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DIMENSIONS OF COMPLEX HILBERT SPACES ARE DETERMINED BY THE COMMUTATIVITY RELATION

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ABSTRACT. Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. Assuming the set-theoretical axiom on generalized continuum hypothesis it is shown that if the commutativity relation in $\mathscr{B}(\mathcal{H})$, the algebra of bounded linear operators on $\mathcal{H}$, is the same as in $\mathscr{B}(\mathcal{K})$, then $\dim \mathcal{H} = \dim \mathcal{K}$.

KEYWORDS: *Hilbert space, Banach algebra, commutativity, commuting graph.*

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1. INTRODUCTION

The essence of the commutativity relation in a given algebra $\mathcal{A}$ is captured in its commuting graph, $\Gamma := \Gamma(\mathcal{A})$. By definition, this is a simple (= undirected, loopless) graph, with vertex set $V(\Gamma)$, consisting of all noncentral elements of algebra $\mathcal{A}$, and where two vertices $x, y \in V(\Gamma)$ form an edge if $xy = yx$ and $x \neq y$. For the algebra $\mathscr{B}(\mathcal{H})$ of bounded operators on a complex Hilbert space, the commuting graph $\Gamma(\mathscr{B}(\mathcal{H}))$ thus consists of all nonscalar operators (an operator is scalar if it is a scalar multiple of the identity), and two distinct nonscalar operators $A, B \in \mathscr{B}(\mathcal{H})$ are connected by an edge (denoted henceforth by $A — B$) if they commute.

Clearly, the commuting graph can be defined on every grupoid (a nonempty set equipped with inner operation). In abelian grupoids, each element is central so its commuting graph is vacuous. However, on nonabelian grupoids, the commuting graph can capture quite a lot of information about the algebraic structure despite its apparent simplicity. For example, it was recently established by Han, Chen, Guo, Abdollahi, Shahverdi, Solomon, and Woldar [1], [13], [17] that commutativity relation alone can distinguish among finite simple nonabelian groups. More precisely, if the commuting graph of a finite simple nonabelian group S is isomorphic to the commuting graph of some group G, then a group S is isomorphic to a group G. In a similar vein, it was shown by Mohammadian [15] that the commutativity relation alone distinguishes $M_2(\mathbb{F})$, the algebra of 2-by-2 matrices over finite field $\mathbb{F}$, among all unital rings.

It is our aim to show that the commutativity relation on $\mathscr{B}(\mathcal{H})$ completely determines the dimension of the underlying complex Hilbert space $\mathcal{H}$ (see Theorem 3.6 below and the paragraph preceding it). We remark that this result is in the spirit of Akbari, Ghandehari, Hadian, and Mohammadian [2] who established a similar conclusion in case of finite semisimple rings.

QUESTION. Recall that $\mathscr{B}(\mathcal{H})$ is an example of a factor von Neumann algebra. It would be interesting to know if Theorem 3.6 also holds in this generality. To put it on precise terms: Assume the commuting graphs of two factor von Neumann algebras are isomorphic. Are the algebras isomorphic?

2. PRELIMINARIES

If $\dim \mathcal{H} = n < \infty$ we identify $\mathcal{H}$ with $\mathbb{C}^n$, the space of column vectors of size n and with $e_1, \dots, e_n$ as a standard basis, and we identify $\mathscr{B}(\mathcal{H})$ with n-by-n matrix algebra $M_n(\mathbb{C})$. Given two matrices $A \in M_n(\mathbb{C})$, $B \in M_m(\mathbb{C})$, we denote the block-diagonal matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \in M_{n+m}(\mathbb{C})$ by $A \oplus B$. The commutant of an operator $A \in \mathscr{B}(\mathcal{H})$ is the algebra

$$A' := \{X \in \mathscr{B}(\mathcal{H}) : AX = XA\} \subseteq \mathscr{B}(\mathcal{H}).$$

Given a matrix $A \in M_n(\mathbb{C})$, let A^{T} denote its transpose. Vectors are considered to be n-by-1 matrices, thus, a typical rank-one matrix in $M_n(\mathbb{C})$ equals xf^{T} where $x, f \in \mathbb{C}^n$ are nonzero. A nonscalar matrix M is maximal if its commutant is maximal in the sense that $X' \supseteq M'$ for nonscalar matrix X implies $X' = M'$. It was shown by Šemrl ([16], Lemma 3.1) that M is maximal if an only if $M = \lambda I + \mu P$ or $M = \lambda I + \nu N$ for some $\lambda, \mu \in \mathbb{C}$, with $\mu \neq 0$, where $P^2 = P \notin \{0, I\}$ is an idempotent, I is the identity matrix, and $N^2 = 0$ but $N \neq 0$. A dimension argument gives that for each nonscalar matrix B there exists a maximal matrix M with $B' \subseteq M'$. Given a subset $\Omega \subseteq \mathbb{C}^n$ we let $\mathrm{Lin}(\Omega) \subseteq \mathbb{C}^n$ be its linear span.

