subset B(H, K)$ is said to act nondegenerately or irreducibly if the corresponding inclusion representation is nondegenerate or irreducible, respectively.

In the following we will use frequently without mention the Haagerup–Paulsen–Wittstock extension theorem ([39], Theorem 8.2), asserting that, if H, K are Hilbert spaces, then the space $B(H, K)$ of bounded linear operators from H to K is injective in the category of operator spaces and completely contractive maps. We will also often use the canonical way, due to Paulsen, to assign to an operator space $X \subset B(H, K)$ an operator system $\mathcal{S}(X) \subset B(K \oplus H)$. This operator system, called the *Paulsen system*, is defined to be the space of operators

$$\left\{ \begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} : x, y \in X, \lambda, \mu \in \mathbb{C} \right\}$$

where I_H and I_K denote the identity operator on H and K, respectively. Any completely contractive map $\phi : X \to Y$ between operator spaces extends canonically to a unital completely positive map $\mathcal{S}(\phi) : \mathcal{S}(X) \to \mathcal{S}(Y)$ defined by

$$\begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} \mapsto \begin{bmatrix} \lambda I_K & \phi(x) \\ \phi(y)^* & \mu I_H \end{bmatrix},$$

see Lemma 8.1 of [39]. (The Paulsen system is defined in Chapter 8 of [39] and Section 1.3 of [9] only in the case when $H = K$. The same proofs from Chapter 8 of [39] and Section 1.3 of [9] apply with no change to this more general situation.)

1.2. DILATIONS OF RECTANGULAR OPERATOR STATES. Suppose that X is an operator space. A *rectangular operator state* on X is a nondegenerate linear map $\phi : X \to B(H, K)$ such that $\|\phi\|_{cb} = 1$. We say that a rectangular operator state $\psi : X \to B(\widetilde{H}, \widetilde{K})$ is a *dilation* of ϕ if there exist linear isometries $v : H \to \widetilde{H}$ and $w : K \to \widetilde{K}$ such that $w^* \psi(x) v = \phi(x)$ for every $x \in X$. The same proof as Theorem 8.4 of [39] gives the following proposition.

PROPOSITION 1.1. *Any rectangular operator state $\phi : T \to B(H, K)$ on a TRO $T \subset B(H_0, K_0)$ can be dilated to a nondegenerate triple morphism $\theta : T \to B(\widetilde{H}, \widetilde{K})$. If H_0, K_0, H, K are finite-dimensional, then one can take $\widetilde{H}$ and $\widetilde{K}$ to be finite-dimensional.*

In order to prove Proposition 1.1 one can proceed as in Theorem 8.4 of [39], by replacing $M_2(A)$ for a given C^*-algebra A with the C^*-algebra generated by $\mathcal{S}(T)$ inside $B(K_0 \oplus H_0)$. It is clear that in Proposition 1.1 one can choose θ and the linear isometries $v : H \to \widetilde{H}$ and $w : K \to \widetilde{K}$ in such a way that $\widetilde{K}$ is the linear span of $\theta(T)\theta(T)^* wK \cup \theta(T)vH$, and $\widetilde{H}$ is the linear span of $\theta(T)^*\theta(T)vH \cup \theta(T)^* wK$. In this case, we call such a dilation θ a *minimal* dilation of ϕ. In the sequel, it will often be convenient to identify H with a subspace of $\widetilde{H}$ and K with a subspace of $\widetilde{K}$.

DEFINITION 1.2. Let $\phi : X \to B(H, K)$ be a rectangular operator state and let $\psi : X \to B(\widetilde{H}, \widetilde{K})$ be a dilation of ϕ. We can assume that $H \subset \widetilde{H}$ and $K \subset \widetilde{K}$. Let p be the orthogonal projection from $\widetilde{H}$ onto H and let q be the orthogonal projection from $\widetilde{K}$ onto K. The dilation ψ is *trivial* if

$$\psi(x) = q\psi(x)p + (1 - q)\psi(x)(1 - p)$$

for every $x \in X$. The operator state ϕ on an operator space X is *maximal* if it has no nontrivial dilation.

It is clear that, when X is an operator system, $\widetilde{H} = \widetilde{K}$, $q = p$, and ψ is a unital completely positive map, the notion of trivial dilation as above recovers the usual notion of trivial dilation.

DEFINITION 1.3. Suppose that X is a subspace of a TRO T such that T is generated as a TRO by X. The operator state ϕ on X has the *unique extension property* if any rectangular operator state $\widetilde{\phi}$ of T whose restriction to X coincides with ϕ is automatically a triple morphism.

We now observe that a rectangular operator state on an operator space is maximal if and only if it has the unique extension property. The analogous fact for unital completely positive maps on operator systems is well known; see [6].

LEMMA 1.4. *For a TRO T, the set of positive elements of the C^*-algebra TT^* is the closed convex cone generated by $\{xx^* : x \in T\}$.*

Proof. Let $C \subset TT^*$ denote the closed convex cone generated by $\{xx^* : x \in T\}$. It is clear that C is contained in the set of positive elements of TT^*. Conversely, suppose that $a \in TT^*$ is positive. By the remarks at the beginning of Section 2.2 of [22], the C^*-algebra TT^* admits a contractive approximate identity (e_i) of elements of the form

$$\sum_j x_j x_j^*$$

where $x_j \in T$. For such an element of TT^* one has that

$$a^{1/2}\left(\sum_j x_j x_j^*\right) a^{1/2} = \sum_j (a^{1/2} x_j)(a^{1/2} x_j)^* \in C,$$

as T is a left TT^*-module. Thus, $a^{1/2} e_i a^{1/2} \in C$ for every i. It follows that $a = \lim_i a^{1/2} e_i a^{1/2} \in C$. ∎

LEMMA 1.5. *Let T be a TRO, let $\psi : T \to B(\widetilde{H}, \widetilde{K})$ be a completely contractive linear map and suppose that $H \subset \widetilde{H}$ and $K \subset \widetilde{K}$ are closed subspaces with corresponding orthogonal projections $p \in B(\widetilde{H})$ and $q \in B(\widetilde{K})$. If the map*

$$\theta : T \to B(H, K), \quad x \mapsto q\psi(x)p,$$

is a non-degenerate triple morphism, then ψ is a trivial dilation of θ.

Proof. By dilating ψ if necessary, we may assume without loss of generality that ψ is a triple morphism. Thus, there exists a unique $*$-homomorphism

$$\sigma : TT^* \to B(\widetilde{K}) \quad \text{such that } \sigma(xy^*) = \psi(x)\psi(y)^* \quad (x, y \in T).$$

The assumption further implies that there exists a $*$-homomorphism

$$\pi : TT^* \to B(K) \quad \text{such that } \pi(xy^*) = q\psi(x)p\psi(y)^* q.$$

Since θ is non-degenerate, π is non-degenerate as well. Consider now the map

$$\varphi : TT^* \to B(K), \quad a \mapsto q\sigma(a)q - \pi(a).$$

We claim that $\varphi = 0$. To this end, observe that if $x \in T$, then

$$\varphi(xx^*) = q\psi(x)(1 - p)\psi(x)^*q \geqslant 0,$$

so φ is a positive map by Lemma 1.4. Let (e_i) be an approximate identity for TT^*. Since π is non-degenerate, $\pi(e_i)$ tends to q in the strong operator topology. Since $q\sigma(e_i)q \leqslant q$, it follows that $\varphi(e_i) = q\sigma(e_i)q - \pi(e_i)$ tends to zero in the strong operator topology. Combining this with positivity of φ, it follows that $\varphi = 0$ (see, e.g. Lemma I.9.5 of [15]).

In particular, we see that for $x \in T$,

$$0 = \varphi(xx^*) = q\psi(x)(1 - p)\psi(x)^*q = [q\psi(x)(1 - p)][q\psi(x)(1 - p)]^*,$$

so that $q\psi(x)(1 - p) = 0$. A similar argument, replacing TT^* with T^*T, shows that $p\psi(x)^*(1 - q) = 0$. Thus, ψ is a trivial dilation of θ.   ∎

PROPOSITION 1.6. *Suppose that $\phi : X \to B(H, K)$ is a rectangular operator state of X, and T is a* TRO *containing X as a generating subspace. Then ϕ is maximal if and only if it has the unique extension property.*

Proof. Suppose initially that ϕ is maximal. Let $\widetilde{\phi} : T \to B(H, K)$ be an extension of ϕ. Let $\theta : T \to B(\widetilde{H}, \widetilde{K})$ be a dilation of $\widetilde{\phi}$ to a triple morphism. We can identify H with a subspace of $\widetilde{H}$ and K with a subspace of $\widetilde{K}$. Let p and q be the orthogonal projections of $\widetilde{H}$ and $\widetilde{K}$ onto H and K, respectively. We have that $\widetilde{\phi}(x) = q\theta(x)|_H$ for every $x \in T$. The restriction of θ to X is a rectangular operator state that dilates ϕ. By maximality of ϕ, we can conclude that $\theta(x) = q\theta(x)p + (1 - q)\theta(x)(1 - p)$ for every $x \in X$. Since X generates T as a TRO and θ is a triple morphism, it follows that this identity holds for every $x \in T$. It follows that $\widetilde{\phi}$ is a triple morphism as well.

Suppose now that ϕ has the unique extension property. Let $\psi : X \to B(\widetilde{H}, \widetilde{K})$ be a dilation of ϕ. As above, we will identify H and K as subspaces of $\widetilde{H}$ and $\widetilde{K}$, respectively, and denote by p and q the corresponding orthogonal projections. We can extend ψ to a rectangular operator state $\psi : T \to B(\widetilde{H}, \widetilde{K})$. Observe that $x \mapsto q\psi(x)|_H$ is a rectangular operator state extending ϕ. Since ϕ has the unique extension property, $x \mapsto q\psi(x)|_H$ is a triple morphism. Hence from Lemma 1.5 we can conclude that ψ is a trivial dilation of ϕ. Since ψ was arbitrary, we can conclude that ϕ is maximal.   ∎

Simple examples show that the implication "unique extension property implies maximal" of Proposition 1.6 may fail if ϕ is a degenerate completely contractive map. Indeed, there are degenerate representations of TROs which have non-trivial dilations.

1.3. BOUNDARY REPRESENTATIONS. Suppose that X is an operator space, and T is a TRO containing X as a generating subspace.

DEFINITION 1.7. A *boundary representation* for X is a rectangular operator state $\phi : X \to B(H, K)$ with the property that any rectangular operator state on T extending X is an irreducible representation of T.

In other words, a rectangular operator state $\phi : X \to B(H, K)$ is a boundary representation for X if and only if it has the unique extension property, and the unique extension of ϕ to T an irreducible representation of T. In the following we will identify a boundary representation of X with its unique extension to an irreducible representation of T. It follows from Proposition 1.6 that the notion of boundary representation does not depend on the concrete realization of X as a space of operators. We remark that our terminology differs slightly from Arveson's original use of the term boundary representation in the context of operator systems [4]. Indeed, for Arveson, a boundary representation is a representation of the C^*-algebra generated by the operator system. More precisely, if S is an operator system that generates the C^*-algebra A, then according to Arveson, a boundary representation for S is an irreducible representation π of A such that π is the unique completely positive extension of $\pi|_S$. We follow the convention, which is for example used in [16], that a boundary representation of S is a unital completely positive map $\phi : S \to B(H)$ such that every extension of ϕ to a completely positive map on A is an irreducible representation of A. Since this notion does not depend on the concrete representation of S, these two points of view are equivalent.

In the rest of this section, we will observe that the boundary representations of X completely norm X. This will be deduced from the corresponding fact about operator systems, proved in [6] in the separable case and in [16] in full generality.

PROPOSITION 1.8. *Suppose $\omega : \mathcal{S}(X) \to B(L_\omega)$ is a boundary representation of the Paulsen system $\mathcal{S}(X)$ associated with X. Then one can decompose L_ω as an orthogonal direct sum $K_\omega \oplus H_\omega$ in such a way that $\omega = \mathcal{S}(\psi)$ for some boundary representation $\psi : X \to B(H_\omega, K_\omega)$ of X.*

Proof. Suppose that $T \subset B(H, K)$ is a TRO containing X as a generating subspace. Let A be the C^*-algebra generated by $\mathcal{S}(X)$ inside $B(K \oplus H)$. Observe that

$$A = \left\{ \begin{bmatrix} x_{11} + \lambda I_K & x_{12} \\ x_{21} & x_{22} + \mu I_H \end{bmatrix} : x_{11} \in TT^*, x_{12} \in T, x_{21} \in T^*, x_{22} \in T^*T, \lambda, \mu \in \mathbb{C} \right\}.$$

Since ω is a boundary representation of $\mathcal{S}(X)$, it extends to an irreducible representation $\omega : A \to B(L_\omega)$. Set

$$q_\omega := \omega\left(\begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix} \right), \quad \text{and} \quad p_\omega := \omega\left(\begin{bmatrix} 0 & 0 \\ 0 & I_H \end{bmatrix} \right).$$

Observe that $p_\omega, q_\omega \in B(L_\omega)$ are orthogonal projections such that $p_\omega + q_\omega = I_{L_\omega}$. Denote by K_ω the range of q_ω and by H_ω the range of p_ω. The fact that ω is a unital $*$-homomorphism implies that, with respect to the decomposition $L_\omega = K_\omega \oplus H_\omega$, one has that

$$\omega = \begin{bmatrix} \sigma & \theta \\ \theta^* & \pi \end{bmatrix}$$

where $\theta : T \to B(H_\omega, K_\omega)$ is a triple morphism.

We claim that θ is irreducible. Suppose that $p \in B(H_\omega)$ and $q \in B(K_\omega)$ are projections such that $q\theta(x) = \theta(x)p$ for every $x \in T$. Since $\sigma(ab^*) = \theta(a)\theta(b)^*$ and $\pi(a^*b) = \theta(a)^*\theta(b)$ for every $a, b \in T$, we can conclude that $(q \oplus p)\omega(x) = \omega(x)(q \oplus p)$ for every $x \in A$. Since ω is an irreducible representation of A, it follows that $q \oplus p = 1$ or $q \oplus p = 0$. This concludes the proof that θ is irreducible.

