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SIMILARITY DEGREE OF TYPE II_1 VON NEUMANN ALGEBRAS WITH PROPERTY Γ

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ABSTRACT. In this paper, we discuss some equivalent definitions of Property Γ for a type II_1 von Neumann algebra. Using these equivalent definitions, we prove that the Pisier's similarity degree of a type II_1 von Neumann algebra with Property Γ is equal to 3.

KEYWORDS: *Property Γ, similarity problem, similarity degree.*

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1. INTRODUCTION

Kadison's similarity problem for a C^*-algebra is a longstanding open problem, which asks whether every bounded representation ρ of a C^*-algebra $\mathcal{A}$ on a Hilbert space H is similar to a $*$-representation, i.e. whether there exists an invertible operator T in $B(H)$, such that $T\rho(\cdot)T^{-1}$ is a $*$-representation of $\mathcal{A}$. Significant progress toward this famous open problem was obtained in [1] and [3]. We will refer to Pisier's book [9] for a wonderful introduction to the problem and many of its recent developments.

Similarity degree for a unital C^*-algebra $\mathcal{A}$, denoted by $d(\mathcal{A})$, was defined by Pisier in [7]. Since its introduction, this new concept has greatly influenced the study of Kadison's similarity problem for C^*-algebras. In fact, it was shown in [7] that Kadison's similarity problem for a unital C^*-algebra $\mathcal{A}$ has an affirmative answer if and only if $d(\mathcal{A}) < \infty$. One of the most surprising results on similarity degree was also obtained by Pisier in [11] where he proved that, for an infinite dimensional unital C^*-algebra $\mathcal{A}$, the similarity degree of $\mathcal{A}$ is equal to 2 if and only if $\mathcal{A}$ is a nuclear C^*-algebra.

Several results on similarity degree for a unital C^*-algebra have now been known. For example, if $\mathcal{A} = B(H)$ for some infinite dimensional Hilbert space H, then $d(\mathcal{A}) = 3$ ([3], [8]). The similarity degree of a type II_1 factor $\mathcal{M}$ with Property Γ is less than or equal to 5 ([8]). This result was later improved in [2] to that the similarity degree of such $\mathcal{M}$ is equal to 3. When $\mathcal{A}$ is a minimal tensor product of two C^*-algebras, one of which is nuclear and contains matrices of any order, it was proved in [12] that $d(\mathcal{A}) \leqslant 5$. Recently, it was shown in [4] that, if $\mathcal{A}$ is $\mathcal{Z}$-stable, then $d(\mathcal{A}) \leqslant 5$. We will recall Property c^*-Γ for C^*-algebras in the beginning of Section 4. In [14], it was shown that, if a separable C^*-algebra $\mathcal{A}$ has Property c^*-Γ, then $d(\mathcal{A}) = 3$.

The definition of Property Γ for type II_1 von Neumann algebras was given in [13] and we will recall it in Section 3. In this paper, we will discuss properties of type II_1 von Neumann algebras with Property Γ and compute the similarity degree for this class of von Neumann algebras. The first result we obtained in the paper is Theorem 3.11, which gives many useful equivalences of Property Γ.

Combining Theorem 3.11 with the results in [14], we are able to calculate the exact value of the similarity degree for a type II_1 von Neumann algebra with Property Γ and obtain the next result (Theorem 4.3) as a generalization of Christensen's result in [2]: *if $\mathcal{M}$ is a type II_1 von Neumann algebra with Property Γ, then the similarity degree $d(\mathcal{M}) = 3$.*

Suppose $\mathcal{A}$ is a unital C^*-algebra. Let $\mathcal{I}$ be some index set and

$$l_\infty(\mathcal{I}, \mathcal{A}) = \Big\{ (x_i)_{i \in \mathcal{I}} : \text{for each } i \in \mathcal{I}, x_i \in \mathcal{A} \text{ and } \sup_{i \in \mathcal{I}} \|x_i\| < \infty \Big\}.$$