We will also require the basic terminology from graph theory. A path in a graph is a finite sequence of vertices $v_0, v_1, \dots, v_k$ such that v_i is connected to v_{i-1} for every $i = 1, 2, \dots, k$, the integer k is the length of this path. A path of length k, which in a commuting graph $\Gamma = \Gamma(\mathcal{A})$ connects noncentral elements v, w, is thus a sequence of $k+1$ noncentral elements $v_0 = v, v_1, \dots, v_k = w$ from $\mathcal{A}$ such that

$$v_i v_{i-1} = v_{i-1} v_i \quad i = 1, \dots, k.$$

Recall that a graph is connected if every two vertices can be joined with a path (containing only finitely many vertices). The length of the shortest possible path between two vertices v, w is their distance and is denoted by $d(v, w)$. This distance makes a connected graph into a metric space. The diameter, $\mathrm{diam}(\Gamma)$, of a connected graph Γ is the supremum of all possible distances. A component of a graph is a maximal connected subgraph. A graph is complete if every two disjoint vertices form an edge. The cardinality, $|\Gamma|$ of a graph Γ is the cardinality of its vertex set.

The Proposition 2.1 below is one of the cornerstones of our proof. It already shows that the commutativity relation alone can distinguish separable from non-separable Hilbert spaces, and can distinguish two-dimensional Hilbert spaces among finite-dimensional ones. The rest of the proof will be devoted to show that commutativity alone can distinguish among finite-dimensional Hilbert spaces of dimension greater than two. Finally, to distinguish among non-separable Hilbert spaces of different cardinalities we will assume the axiom on generalized continuum hypothesis (abbreviated GCH). Recall that GCH axiom states that, given any infinite set Ω, there is no set whose cardinality would lie strictly between $|\Omega|$ and $|2^{\Omega}|$, the latter denoting the cardinality of the power set of Ω. By Gödel's and Cohen's results, GCH is independent of the standard ZFC axioms of set theory (see [12] and the book [6]). By assuming GCH we assured that the power function is injective among the cardinals in the sense that

$$\aleph < \bar{\aleph} \quad \text{implies} \quad 2^{\aleph} < 2^{\bar{\aleph}}$$

(to see this note that, under GCH, $2^{\aleph} \leqslant \bar{\aleph} \lneqq 2^{\bar{\aleph}}$). We remark that this argument fails if GCH does not hold since then there exists a cardinal $\bar{\aleph}$ with $\aleph < \bar{\aleph} < 2^{\aleph}$. Worse still, Easton's theorem [11] (see also Theorem 15.18 of [14] for modern treatment) shows that on regular cardinals the power function can be wild. Our basic reference for cardinal arithmetic is a book by Dugundji [10].

PROPOSITION 2.1. *The following holds for the commuting graph $\Gamma = \Gamma(\mathscr{B}(\mathcal{H}))$ of the algebra of bounded operators on a complex Hilbert space $\mathcal{H}$.*

(i) *If* $\dim \mathcal{H} = 2$, *then Γ is disconnected. Each of its component is a complete graph of infinite cardinality.*

(ii) *If* $2 < \dim \mathcal{H} < \infty$, *then Γ is connected and* $\operatorname{diam}(\Gamma) = 4$.

(iii) *If $\mathcal{H}$ is separable and* $\dim \mathcal{H} = \infty$, *then Γ is disconnected and contains a component (generated by rank-one operators) which is not a complete graph.*

(iv) *If $\mathcal{H}$ is non-separable, then Γ is connected and* $\operatorname{diam}(\Gamma) = 2$.

Proof. (i) Identify $\mathscr{B}(\mathcal{H}) = M_2(\mathbb{C})$ and observe that every nonzero matrix $A \in M_2(\mathbb{C})$ is a sum of a rank-one matrix and a scalar matrix. As far as commutativity is concerned we may thus assume $\operatorname{rank} A = 1$. By applying a suitable similarity matrix and multiplying A with a suitable scalar we may further assume that $A = E_{11} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $A = E_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. In both cases, if $\lambda, \mu \in \mathbb{C}$ and $\mu \neq 0$, then $A' = (\lambda I + \mu A)' = \mathbb{C}I + \mathbb{C}A$ is an abelian algebra. With scalar matrices removed it forms a complete subgraph of Γ which is simultaneously a component of Γ (see also Remark 2 of [4]).

Item (ii) was proved in Corollary 7 of [3] (but see also Corollary 2.2 of [9] for a short proof that $\operatorname{diam}(\Gamma) \leqslant 4$). Items (iii)–(iv) were proved in [5]. ∎

3. MAIN RESULTS

To distinguish among finite-dimensional Hilbert spaces we will rely on five lemmas which were motivated by our recent results in [8]. The first Lemma 3.1 was already proved in Lemma 2.3 of [8]; but for the sake of convenience we choose to present its full proof.

Recall that a Cauchy matrix, $C = (1/(x_i - y_j))_{ij} \in M_n(\mathbb{C})$ where $x_i, y_j \in \mathbb{C}$ are distinct scalars, has all its minors nonzero. In particular, if $e_1, \dots, e_n$ is a standard basis of $\mathbb{C}^n$ and c_i is any column of C, then $\{e_1, \dots, e_{n-1}, c_i\}$ are linearly independent vectors, hence a basis for $\mathbb{C}^n$.