Denote by ψ the restriction of θ to X. Observe that $\omega = \mathcal{S}(\psi)$. We claim that ψ is maximal. Indeed, let $\phi : X \to B(\widetilde{H}, \widetilde{K})$ be a dilation of ψ. We can identify H and K as subspaces of $\widetilde{H}$ and $\widetilde{K}$, with corresponding orthogonal projections p and q. Then $\mathcal{S}(\phi) : \mathcal{S}(X) \to B(\widetilde{K} \oplus \widetilde{H})$ is a dilation of ω. By maximality of ω, we have that

$$\begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \omega(x) = \omega(x) \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

for every $x \in \mathcal{S}(X)$. It follows that $q\phi(x) = \phi(x)p$ for every $x \in X$. This shows that ϕ is a trivial dilation of ψ, concluding the proof that ψ is maximal. ∎

The following result is now an immediate consequence of Proposition 1.8 and Theorem 3.4 of [16].

THEOREM 1.9. *Suppose that X is an operator space. Then X is completely normed by its boundary representations.*

1.4. RECTANGULAR EXTREME POINTS AND PURE UNITAL COMPLETELY POSITIVE MAPS. Suppose that X is an operator space, and $\phi : X \to B(H, K)$ is a completely contractive linear map. A *rectangular operator convex combination* is an expression $\phi = \alpha_1^*\phi_1\beta_1 + \cdots + \alpha_n^*\phi_n\beta_n$, where $\beta_i : H \to H_i$ and $\alpha_i : K \to K_i$ are linear maps, and $\phi_i : X \to B(H_i, K_i)$ are completely contractive linear maps for $i = 1, 2, \ldots, \ell$ such that $\alpha_1^*\alpha_1 + \cdots + \alpha_n^*\alpha_n = 1$, and $\beta_1^*\beta_1 + \cdots + \beta_n^*\beta_n = 1$. Such a rectangular convex combination is *proper* if α_i, β_i are surjective, and *trivial* if $\alpha_i^*\alpha_i = \lambda_i 1$, $\beta_i^*\beta_i = \lambda_i 1$, and $\alpha_i^*\phi_i\beta_i = \lambda_i\phi$ for some $\lambda_i \in [0, 1]$.

DEFINITION 1.10. A completely contractive map $\phi : X \to B(H, K)$ is a *rectangular operator extreme point* if any proper rectangular operator convex combination $\phi = \alpha_1^*\phi_1\beta_1 + \cdots + \alpha_n^*\phi_n\beta_n$ is trivial.

Suppose now that X is an operator system. An *operator state* on X is a unital completely positive map $\phi : X \to B(H)$. An *operator convex combination* is an expression $\phi = \alpha_1^*\phi_1\alpha_1 + \cdots + \alpha_n^*\phi_n\alpha_n$, where $\alpha_i : H \to H_i$ are linear maps, and $\phi_i : X \to B(H_i)$ are operator states for $i = 1, 2, \ldots, \ell$ such that $\alpha_1^*\alpha_1 + \cdots +$

$\alpha_n^* \alpha_n = 1$. Such an operator convex combination is *proper* if α_i is right invertible for $i = 1, 2, \ldots, \ell$, and *trivial* if $\alpha_i^* \alpha_i = \lambda_i 1$ and $\alpha_i^* \phi_i \alpha_i = \lambda_i \phi$ for some $\lambda_i \in [0, 1]$.

We say that ϕ is an *operator extreme point* if any proper operator convex combination $\phi = \alpha_1^* \phi_1 \alpha_1 + \cdots + \alpha_n^* \phi_n \alpha_n$ is trivial. The proof of Theorem B of [23] shows that an operator state is an operator extreme point if and only if it is a pure element in the cone of completely positive maps.

When H is finite-dimensional, the notion of proper operator convex combination coincides with the notion of proper matrix convex combination from [46]. In this case, the notion of operator extreme point coincides with the notion of matrix extreme point from Definition 2.1 of [46].

LEMMA 1.11. *Suppose that $\phi : X \to B(H, K)$ is a completely contractive linear map, $\Psi : \mathcal{S}(X) \to B(K \oplus H)$ is a completely positive map such that $\mathcal{S}(\phi) - \Psi$ is completely positive. Suppose that $\Psi(1)$ is an invertible element of $B(K \oplus H)$. Then there exist positive invertible elements $a \in B(K)$ and $b \in B(H)$, and a completely contractive map $\psi : X \to B(H, K)$ such that*

$$\Psi\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \lambda I_K & \psi(x) \\ \psi(y)^* & \mu I_H \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Proof. Fix a concrete representation $X \subset B(L_0, L_1)$ of X. In this case $\mathcal{S}(X) \subset B(L_1 \oplus L_0)$. Set $\Phi := \mathcal{S}(\phi)$ and let $T \subset B(L_0, L_1)$ denote the TRO generated by X. By Arveson's extension theorem, we may extend Φ and Ψ to the C^*-algebra A generated by $\mathcal{S}(X)$ inside $B(L_1 \oplus L_0)$ in such a way that $\Phi - \Psi$ is still completely positive. In the following we regard Φ, Ψ as maps from A to $B(K \oplus H)$. Since $\Phi - \Psi$ is completely positive, the argument in the proof of Theorem 8.3 in [39] shows that there exist linear maps $\varphi_1 : TT^* + \mathbb{C}I_{L_1} \to B(K)$ and $\varphi_0 : T^*T + \mathbb{C}I_{L_0} \to B(H)$ such that

$$\Psi\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \varphi_1(x) & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Psi\left(\begin{bmatrix} 0 & 0 \\ 0 & y \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & \varphi_0(y) \end{bmatrix}.$$

In particular, $w = \Psi(1)$ is a diagonal element, and the unital completely positive map $\Psi_0 = w^{-1/2} \Psi w^{-1/2}$ satisfies

$$\Psi_0\left(\begin{bmatrix} I_{L_1} & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \Psi_0\left(\begin{bmatrix} 0 & 0 \\ 0 & I_{L_0} \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & I_H \end{bmatrix}.$$

It follows that the two projections $\begin{bmatrix} I_{L_1} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & I_{L_0} \end{bmatrix}$ belong to the multiplicative domain of Ψ_0 ([9], Proposition 1.3.11), so that there exists a completely contractive map $\psi : X \to B(H, K)$ such that

$$\Psi_0\left(\begin{bmatrix} \lambda I_{L_1} & x \\ y^* & \mu I_{L_0} \end{bmatrix} \right) = \begin{bmatrix} \lambda I_K & \psi(x) \\ \psi(y)^* & \mu I_H \end{bmatrix},$$

which finishes the proof. ∎

PROPOSITION 1.12. *Suppose that $\phi : X \to B(H, K)$ is a completely contractive map and $\mathcal{S}(\phi) : \mathcal{S}(X) \to B(K \oplus H)$ is the associated unital completely positive map defined on the Paulsen system. The following assertions are equivalent:*

(i) $\mathcal{S}(\phi)$ *is a pure completely positive map;*

(ii) $\mathcal{S}(\phi)$ *is an operator extreme point;*

(iii) ϕ *is a rectangular operator extreme point.*

Proof. We have already observed that the equivalence of (i) and (ii) holds, as the argument in the proof of Theorem B in [23] shows.

(ii) $\Rightarrow$ (iii) Suppose that $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_\ell^* \phi_\ell \beta_\ell$ is a proper rectangular matrix convex combination. Define $\gamma_i = \alpha_i \oplus \beta_i$ for $i = 1, 2, \ldots, \ell$. Then we have that $\mathcal{S}(\phi) = \gamma_1^* \mathcal{S}(\phi_1) \gamma_1 + \cdots + \gamma_\ell^* \mathcal{S}(\phi_\ell) \gamma_\ell$ is a proper matrix convex combination. Since by assumption $\mathcal{S}(\phi)$ is an operator extreme point in the state space of $\mathcal{S}(X)$, we can conclude that the proper matrix convex combination $\gamma_1^* \mathcal{S}(\phi_1) \gamma_1 + \cdots + \gamma_\ell^* \mathcal{S}(\phi_\ell) \gamma_\ell$ is trivial. This implies that the proper rectangular matrix convex combination $\alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_\ell^* \phi_\ell \beta_\ell$ is trivial as well.

(iii) $\Rightarrow$ (i) Suppose that $\mathcal{S}(\phi) = \Psi_1 + \Psi_2$ for some completely positive maps $\Psi_1, \Psi_2 : \mathcal{S}(X) \to B(K \oplus H)$. Fix $\varepsilon > 0$ and define $\Xi_i = (1 - \varepsilon)\Psi_i + (\varepsilon/2)\mathcal{S}(\phi)$ for $i = 1, 2$. Then $\Xi_1, \Xi_2 : \mathcal{S}(X) \to B(K \oplus H)$ are completely positive maps such that $\Xi_1 + \Xi_2 = \mathcal{S}(\phi)$ and $\Xi_i(1)$ is invertible for $i = 1, 2$; cf. the proof of Lemma 2.3 in [16]. By Lemma 1.11 we have that, for $i = 1, 2$,

$$\Xi_i\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \right) = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \begin{bmatrix} \lambda 1 & \psi_i(x) \\ \psi_i(y)^* & \mu 1 \end{bmatrix} \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

for some positive invertible elements $a_i \in B(K)$, $b_i \in B(H)$, and completely contractive $\psi_i : X \to B(H, K)$. Thus we have that $\phi = a_1 \psi_1 b_1 + a_2 \psi_2 b_2$ is a proper rectangular operator convex combination. By assumption, we have that $a_i^2 = t_i 1$, $b_i^2 = t_i 1$, and $a_i \psi_i b_i = t_i \phi$ for some $t_i \in [0, 1]$ and $i = 1, 2$. It follows that $\Xi_i = t_i \mathcal{S}(\phi)$ for $i = 1, 2$. Since this is true for every ε, it follows that the Ψ_i are also scalar multiples of $\mathcal{S}(\phi)$. This concludes the proof that $\mathcal{S}(\phi)$ is pure. ∎

The following corollary is an immediate consequence of Proposition 1.12, Proposition 1.8, and Theorem 2.4 of [16].

COROLLARY 1.13. *Suppose that $\phi : X \to B(H, K)$ is a rectangular operator state. If ϕ is rectangular operator extreme, then ϕ admits a dilation to a boundary representation of X.*

Suppose that X is a Banach space. We regard X as an operator space endowed with its canonical *minimal* operator space structure obtained from the canonical inclusion of X in the C^*-algebra $C(\text{Ball}(X'))$, where X' denotes the dual space of X. In [2] Alfsen and Effros considered the following notion. Suppose that ϕ_0, ϕ_1 are nonzero contractive linear functionals on X. Then ϕ_0, ϕ_1 are *codirectional* if $\|\phi_0 + \phi_1\| = \|\phi_0\| + \|\phi_1\|$. This is equivalent to the assertion that $\|\phi_0 + \phi_1\| \mathcal{S}((\phi_0 + \phi_1)/(\|\phi_0 + \phi_1\|)) = \|\phi_0\| \mathcal{S}(\phi_0/\|\phi_0\|) + \|\phi_1\| \mathcal{S}(\phi_1/\|\phi_1\|)$. The

relation $\prec$ on nonzero contractive linear functionals is defined by setting $\phi \prec \psi$ if and only if either $\phi = \psi$ or ϕ and $\psi - \phi$ are codirectional. This is equivalent to the assertion that $\|\phi\|\mathcal{S}(\phi/\|\phi\|) \leqslant \|\psi\|\mathcal{S}(\psi/\|\psi\|)$.

PROPOSITION 1.14. *Suppose that X is a Banach space and ϕ is a bounded linear functional on X of norm 1. The following statements are equivalent:*

(i) *ϕ is an extreme point of* $\mathrm{Ball}(X')$;

(ii) *if $\psi \in \mathrm{Ball}(X')$ is a nonzero linear functional such that $\psi \prec \phi$, then ψ is a scalar multiple of ϕ;*

(iii) *ϕ is a rectangular extreme point.*

Proof. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are straightforward.

(ii) $\Rightarrow$ (iii) Suppose that, for every nonzero $\psi \in \mathrm{Ball}(X')$, $\psi \prec \phi$ implies that ψ is a scalar multiple of ϕ. The proof of Proposition 1.12 shows that it suffices to show that every proper rectangular convex combination of two elements is trivial. Thus, consider a rectangular convex combination $\phi = s_0 \bar{t}_0 \phi_0 + s_1 \bar{t}_1 \phi_1$ for some $\phi_0, \phi_1 \in \mathrm{Ball}(X')$ and non-zero $s_0, s_1, t_0, t_1 \in \mathbb{C}$ such that $|s_0|^2 + |s_1|^2 = 1$ and $|t_0|^2 + |t_1|^2 = 1$. Observe that

$$1 = \|s_0 \bar{t}_0 \phi_0 + s_1 \bar{t}_1 \phi_1\| \leqslant \|s_0 \bar{t}_0 \phi_0\| + \|s_1 \bar{t}_1 \phi_1\|$$
$$\leqslant |s_0 \bar{t}_0|\|\phi_0\| + |s_1 \bar{t}_1|\|\phi_1\| \leqslant |s_0 \bar{t}_0| + |s_1 \bar{t}_1| \leqslant 1.$$

Hence $\|s_0 \bar{t}_0 \phi_0\| + \|s_1 \bar{t}_1 \phi_1\| = \|\phi_0\| = \|\phi_1\| = 1$ and $|s_0| = |t_0|$ and $|s_1| = |t_1|$. By hypothesis we have that $s_0 \bar{t}_0 \phi_0 = \rho_0 \phi$ and $s_1 \bar{t}_1 \phi_1 = \rho_1 \phi$ for some $\rho_0, \rho_1 \in \mathbb{C}$. In particular, $|\rho_i| = |s_i|^2 = |t_i|^2$ for $i = 0, 1$. Since $\phi = s_0 \bar{t}_0 \phi_0 + s_1 \bar{t}_1 \phi_1$, it follows that $\rho_0 + \rho_1 = 1$. Combined with $|\rho_0| + |\rho_1| = 1$, this implies that $\rho_0, \rho_1 \in [0, 1]$, so that the rectangular convex combination was trivial. ∎

1.5. TRO-EXTREME POINTS. Suppose that X is an operator space and $\varphi : X \to B(H, K)$ is a completely contractive map. We say that φ is a TRO-extreme point if whenever $\varphi = \alpha_1^* \varphi_1 \beta_1 + \cdots + \alpha_\ell^* \varphi_\ell \beta_\ell$ is a proper rectangular matrix convex combination such that $\varphi_i : X \to B(H, K)$ for $i = 1, 2, \ldots, \ell$, then $\alpha_i^* \alpha_i = t_i 1$, $\beta_i^* \beta_i = t_i 1$, and $\alpha_i^* \varphi_i \beta_i = t_i \varphi$ for some $t_i \in [0, 1]$. When H, K are finite-dimensional, this is equivalent to requiring that there exist unitaries $u_i \in M_n(\mathbb{C})$ and $w_i \in M_m(\mathbb{C})$ such that $\varphi_i = u_i^* \varphi w_i$. This can be seen arguing as in the proof of Lemma 2.7 below. The notion of TRO-extreme point can be seen as the operator space analog of the notion of C^*-extreme point considered in [25], [26], [32].