We apply Theorem 4.3 to calculate values of similarity degrees for two classes of C^*-algebras, which were also considered by Pisier in [8]: first we obtain that, *if $\mathcal{M}$ is a type II_1 factor with Property Γ, then $d(l_\infty(\mathcal{I}, \mathcal{M})) = 3$ for any index set $\mathcal{I}$.* On the other hand, *let $C = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$ (infinite C^*-tensor product of 2×2 matrix algebras). Then, for any infinite index set $\mathcal{I}$, $d(l_\infty(\mathcal{I}, C)) = 3$.*

The organization of this paper is as follows. In Section 2, we give some preliminaries on direct integrals of separable Hilbert spaces and von Neumann algebras acting on separable Hilbert spaces. In Section 3, we give a characterization of type II_1 von Neumann algebras with Property Γ and obtain some equivalent definitions. In Section 4, by showing that every finite subset F of a type II_1 von Neumann algebra $\mathcal{M}$ with Property Γ is contained in a separable unital C^*-subalgebra with Property c^*-Γ, we obtain that $d(\mathcal{M}) = 3$.

2. PRELIMINARIES

2.1. DIXMIER APPROXIMATION THEOREM. We will need the following Dixmier approximation theorem in the paper.

LEMMA 2.1 (Dixmier approximation theorem). *Let $\mathcal{M}$ be a finite von Neumann algebra with center $\mathcal{Z}$. Let τ be the center-valued trace on $\mathcal{M}$. If $a \in \mathcal{M}$, then*

$$\{\tau(a)\} = \mathcal{Z} \cap (\mathrm{conv}(a)^=),$$

where $\mathrm{conv}(a)^=$ is the norm closure of the convex hull of $\{uau^* : u$ is a unitary in $\mathcal{M}\}$.

2.2. DIRECT INTEGRAL THEORY. General knowledge about direct integrals of separable Hilbert spaces and von Neumann algebras acting on separable Hilbert spaces can be found in [18] and [5]. Here we list a few lemmas that will be needed in this paper.

LEMMA 2.2 ([5], Theorem 14.2.2, Corollary 14.2.3). *Suppose $\mathcal{M}$ is a von Neumann algebra acting on a separable Hilbert space H. Let $\mathcal{Z}$ be the center of $\mathcal{M}$. Then there is a direct integral decomposition of $\mathcal{M}$ relative to $\mathcal{Z}$, i.e. there exists a locally compact complete separable metric measure space (X, μ) such that:*

(i) *H is (unitarily equivalent to) the direct integral of $\{H_s : s \in X\}$ over (X, μ), where each H_s is a separable Hilbert space, $s \in X$;*

(ii) *$\mathcal{M}$ is (unitarily equivalent to) the direct integral of $\{\mathcal{M}_s\}$ over (X, μ), where $\mathcal{M}_s$ is a factor in $B(H_s)$ almost every where. Also, if $\mathcal{M}$ is of type I_n (n could be infinite), II_1, II_∞ or III, then the components $\mathcal{M}_s$ are, almost everywhere, of type I_n, II_1, II_∞ or III, respectively.*

Moreover, the center $\mathcal{Z}$ is (unitarily equivalent to) the algebra of diagonalizable operators relative to this decomposition.

The following lemma gives a decomposition of a normal state on a direct integral of von Neumann algebras.

LEMMA 2.3 ([5], Lemma 14.1.19). *Suppose H is the direct integral of separable Hilbert spaces $\{H_s\}$ over (X, μ), $\mathcal{M}$ is a decomposable von Neumann algebra on H (i.e., every operator in $\mathcal{M}$ is decomposable relative to the direct integral decomposition, see Definition* 14.1.6 *in* [5]*) and ρ is a normal state on $\mathcal{M}$. There is a positive normal linear functional ρ_s on $\mathcal{M}_s$ for every $s \in X$ such that $\rho(a) = \int_X \rho_s(a(s))\mathrm{d}\mu$ for each a in $\mathcal{M}$. If $\mathcal{M}$ contains the algebra $\mathcal{C}$ of diagonalizable operators and $\rho|_{E\mathcal{M}E}$ is faithful or tracial, for some projection E in $\mathcal{M}$, then $\rho_s|_{E(s)\mathcal{M}_sE(s)}$ is, accordingly, faithful or tracial almost everywhere.*