LEMMA 3.1. *Let $n \geqslant 3$. Define matrices $B_1, \dots, B_n$ by $B_i e_k = e_{k+1}$ for $k = 1, \dots, (n-2)$, $B_i e_{n-1} = c_i$, and $B_i c_i = 0$, where c_i is the i-th column of Cauchy matrix C. Then, in the commuting graph, the distance $d(B_i, B_j) = 4$ for $i \neq j$.*

Proof. Write the i-th column of Cauchy matrix as $c_i = \sum_j c_{ij} e_j$. Observe that the matrix B_i^{n-1} annihilates $\{e_2, \dots, e_{n-1}, c_i\}$ and maps e_1 to c_i. Hence B_i is nonderogatory (i.e., its minimal and characteristic polynomials coincide) and B_i^{n-1} is a trace-zero matrix of the form

$$B_i^{n-1} = c_i (e_1 + \beta_i e_n)^{\mathrm{T}},$$

where β_i is such that $(e_1 + \beta_i e_n)^{\mathrm{T}} c_i = c_{i1} + \beta_i c_{in} = 0$. Because all minors of C are nonzero, the vectors $(c_{i1}, c_{in})^{\mathrm{T}}$ and $(c_{j1}, c_{jn})^{\mathrm{T}}$ are linearly independent for $i \neq j$. Hence, the functional $(1, \beta_i)$ which annihilates the first vector cannot annihilate also the second one, i.e., $(c_{j1}, c_{jn})^{\mathrm{T}}$. Therefore,

$$B_i^{n-1} c_j \neq 0, \quad i \neq j. \tag{3.1}$$

We will now show that $d(B_j, B_i) = 4$ for $i \neq j$. Assume otherwise. Then there exists a path $B_j — X — Y — B_i$ of length 3 in the commuting graph. That is,

$$B_j X = X B_j, \quad \text{and} \quad XY = YX, \quad \text{and} \quad Y B_i = B_i Y$$

holds. With no loss of generality X is a maximal matrix (i.e., its commutant is maximal possible) for otherwise we can find a maximal matrix with greater commutant than X and replace X with it. Now, X commutes with the nonderogatory B_j so it is a polynomial in B_j (see for example Proposition 2.3 of [7]). Also, X is maximal, hence by Lemma 3.1 of [16] takes the form $X = \lambda I + N \in \operatorname{Poly}(B_j)$ for some nonzero N with $N^2 = 0$ and by subtracting from it λI and multiplying it with a suitable nonzero scalar we can assume it is of the form

$$X = B_j^{k_j} + \sum_{k=k_j+1}^{n-1} \lambda_{jk} B_j^k$$

for some $k_j \geqslant n/2$. In particular, the image, $\operatorname{Im} X = \operatorname{Lin}\{e_{k_j+1}, e_{k_j+2}, \ldots, e_{n-1}, c_j\}$. Likewise we can assume that

$$Y = B_i^{k_i} + \sum_{k=k_i+1}^{n-1} \lambda_{ik} B_i^k$$

for some $k_i \geqslant n/2$. Observe that Y annihilates all the vectors $e_{k_j+1}, \ldots, e_{n-1} \in \operatorname{Im} X$ but does not annihilate c_j because if x is a vector with $Xx = c_j$, then, by (3.1),

$$B_i^{n-k_i-1} YXx = B_i^{n-1} Xx = B_i^{n-1} c_j \neq 0.$$

Hence, YX is of rank-one. Likewise, XY is of rank-one and it does not annihilate c_i.

Now, we assume $YX = XY$. If $k_i > k_j$, then $0 \neq Yc_j = YXe_{n-k_j} = XYe_{n-k_j} = X(0) = 0$, a contradiction. Likewise if $k_i < k_j$. Lastly, suppose $k_i = k_j$. Then

$$Yc_j = YXe_{n-k_j} = XYe_{n-k_j} = Xc_i. \tag{3.2}$$

The equation $XY = YX$ implies that the kernel, $\ker Y$ is invariant for X. Since $c_i \in \ker Y = \operatorname{Lin}\{e_{n-k_i+1}, \ldots, e_{n-1}, c_i\}$ the vector Xc_i is spanned by the vectors $e_{n-k_i+1}, \ldots, e_{n-1}, c_i$. Hence, as $k_i \geqslant n/2$ implies $k_i \geqslant 2$ and therefore $e_1, e_n \notin \operatorname{Lin}\{e_{n-k_i+1}, \ldots, e_{n-1}\}$ we see that $\operatorname{pr}_2(Xc_i)$ is a scalar multiple of $\operatorname{pr}_2(c_i)$, where $\operatorname{pr}_2\colon (x_1, \ldots, x_n)^{\mathrm{T}} \mapsto (x_1, x_n)^{\mathrm{T}}$ is a projection of the vector onto its first and last components. Likewise, $\operatorname{pr}_2(Yc_j)$ is a scalar multiple of $\operatorname{pr}_2(c_j)$. But, since C is a Cauchy matrix, $\operatorname{pr}_2(c_j)$ and $\operatorname{pr}_2(c_i)$ are linearly independent, a contradiction to (3.2). Hence $XY \neq YX$. So, $d(B_j, B_i) > 3$ and by (ii) of Proposition 2.1, $d(B_j, B_i) = 4$ for $j \neq i$. ∎