A similar proof as Theorem B of [23] gives the following lemma.

LEMMA 1.15. *Let X be an operator space, and A be the C^*-algebra generated by $\mathcal{S}(X)$. If $\varphi : X \to B(H, K)$ is a TRO-extreme point such that the range of φ is an irreducible subspace of $B(H, K)$, then there exists a pure unital completely positive map $\Phi : A \to B(K \oplus H)$ that extends $\mathcal{S}(\varphi)$.*

Using this lemma, one can prove similarly as Theorem C of [23] the following fact.

PROPOSITION 1.16. *Suppose that T is a TRO, and $\varphi : T \to B(H, K)$ is a TRO-extreme point. Then there exist pairwise orthogonal projections $(p_i)_{i \in I}$ in $B(H)$ and $(q_i)_{i \in I}$ in $B(K)$ such that $p_i \varphi q_i$ is rectangular operator extreme for every i, and $p_i \varphi q_j = 0$ for every $i \neq j$.*

1.6. THE SHILOV BOUNDARY OF AN OPERATOR SPACE. Suppose that X is an operator space. A *triple cover* of X is a pair (ι, T) where T is a TRO and $\iota : X \to T$ is a completely isometric linear map whose range is a subspace of T that generates T as a TRO. Among the triple covers there exists a canonical one: the *triple envelope*. This is the (unique) triple cover $(\iota_e, \mathcal{T}_e(X))$ with the property that for any other triple cover (ι, T) of X, there exists a triple morphism $\theta : T \to \mathcal{T}_e(X)$ such that $\theta \circ \iota = \iota_e$. The existence of the triple envelope was established by Hamana in [28] using the construction of the injective envelope of an operator space; see also Section 4.4 of [9].

The triple envelope of an operator space is referred to as the *(noncommutative) Shilov boundary* in [8]. It is remarked in [8] at the beginning of Section 4, referring to the Shilov boundary of an operator space, that "the spaces above are not at the present time defined canonically", and "this lack of canonicity is always a potential source of blunders in this area, if one is not careful about various identifications". We remark here that the theory of boundary representations provides a canonical construction of the Shilov boundary of an operator space X. Indeed one can consider $\iota_e : X \to B(H, K)$ to be the direct sum of all the boundary representations for X, and then let $(\iota_e, \mathcal{T}_e(X))$ be the subTRO of $B(H, K)$ generated by the image of ι_e. Proposition 1.6 implies that ι_e is maximal, so we may argue as in the proof of Theorem 4.1 in [17] that $\mathcal{T}_e(X)$ is indeed the triple envelope of X. It also follows that if X has a completely isometric boundary representation $\theta : X \to B(H)$, then the triple envelope of X is the TRO generated by the range of θ inside $B(H)$.

1.7. THE SHILOV BOUNDARY OF A BANACH SPACE. A TRO T is *commutative* if $xy^*z = zy^*x$ for every $x, y, z \in T$. Several equivalent characterizations of commutative TROs are provided in Proposition 8.6.5 of [9]. Suppose that E is a locally trivial line bundle over a locally compact Hausdorff space U. Then the space $\Gamma_0(E)$ of continuous sections of E that vanish at infinity is a commutative TRO such that $\Gamma_0(E)^* \Gamma_0(E) = C_0(E)$. Conversely, it is observed in Section 4 of [8] (see also [18]) that any commutative TRO is of this form. One can also describe the commutative TROs as the C_σ-spaces from the Banach space literature.

Suppose that E is a locally trivial line bundle over a locally compact Hausdorff space U with point at infinity ∞ and $X \subset \Gamma_0(E)$ be a closed subspace. Assume that the set of elements $\{\langle x, y \rangle : x, y \in X\}$ of $C_0(U)$ separates the points of U and does not identitically vanish at any point of U. This is equivalent to the assertion that X generates $\Gamma_0(E)$ as a TRO, as proved in Theorem 4.20 of [8]. An irreducible representation of $\Gamma_0(E)$ is of the form $x \mapsto x(\omega_0)$ for some $\omega_0 \in U$. A linear map from $\Gamma_0(E)$ to $\mathbb{C}$ of norm 1 has the form $x \mapsto \int x(\omega) \mathrm{d}\mu(\omega)$ for

some Borel probability measure μ on U. We say that μ is a *representing measure* for $\omega_0 \in U$ if

$$\int x(\omega) \mathrm{d}\mu(\omega) = x(\omega_0) \quad \text{for every } x \in X.$$

A point $\omega_0 \in U$ is a *Choquet boundary point* if the point mass at ω_0 is the unique representing measure for ω_0. It follows from the observations above that ω_0 is a Choquet boundary point for X if and only if the map $x \mapsto x(\omega_0)$ is a boundary representation for X. The Choquet boundary $\mathrm{Ch}(X)$ of X is the set of Choquet boundary points of X.

Suppose that $\partial_S X \cup \{\infty\}$ is the closure of $\mathrm{Ch}(X) \cup \{\infty\}$ inside $U \cup \{\infty\}$. Then it follows from Theorem 1.9 that $E|_{\partial_S X}$ is the Shilov boundary of X in the sense of Theorem 4.25 in [8]. This means that the linear map $X \to \Gamma_0(E|_{\partial_S X})$, $x \mapsto x|_{\partial_S X}$ is isometric, and for any locally trivial line bundle over a locally compact Hausdorff space V and linear isometry $J : X \to \Gamma_0(V)$ with the property that the set $\{J(x)^* J(y) : x, y \in X\}$ separates the points of V and does not identically vanish at any point of V, there exists a proper continuous injection $\varphi : \partial_S X \to V$ with the property that $J(x) \circ \varphi = x|_{\partial_S X}$ for every $x \in X$. This gives a canonical construction of the Shilov boundary of a Banach space, analogous to the canonical construction of a Shilov boundary of a unital function space; see Section 4.1 of [9].

1.8. THE RECTANGULAR BOUNDARY THEOREM. Arveson's boundary theorem ([5], Theorem 2.1.1) asserts that if $S \subset B(H)$ is an operator system which acts irreducibly on H such that the C^*-algebra $C^*(S)$ contains the algebra of compact operators $\mathcal{K}(H)$, then the identity representation of $C^*(S)$ is a boundary representation for S if and only if the quotient map $B(H) \to B(H)/\mathcal{K}(H)$ is not completely isometric on S.

The following result is a rectangular generalization of Arveson's boundary theorem.

THEOREM 1.17. *Let $X \subset B(H, K)$ be an operator space such that the* TRO *T generated by X acts irreducibly and such that $T \cap \mathcal{K}(H, K) \neq \{0\}$. Then the identity representation of T is a boundary representation for X if and only if the quotient map $B(H, K) \to B(H, K)/\mathcal{K}(H, K)$ is not completely isometric on X.*

Proof. Suppose first that the quotient map π is completely isometric on X. Then π, regarded as a map from $X \to \pi(X)$, admits a completely isometric inverse, which extends to a complete contraction $\psi : B(H, K)/\mathcal{K}(H, K) \to B(H, K)$. Clearly, $\psi \circ \pi$ is a completely contractive map which extends the inclusion of X into $B(H, K)$, but it does not extend the inclusion of T into $B(H, K)$, since it annihilates the compact operators.

Conversely, suppose that the quotient map is not completely isometric on X. Let $\mathcal{S}(X) \subset B(K \oplus H)$ denote the Paulsen system associated with X. We will verify that $\mathcal{S}(X)$ satisfies the assumptions of Arveson's boundary theorem.

To see that $\mathcal{S}(X)$ acts irreducibly, suppose that p is an orthogonal projection on $K \oplus H$ which commutes with $\mathcal{S}(X)$. In particular, p commutes with $I_K \oplus 0$ and

$0 \oplus I_H$, from which we deduce that $p = p_1 \oplus p_2$, where $p_1 \in B(K)$ and $p_2 \in B(H)$ are orthogonal projections. For $x \in X$, we therefore have

$$\begin{bmatrix} 0 & p_1 x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} = \begin{bmatrix} 0 & xp_2 \\ 0 & 0 \end{bmatrix},$$

hence $p_1 x = x p_2$ for all $x \in X$. Since X acts irreducibly, it follows that either $p_1 \oplus p_2 = 0$ or $p_1 \oplus p_2 = I_{K \oplus H}$, so that $\mathcal{S}(X)$ acts irreducibly. The assumption that T contains a non-zero compact operator implies that $\mathcal{S}(X)$ contains a non-zero compact operator, hence by irreducibility of $\mathcal{S}(X)$, we see that $\mathcal{K}(K \oplus H) \subset C^*(\mathcal{S}(X))$.

Since the quotient map $B(H, K) \to B(H, K)/\mathcal{K}(H, K)$ is not completely isometric on X, there exists $x \in M_n(X)$ and $k \in M_n(\mathcal{K}(H, K))$ such that $\|x - k\| < \|x\|$. Regarding x as an element of $M_n(\mathcal{S}(X))$ in the canonical way and correspondingly k as an element of $M_n(\mathcal{K}(K \oplus H))$, we see that the quotient map $B(K \oplus H) \to B(K \oplus H)/\mathcal{K}(K \oplus H)$ is not completely isometric on $\mathcal{S}(X)$. Thus, Arveson's boundary theorem implies that the identity representation is a boundary representation of $\mathcal{S}(X)$. According to Proposition 1.8, there exists a boundary representation $\psi : X \to B(L_1, L_2)$ of X such that $S(\psi)$ is the inclusion of $\mathcal{S}(X)$ into $B(K \oplus H)$. It easily follows now that $L_1 = H, L_2 = K$ and that ψ is the inclusion of X into $B(H, K)$, which finishes the proof. ∎

1.9. RECTANGULAR MULTIPLIERS. A *reproducing kernel Hilbert space H* on a set X is a Hilbert space of functions on X such that for every $x \in X$, the functional

$$H \to \mathbb{C}, \quad f \mapsto f(x),$$

is bounded. The unique function $k : X \times X \to \mathbb{C}$ which satisfies $k(\cdot, x) \in H$ for all $x \in X$ and

$$\langle f, k(\cdot, x) \rangle = f(x)$$

for all $x \in X$ and $f \in H$ is called the *reproducing kernel* of H. We will always assume that H has no common zeros, meaning that there does not exist $x \in X$ such that $f(x) = 0$ for all $x \in X$. Equivalently, $k(x, x) \neq 0$ for all $x \in X$. We refer the reader to the books [1] and [40] for background material on reproducing kernel Hilbert spaces.

If H and K are reproducing kernel Hilbert spaces on the same set X, we define the multiplier space

$$\text{Mult}(H, K) = \{\varphi : X \to \mathbb{C} : \varphi \cdot f \in K \text{ for all } f \in H\},$$

where $(\varphi \cdot f)(x) = \varphi(x)f(x)$ for $x \in X$; see Section 5.7 of [40]. By Theorem 5.21 of [40], every $\varphi \in \text{Mult}(H, K)$ induces a bounded multiplication operator $M_\varphi : H \to K$. Moreover, since K has no common zeros, every multiplier φ is uniquely determined by its associated multiplication operator M_φ. We may thus regard $\text{Mult}(H, K)$ as a subspace of $B(H, K)$.

The best studied case occurs when $H = K$, in which case $\text{Mult}(H) = \text{Mult}(H, H)$ is an algebra, called the multiplier algebra of H ([1], Section 2.3).

Nevertheless, the rectangular case of two different reproducing kernel Hilbert spaces has been studied as well, see for example [44] and [45], where multipliers between weighted Dirichlet spaces are investigated.

We say that a reproducing kernel Hilbert space K on X with reproducing kernel k is *irreducible* if X cannot be partitioned into two non-empty sets X_1 and X_2 such that $k(x, y) = 0$ for all $x \in X_1$ and $y \in X_2$. This definition is more general than the definition of irreducibility in Definition 7.1 of [1], but it suffices for our purposes.

For the next lemma, we observe that if H contains the constant function 1, then $\mathrm{Mult}(H, K)$ is contained in K.