REMARK 2.4. From the proof of Lemma 14.1.19 in [5], we obtain that if $\rho = \sum_{n=1}^{\infty} \omega_{y_n}$ on $\mathcal{M}$, where $\{y_n\}$ is a sequence of vectors in H such that $\sum_{n=1}^{\infty} \|y_n\|^2 = 1$ and ω_y is defined on $\mathcal{M}$ such that $\omega_y(a) = \langle ay, y\rangle$ for any $a \in \mathcal{M}, y \in H$, then ρ_s can be chosen to be $\sum_{n=1}^{\infty} \omega_{y_n(s)}$ for each $s \in X$.

3. SOME EQUIVALENT DEFINITIONS OF PROPERTY Γ FOR TYPE II_1 VON NEUMANN ALGEBRAS

We recall the definition of Property Γ for general type II_1 von Neumann algebras in [13].

DEFINITION 3.1 ([13]). Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with a predual $\mathcal{M}_\sharp$. Suppose $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$ is the weak* topology on $\mathcal{M}$ induced from $\mathcal{M}_\sharp$. We say that $\mathcal{M}$ has *Property Γ* if and only if $\forall\, a_1, a_2, \ldots, a_k \in \mathcal{M}$ and $\forall\, n \in \mathbb{N}$, there exist a partially ordered set Λ and a family of projections

$$\{p_{i\lambda} : 1 \leqslant i \leqslant n; \lambda \in \Lambda\} \subseteq \mathcal{M}$$

satisfying:

(i) for each $\lambda \in \Lambda$, $p_{1\lambda}, p_{2\lambda}, \ldots, p_{n\lambda}$ are mutually orthogonal equivalent projections in $\mathcal{M}$ with sum I;

(ii) for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$,

$$\lim_{\lambda}(p_{i\lambda}a_j - a_j p_{i\lambda})^*(p_{i\lambda}a_j - a_j p_{i\lambda}) = 0 \quad \text{in } \sigma(\mathcal{M}, \mathcal{M}_\sharp) \text{ topology.}$$

We remark that this definition coincides with Murray and von Neumann's definition when $\mathcal{M}$ is a type II_1 factor.

In this section, we will give some equivalent definitions of Property Γ for type II_1 von Neumann algebras. The following two lemmas are well-known. We give brief proofs here for the purpose of completeness.

LEMMA 3.2. *Suppose that $\mathcal{M}$ is a type* II_1 *von Neumann algebra. Then the following are true:*

(i) *for any nonzero element $x \in \mathcal{M}$, there exists a normal tracial state ρ on $\mathcal{M}$ such that $\rho(x^*x) \neq 0$;*

(ii) *there exists a non-zero central projection q of $\mathcal{M}$, such that $q\mathcal{M}$ is a countably decomposable type* II_1 *von Neumann algebra.*

Proof. Assume that $\mathcal{M}$ acts on a Hilbert space H.

(i) Let $\mathcal{Z}$ be the center of $\mathcal{M}$ and τ be the unique center-valued trace on $\mathcal{M}$ (see Theorem 8.2.8 in [5]). Let $\widehat{\rho}$ be a normal state on $\mathcal{Z}$ such that $\widehat{\rho}(\tau(x^*x)) \neq 0$. Therefore $\rho = \widehat{\rho} \circ \tau$ is a normal tracial state satisfying the required condition.