LEMMA 3.2. *Let $n \geqslant 3$. Let $A = I_r \oplus 0_{n-r} \in M_n(\mathbb{C})$ be an idempotent with $1 \leqslant r = \operatorname{rank} A \leqslant n/2$ and let $B_1, \ldots, B_{n-1}$ be nilpotent matrices defined in Lemma 3.1. If $X_i \in B_i'$ are nonscalar matrices, then we have $\{A, X_1, \ldots, X_{n-1}\}' = \mathbb{C}I$.*

Proof. We can assume with no loss of generality that X_i are already maximal for otherwise we can find maximal matrices $\widehat{X}_i$ with $\widehat{X}_i' \supseteq X_i'$, and if we have $\{A, \widehat{X}_1, \ldots, \widehat{X}_{n-1}\}' = \mathbb{C}I$ then also $\{A, X_1, \ldots, X_{n-1}\}' = \mathbb{C}I$. Since X_i commutes with the nonderogatory nilpotent B_i it is a polynomial in B_i, and since X_i is also maximal, it equals $\lambda I + N$, $N^2 = 0$. So after subtracting λI and multiplying with a suitable scalar we may assume that

$$X_i = B_i^{k_i} + \sum_{k=k_i+1}^{n-1} \lambda_{ik} B_i^k \quad \text{for some } k_i \geqslant \frac{n}{2}. \tag{3.3}$$

Assume there exists a nonscalar $T \in \{A, X_1, \ldots, X_{n-1}\}'$. Again we may assume that T is maximal. Then, $X_i T = T X_i$ implies that

$$Tc_i = TX_i e_{n-k_i} = X_i T e_{n-k_i} \in \operatorname{Im} X_i = \operatorname{Im} B_i^{k_i} = \operatorname{Lin}\{e_{k_i+1}, \ldots, e_{n-1}, c_i\}. \tag{3.4}$$

Moreover, T commutes with A so it has a form $T = \dot{T} \oplus \ddot{T} \in M_r(\mathbb{C}) \oplus M_{n-r}(\mathbb{C})$. Together with (3.4) we get $Tc_i = (\dot{T}\dot{c}_i) \oplus (\ddot{T}\ddot{c}_i) \in \xi_i(\dot{c}_i \oplus \ddot{c}_i) + \operatorname{Lin}\{e_{k_i+1}, \dots, e_{n-1}\}$ for some $\xi_i \in \mathbb{C}$. Recall that $k_i + 1 > n/2 \geqslant r$ so $e_{k_i+1}, \dots, e_{n-1} \in 0_r \oplus \mathbb{C}^{n-r}$, and therefore $(\dot{T}\dot{c}_i) = \xi_i \dot{c}_i$. Hence, each $\dot{c}_i$ is an eigenvector of $\dot{T}$. Since each minor of a Cauchy matrix is nonzero, we have that $\dot{c}_1, \dots, \dot{c}_r$ are linearly independent. Hence, $\dot{T}$ is diagonalizable. We claim $\dot{T}$ is a scalar matrix. This is obvious if $r = 1$. Otherwise, $2 \leqslant r \leqslant n/2$ and as $n \geqslant 3$ we have that $\dot{c}_1, \dots, \dot{c}_{n-1}$ are $n - 1 > n/2 \geqslant r$ vectors in $\mathbb{C}^r$, so they are linearly dependent. Then, $\dot{c}_{n-1} = \sum_{k=1}^{r} \alpha_k \dot{c}_k$. This linear combination has all coefficients nonzero, for otherwise, if say $\dot{c}_{n-1}$ is a linear combination of $\dot{c}_1, \dots, \dot{c}_{r-1}$ then the vectors $\dot{c}_1, \dots, \dot{c}_{r-1}, \dot{c}_{n-1}$ would be linearly dependent, and hence the corresponding $r \times r$ minor of a Cauchy matrix C would be zero, a contradiction. From this it easily follows that all eigenvalues of $\dot{T}$ are the same: $\xi_{n-1} \sum_{k=1}^{r} \alpha_k \dot{c}_k = \xi_{n-1} \dot{c}_{n-1} = \dot{T}\dot{c}_{n-1} = \sum_k \alpha_k \dot{T}\dot{c}_k = \sum_k \alpha_k \xi_k \dot{c}_k$, and since $\alpha_k \neq 0$ we get $\xi_k = \xi_{n-1}$ for every k.