LEMMA 1.18. *Let H and K be reproducing kernel Hilbert spaces on the same set. Suppose that:*

(i) *H contains the constant function* 1;

(ii) $\mathrm{Mult}(H, K)$ *is dense in K; and*

(iii) *K is irreducible.*

Then $\mathrm{Mult}(H, K) \subset B(H, K)$ *acts irreducibly.*

Proof. Suppose that $p \in B(H)$ and $q \in B(K)$ are orthogonal projections which satisfy $q M_\varphi = M_\varphi p$ for all $\varphi \in \mathrm{Mult}(H, K)$. Define $\psi = p1 \in H$. Then for all $\varphi \in \mathrm{Mult}(H, K)$, the identity

$$q\varphi = q M_\varphi 1 = M_\varphi p1 = \psi\varphi$$

holds. Since $\mathrm{Mult}(H, K)$ is dense in K, we deduce that $\psi \in \mathrm{Mult}(K)$ and that $q = M_\psi$. We claim that ψ is necessarily constant. To this end, let k denote the reproducing kernel of K. Note that M_ψ is in particular selfadjoint, so that

$$\psi(x)k(x, y) = \langle M_\psi k(\cdot, y), k(\cdot, x)\rangle = \langle k(\cdot, y), M_\psi k(\cdot, x)\rangle = \overline{\psi(y)}k(x, y)$$

for all $x, y \in X$. Hence $\psi(x)$ is real for all $x \in X$ and $\psi(x) = \psi(y)$ if $k(x, y) \neq 0$. Fix $x_0 \in X$, and suppose for a contradiction that

$$X_1 = \{x \in X : \psi(x) = \psi(x_0)\}$$

is a proper subset of X and let $X_2 = X \setminus X_1$. If $x \in X_1$ and $y \in X_2$, then $\psi(y) \neq \psi(x_0) = \psi(x)$, hence $k(x, y) = 0$. This contradicts irreducibility of K, so that ψ is constant. Moreover, since M_ψ is a projection, we necessarily have $\psi = 1$ or $\psi = 0$. If $\psi = 1$, then $q = M_\psi = I_K$ and $M_\varphi(I_H - p) = 0$ for all $\varphi \in \mathrm{Mult}(H, K)$. Similarly, if $\psi = 0$, then $q = 0$ and $M_\varphi p = 0$ for all $\varphi \in \mathrm{Mult}(H, K)$. We may thus finish the proof by showing that

$$\bigcap_{\varphi \in \mathrm{Mult}(H,K)} \ker(M_\varphi) = \{0\}.$$

To this end, note that if $x \in X$, then $\{f \in K : f(x) = 0\}$ is a proper closed subspace of K, as K has no common zeros. Since $\mathrm{Mult}(H, K)$ is dense in K, it cannot be contained in such a subspace, thus for every $x \in X$, there exists $\varphi \in$

$\mathrm{Mult}(H, K)$ such that $\varphi(x) \neq 0$. If $f \in H$ satisfies $M_\varphi f = 0$ for all $\varphi \in \mathrm{Mult}(H, K)$, it therefore follows that $f(x) = 0$ for all $x \in X$, that is, $f = 0$, as desired. ∎

We can now use the rectangular boundary theorem to show that for many multiplier spaces, the identity representation is always a boundary representation.

PROPOSITION 1.19. *Let H and K be reproducing kernel Hilbert spaces on the same set and let $M = \mathrm{Mult}(H, K)$. Suppose that:*

 (i) *H contains the constant function 1;*
 (ii) *M is dense in K;*
 (iii) *K is irreducible; and*
 (iv) *M contains a non-zero compact operator.*
Then the identity representation is a boundary representation of M. In particular, the triple envelope of M is the TRO *generated by M.*

Proof. Lemma 1.18 shows that M acts irreducibly. Moreover, the quotient map by the compacts is not isometric on M since M contains a non-zero compact operator. An application of the rectangular boundary theorem (Theorem 1.17) now finishes the proof. ∎

For $s \in \mathbb{R}$, let

$$H_s = \Big\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \|f\|_{H_s^2} = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{-s} < \infty \Big\}.$$

This is a reproducing kernel Hilbert space on the open unit disc $\mathbb{D}$ with reproducing kernel

$$k_s(z, w) = \sum_{n=0}^{\infty} (n+1)^s (z\overline{w})^n.$$

This scale of spaces is a frequent object of study in the theory of reproducing kernel Hilbert spaces. The space H_0 is the classical Hardy space H^2, the space H_{-1} is the Dirichlet space, and the space H_1 is the Bergman space.

The elements of $\mathrm{Mult}(H_s, H_t)$ were characterized in [44] and [45]. We remark that the spaces $\mathcal{D}_\alpha$ of [45] are related to the spaces above via the formula $\mathcal{D}_\alpha = H_{-\alpha}$. In [44], a slightly different convention is used. There, $\mathcal{D}_\alpha = H_{-2\alpha}$, at least with equivalent norms. Theorem 4 of [45] shows that $\mathrm{Mult}(H_s, H_t) = \{0\}$ if $s > t$. On the other hand, if $s \leqslant t$, then $H_s \subset H_t$, hence $\mathrm{Mult}(H_s) \subset \mathrm{Mult}(H_s, H_t)$. Since $\mathrm{Mult}(H_s)$ at least contains the polynomials, the same is true for $\mathrm{Mult}(H_s, H_t)$.

In the square case $s = t$, boundary representations of operator spaces related to the algebras $\mathrm{Mult}(H_s)$, and their analogs on higher dimensional domains, were studied in [13], [27], [33]; see in particular Section 2 of [27] and Section 5.2 of [33]. It is well known that if $s \geqslant 0$, then $\mathrm{Mult}(H_s) = H^\infty$, the algebra of all bounded analytic functions on the unit disc, endowed with the supremum norm. This can be deduced, for example, from Proposition 26(ii) of [43]. In particular, the

C^*-envelope of $\mathrm{Mult}(H_s)$ is commutative, so that the identity representation of $\mathrm{Mult}(H_s)$ on H_s is not a boundary representation. On the other hand, if $s < 0$, then the identity representation of $\mathrm{Mult}(H_s)$ on H_s is a boundary representation. This follows, for instance, from Corollary 2 in Section 2 of [5] and its proof.

We use the results above to prove that, in the rectangular case, the identity representation is always a boundary representation.

COROLLARY 1.20. *If $s < t$, then the identity representation is a boundary representation of $\mathrm{Mult}(H_s, H_t)$.*

Proof. We verify that the pair (H_s, H_t) satisfies the assumptions of Proposition 1.19. It is clear that H_s contains the constant function 1. By the remark above, $\mathrm{Mult}(H_s, H_t)$ contains the polynomials, and is therefore dense in K. Moreover, since $k_t(0, w) = 1$ for all $w \in \mathbb{D}$, the space H_t is irreducible. Finally, since

$$\frac{\|z^n\|_{H_t}^2}{\|z^n\|_{H_s}^2} = (n+1)^{s-t},$$

which tends to zero as $n \to \infty$, the inclusion $H_s \subset H_t$ is compact, so that $M_1 \in \mathrm{Mult}(H_s, H_t)$ is a compact operator. Therefore, the result follows from Proposition 1.19. ∎

2. OPERATOR SPACES AND RECTANGULAR MATRIX CONVEX SETS

In the following we will use notation from [21] and [46]. In particular, if V and V' are vector spaces in duality via a bilinear map $\langle \cdot, \cdot \rangle$, $x = [x_{ij}] \in M_{n,m}(V)$, and $\psi = [\psi_{\alpha\beta}] \in M_{r,s}(V')$, then we let $\langle\langle x, \psi \rangle\rangle$ be the element $[\langle x_{ij}, \psi_{\alpha\beta} \rangle]$ of $M_{nr,ms}(\mathbb{C})$, where the rows of $\langle\langle x, \psi \rangle\rangle$ are indexed by (i, α) and the columns of $\langle\langle x, \psi \rangle\rangle$ are indexed by (j, β). We also let $\psi^{(n,m)}$ be the map $\langle\langle \cdot, \psi \rangle\rangle : M_{n,m}(V) \to M_{nr,ms}(\mathbb{C})$.

2.1. RECTANGULAR MATRIX CONVEX SETS.

DEFINITION 2.1. A *rectangular matrix convex set* in a vector space V is a collection $\mathbb{K} = (K_{n,m})$ of subsets of $M_{n,m}(V)$ with the property that for any $\alpha_i \in M_{n_i,n}(\mathbb{C})$ and $\beta_i \in M_{m_i,m}(\mathbb{C})$ and $v_i \in K_{n_i,m_i}$ for $1 \leqslant i \leqslant \ell$ such that

$$\|\alpha_1^*\alpha_1 + \cdots + \alpha_\ell^*\alpha_\ell\| \|\beta_1^*\beta_1 + \cdots + \beta_\ell^*\beta_\ell\| \leqslant 1$$

one has that $\alpha_1^* v_1 \beta_1 + \cdots + \alpha_\ell^* v_\ell \beta_\ell \in K_{n,m}$.

When V is a topological vector space, we say that $\mathbb{K}$ is compact if $K_{n,m}$ is compact for every n, m. The following characterization of rectangular matrix convex sets can be easily verified using Proposition 1.1 and the fact that any finite-dimensional representation of $M_{n,m}(\mathbb{C})$ as a TRO is unitarily conjugate to a finite direct sum of copies of the identity representation ([10], Lemma 3.2.3).

LEMMA 2.2. *Suppose that* $\mathbb{K} = (K_{n,m})$ *where* $K_{n,m} \subset M_{n,m}(V)$. *The following assertions are equivalent:*

(i) $\mathbb{K}$ *is a rectangular convex set;*

(ii) $x \oplus y \in K_{n+m,r+s}$ *for any* $x \in K_{n,r}$ *and* $y \in K_{m,s}$, *and* $\alpha^* x \beta \in K_{r,s}$ *for any* $x \in K_{n,m}$, $\alpha \in M_{n,r}(\mathbb{C})$ *and* $\beta \in M_{m,s}(\mathbb{C})$ *with* $\|\alpha^* \alpha\| \|\beta^* \beta\| \leqslant 1$;

(iii) $x \oplus y \in K_{n+m,r+s}$ *for any* $x \in K_{n,r}$ *and* $y \in K_{m,s}$, *and* $(\sigma \otimes \mathrm{id}_V)[K_{n,m}] \subset K_{r,s}$ *for any completely contractive map* $\sigma : M_{n,m}(\mathbb{C}) \to M_{r,s}(\mathbb{C})$.

It is clear that, if $\mathbb{K}$ is a rectangular matrix convex set, then $(K_{n,n})$ is a matrix convex set in the sense of [48]. Furthermore if $\mathbb{K}$ and $\mathbb{T}$ are rectangular matrix convex sets such that $T_n = K_n$ for every $n \in \mathbb{N}$ then $T_{n,m} = K_{n,m}$ for every $n, m \in \mathbb{N}$. If $\mathbb{S} = (S_{n,m})$ is a collection of subsets of a (topological) vector space V, the (closed) rectangular matrix convex hull of $\mathbb{S}$ is the smallest (closed) rectangular matrix convex set containing $\mathbb{S}$.

EXAMPLE 2.3. Suppose that X is an operator space. Set $K_{n,m}$ to be space of completely contractive maps from X to $M_{n,m}(\mathbb{C})$. Then $\mathrm{CBall}(X) = (K_{n,m})$ is a rectangular matrix convex set.

2.2. THE RECTANGULAR POLAR THEOREM. Suppose that V and V' are vector spaces in duality. We endow both V and V' with the weak topology induced from such a duality. Let $\mathbb{S} = (S_{n,m})$ be a collection of subsets $S_{n,m} \subset M_{n,m}(V)$. We define the *rectangular matrix polar* $\mathbb{S}^\rho$ to be the closed rectangular matrix convex subset of V' such that $f \in S_{n,m}^\rho$ if and only if $\|\langle\langle v, f \rangle\rangle\| \leqslant 1$ for every $r, s \in \mathbb{N}$ and every $v \in S_{r,s}$. The same proof as Lemma 5.1 of [21] shows that $f \in S_{n,m}^\rho$ if and only if $\|\langle\langle v, f \rangle\rangle\| \leqslant 1$ for every $v \in S_{n,m}$.

If $A \subset V$, then its *absolute polar* A° is the set of $f \in V'$ such that $|\langle v, f \rangle| \leqslant 1$ for every $v \in A$. The classical bipolar theorem asserts that the absolute bipolar $A^{\circ\circ}$ is the closed absolutely convex hull of A ([14], Theorem 8.1.12). We will prove below the rectangular analog of this fact. The proof is analogous to the one of Theorem 5.4 in [21].

THEOREM 2.4. *If* $\mathbb{S} = (S_{n,m})$ *is a collection of subsets* $S_{n,m} \subset M_{n,m}(V)$, *then the rectangular matrix bipolar* $\mathbb{S}^{\rho\rho}$ *is the closed rectangular matrix convex hull of* $\mathbb{S}$.

The proof of Theorem C in [20] shows that if $\mathbb{K}$ is a rectangular matrix convex set in a vector space V, and F is a linear functional on $M_{n,m}(V)$ satisfying $\|F|_{K_{n,m}}\| \leqslant 1$, then

(i) there exist states p on $M_n(\mathbb{C})$ and q on $M_m(\mathbb{C})$ such that $|F(\alpha^* v \beta)|^2 \leqslant p(\alpha^* \alpha) q(\beta^* \beta)$ for every $r, s \in \mathbb{N}$, $\alpha \in M_{n,r}(\mathbb{C})$, $\beta \in M_{m,s}(\mathbb{C})$, and $v \in M_{r,s}(V)$, and

(ii) there exist matrices $\gamma \in M_{n^2,1}(\mathbb{C})$, $\delta \in M_{m^2,1}(\mathbb{C})$, and a map $\varphi : V \to M_{n,m}(\mathbb{C})$, such that $F(w) = \gamma^* \langle\langle w, \varphi \rangle\rangle \delta$ for every $w \in M_{n,m}(W)$ and $\|\langle\langle w, \varphi \rangle\rangle\| \leqslant 1$ for every $r, s \in \mathbb{N}$ and $w \in K_{r,s}$.

From this one can easily deduce the following proposition, which gives the rectangular matrix bipolar theorem as an easy consequence.