(ii) Let ρ be a normal tracial state on $\mathcal{M}$ and $\mathcal{I} = \{a \in \mathcal{M} : \rho(a^*a) = 0\}$. It follows from Proposition III.3.12 in [17] and Proposition 1.10.5 in [16] that there exists a central projection q in $\mathcal{Z}$ such that $\mathcal{I} = (1-q)\mathcal{M}$. It is easy to verify that $q\mathcal{M}$ is countably decomposable. ∎

LEMMA 3.3. *Suppose $\mathcal{M}$ is a type* II_1 *von Neumann algebra. Then there is a family of orthogonal central projections $\{q_\alpha : \alpha \in \Omega\}$ in $\mathcal{M}$ with sum I such that $q_\alpha\mathcal{M}$ is countably decomposable for each $\alpha \in \Omega$.*

Proof. By Lemma 3.2 and Zorn's lemma, there exists an orthogonal family $\{q_\alpha\}$ of non-zero central projections in $\mathcal{M}$, which is maximal with respect to the property that $q_\alpha\mathcal{M}$ is countable decomposable for each α. Let $Q = \sum q_\alpha$. We claim that $Q = I$, where I is the identity of $\mathcal{M}$. Assume, to the contrary, that $Q \neq I$. Then by Lemma 3.2, there is a nonzero central projection q in $(I-Q)\mathcal{M}$ such that $q\mathcal{M}$ is countably decomposable. The existence of such q contradicts the maximality of the family $\{q_\lambda\}$. Therefore $I = \sum q_\alpha$ and the proof of the lemma is completed. ∎

REMARK 3.4. Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with Property Γ. Let q be a central projection of $\mathcal{M}$. Then it follows directly from the definition of Property Γ that $q\mathcal{M}$ also has Property Γ.

LEMMA 3.5. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra acting on a separable Hilbert space H and $\mathcal{Z}_\mathcal{M}$ the center of $\mathcal{M}$. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(z) = z$ for any $z \in \mathcal{Z}_\mathcal{M}$. Let $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s \mathrm{d}\mu$ and $H = \int_X \bigoplus H_s \mathrm{d}\mu$ be the direct integral decompositions of $\mathcal{M}$ and H relative to $\mathcal{Z}_\mathcal{M}$ as in Lemma* 2.2*. Assume that $\mathcal{M}_s$ is a type II_1 factor with a trace τ_s for each $s \in X$. Then for any $a \in \mathcal{M}$,*

$$\tau(a)(s) = \tau_s(a(s))I_s$$

for almost every $s \in X$.

Proof. Fix $a \in \mathcal{M}$. By the Dixmier approximation theorem, for each $t \in \mathbb{N}$, there exist a positive integer k_t, a family of unitaries $\{v_j^{(t)} : t \in \mathbb{N}, 1 \leqslant j \leqslant k_t\}$ in $\mathcal{M}$ and scalars $\{\lambda_j^{(t)} : t \in \mathbb{N}, 1 \leqslant j \leqslant k_t\} \subseteq [0,1]$ such that:

(i) for each $t \in \mathbb{N}$, $\sum_{1\leqslant j\leqslant k_t} \lambda_j^{(t)} = 1$;

(ii) $\lim_{t\to\infty} \Big\| \sum_{1\leqslant j\leqslant k_t} \lambda_j^{(t)} (v_j^{(t)})^* a v_j^{(t)} - \tau(a) \Big\| = 0$.

Since $\{v_j^{(t)} : t \in \mathbb{N}, 1 \leqslant j \leqslant k_t\}$ is a countable set, we may assume that, for every $s \in X$, $v_j^{(t)}(s)$ is a unitary in $\mathcal{M}_s$ for any $t \in \mathbb{N}$ and any $1 \leqslant j \leqslant k_t$. By Proposition 14.1.9 in [5], for any $t \in \mathbb{N}$, we may assume that

$$\Big\| \sum_{1\leqslant j\leqslant k_t} \lambda_j^{(t)} (v_j^{(t)})^* a v_j^{(t)} - \tau(a) \Big\| = \sup_{s\in X} \Big\| \sum_{1\leqslant j\leqslant k_t} \lambda_j^{(t)} (v_j^{(t)}(s))^* a(s) v_j^{(t)}(s) - \tau(a)(s) \Big\|.$$