By subtracting a suitable scalar matrix from T we may hence assume without loss of generality that $T = 0 \oplus \ddot{T}$. Now, since each component of a vector c_i is nonzero while $k_i \geqslant n/2 \geqslant r$ we have

$$
\begin{aligned}
0 \oplus \ddot{T}\ddot{c}_i &= Tc_i = TX_i e_{n-k_i} = X_i T e_{n-k_i} \in \operatorname{Im}(0 \oplus \ddot{T}) \cap \operatorname{Im} X_i \\
&\subseteq \operatorname{Lin}\{e_{r+1}, \dots, e_{n-1}, e_n\} \cap \operatorname{Lin}\{e_{k_i+1}, \dots, e_{n-1}, c_i\} \qquad (3.5) \\
&= \operatorname{Lin}\{e_{k_i+1}, \dots, e_{n-1}\}.
\end{aligned}
$$

But since each $(n-r) \times (n-r)$ minor of a Cauchy matrix is nonzero, the $n - 1$ vectors $\ddot{c}_1, \dots, \ddot{c}_{n-1}$ span the whole linear space $\mathbb{C}^{n-r}$. Hence, it follows from (3.5) that $\operatorname{Im} \ddot{T} \subseteq \operatorname{Lin}\{\ddot{e}_{k_i+1}, \dots, \ddot{e}_{n-1}\}$, so the last row of $\ddot{T}$ vanishes. Also, $\operatorname{Im} T = 0_r \oplus \operatorname{Im} \ddot{T} \subseteq \operatorname{Lin}\{e_{k_i+1}, \dots, e_{n-1}\} \subseteq \operatorname{Lin}\{e_{\lfloor (n+1)/2 \rfloor + 1}, \dots, e_{n-1}\} \subseteq \ker X_i$, giving that $X_i T = 0$ for each $i = 1, \dots, n-1$. Hence, $TX_i = X_i T = 0$, so $T \operatorname{Im} X_i = 0$ and in particular $Tc_i = 0$ for each i, and hence $\ddot{T}\ddot{c}_i = 0$ for each $i = 1, \dots, n-1$. Since $\ddot{c}_1, \dots, \ddot{c}_{n-1}$ span the whole linear space $\mathbb{C}^{n-r}$ we have $\ddot{T} = 0$ and hence $T = 0$, a contradiction since T was not a scalar matrix. ∎

The next lemma gives a similar result in the case of a square-zero matrix A. It is understood that if $r = n/2$, then $A = \begin{pmatrix} 0_r & 0 \\ I_r & 0_r \end{pmatrix}$.

LEMMA 3.3. *Let* $n \geqslant 3$. *Let* $A = \begin{pmatrix} 0_r & 0 & 0 \\ 0 & 0_{n-2r} & 0 \\ I_r & 0 & 0_r \end{pmatrix} \in M_n(\mathbb{C})$ *be a square-zero matrix with* $1 \leqslant r = \operatorname{rank} A \leqslant n/2$ *and let* $B_1, \dots, B_{n-1}$ *be nilpotent matrices defined in Lemma* 3.1. *If* $X_i \in B_i'$ *are nonscalar matrices, then we have* $\{A, X_1, \dots, X_{n-1}\}' = \mathbb{C}I$.

Proof. Assume there exists a nonscalar $T \in \{A, X_1, \dots, X_{n-1}\}'$. Since $T \in A'$ it easily follows that $T = \begin{pmatrix} \dot{T} & 0 & 0 \\ * & * & 0 \\ * & * & \dot{T} \end{pmatrix}$ with blocks of appropriate size. Arguing as in the proof of Lemma 3.2 and keeping the same notations for X_i and $\dot{c}_i$ we can assume that X_i are square-zero matrices of the form (3.3) and by (3.4) the first r components of c_i are scalar multiples of c_i, i.e., $\dot{T}\dot{c}_i \in \mathbb{C}\dot{c}_i$. Hence, as before, $\dot{T}$ is a scalar matrix and by subtracting a suitable scalar from T we may assume $\dot{T} = 0$.

Similarly as in (3.5) we obtain that $Tc_i \in \mathrm{Lin}\{e_{\lfloor (n+1)/2 \rfloor + 1}, \dots, e_{n-1}\} \subseteq \ker X_j$ for each $i, j \leqslant n-1$. Since the last $r \geqslant 1$ columns of T vanish we further have $Te_{n-r} = \cdots = Te_n = 0$. Note that $c_1, \dots, c_{n-1}, e_n$ is a basis for $\mathbb{C}^n$ — this follows by expanding the determinant of the matrix $[c_1|\cdots|c_{n-1}|e_n]$ by the last column and noting that each $(n-1)\times(n-1)$ minor of a Cauchy matrix is nonzero. Therefore, $\mathrm{Im}\, T = \mathrm{Lin}\{Tc_1, \dots, Tc_{n-1}, Te_n\} = \mathrm{Lin}\{Tc_1, \dots, Tc_{n-1}, 0\}$, so $X_i T = 0$. Then also $TX_i = X_i T = 0$, so in particular, $Tc_i = 0$ for each i, and, as already noted, $Te_n = 0$. Hence $T = 0$, a contradiction. ∎

From the last two lemmas we obtain the following classification of dimension via a commuting graph.

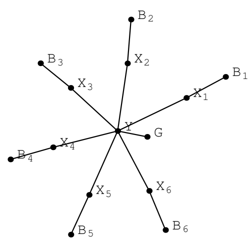

FIGURE 1. Visualization of Lemmas 3.4–3.5. By Lemma 3.5 such subgraph exists in $\Gamma(M_8(\mathbb{C}))$, but not in $\Gamma(M_7(\mathbb{C}))$, for every $B_1, \dots, B_6$ pairwise at distance 4.