PROPOSITION 2.5. *Suppose that V and V' are vector spaces in duality, and $\mathbb{K}$ is a compact rectangular convex space in V. If $v_0 \in M_{n,m}(V) \backslash K_{n,m}$, then there exists $\varphi \in K_{n,m}^\rho$ such that $\|\langle\langle v_0, \varphi \rangle\rangle\| > 1$.*

Proof. By the classical bipolar theorem there exists a continuous linear functional F on $M_{n,m}(V)$ such that $\|F|_{K_{n,m}}\| \leqslant 1$ and $|F(v_0)| > 1$. By the remarks above there exists $\varphi \in K_{n,m}^\rho$ and contractive $\gamma \in M_{n^2 \times 1}(\mathbb{C})$ and $\delta \in M_{m^2 \times 1}(\mathbb{C})$ such that $F(v) = \gamma^* \langle\langle v, \varphi \rangle\rangle \delta$. Thus we have

$$\|\langle\langle v_0, \varphi \rangle\rangle\| \geqslant \|\gamma^* \langle\langle v_0, \varphi \rangle\rangle \delta\| = \|F(v_0)\| > 1. \quad \blacksquare$$

2.3. REPRESENTATION OF RECTANGULAR CONVEX SETS. Suppose that $\mathbb{K}$ is a rectangular matrix convex set in a vector space V. A *rectangular matrix convex combination* in a rectangular convex set $\mathbb{K}$ is an expression of the form $\alpha_1^* v_1 \beta_1 + \cdots + \alpha_\ell^* v_\ell \beta_\ell$ for $v_i \in K_{n_i, m_i}$, $\alpha_i \in M_{n_i, n}(\mathbb{C})$, and $\beta_i \in M_{m_i, m}(\mathbb{C})$ such that $\alpha_1^* \alpha_1 + \cdots + \alpha_\ell^* \alpha_\ell = 1$, and $\beta_1^* \beta_1 + \cdots + \beta_\ell^* \beta_\ell = 1$. A *proper rectangular matrix convex combination* is a rectangular convex combination $\alpha_1^* v_1 \beta_1 + \cdots + \alpha_\ell^* v_\ell \beta_\ell$ where furthermore $\alpha_1, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_\ell$ are right invertible. Observe that these notions are a particular instance of the notions of (proper) rectangular operator convex combination introduced in Subsection 1.4.

DEFINITION 2.6. A *rectangular matrix affine mapping* from a rectangular convex set $\mathbb{K}$ to a rectangular convex set $\mathbb{T}$ is a sequence $\boldsymbol{\theta}$ of maps $\theta_{n,m} : K_{n,m} \to T_{n,m}$ that preserves rectangular matrix convex combinations.

When $\mathbb{K}$ and $\mathbb{T}$ are compact rectangular convex sets, we say that $\boldsymbol{\theta}$ is continuous (respectively, a homeomorphism) when $\theta_{n,m}$ is continuous (respectively, a homeomorphism) for every $n, m \in \mathbb{N}$.

Given a compact rectangular matrix convex set $\mathbb{K}$ we let $A_\rho(\mathbb{K})$ be the complex vector space of continuous rectangular matrix affine mappings from $\mathbb{K}$ to CBall$(\mathbb{C})$. Here CBall$(\mathbb{C})$ is the compact rectangular matrix convex set defined as in Example 2.3, where $\mathbb{C}$ is endowed with its canonical operator space structure. The space $A_\rho(\mathbb{K})$ has a natural operator space structure where $M_{n,m}(A_\rho(\mathbb{K}))$ is identified isometrically with a subspace of $C(K_{n,m}, M_{n,m}(\mathbb{C}))$ endowed with the supremum norm. More generally if Y is any operator space, then we define $A_\rho(\mathbb{K}, Y)$ to be the operator space of continuous rectangular affine mappings from $\mathbb{K}$ to CBall(Y). Observe that $M_{n,m}(A_\rho(\mathbb{K}))$ is completely isometric to $A_\rho(\mathbb{K})$.

Starting from the operator space $A_\rho(\mathbb{K})$ one can consider the compact rectangular matrix convex set CBall$(A_\rho(\mathbb{K})')$ as in Example 2.3. Here $A_\rho(\mathbb{K})'$ denotes the dual space of the operator space $A_\rho(\mathbb{K})$, endowed with its canonical operator space structure. There is a canonical rectangular matrix affine mapping $\boldsymbol{\theta}$ from $\mathbb{K}$ to CBall$(A_\rho(\mathbb{K})')$ given by point evaluations. It is clear that this map

is injective. It is furthermore surjective in view of the rectangular bipolar theorem. The argument is similar to the one of the proof of Proposition 3.5 in [46]. This shows that the map θ is indeed a rectangular matrix affine homeomorphism from $\mathbb{K}$ onto $\mathrm{CBall}(A_\rho(\mathbb{K})')$. This implies that the assignment $X \mapsto \mathrm{CBall}(X')$ is a 1:1 correspondence between operator spaces and rectangular convex sets. It is also not difficult to verify that this correspondence is in fact an equivalence of categories, where morphisms between operator spaces are completely contractive linear maps, and morphisms between rectangular convex sets are continuous rectangular matrix affine mappings.

2.4. THE RECTANGULAR KREIN–MILMAN THEOREM. The notion of (proper) rectangular convex combination yields a natural notion of extreme point in a rectangular convex set. An element v of a rectangular convex set $\mathbb{K}$ is a *rectangular matrix extreme point* if for any proper rectangular convex combination $\alpha_1^* v_1 \beta_1 + \cdots + \alpha_\ell^* v_\ell \beta_\ell = v$ for $v_i \in K_{n_i,m_i}$ one has that, for every $1 \leqslant i \leqslant \ell$, $n_i = n$, $m_i = m$, and $v_i = u_i^* v w_i$ for some unitaries $u_i \in M_n(\mathbb{C})$ and $w_i \in K_m$. We now observe that the notion of rectangular extreme point coincides with the notion of rectangular operator extreme operator state from Definition 1.10. The argument is borrowed from the proof of Theorem B in [23].

LEMMA 2.7. *Suppose that X is an operator space, $\mathbb{K} = \mathrm{CBall}(X)$, and $\phi \in K_{n,m}$. Then ϕ is a rectangular matrix extreme point of $\mathbb{K}$ if and only if it is a rectangular operator extreme operator state of X.*

Proof. It is clear that a rectangular operator extreme point is a rectangular matrix extreme point. We prove the converse implication. Suppose that ϕ is a rectangular matrix extreme point. Let $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_\ell^* \phi_\ell \beta_\ell$ be a proper rectangular matrix convex combination, where $\phi_i \in K_{n_i,m_i}$ for $i = 1, 2, \ldots, \ell$. By assumption, we have that $n_i = n$ and $m_i = m$ for $i = 1, 2, \ldots, m$, and there exist unitaries $u_i \in M_n(\mathbb{C})$ and $w_i \in M_m(\mathbb{C})$ such that $\phi_i = u_i^* \phi w_i$ for $i = 1, 2, \ldots, \ell$. Therefore we have that

$$(2.1) \qquad \phi = (u_1 \alpha_1)^* \phi (w_1 \beta_1) + \cdots + (u_\ell \alpha_\ell)^* \phi_\ell (w_\ell \beta_\ell).$$

Define $R \subset M_{m+m}(\mathbb{C})$ to be the range of $\mathcal{S}(\phi)$. Observe that it follows from the fact that ϕ is a rectangular extreme point that the commutant of R is one-dimensional. Set

$$A_i = \begin{bmatrix} u_i \alpha_i & 0 \\ 0 & w_i \beta_i \end{bmatrix}$$

for $i = 1, 2, \ldots, n$. Define the unital completely positive map $\Psi : M_{n+m}(\mathbb{C}) \to M_{n+m}(\mathbb{C})$, $z \mapsto A_1^* z A_1 + \cdots + A_\ell^* z A_\ell$. By equation (2.1) we have that $\Psi(z) = z$ for every $z \in R$. It follows from this and Theorem 2.11 of [5] that $\Psi(z) = z$ for every $z \in M_{n+m}(\mathbb{C})$. By the uniqueness statement in the Choi's representation of a unital completely positive map [12], we deduce that there exist $\lambda_i \in \mathbb{C}$ such that $A_i = \lambda_i 1$ for $i = 1, 2, \ldots, \ell$. Therefore $\alpha_i^* \alpha_i = (u_i \alpha_i)^* (u_i \alpha_i) = |\lambda_i|^2 1$, $\beta_i^* \beta_i =$

$(w_i\beta_i)^*(w_i\beta_i) = |\lambda_i|^2 1$, and $\alpha_i^*\phi_i\beta_i = (u_i\alpha_i)^*\phi(w_i\beta_i) = |\lambda_i|^2\phi$. This concludes the proof that ϕ is a rectangular operator extreme point. ∎

We denote by $\partial_\rho\mathbb{K} = (\partial_\rho K_{n,m})$ the set of rectangular matrix extreme points of $\mathbb{K}$. Recall that the Krein–Milman theorem asserts that, if $K \subset V$ is a compact convex subset of a topological vector space V, then K is the closed convex hull of the set of its extreme points. The following is the natural analog of the Krein–Milman theorem for compact rectangular matrix convex sets. The proof is analogous to the proof of the Krein–Milan theorem for compact matrix convex sets ([46], Theorem 4.3).

THEOREM 2.8. *Suppose that $\mathbb{K}$ is a compact rectangular convex set. Then $\mathbb{K}$ is the closed rectangular matrix convex hull of $\partial_\rho\mathbb{K}$.*

Proof. Suppose that $\mathbb{K}$ is a compact rectangular convex set. In view of the representation theorem from Subsection 2.3, we can assume without loss of generality that $\mathbb{K} = \mathrm{CBall}(X')$ for some operator space X. We will assume that X is concretely represented as a subspace of $B(H)$ for some Hilbert space H. We will also canonically identify $M_{n,r}(X')$ with the space of bounded linear functionals on $M_{n,r}(X)$.

Fix $n, m \in \mathbb{N}$. Let $\widetilde{X}$ be the space of operators of the form

$$\begin{bmatrix} \lambda I^{\oplus n} & x \\ y^* & \mu I^{\oplus m} \end{bmatrix}$$

for $\lambda, \mu \in \mathbb{C}$ and $x, y \in M_{n,m}(X)$, where $I^{\oplus n}$ and $I^{\oplus m}$ are the identity operator on, respectively, the n-fold and m-fold Hilbertian sum of H by itself. If $\varphi \in M_{r,s}(X')$, then we denote by $\widetilde{\varphi} \in M_{nr+ms}(\widetilde{X}')$ the element defined by

$$\begin{bmatrix} \lambda I^{\oplus n} & x \\ y^* & \mu I^{\oplus n} \end{bmatrix} \mapsto \begin{bmatrix} \lambda I_{rn} & \varphi^{(n,m)}(x) \\ \varphi^{(n,m)}(y)^* & \mu I_{ms} \end{bmatrix}$$

where I_{rn} and I_{ms} denote the identity $rn \times rn$ and $ms \times ms$ matrices. If $\xi \in M_{r,n}(\mathbb{C})$ and $\eta \in M_{s,m}(\mathbb{C})$ we also set

$$\xi \odot \eta := \begin{bmatrix} I_n \otimes \xi & 0 \\ 0 & I_m \otimes \eta \end{bmatrix}.$$

We let $\Delta_{n,m}$ be the set of elements of $M_{n^2+m^2}(\widetilde{X}')$ of the form $(\xi \odot \eta)^*\widetilde{\varphi}(\xi \odot \eta)$ for $r, s \in \mathbb{N}$, $\varphi \in K_{r,s}$, $\xi \in M_{r,n}(\mathbb{C})$ and $\eta \in M_{s,m}(\mathbb{C})$ such that $\|\xi\|_2 = \|\eta\|_2 = 1$. It is not difficult to verify as in Section 4 of [46] that one can assume without loss of generality that $r \leqslant n, s \leqslant m$, and ξ, η are right invertible. The computation below shows that $\Delta_{n,m}$ is convex. If $t_1, t_2 \in [0,1]$ are such that $t_1 + t_2 = 1$ then

$$t_1(\xi_1 \odot \eta_1)^*\widetilde{\varphi}_1(\xi_1 \odot \eta_1) + t_2(\xi_1 \odot \eta_1)^*\widetilde{\varphi}_2(\xi_2 \odot \eta_2) = (\xi \odot \eta)^*\widetilde{\varphi}(\xi \odot \eta)$$

where

$$\xi = \begin{bmatrix} t_1\xi_1 \\ t_2\xi_2 \end{bmatrix}, \quad \eta = \begin{bmatrix} t_1\eta_1 \\ t_2\eta_2 \end{bmatrix}, \quad \text{and} \quad \varphi = \begin{bmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{bmatrix}.$$

Thus $\Delta_{n,m}$ is a compact convex subset of the space of unital completely positive maps from $\widetilde{X}$ to $M_{n^2+m^2}(\mathbb{C})$. Consider now an element $(\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$ of $\Delta_{n,m}$, where $\xi \in M_{r,n}(\mathbb{C})$ and $\eta \in M_{s,m}(\mathbb{C})$ are right invertible and $\varphi \in K_{n,m}$. Assume that $(\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$ is an extreme point of $\Delta_{n,m}$. We claim that this implies that φ is a rectangular extreme point of $\mathbb{K}$. Indeed suppose that, for some $s_k, r_k \in \mathbb{N}$, $\varphi_k \in K_{r_k,s_k}$, $\delta_k \in M_{s_k,s}(\mathbb{C})$, and $\gamma_k \in M_{r_k,r}(\mathbb{C})$, $\gamma_1^* \varphi_1 \delta_1 + \cdots + \gamma_\ell^* \varphi_\ell \delta_\ell$ is a proper rectangular convex combination in $\mathbb{K}$ that equals φ. Then we have that

$$(\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta) = (\gamma_1 \xi \odot \delta_1 \eta)^* \widetilde{\varphi}_1(\gamma_1 \xi \odot \delta_1 \eta) + \cdots + (\gamma_1 \xi \odot \delta_1 \eta)^* \widetilde{\varphi}_\ell(\gamma_1 \xi \odot \delta_1 \eta).$$

Let $t_k = \|(\gamma_k \xi \odot \delta_k \eta)\|_2$ for $k = 1, 2, \ldots, \ell$. Observe that

$$\sum_{k=1}^{\ell} t_k^2 = \frac{1}{r+s} \sum_{k=1}^{\ell} (r\mathrm{Tr}(\xi^* \gamma_k^* \gamma_k \xi) + s\mathrm{Tr}(\eta^* \delta_k^* \delta_k \eta)) = 1.$$

Therefore, if we set

$$\psi_k := t_k^{-2}(\xi \gamma_1 \odot \eta \delta_1)^* \widetilde{\varphi}_k(\xi \gamma_1 \odot \eta \delta_1)$$

for $k = 1, 2, \ldots, \ell$, we obtain elements $\psi_1, \ldots, \psi_\ell$ of $\Delta_{n,m}$ such that $t_1^2 \psi_1 + \cdots + t_\ell^2 \psi_\ell = (\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$. Since by assumption $(\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$ is an extreme point of $\Delta_{n,m}$, we can conclude that $\psi_k = (\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$ for $k = 1, 2, \ldots, \ell$. The fact that ξ and η are right invertible now easily implies that $r_k = r$, $s_k = s$, $\gamma_i^* \gamma_i = t_i^2 1$, $\delta_i^* \delta_i = t_i^2 1$, and $\gamma_i^* \varphi_i \delta_i = t_i^2 \varphi$. This conclude the proof that φ is a rectangular extreme point of $\mathbb{K}$.