It follows that

$$\lim_{t\to\infty} \Big\| \sum_{1\leqslant j\leqslant k_t} \lambda_j^{(t)} (v_j^{(t)}(s))^* a(s) v_j^{(t)}(s) - \tau(a)(s) \Big\| = 0 \tag{3.1}$$

for almost every $s \in X$. Again, by the Dixmier approximation theorem and the fact that each $\mathcal{M}_s$ is a type II_1 factor, (3.1) gives that

$$\tau(a)(s) = \tau_s(a(s))I_s$$

for almost every $s \in X$. ∎

LEMMA 3.6. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra with center $\mathcal{Z}_\mathcal{M}$. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_\mathcal{M}$. Suppose $\varepsilon > 0$, $x \in \mathcal{M}$ and $\tau(x^*x) < \varepsilon I$. Then for any tracial state ρ on $\mathcal{M}$,*

$$\rho(x^*x) < 2\varepsilon.$$

Proof. Note that $\tau(x^*x) < \varepsilon I$. It follows from the Dixmier approximation theorem that there exist a positive integer $n \in \mathbb{N}$, a family of unitaries $\{v_1, v_2, \ldots, v_n\}$ in $\mathcal{M}$ and a family of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq [0,1]$ such that:

(i) $\sum_{1\leqslant i\leqslant n} \alpha_i = 1$;

(ii) $\left\| \tau(x^*x) - \sum_{1\leqslant i\leqslant n} \alpha_i v_i^* x^* x v_i \right\| < \varepsilon$.

Since ρ is tracial, it follows from (i) and (ii) the following that completes the proof:

$$\rho(x^*x) = \rho\Big(\sum_{1\leqslant i\leqslant n} \alpha_i v_i^* x^* x v_i\Big) = \rho\Big(\sum_{1\leqslant i\leqslant n} \alpha_i v_i^* x^* x v_i\Big) - \tau(x^*x) + \rho(\tau(x^*x)) < 2\varepsilon. \quad \blacksquare$$

PROPOSITION 3.7. *Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra acting on a separable Hilbert space H. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_{\mathcal{M}}$, where $\mathcal{Z}_{\mathcal{M}}$ is the center of $\mathcal{M}$. Suppose that $\mathcal{M}$ has Property Γ. Then, for $a_1, a_2, \ldots, a_k \in \mathcal{M}$, any $n \in \mathbb{N}$, any $\varepsilon > 0$, there exist n orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ with sum I such that*

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

Proof. Suppose $\mathcal{M}$ has Property Γ. Let $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s \mathrm{d}\mu$ and $H = \int_X \bigoplus H_s \mathrm{d}\mu$ be the direct integral decompositions of $\mathcal{M}$ and H relative to the center $\mathcal{Z}_{\mathcal{M}}$ as in Lemma 2.2. We might assume that $\mathcal{M}_s$ is a type II_1 factor with a trace τ_s for each $s \in X$.

Fix $a_1, a_2, \ldots, a_k \in \mathcal{M}$, $n \in \mathbb{N}$, and $\varepsilon > 0$. By Corollary 4.2 in [13], there exist n orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ with sum I such that

$$\|p_i(s)a_j(s) - a_j(s)p_i(s)\|_{2,s} \leqslant \frac{\varepsilon}{2}, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k, \tag{3.2}$$

for almost every $s \in X$, where $\|\cdot\|_{2,s}$ is the trace norm induced by τ_s on $\mathcal{M}_s$ for each $s \in X$.

For any $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k$, Lemma 3.5 gives

$$\begin{aligned} &\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i))(s) \\ &\qquad = \tau_s((p_i(s)a_j(s) - a_j(s)p_i(s))^*(p_i(s)a_j(s) - a_j(s)p_i(s)))I_s \end{aligned} \tag{3.3}$$

for almost every $s \in X$.