LEMMA 3.4. *Let $3 \leqslant n = \dim \mathcal{H} < \infty$. If $G \in \Gamma(\mathscr{B}(\mathcal{H}))$, then we can find $n-1$ operators $B_1, \dots, B_{n-1} \in \Gamma(\mathscr{B}(\mathcal{H}))$, pairwise at distance 4 in a commuting graph, such that there does not exist $Y \in \Gamma(\mathscr{B}(\mathcal{H}))$ for which we would simultaneously have $\binom{n-1}{2}$ star-like paths*

$$\begin{array}{ccccccccc} B_i & — & X_i & — & Y & — & X_j & — & B_j \\ & & & & | & & & & \\ & & & & G & & & & \end{array} \quad \textit{where } 1 \leqslant i < j \leqslant (n-1).$$

Proof. The statement about nonexistence of Y is clearly equivalent to the fact that for every choice of nonscalar operators $X_i \in B_i'$ the only operators $Y \in \{G, X_1, \dots, X_{n-1}\}'$ are the scalar ones.

Identify $\mathscr{B}(\mathcal{H})$ with $M_n(\mathbb{C})$. By dimension argument there exists a maximal matrix $A \in M_n(\mathbb{C})$ with $G' \subseteq A'$. As already mentioned in the introduction, by subtracting a suitable scalar matrix from A and dividing with a suitable scalar, we can achieve that either $A^2 = A$ is an idempotent or that $A^2 = 0$ (see

Lemma 3.1 of [16]). Hence, after a suitably chosen similarity we may further assume with no loss of generality that $A = I_r \oplus 0_{n-r}$ or that $A = \begin{pmatrix} 0_r & 0 & 0 \\ 0 & 0_{n-2r} & 0 \\ I_r & 0 & 0_r \end{pmatrix}$. In both cases take matrices $B_1, \ldots, B_{n-1}$ from Lemmas 3.2–3.3 to conclude that $\{G, X_1, \ldots, X_{n-1}\}' \subseteq \{A, X_1, \ldots, X_{n-1}\}' = \mathbb{C}I$ whenever $X_i \in B_i'$ are nonscalar. This is clearly equivalent to the statement of the lemma. ∎

In contrast, with less operators B_i such paths are possible, provided the operator G is suitably chosen. Note that the lemma below makes sense only if $\dim \mathcal{H} \geqslant 4$.

LEMMA 3.5. *Let $4 \leqslant n = \dim \mathcal{H} < \infty$. If $G \in \Gamma(\mathscr{B}(\mathcal{H}))$ has rank-one, then for every choice of $n-2$ operators $B_1, \ldots, B_{n-2} \in \Gamma(\mathscr{B}(\mathcal{H}))$, pairwise at distance 4, we can find $Y \in \Gamma(\mathscr{B}(\mathcal{H}))$ together with $\binom{n-2}{2}$ star-like paths*

$$\begin{array}{ccccccccc} B_i & - & X_i & - & Y & - & X_j & - & B_j \\ & & & & | & & & & \\ & & & & G & & & & \end{array} \quad \text{where } 1 \leqslant i < j \leqslant (n-2).$$

Proof. Write $G = x_G f_G^{\mathrm{T}}$ for appropriate nonzero vectors $x_G, f_G \in \mathbb{C}^n$. Consider any $n-2$ nonscalar matrices $B_1, \ldots, B_{n-2} \in M_n(\mathbb{C})$ pairwise at distance 4. For each $i = 1, \ldots, n-2$, choose an eigenvector x_i for B_i and an eigenvector f_i for B_i^{T}, which both correspond to the same eigenvalue. Then, the rank-one matrix $X_i = x_i f_i^{\mathrm{T}}$ clearly commutes with B_i for $i = 1, \ldots, n-2$. Now, the space $\mathrm{Lin}\{x_1, \ldots, x_{n-2}, x_G\}$ is at most $(n-1)$-dimensional, so there exists a nonzero vector f such that f^{T} annihilates this space. Likewise there exists a nonzero vector y such that $f_G^{\mathrm{T}} y = 0$ and $f_i^{\mathrm{T}} y = 0$ for each $i = 1, \ldots, n-2$. This gives a rank-one matrix $Y = y f^{\mathrm{T}}$ which commutes with G and with each $X_i = x_i f_i^{\mathrm{T}} \in B_i'$, and establishes the desired paths. ∎

The following is our main result. It shows that if the commutativity relation on $\mathscr{B}(\mathcal{H})$ is indistinguishable from that on $\mathscr{B}(\mathcal{K})$, then $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$, that is, the two algebras are isomorphic. Note that the centers of $\mathscr{B}(\mathcal{K})$ and $\mathscr{B}(\mathcal{H})$ are both isomorphic to $\mathbb{C}$ so the indistinguishability of the commutativity relations is exactly the fact that the two commuting graphs are isomorphic.