We are now ready to conclude the proof that $\mathbb{K}$ is the rectangular convex hull of $\partial_\rho \mathbb{K}$. In view of the rectangular bipolar theorem, it is enough to prove that if $n, m \in \mathbb{N}$ and $z \in M_{n,m}(X)$ are such that $\|\varphi^{(n,m)}(z)\| \leqslant 1$ for every $r \leqslant n, s \leqslant m$, and $\varphi \in \partial_\rho K_{s,t}$, then $\|\psi^{(n,m)}(z)\| \leqslant 1$ for every $\psi \in K_{n,m}$. If $x \in M_{n,m}(X)$ then we let $\widetilde{x}$ be the element

$$\begin{bmatrix} I^{\oplus n} & x \\ x^* & I^{\oplus m} \end{bmatrix}$$

of $\widetilde{X}$. Observe that if $\varphi \in M_{r,s}(X')$ and $x \in M_{n,m}(X)$, then

$$\widetilde{\varphi}^{(n,m)}(x) = \begin{bmatrix} I_{nr} & \varphi^{(n,m)}(x) \\ \varphi^{(n,m)}(x)^* & I_{ms} \end{bmatrix} \in M_{nr+ms}(\mathbb{C}).$$

If furthermore $\xi \in M_{r,n}(\mathbb{C})$ and $\eta \in M_{s,m}(\mathbb{C})$ then

$$(\xi \odot \eta)^* \widetilde{\varphi}(x)(\xi \odot \eta) = \begin{bmatrix} I_n \otimes \xi^* \xi & (\xi^* \varphi \eta)^{(n,m)}(x) \\ (\xi^* \varphi \eta)^{(n,m)}(x)^* & I_m \otimes \eta^* \eta \end{bmatrix}.$$

Let now $(\xi \odot \eta)^* \widetilde{\varphi}(\xi \odot \eta)$ be an extreme point of $\Delta_{n,m}$, where $\xi \in M_{r,n}(\mathbb{C})$ and $\eta \in M_{s,m}(\mathbb{C})$ are right invertible and such that $\|\xi\|_2 = \|\eta\|_2 = 1$, and $\varphi \in \partial K_{r,s}$. By assumption we have that $\|(\mathrm{id}_{M_{n,m}(\mathbb{C})} \otimes \varphi)(z)\| \leqslant 1$. Thus by Lemma 3.1 of

[39] we have

$$
\begin{bmatrix}
I_{nr} & \varphi^{(n,m)}(z) \\
\varphi^{(n,m)}(z)^* & I_{ms}
\end{bmatrix} \geqslant 0
$$

and hence

$$
(\xi \odot \eta)^* \widetilde{\varphi}(z)(\xi \odot \eta) =
\begin{bmatrix}
I_n \otimes \xi^* \xi & (\xi^* \varphi \eta)^{(n,m)}(z) \\
(\xi^* \varphi \eta)^{(n,m)}(z)^* & I_m \otimes \eta^* \eta
\end{bmatrix} \geqslant 0.
$$

It follows from this and the classical Krein–Milman theorem that $\psi(z) \geqslant 0$ for any $\psi \in \Delta_{n,m}$. Let us fix $\varphi \in K_{n,m}$. If $\xi = I_n$ and $\eta = I_m$ then $(\xi \odot \eta)^* \varphi(\xi \odot \eta) \in \Delta_{n,m}$ and

$$
(\xi \odot \eta)^* \widetilde{\varphi}(z)(\xi \odot \eta) =
\begin{bmatrix}
I_n \otimes \xi^* \xi & (\xi^* \varphi \eta)^{(n,m)}(z) \\
(\xi^* \varphi \eta)^{(n,m)}(z)^* & I_m \otimes \eta^* \eta
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
I_{nr} & (\varphi^{(n,m)})(z) \\
\varphi^{(n,m)}(z)^* & I_{ms}
\end{bmatrix} \geqslant 0.
$$

This implies again by Lemma 3.1 of [39] that $\|\varphi^{(n,m)}(z)\| \leqslant 1$. Since this is valid for an arbitrary element of $K_{n,m}$, the proof is concluded. ∎

REMARK 2.9. The proof of Theorem 2.8 shows something more precise: if $\mathbb{K}$ is a rectangular convex set, then for every $n, m \in \mathbb{N}$, $K_{n,m}$ is equal to the closed rectangular convex hull of $K_{r,s}$ for $r \leqslant n$ and $s \leqslant m$.

The following is an immediate corollary of the rectangular Krein–Milman theorem as formulated in Remark 2.9.

COROLLARY 2.10. *Suppose that X is an operator space, and $\mathbb{K} = \mathrm{CBall}(X')$ is the corresponding compact rectangular matrix convex set. If $n, m \in \mathbb{N}$ and $x \in M_{n,m}(X)$, then $\|x\|$ is the supremum of $\|\varphi^{(n,m)}(x)\|$ where φ ranges among all the rectangular extreme points of $K_{r,s}$ for $r \leqslant n$ and $s \leqslant m$.*

3. BOUNDARY REPRESENTATIONS AND THE C^*-ENVELOPE OF A MATRIX-GAUGED SPACE

3.1. SELFADJOINT OPERATOR SPACES. By a concrete *selfadjoint operator space* we mean a closed selfadjoint subspace X of $B(H)$. Any selfadjoint operator space is endowed with a canonical involution, matrix norms, and matrix positive cones inherited from $B(H)$.

An (abstract) matrix-ordered matrix-normed ∗-vector space (see Subsection 3.1 of [41]) is a vector space V endowed with

- (•) a conjugate-linear involution $v \mapsto v^*$,
- (•) a complete norm in $M_n(V)$ for every $n \in \mathbb{N}$,
- (•) a distinguished positive cone $M_n(V)_+ \subset M_n(V)$,

such that, for every $n, k \in \mathbb{N}$, $x \in M_n(X)$, and $a, b \in M_{n,k}(\mathbb{C})$,

(i) $M_n(V)_+$ is proper, i.e. $M_n(V)_+ \cap (-M_n(V)_+) = \{0\}$,

(ii) $M_n(V)_+$ is closed in the topology induced by the norm,

(iii) $\|a^*xb\| \leqslant \|a\|\|x\|\|b\|$, and

(iv) $a^*xa \in M_k(V)_+$, when $x \in M_n(V)_+$.

A matrix-ordered matrix-normed $*$-vector space V is *normal* if, for every $n \in \mathbb{N}$ and $x, y, z \in M_n(V)$, $x \leqslant y \leqslant z$ implies that $\|y\| \leqslant \max\{\|x\|, \|z\|\}$. It is essentially proved in [47] (see also Theorem 3.2 of [41] and Theorem 5.6 of [42]) that a matrix-ordered matrix-normed $*$-vector space V is normal if and only if there exists a completely positive completely isometric selfadjoint linear map $\phi : V \to B(H)$, where H is a Hilbert space and the space $B(H)$ of bounded linear operators on H is endowed with its canonical matrix-ordered matrix-normed $*$-vector space structure.

3.2. MATRIX-GAUGED SPACES. Suppose that V is a real vector space. A *gauge* over V is a subadditive and positively-homogeneous function $\nu : V \to [0, +\infty)$. The conjugate gauge $\overline{\nu}$ is defined by $\overline{\nu}(x) = \nu(-x)$. The seminorm $\|x\|_\nu$ corresponding to a gauge is given by $\|x\|_\nu = \max\{\nu(x), \overline{\nu}(x)\}$. A gauge is *proper* if the seminorm $\|\cdot\|_\nu$ is a norm. The positive cone associated with a gauge ν is the set $V_{+,\nu} = \{x \in V : \overline{\nu}(x) = 0\}$.

The following notion is considered in [41] under the name of L^∞-matricially ordered vector space.

DEFINITION 3.1. A *matrix-gauged* space is a $*$-vector space V endowed with a sequence of proper gauges $\nu_n : M_n(V)_{\text{sa}} \to [0, +\infty)$ for $n \in \mathbb{N}$ with the property that, for every $n, k \in \mathbb{N}$, $x \in M_n(V)$, $y \in M_k(V)$, and $a \in M_{n,k}(\mathbb{C})$, one has that

$$\nu_k(a^*xa) \leqslant \|a\|^2 \nu_n(x) \quad \text{and} \quad \nu_{n+k}(x \oplus y) = \max\{\nu_n(x), \nu_k(y)\}.$$

A linear map $\phi : V \to W$ between matrix-gauged spaces is *completely gauge-contractive* if it is selfadjoint and $\nu(\phi^{(n)}(x)) \leqslant \nu(x)$ for every $n \in \mathbb{N}$ and $x \in M_n(X)_{\text{sa}}$, and *completely gauge-isometric* if it is selfadjoint and $\nu(\phi^{(n)}(x)) = \nu(x)$ for every $n \in \mathbb{N}$ and $x \in M_n(X)_{\text{sa}}$; see Definition 3.11 of [41]. Matrix-gauged spaces naturally form a category, where the morphisms are the completely gauge-contractive maps, and isomorphism are completely gauge-isometric surjective maps. In the following we will consider matrix-gauged spaces as objects in this category. By Corollary 3.10 of [41], every matrix-gauged space is completely gauge-isometrically isomorphic to a concrete selfadjoint operator space.

Any matrix-gauged space V has a canonical *normal* matrix-ordered matrix-normed $*$-vector space structure, obtained by considering the gauge norms and the gauge cones associated with the given matrix-gauges. Conversely, suppose that V is a normal matrix-ordered matrix-normed $*$-vector space. Letting $\nu_n(x)$ be the distance of x from $-M_n(V)_+$ for every $x \in M_n(V)_{\text{sa}}$ defines a canonical matrix-gauged structure on X. This matrix-gauge structure induces the original matrix-order and matrix-norms on V that one started from; see Proposition 3.5 of

[41]. Furthermore a selfadjoint linear map $\phi : V \to B(H)$ is completely positive and completely contractive if and only if it is completely gauge-contractive with respect to these specific matrix-gauges. However, there might be different matrix-gauges on V that induce the same matrix-order and matrix-norms on V.

Suppose now that S is an operator system with order unit 1. Then, in particular, S is a normal matrix-ordered matrix-normed $*$-vector space. Furthermore, it admits a unique matrix-gauge structure compatible with its matrix-order and matrix-norms. These matrix-gauges are defined by $\nu_n(x) = \inf\{t > 0 : x \leqslant t1\}$ for a selfadjoint $x \in S$. Uniqueness can be deduced from Arveson's extension theorem ([11], Theorem 1.6.1), as proved in Theorem 6.9 of [42]. In the following we will regard an operator system as a matrix-gauge space with such canonical matrix-gauges. A unital selfadjoint linear map between operator systems is completely positive if and only if it is completely gauge-contractive, and completely isometric if and only if it is completely gauge-isometric.

It is proved in Subsection 3.3 of [41] that any matrix-gauged space W admits a completely gauge isometric embedding as a subspace of codimension 1 into an operator system $W^\dagger$, called the unitization of W, that satisfies the following universal property: any completely gauge-contractive map from W to an operator system V admits a unique extension to a unital completely positive map from $W^\dagger$ to V. The unitization $W^\dagger$ of W is uniquely characterized by the above. If W is a normal matrix-ordered matrix-normed space, then we define the unitization of W to be the unitization of W endowed with the canonical matrix-gauges described above.

Suppose that A is a (not necessarily unital) C^*-algebra. Then A is endowed with canonical matrix-gauges, obtained by setting $\nu_n(x) = \|x_+\|$ for a selfadjoint $x \in A$, where x_+ denotes the positive part of x. In the following we will consider a C^*-algebra as a matrix-gauged space with these canonical matrix-gauges. It follows from the unitization construction that any matrix-gauged space admits a completely gauge-isometric embedding into $B(H)$. The following result can also be found in Proposition 2.2.1 of [11] with a different proof.

LEMMA 3.2. *Suppose that $A \subset B(H)$ is a C^*-algebra such that the identity 1 of $B(H)$ does not belong to A. Let Y be an operator system. Then for any completely positive completely contractive map $\phi : A \to Y$ there exists a unital completely positive map $\psi : \mathrm{span}\{A, 1\} \to Y$ extending ϕ.*

Proof. Let $Z := \mathrm{span}\{A, 1\} \subset B(H)$ and assume that $Y \subset B(L)$ for some Hilbert space L. We have to prove that if $z = x + \alpha 1 \in M_n(Z)$ for $\alpha \in M_n(\mathbb{C})$ is positive, then $\phi^{(n)}(x) + \alpha 1 \in M_n(Y)$ is positive. Since $1 \notin A$, we have that α is positive. Without loss of generality, we can assume that α is invertible. After replacing z with $\alpha^{-1/2} z \alpha^{-1/2}$ we can assume that $\alpha = 1$. By Stinespring's theorem ([7], Theorem II.6.9.7) there exist a Hilbert space K, a $*$-homomorphism $\pi : A \to B(K)$, and a linear map $v : L \to K$ such that $\|v\| = 1$ and $\phi(x) = v^* \pi(x) v$ for every $x \in A$. Observe that π extends to a unital $*$-homomorphism from Z into

$B(L)$, which we still denote by π. Let $v^{(n)} : H^{(n)} \to K^{(n)}$ be the map $v \oplus \cdots \oplus v$. Then we have that

$$
\begin{aligned}
\phi^{(n)}(x) + 1 &= v^{(n)*}\pi^{(n)}(x)v^{(n)} + 1 \\
&\geqslant v^{(n)*}\pi^{(n)}(x)v^{(n)} + v^{(n)*}\pi^{(n)}(1)v^{(n)} = v^{(n)*}\pi^{(n)}(x+1)v^{(n)} \geqslant 0.
\end{aligned}
$$

This concludes the proof. ∎

It follows from the previous lemma that the unitization of a C^*-algebra A as a matrix-gauged space coincides with the unitization of A as a C^*-algebra; see also Corollary 4.17 of [47]. Furthermore, it follows from Lemma 3.2, Theorem 6.9 of [42], and Arveson's extension theorem that a C^*-algebra admits unique compatible matrix-gauges. One can then deduce from Theorem 3.16 of [41] that a linear map between C^*-algebras or operator systems is completely gauge-contractive if and only if it is completely positive contractive.