For any $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k$, from (3.2), (3.3) and Proposition 14.1.9 in [5], it follows that

$$\|\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i))\| \leqslant \frac{\varepsilon}{2}$$

and, thus we have the following that finishes the proof:

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I. \quad \blacksquare$$

LEMMA 3.8. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra with a center $\mathcal{Z}_{\mathcal{M}}$. Let $\mathcal{M}_1$ be a von Neumann subalgebra of $\mathcal{M}$ and $\mathcal{Z}_{\mathcal{M}_1}$ be the center of $\mathcal{M}_1$. Suppose $\tau_{\mathcal{M}}$*

and $\tau_{\mathcal{M}_1}$ are the center-valued traces of $\mathcal{M}$, and $\mathcal{M}_1$ respectively. For any $x \in \mathcal{M}_1$, we have $\|\tau_{\mathcal{M}}(x)\| \leqslant \|\tau_{\mathcal{M}_1}(x)\|$.

Proof. Let x be an element in $\mathcal{M}_1$. For any $\varepsilon > 0$, by the Dixmier approximation theorem, there exist a positive integer k, a family of unitaries $\{v_j : 1 \leqslant j \leqslant k\}$ in $\mathcal{M}_1$ and scalars $\{\lambda_j : 1 \leqslant j \leqslant k\} \subseteq [0,1]$ such that (i) $\sum_{1\leqslant j\leqslant k} \lambda_j = 1$ and (ii) $\Big\| \sum_{1\leqslant j\leqslant k} \lambda_j v_j^* x v_j - \tau_{\mathcal{M}_1}(x) \Big\| \leqslant \varepsilon$.

Hence,

$$\begin{aligned}\|\tau_{\mathcal{M}}(x)\| &= \Big\|\tau_{\mathcal{M}}\Big(\sum_{1\leqslant j\leqslant k} \lambda_j v_j^* x v_j\Big)\Big\| \\ &\leqslant \Big\|\tau_{\mathcal{M}}\Big(\sum_{1\leqslant j\leqslant k} \lambda_j v_j^* x v_j\Big) - \tau_{\mathcal{M}}(\tau_{\mathcal{M}_1}(x))\Big\| + \|\tau_{\mathcal{M}}(\tau_{\mathcal{M}_1}(x))\| \leqslant \varepsilon + \|\tau_{\mathcal{M}_1}(x)\|.\end{aligned}$$

Since ε is arbitrary, we have $\|\tau_{\mathcal{M}}(x)\| \leqslant \|\tau_{\mathcal{M}_1}(x)\|$. ∎

PROPOSITION 3.9. *Suppose $\mathcal{M}$ is a countably decomposable type II_1 von Neumann algebra. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_{\mathcal{M}}$, where $\mathcal{Z}_{\mathcal{M}}$ is the center of $\mathcal{M}$. Suppose that $\mathcal{M}$ has Property Γ. Then, for $a_1, a_2, \dots, a_k \in \mathcal{M}$, any $n \in \mathbb{N}$, any $\varepsilon > 0$, there exist n orthogonal equivalent projections $p_1, p_2, \dots, p_n$ in $\mathcal{M}$ with sum I such that*

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

Proof. Let $a_1, a_2, \dots, a_k$ be in $\mathcal{M}$. By Lemma 3.6 in [14], there is a type II_1 von Neumann algebra $\mathcal{M}_1$ with separable predual and Property Γ such that $\{a_1, \dots, a_k\} \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$. From Proposition 3.7, it follows that there exist n orthogonal equivalent projections $p_1, p_2, \dots, p_n$ in $\mathcal{M}_1$ with sum I such that

$$\tau_{\mathcal{M}_1}((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k,$$

where $\tau_{\mathcal{M}_1}$ is the center-valued trace on $\mathcal{M}_1$. By Lemma 3.8, we obtain that

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k. \quad ∎$$

REMARK 3.10. Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with center $\mathcal{Z}_{\mathcal{M}}$. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_{\mathcal{M}}$. Suppose $\{q_\alpha : \alpha \in \Omega\}$ is a family of nonzero orthogonal central projections in $\mathcal{M}$ with sum I. Therefore $q_\alpha \mathcal{M}$ is a type II_1 von Neumann algebra with center $q_\alpha \mathcal{Z}_{\mathcal{M}}$. Let τ_α be the center-valued trace on $q_\alpha \mathcal{M}$ such that $\tau_\alpha(a) = a$ for any $a \in q_\alpha \mathcal{Z}_{\mathcal{M}}$. We have

$$\tau(a) = \sum_{\alpha\in\Omega} \tau_\alpha(q_\alpha a), \quad \forall\, a \in \mathcal{M}.$$