THEOREM 3.6. *Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces of dimension at least two. If $\Gamma(\mathscr{B}(\mathcal{H}))$ is isomorphic to $\Gamma(\mathscr{B}(\mathcal{K}))$, then $\dim \mathcal{H} = \dim \mathcal{K}$ and hence $\mathscr{B}(\mathcal{H}) \sim \mathscr{B}(\mathcal{K})$.*

Proof. By Proposition 2.1, if $\mathcal{H}$ is separable (finite or infinite-dimensional) while $\mathcal{K}$ is not, the commuting graphs cannot be isomorphic and the same conclusion holds if $\mathcal{H}$ is finite-dimensional, while $\mathcal{K} = \ell^2$ (square-summable sequences) is separable but of infinite-dimension. It only remains to see that the two graphs are not isomorphic if (i) $3 \leqslant \dim \mathcal{H} < \dim \mathcal{K} < \infty$ or if (ii) $\dim \mathcal{H} < \dim \mathcal{K}$ and both $\mathcal{H}$ and $\mathcal{K}$ are non-separable.

Assume (i) and let $n = \dim\mathcal{K} \geqslant 4$. By Lemma 3.5 there exists a vertex $G \in \Gamma(\mathscr{B}(\mathcal{K}))$ (of rank-one) such that for every $n-2$ vertices $B_1, \dots, B_{n-2} \in \Gamma(\mathscr{B}(\mathcal{K}))$, pairwise at distance 4, we can find a vertex $Y \in \Gamma(\mathscr{B}(\mathcal{K}))$ and $\binom{n-2}{2}$ star-like paths

$$\begin{array}{ccccccccc} B_i & - & X_i & - & Y & - & X_j & - & B_j \\ & & & & | & & & & \\ & & & & G & & & & \end{array}.$$

Since $m := \dim\mathcal{H} < n$, Lemma 3.4 implies that the graph $\Gamma(\mathscr{B}(\mathcal{H}))$ contains no vertex G with this properties (a vertex $G \in \Gamma(\mathscr{B}(\mathcal{H}))$ can induce star-like paths with at most $m-2$ vertices B_i, which is less than what can be obtaind in $\Gamma(\mathscr{B}(\mathcal{K}))$). The two graphs are therefore not isomorphic.

Finally assume (ii) and let $\mathcal{H}$ be non-separable with $\aleph = \dim\mathcal{H}$ the cardinality of its dimension. This means that there exists an orthonormal basis $B \subseteq \mathcal{H}$ with $|B| = \aleph$. Let us show that

$$|\mathscr{B}(\mathcal{H})\backslash\mathbb{C}I| = 2^{\aleph}. \tag{3.6}$$

To this end, note that every $x \in \mathcal{H}$ can be uniquely expanded by a fixed orthonormal basis $B \subseteq \mathcal{H}$ and, by Bessel inequality, all but at most countably many coefficients of this expansion vanish. Thus, to each vector $x \in \mathcal{H}$ there corresponds a unique finite or at most countably infinite subset $B_x \subseteq B$ such that x is expanded by orthonormal vectors from B_x and no coefficient of this expansion vanishes; we agree here that 0 can be expanded on the empty subset of B. Denote by $\aleph_0 = |\mathbb{N}|$ the cardinality of positive integers $\mathbb{N}$ and by $\mathfrak{c} := |\mathbb{C}| = 2^{\aleph_0}$ the cardinality of continuum. Then, there are at least $\mathfrak{c}$ and at most $|\ell^2| \leqslant \mathfrak{c}^{\aleph_0} = \mathfrak{c}$ (see Example 7, p. 53 of [10] for the last identity), vectors which can be expanded on a given at most countable nonempty subset of B. Clearly then, for each at most countable nonempty subset $B_\lambda \subseteq B$, the set $\Xi_{B_\lambda} \subseteq \mathcal{H}$ of vectors which can be expanded by B_λ such that each coefficient of this expansion is nonzero, has again cardinality $\mathfrak{c}$. Since $\mathcal{H}\backslash\{0\}$ is a disjoint union of different Ξ_{B_λ} we deduce that $|\mathcal{H}| = \mathfrak{b}\cdot\mathfrak{c} + 1 = \mathfrak{b}\cdot\mathfrak{c}$, where $\mathfrak{b}$ is the cardinality of the set $\mathscr{C}_\omega(B)$ of at most countable nonempty subsets of B. Note that there is a one-to-one, onto correspondence between $\mathscr{C}_\omega(B)$ and the set $B^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to B — taking images of functions, $\mathrm{Im}\colon B^{\mathbb{N}} \to \mathscr{C}_\omega(B)$ is a surjection, while taking graphs of functions, $\mathscr{G}\colon B^{\mathbb{N}} \to \mathscr{C}_\omega(\mathbb{N}\times B) \sim \mathscr{C}_\omega(B)$ is an injection. Thus, $\mathfrak{b} = \aleph^{\aleph_0}$. Clearly, $\aleph_0 < \aleph$, so by Example 9, p. 54 of [10],