3.3. THE INJECTIVE ENVELOPE OF A MATRIX-GAUGED SPACE. We say that a matrix-gauged space is *injective* if it is injective in the category of matrix-gauged spaces and completely gauge-contractive maps. Theorem 3.14 of [41] shows that $B(H)$ is an injective matrix-gauged space when endowed with its canonical matrix-gauges. It follows from this that the unitization functor $W \mapsto W^\dagger$ is an injective functor from the category of matrix-gauged spaces and gauge-contractive maps to the category of operator systems and unital completely positive maps.

Our goal now is to show that any injective matrix-gauged space is (completely gauge-isometrically isomorphic to) a unital C^*-algebra. This is a generalization of a theorem of Choi and Effros; see Theorem 15.2 of [39].

PROPOSITION 3.3. *Let X be an injective matrix-gauged space. Then X is completely gauge-isometrically isomorphic to a unital C^*-algebra.*

Proof. We may assume that $X \subset B(H)$ is concretely represented as a self-adjoint operator space. Since X is injective, there exists a gauge-contractive and hence completely contractive and completely positive projection $\Phi : B(H) \to X$. We define the Choi–Effros product on X by

$$
x \circ_\Phi y = \Phi(xy).
$$

As in the proof of Theorem 15.2 in [39], one shows that

$$
\Phi(\Phi(a)x) = \Phi(ax) \quad \text{and} \quad \Phi(x\Phi(a)) = \Phi(xa)
$$

holds for all $x \in X$ and all $a \in B(H)$. Indeed, the proof only requires the Schwarz inequality for unital completely positive maps, which remains valid for completely positive completely contractive maps. In particular, we see that $e := \Phi(I_H) \in X$ is a unit for the Choi–Effros product. Moreover, the proof of Theorem 15.2 in [39] shows that $(X, \circ_\Phi)$, endowed with the norm and involution of $B(H)$, is a C^*-algebra.

It is clear from the above that the identity map from X onto $(X, \cdot_\Phi)$ is an isometry. To see that it is an order isomorphism, suppose that $x \in X$ is positive with respect to the order on $B(H)$. Then $\|cI_H - x\| \leqslant c$ for all $c \geqslant \|x\|$, so since Φ is contractive $\|ce - x\| \leqslant c$ for all $c \geqslant \|x\|$, thus x is a positive element of the C^*-algebra $(X, \circ_\Phi)$. Conversely, if x is positive in the C^*-algebra $(X, \circ_\Phi)$, then there exists $y \in X$ such that $x = y^* \circ_\Phi y$, hence $x = \Phi(y^*y)$ is positive in $B(H)$. Moreover, the argument at the end of the proof of Theorem 15.2 in [39] shows that $M_n(X)$, endowed with the Choi–Effros product $\circ_{\Phi^{(n)}}$, is the C^*-tensor product of $(X, \circ_\Phi)$ with $M_n(\mathbb{C})$. By the above, the identity map is an isometry and an order isomorphism between $M_n(X) \subset M_n(B(H))$ and $(M_n(X), \circ_\Phi)$. Therefore, the identity map from X onto $(X, \circ_{\Phi^{(n)}})$ is a selfadjoint complete isometry and complete order isomorphism.

To see that the identity map from X onto $(X, \circ_\Phi)$ is in fact a complete gauge isometry, observe that Φ is a unital completely positive map from $B(H)$ onto $(X, \circ_\Phi)$ by the preceding paragraph, so it is completely gauge contractive. Conversely, if $x \in M_n(X)$ is self-adjoint and satisfies $\|x_+\| \leqslant 1$, where the positive part is taken in the C^*-algebra $(M_n(X), \circ_\Phi)$, then $x \leqslant e^{(n)}$ in $(M_n(X), \circ_\Phi)$, hence $x \leqslant I_{\mathbb{C}^n} \otimes I_H$ in $M_n(B(H))$ by the preceding paragraph, so that the identity map from $(X, \circ_\Phi)$ to X is completely gauge contractive as well. ∎

In particular, we see that every injective matrix-gauged space is (completely gauge-isometrically isomorphic to) an injective operator system. Conversely, since the unitization functor is injective, every operator system that is injective in the category of operator systems and unital completely positive maps is also injective as a matrix-gauged space, when endowed with the unique compatible matrix-gauge structure.

The usual proof of the existence of the injective envelope of an operator system yields the existence of a gauge analog of Hamana's injective envelope of operator spaces. Let us say that a gauge-extension of a matrix-gauged space X is a pair (Y, i) where Y is a matrix-gauged space and $i : X \to Y$ is a completely gauge-isometric map. As in the case of operator systems, we say that such a gauge-extension is:

(i) *rigid* if the identity map of Y is the unique gauge-contractive map $\phi : Y \to Y$ such that $\phi \circ i = i$;

(ii) *essential* if whenever $u : Y \to Z$ is a gauge-contractive map to a matrix-gauged space Z such that $u \circ i$ is a completely gauge-isometric, then u is a completely gauge-isometric;

(iii) an *injective envelope* if Y is injective, and there is no proper injective subspace of Y that contains X.

The same proof as Lemma 4.2.4 of [9] shows that if X is a matrix-gauged space, and (Y, i) is a gauge-extension of X such that Y is injective, then the following assertions are equivalent:

(1) (Y, i) is an injective envelope of X;
(2) (Y, i) is essential;
(3) (Y, i) is rigid.

To this purpose one can consider the gauge analog of the notion of projections and seminorms from Subsection 4.2.1 of [9].

Suppose that W is a matrix-gauged space, and X is a selfadjoint subspace of W. A completely gauge-contractive X-projection on W is an idempotent completely gauge-contractive map $u : W \to W$ that restricts to the identity on X. A gauge X-seminorm on W is a seminorm of the form $p(x) = \|u(x)\|$ for some completely gauge-contractive X-projection u on W. One can define an order on completely gauge-contractive X-projections by $u \leqslant v$ if and only if $u \circ v = v \circ u = u$, while gauge X-seminorms are ordered by pointwise comparison. The same proof as Lemma 4.2.2 of [9] shows that any gauge X-seminorm majorizes a minimal gauge X-seminorm, and if p is a minimal gauge X-seminorm and $u : W \to W$ is a completely gauge-contractive map that restricts to the identity on X, then u is a minimal gauge X-projection. To this purpose, it is enough to observe that the set of completely gauge-contractive selfadjoint maps from W to $B(H)$ is closed in the weak* topology of the space of $\mathrm{CB}(W, B(H))$ of completely bounded maps from W to $B(H)$. Indeed $\phi : W \to B(H)$ is completely gauge-contractive if and only if it is selfadjoint and $\langle \phi^{(n)}(x)\xi, \xi \rangle \leqslant \nu(x)$ for every $n \in \mathbb{N}$, $\xi \in H^{\oplus n}$, and $x \in M_n(W)_{\mathrm{sa}}$.

The proof of Lemma 4.2.4 in [9] can now be easily adapted to prove the claim above, by replacing X-projections with gauge X-projections and X-seminorms with gauge X-seminorms. Similarly the same proof as Theorem 4.2.6 of [9] shows that if a matrix-gauged space X is contained in an injective matrix-gauged space W, then there exists an injective envelope $X \subset Z \subset W$. Furthermore the injective envelope of X is essentially unique. We denote by $I(X)$ the injective envelope of a matrix-gauged space X, and we identify X with a selfadjoint subspace of $I(X)$. It is clear that, when X is an operator system endowed with its canonical matrix-gauges, the injective envelope of X as a matrix-gauged space coincides with the injective envelope of X as an operator system (endowed with the canonical matrix-gauges). Furthermore, it is a consequence of Proposition 3.3 that the unitization of a matrix-gauged space X is $\mathrm{span}\{X, 1\} \subset I(X)$, where 1 denotes the identity of the unital C^*-algebra $I(X)$.

3.4. BOUNDARY REPRESENTATIONS. Most fundamental notions in dilation theory admit straightforward versions in the setting of matrix-gauged spaces. Suppose that X is a matrix-gauged space. An *operator state* on X is a completely gauge-contractive map $\phi : X \to B(H)$. We say that an operator state $\psi : X \to B(\widetilde{H})$ is a *dilation* of ϕ if there exists a linear isometry $v : H \to \widetilde{H}$ such that $v^* \psi(x) v = \phi(x)$ for every $x \in X$. It follows from Stinespring's dilation theorem ([7], Theorem II.6.9.7) that if A is a C^*-algebra, then an operator state on A admits a dilation which is a $*$-homomorphism. A dilation ψ of an operator state

$\phi : x \mapsto v^* \psi(x) v$ on X is *trivial* if

$$\psi(x) = vv^* \psi(x) vv^* + (1 - vv^*) \psi(x)(1 - vv^*).$$

We say that ϕ is *maximal* if it has no nontrivial dilation. As in the case of operator systems, one can prove that an operator state $\phi : X \to B(H)$ is maximal if and only if for any dilation ψ of ϕ one has that $\|\psi(x)\xi\| = \|\phi(x)\xi\|$ for every $x \in X$, and $\xi \in H$.

Suppose that X is a selfadjoint subspace of a C^*-algebra A such that A is generated as a C^*-algebra by X. An operator state ϕ on X has the *unique extension property* if any completely positive contractive map $\widetilde{\phi} : A \to B(H)$ whose restriction to X coincides with ϕ is automatically a $*$-homomorphism. The same argument as in the operator systems setting shows that an operator state is maximal if and only if it has the unique extension property; see [6].

DEFINITION 3.4. A *boundary representation* for a matrix-gauged space $X \subset B(H)$ is an operator state $\phi : X \to B(H)$ with the property that any completely positive contractive map $\psi : C^*(X) \to B(H)$ extending X is an irreducible representation of $C^*(X)$.

In the following we will identify a boundary representation of X with its unique extension to an irreducible representation of $C^*(X)$. It follows from the remarks above that the notion of boundary representation does not depend on the concrete realization of X as a selfadjoint space of operators. In the following we will assume that A is a C^*-algebra, and $X \subset A$ is a selfadjoint subspace that generates A as a C^*-algebra. We regard X as a matrix-gauged space endowed with the matrix-gauges induced by A.

PROPOSITION 3.5. *Suppose that $\phi : X \to B(H)$ is an operator state of X, and $\phi^\dagger : X^\dagger \to B(H)$ is its canonical unital completely positive extension to the unitization of X. If $\phi^\dagger$ is a boundary representation for $X^\dagger$, then ϕ is a boundary representation for X.*

Proof. Let $\Phi : A \to B(H)$ be a completely positive contractive map extending ϕ. Extend Φ to a unital completely positive $\Phi^\dagger : A^\dagger \to B(H)$. Then since by assumption $\phi^\dagger$ is a boundary representation, we conclude that $\Phi^\dagger$ is an irreducible representation for $A^\dagger$. Therefore $\Phi|_A$ is an irreducible representation of A. This concludes the proof. ∎

The following result is then a consequence of Proposition 3.5 and Theorem 3.1 of [16].

THEOREM 3.6. *Suppose that X is a matrix-gauged space. Then the matrix-gauges of X are completely determined by the boundary representations of X. Precisely, if $x \in M_n(X)$, then $\nu_n(x)$ is the supremum of $\|\phi^{(n)}(x)_+\|$ where ϕ ranges among all the boundary representations of X.*

Suppose that X is a matrix-gauged space, and $\phi : X \to B(H)$ is an operator state. An *operator convex combination* is an expression $\phi = \alpha_1^* \phi_1 \alpha_1 + \cdots + \alpha_n^* \phi_n \alpha_n$,

where $\alpha_i : H \to H_i$ are linear maps, and $\phi_i : X \to B(H_i)$ are operator states for $i = 1, 2, \ldots, \ell$. Such a rectangular convex combination is *proper* if the α_i's are right invertible and $\alpha_1^* \alpha_1 + \cdots + \alpha_n^* \alpha_n = 1$ and *trivial* if

$$\alpha_i^* \alpha_i = \lambda_i 1 \quad \text{and} \quad \alpha_i^* \phi_i \alpha_i = \lambda_i \phi \quad \text{for some} \quad \lambda_i \in [0, 1].$$

DEFINITION 3.7. An operator state $\phi : X \to B(H)$ is an *operator extreme point* if for any proper operator convex combination $\phi = \alpha_1^* \phi_1 \alpha_1 + \cdots + \alpha_n^* \phi_n \alpha_n$ is trivial.

Observe that the map $\phi \mapsto \phi^\dagger$ establishes a 1:1 correspondence between operator states on X and operator states on the operator system $X^\dagger$. Furthermore this correspondence is *operator affine* in the sense that it preserves operator convex combinations. The following proposition is then an immediate consequence of this observation and (the proof of) Theorem B in [23].

PROPOSITION 3.8. *Suppose that $\phi : X \to B(H)$ is an operator state, and let $\phi^\dagger : X^\dagger \to B(H)$ be its unital extension to the unitization of X. The following assertions are equivalent:*

 (i) ϕ is a pure element in the cone of completely gauge-contractive maps from X to $B(H)$;

 (ii) ϕ is an operator extreme point;

 (iii) $\phi^\dagger$ is an operator extreme point.

The following corollary is an immediate consequence of Proposition 3.8, Proposition 3.5, and [16].

COROLLARY 3.9. *Suppose that X is a matrix-gauged space, and ϕ is an operator state on X. If ϕ is operator extreme, then ϕ admits a dilation to a boundary representation of X.*

3.5. THE C^*-ENVELOPE OF A MATRIX-GAUGED SPACE. Suppose that X is a matrix-gauged space. A pair (A, i) is a C^*-cover if A is a C^*-algebra and $i : X \to A$ is a completely gauge-isometric map whose range generates A as a C^*-algebra.

DEFINITION 3.10. A C^*-*envelope* $(C_e^*(X), i)$ of X is a C^*-cover of X if it has the following universal property: for any C^*-cover (B, j) of X, there exists a *-homomorphism $\theta : B \to C_e^*(X)$ such that $\theta \circ j = i$.

It is clear that the C^*-envelope of a matrix-gauged space, if it exists, is essentially unique. We will prove below that any matrix-gauged space has a C^*-envelope. The proof is essentially the same as the one for the existence of the C^*-envelope of an operator system.

Suppose that X is a matrix-gauged space. Let $X \subset I(X)$ be the injective envelope of X. By Proposition 3.3, $I(X)$ is a unital C^*-algebra. Let A be the C^*-subalgebra of $I(X)$ generated by X. As in the proof of Theorem 15.16 in [39], one sees that (i, A) where $i : X \to A$ is the inclusion map, is the C^*-envelope of X.

It follows from the construction that the C^*-envelope (as defined above) of an operator system regarded as matrix-gauged space with its unique compatible matrix-gauges coincides with the usual notion of C^*-envelope of an operator system.

Alternatively, one can construct the C^*-envelope of a matrix-gauged space using boundary representations, as for the C^*-envelope of an operator system. Indeed let X be a matrix-gauged space. Define $\iota_e : X \to B(H)$ to be the direct sum of all the boundary representations for X, and then let A be the C^*-subalgebra of $B(H)$ generated by the image of ι_e. It follows from the unique extension property of boundary representations that (ι_e, A) is indeed the C^*-envelope of X. In particular, this construction shows that, for any C^*-algebra B, $C_e^*(B) = B$.

As in the case of operator systems and operator spaces, one can define a maximal or universal C^*-algebra that contains a given ordered operator space as a generating subset. Explicitly, the *maximal* C^*-algebra $C_{\max}^*(X)$ of an ordered operator space is a C^*-cover (i, A) of X that has the following universal property: given any other completely gauge-contractive map $f : X \to B$, where B is a C^*-algebra, there exists a $*$-homomorphism $\theta : A \to B$ such that $\theta \circ i = f$. In order to see that such a maximal C^*-algebra exists, one can consider the collection $\mathcal{F}$ of all completely gauge-contractive maps from X to $M_n(\mathbb{C})$ for $n \in \mathbb{N}$. Then let i be the direct sum of the elements $s : X \to M_{n_s}(\mathbb{C})$ of $\mathcal{F}$, and then A to be the C^*-subalgebra of $\bigoplus_{s \in \mathcal{F}}^{\infty} M_{n_s}(\mathbb{C})$ generated by the image of i. The same proof as Proposition 8 of [34] shows that such a C^*-cover satisfies the required universal property.

3.6. SELFADJOINT ORDERED OPERATOR SPACES AND COMPACT MATRIX CONVEX SETS. We want to conclude by observing that selfadjoint operator spaces are in canonical 1:1 correspondence with compact matrix convex sets with a distinguished extreme point.

Suppose that $\mathbb{K} = (K_n)$ is a compact matrix convex set, and $e \in K_1$ is a matrix extreme point. Define $A_0(\mathbb{K}, e)$ to be the set of continuous matrix-affine functions from $\mathbb{K}$ to $(M_n(\mathbb{C}))_{n \in \mathbb{N}}$ that vanish at e. Then $A_0(\mathbb{K}, e)$ is a selfadjoint subspace of codimension 1 of the operator system $A(\mathbb{K})$.

Conversely suppose that $X \subset B(H)$ is a selfadjoint operator space. Consider X as a normal matrix-ordered and matrix-normed space with respect to the induced matrix-cones and matrix-norms, and let $X^\dagger$ be the unitization of X. Let, for $n \in \mathbb{N}$, K_n be the space of completely positive completely contractive selfadjoint maps from X to $M_n(\mathbb{C})$, endowed with the topology of pointwise convergence. Observe that $\mathbb{K} = (K_n)$ is a compact matrix-convex set, and K_n can be identified with the space of unital completely positive maps from $X^\dagger$ to $M_n(\mathbb{C})$. Let $e \in K_1$ be the zero functional on X. We have a canonical unital complete order isomorphism $X^\dagger \cong A(\mathbb{K})$. Under this isomorphism X is mapped into $A_0(\mathbb{K}, e)$. Since X has codimension 1 in $X^\dagger$, such an isomorphism in fact maps X onto $A_0(\mathbb{K}, e)$.

The above construction shows that one can identify the unitization of $A_0(\mathbb{K}, e)$ with the operator system $A(\mathbb{K})$. Furthermore it is easy to see that the correspondence $(K, e) \mapsto A_0(\mathbb{K}, e)$ is a contravariant equivalence of categories from the category of compact matrix convex sets $\mathbb{K}$ with a distinguished matrix extreme point $e \in K_1$, where morphisms are continuous matrix-affine maps that preserve the distinguished point, to the category of selfadjoint operator spaces and completely positive completely contractive selfadjoint maps.

The commutative analog of the argument above establishes a correspondence between compact convex sets with a distinguished extreme point and selfadjoint operator spaces that can be represented inside an abelian C^*-algebra (selfadjoint function spaces).

REFERENCES

[1] J. AGLER, J.E. MCCARTHY, *Pick Interpolation and Hilbert Function Spaces*, Grad. Stud. Math., vol. 44, Amer. Math. Soc., Providence, RI 2002.

[2] E.M. ALFSEN, E.G. EFFROS, Structure in real Banach spaces. I, *Ann. of Math. Sec. Ser.* **96**(1972), 98–128.

[3] E.M. ALFSEN, E.G. EFFROS, Structure in real Banach spaces. II, *Ann. of Math. Sec. Ser.* **96**(1972), 129–173.

[4] W.B. ARVESON, Subalgebras of C^*-algebras. I, *Acta Math.* **123**(1969), 141–224.

[5] W.B. ARVESON, Subalgebras of C^*-algebras. II, *Acta Math.* **128**(1972), 271–308.

[6] W.B. ARVESON, The noncommutative Choquet boundary, *J. Amer. Math. Soc.* **21**(2008), 1065–1084.

[7] B. BLACKADAR, *Operator Algebras*, Encyclopaedia Math. Sci., vol. 122, Springer-Verlag, Berlin 2006.

[8] D.P. BLECHER, The Shilov boundary of an operator space and the characterization theorems, *J. Funct. Anal.* **182**(2001), 280–343.

[9] D.P. BLECHER, C. LE MERDY, *Operator Algebras and their Modules — An Operator Space Approach*, London Math. Soc. Monogr. (N. S.), vol. 30, Oxford Univ. Press, Oxford 2004.

[10] D. BOHLE, K-theory for ternary structures, Ph.D. Dissertation, Westfälische Wilhelms-Universität Münster, Münster 2011.

[11] N.P. BROWN, N. OZAWA, *C*-Algebras and Finite-dimensional Approximations*, Grad. Stud. Math., vol. 88, Amer. Math. Soc., Providence, RI 2008.

[12] M.D. CHOI, Completely positive linear maps on complex matrices, *Linear Algebra Appl.* **10**(1975), 285–290.

[13] R. CLOUÂTRE, M. HARTZ, Multiplier algebras of complete Nevanlinna–Pick spaces: dilations, boundary representations and hyperrigidity, in preparation.

[14] J.B. CONWAY, *A Course in Functional Analysis*, second ed., Grad. Texts in Math., vol. 96, Springer-Verlag, New York 1990.

[15] K.R. DAVIDSON, *C*-Algebras by Example*, Fields Inst. Monogr., vol. 6, Amer. Math. Soc., Providence, RI 1996.

[16] K.R. DAVIDSON, M. KENNEDY, The Choquet boundary of an operator system, *Duke Math. J.* **164**(2015), 2989–3004.

[17] M.A. DRITSCHEL, S.A. MCCULLOUGH, Boundary representations for families of representations of operator algebras and spaces, *J. Operator Theory* **53**(2005), 159–167.

[18] M.J. DUPRÉ, R.M. GILLETTE, *Banach Bundles, Banach Modules and Automorphisms of C*-Algebras*, Res. Notes Math., vol. 92, Pitman (Adv. Publ. Program), Boston, MA 1983.

[19] E.G. EFFROS, Aspects of noncommutative order, in *C*-Algebras and Applications to Physics (Proc. Second Japan-U.S.A. Sem., Los Angeles, CA., 1977)*, Lecture Notes in Math., vol. 650, Springer, Berlin 1978, pp. 1–40.

[20] E.G. EFFROS, Z.-J. RUAN, On the abstract characterization of operator spaces, *Proc. Amer. Math. Soc.* **119**(1993), 579–584.

[21] E.G. EFFROS, S. WINKLER, Matrix convexity: operator analogues of the bipolar and Hahn–Banach theorems, *J. Funct. Anal.* **144**(1997), 117–152.

[22] G.K. ELEFTHERAKIS, E.T.A. KAKARIADIS, Strong morita equivalence of operator spaces, *J. Math. Anal. Appl.* **446**(2017), 1632–1653.

[23] D. FARENICK, Extremal matrix states on operator systems, *J. London Math. Soc.* **61**(2000), 885–892.

[24] D. FARENICK, Pure matrix states on operator systems, *Linear Algebra Appl.* **393**(2004), 149–173.

[25] D. FARENICK, P. MORENZ, *C*-extreme points in the generalized state spaces of a C*-algebra, *Trans. Amer. Math. Soc.* **349**(1997), 1725–1748.

[26] D. FARENICK, P.B. MORENZ, *C*-extreme points of some compact C*-convex sets, *Proc. Amer. Math. Soc.* **118**(1993), 765–775.

[27] K. GUO, J. HU, X. XU, Toeplitz algebras, subnormal tuples and rigidity on reproducing $\mathbf{C}[z_1, \ldots, z_d]$-modules, *J. Funct. Anal.* **210**(2004), 214–247.

[28] M. HAMANA, Injective envelopes of dynamical systems, in *Operator Algebras and Operator Theory*, Pitman Res. Notes Math. Ser., vol. 271, Longman Sci. Tech., Harlow 1992.

[29] M. HAMANA, Triple envelopes and Shilov boundaries of operator spaces, *Math. J. Toyama Univ.* **22**(1999), 77–93.

[30] J.W. HELTON, S. MCCULLOUGH, Every convex free basic semi-algebraic set has an LMI representation, *Ann. Math. Second Ser.* **176**(2012), 979–1013.

[31] W. HELTON, I. KLEP, S. MCCULLOUGH, Free convex algebraic geometry, in *Semidefinite Optimization and Convex Algebraic Geometry*, MOS-SIAM Ser. Optim., vol. 13, SIAM, Philadelphia, PA 2013, pp. 341–405.

[32] A. HOPENWASSER, R.L. MOORE, V.I. PAULSEN, *C*-extreme points, *Trans. Amer. Math. Soc.* **266**(1981), 291–307.

[33] M. KENNEDY, O.M. SHALIT, Essential normality, essential norms and hyperrigidity, *J. Funct. Anal.* **268**(2015), 2990–3016.

[34] E. KIRCHBERG, S. WASSERMANN, C^*-algebras generated by operator systems, *J. Funct. Anal.* **155**(1998), 324–351.

[35] A.J. LAZAR, J. LINDENSTRAUSS, On Banach spaces whose duals are L_1-spaces, *Israel J. Math.* **4**(1966), 205–207.

[36] A.J. LAZAR, J. LINDENSTRAUSS, Banach spaces whose duals are L_1-spaces and their representing matrices, *Acta Math.* **126**(1971), 165–193.

[37] M. LUPINI, Uniqueness, universality, and homogeneity of the noncommutative Gurarij space, *Adv. in Math.* **298**(2016), 286–324.

[38] T. OIKHBERG, The non-commutative Gurarii space, *Arch. Math.* **86**(2006), 356–364.

[39] V.I. PAULSEN, *Completely Bounded Maps and Operator Algebras*, Cambridge Stud. Adv. Math., vol. 78, Cambridge Univ. Press, Cambridge 2002.

[40] V.I. PAULSEN, M. RAGHUPATHI, *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*, Cambridge Stud. Adv. Math., Cambridge Univ. Press, Cambridge 2016.

[41] T.B. RUSSELL, Characterizations of ordered self-adjoint operator spaces, arXiv:1508.06272 (2015).

[42] T.B. RUSSELL, Characterizations of ordered operator spaces, arXiv:1608.00939 (2016).

[43] A.L. SHIELDS, Weighted shift operators and analytic function theory, in *Topics in Operator Theory*, Math. Surveys, vol. 13, Amer. Math. Soc., Providence, R.I. 1974, pp. 49–128.

[44] D.A. STEGENGA, Multipliers of the Dirichlet space, *Illinois J. Math.* **24**(1980), 113–139.

[45] G.D. TAYLOR, Multipliers on D_α, *Trans. Amer. Math. Soc.* **123**(1966), 229–240.

[46] C. WEBSTER, S. WINKLER, The Krein–Milman theorem in operator convexity, *Trans. Amer. Math. Soc.* **351**(1999), 307–322.

[47] W. WERNER, Subspaces of $L(H)$ that are $*$-invariant, *J. Funct. Anal.* **193**(2002), 207–223.

[48] G. WITTSTOCK, On matrix order and convexity, in *Functional Analysis: Surveys and Recent Results, III (Paderborn, 1983)*, North-Holland Math. Stud., vol. 90, North-Holland, Amsterdam 1984, pp. 175–188.

ADAM H. FULLER, DEPT. OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OH 45701, U.S.A.
E-mail address: fullera@ohio.edu

MICHAEL HARTZ, DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY IN ST LOUIS, ONE BROOKINGS DRIVE, ST. LOUIS, MO 63130, U.S.A.
E-mail address: mphartz@wustl.edu

MARTINO LUPINI, MATHEMATICS DEPARTMENT, CALIFORNIA INSTITUTE OF TECHNOLOGY, 1200 E. CALIFORNIA BLVD, MC 253-37, PASADENA, CA 91125, U.S.A.
E-mail address: lupini@caltech.edu