$$\aleph \leqslant \mathfrak{b} = \aleph^{\aleph_0} \leqslant 2^{\aleph}.$$

This gives that $|\mathcal{H}| = \mathfrak{c}\cdot\aleph^{\aleph_0} = \max\{\mathfrak{c}, \aleph^{\aleph_0}\}$. As for the cardinality of $\mathscr{B}(\mathcal{H})$, note that each linear bounded operator is uniquely determined by prescribing its values on vectors from orthonormal basis. It can map each basis vector to any vector from $\mathcal{H}$, as long as it remains bounded. Thus, $|\mathscr{B}(\mathcal{H})| \leqslant |\mathcal{H}|^{\aleph} = \max\{\mathfrak{c}^{\aleph}, (\aleph^{\aleph_0})^{\aleph}\}$. Using $\mathfrak{c}^{\aleph} = (2^{\aleph_0})^{\aleph} = 2^{\aleph_0\cdot\aleph} = 2^{\aleph}$ and using $(\aleph^{\aleph_0})^{\aleph} = \aleph^{\aleph_0\aleph} =$

$\aleph^{\aleph} = 2^{\aleph}$ we see that $|\mathscr{B}(\mathcal{H})| \leqslant 2^{\aleph}$. On the other hand, each bijection on orthonormal basis induces a unique unitary (hence bounded) operator on $\mathcal{H}$. We will show below that there are $2^{|B|} = 2^{\aleph}$ bijections on B so there are at least $2^{\aleph}$ elements in $\mathscr{B}(\mathcal{H})$. Combined with the previous estimate we now see that $|\mathscr{B}(\mathcal{H})| = 2^{\aleph}$ and (3.6) follows easily.

As for the cardinality of the set of all bijections on B, each $A \subseteq B$ induces a bijection $\phi_A\colon B \times \{0,1\} \to B \times \{0,1\}$ by $\phi_A(a,i) = (a,i)$ if $a \in A$ and $\phi_A(x,0) = (x,1)$, $\phi_A(x,1) = (x,0)$ if $x \notin A$; so there are at least as many bijections on $(B \times \{0,1\}) \sim B$ as there are subsets of B. The converse estimate is obtained by considering a graph of a bijection in $(B \times B) \sim B$.

Consequently, assuming the generalized continuum hypothesis, we have for non-separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ that $\dim \mathcal{H} < \dim \mathcal{K}$ implies $|\Gamma(\mathscr{B}(\mathcal{H}))| = 2^{\dim \mathcal{H}} < 2^{\dim \mathcal{K}} = |\Gamma(\mathscr{B}(\mathcal{K}))|$ and consequently the two commuting graphs are not isomorphic. ∎

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REFERENCES

[1] A. ABDOLLAHI, H. SHAHVERDI, Characterization of the alternating group by its non-commuting graph, *J. Algebra* **357**(2012), 203–207.

[2] S. AKBARI, M. GHANDEHARI, H. HADIAN, A. MOHAMMADIAN, On commuting graphs of semisimple rings, *Linear Algebra Appl.* **390**(2004), 345–355.

[3] S. AKBARI, A. MOHAMMADIAN, H. RADJAVI, P. RAJA, On the diameters of commuting graphs, *Linear Algebra Appl.* **418**(2006), 161–176.

[4] S. AKBARI, P. RAJA, Commuting graphs of some subsets in simple rings, *Linear Algebra Appl.* **416**(2006), 1038–1047.

[5] C. AMBROZIE, J. BRAČIČ, B. KUZMA, V. MÜLLER, The commuting graph of bounded linear operators on a Hilbert space, *J. Funct. Anal.* **264**(2013), 1068–1087.

[6] P.J. COHEN, *Set Theory and the Continuum Hypothesis*, W.A. Benjamin Inc., New York 1966.

[7] G. DOLINAR, A. GUTERMAN, B. KUZMA, P. OBLAK, Extremal matrix centralizers, *Linear Algebra Appl.* **438**(2013), 2904–2910.

[8] G. DOLINAR, B. KUZMA, Homomorphisms of commutativity relation, *Linear Multilinear Algebra* **64**(2016), 897–922.

[9] G. DOLINAR, B. KUZMA, P. OBLAK, On maximal distances in a commuting graph, *Electron. J. Linear Algebra* **23**(2012), 243–256.

[10] J. DUGUNDJI, *Topology*, Wm.C. Brown Publ., Dubuque, Iowa 1966.

[11] W. EASTON, Powers of regular cardinals, *Ann. Math. Logic* **1**(1970), 139–178.

[12] K. GÖDEL, Consistency-proof for the generalized continuum-hypothesis, *Proc. Nat. Acad. Sci. U.S.A.* **25**(1939), 220–224.

[13] Z. HAN, G. CHEN, X. GUO, A characterization theorem for sporadic simple groups, *Sib. Math. J.* **49**(2008), 1138–1146.

[14] T. JECH, *Set Theory*, Springer-Verlag, Berlin 2003.

[15] A. MOHAMMADIAN, On commuting graphs of finite matrix rings, *Commun. Algebra* **38**(2010), 988–994.

[16] P. ŠEMRL, Non-linear commutativity preserving maps, *Acta Sci. Math. (Szeged)* **71**(2005), 781–819.

[17] R. SOLOMON, A. WOLDAR, Simple groups are characterized by their non-commuting graphs, *J. Group Theory* **16**(2013), 793–824.

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