THEOREM 3.11. *Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra and $\mathcal{Z}_{\mathcal{M}}$ is the center of $\mathcal{M}$. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_{\mathcal{M}}$. Then the following statements are equivalent:*

(i) *$\mathcal{M}$ has Property Γ.*

(ii) *There exists a family of nonzero orthogonal central projections $\{q_\alpha : \alpha \in \Omega\}$ in $\mathcal{M}$ with sum I such that $q_\alpha\mathcal{M}$ is a countably decomposable type* II_1 *von Neumann algebra with Property Γ for each $\alpha \in \Omega$.*

(iii) *For any $n \in \mathbb{N}$, any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist n orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ with sum I such that*

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

(iv) *There exists a positive integer $n_0 \geqslant 2$ satisfying that for any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist n_0 orthogonal equivalent projections $p_1, p_2, \ldots, p_{n_0}$ in $\mathcal{M}$ with sum I satisfying*

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n_0, 1 \leqslant j \leqslant k.$$

(v) *For any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exists a unitary u in $\mathcal{M}$ such that*

(a) *$\tau(u) = 0$;*

(b) *$\tau((ua_j - a_j u)^*(ua_j - a_j u)) < \varepsilon I, \quad \forall\, 1 \leqslant j \leqslant k$.*

(vi) *For any $n \in \mathbb{N}$, any normal tracial state ρ on $\mathcal{M}$, any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist n orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ with sum I such that*

$$\|p_i a_j - a_j p_i\|_{2,\rho} < \varepsilon, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k,$$

where $\|\cdot\|_{2,\rho}$ is the 2*-norm on $\mathcal{M}$ induced by ρ.*

(vii) *There exists a positive integer $n_0 \geqslant 2$ satisfying that for any normal tracial state ρ on $\mathcal{M}$, any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist n_0 orthogonal equivalent projections $p_1, p_2, \ldots, p_{n_0}$ in $\mathcal{M}$ with sum I satisfying*

$$\|p_i a_j - a_j p_i\|_{2,\rho} < \varepsilon, \quad \forall\, 1 \leqslant i \leqslant n_0, 1 \leqslant j \leqslant k,$$

where $\|\cdot\|_{2,\rho}$ is the 2*-norm on $\mathcal{M}$ induced by ρ.*

(viii) *For any normal tracial state ρ on $\mathcal{M}$, any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exists a unitary u in $\mathcal{M}$ such that*

(a) *$\tau(u) = 0$;*

(b) *$\|ua_j - a_j u\|_{2,\rho} < \varepsilon$, for all $1 \leqslant j \leqslant k$, where $\|\cdot\|_{2,\rho}$ is the* 2*-norm on $\mathcal{M}$ induced by ρ.*

Proof. We will prove the result by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (vi) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii) It follows from Lemma 3.3 and Remark 3.4.

(ii) $\Rightarrow$ (iii) Assume that there exists a family of nonzero orthogonal central projections $\{q_\alpha : \alpha \in \Omega\}$ with sum I such that $q_\alpha\mathcal{M}$ is a countably decomposable type II_1 von Neumann algebra with Property Γ for each $\alpha \in \Omega$. Fix $n \in \mathbb{N}$, any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$. Then

$$a_j = \sum_\alpha q_\alpha a_j, \quad \forall\, 1 \leqslant j \leqslant n.$$

For each $\alpha \in \Omega$, by Proposition 3.9, there exist n orthogonal equivalent projections $p_1^{(\alpha)}, p_2^{(\alpha)}, \ldots, p_n^{(\alpha)}$ in $q_\alpha\mathcal{M}$ with sum q_α such that

$$\tau_\alpha((p_i^{(\alpha)}(q_\alpha a_j) - (q_\alpha a_j)p_i^{(\alpha)})^*(p_i^{(\alpha)}(q_\alpha a_j) - (q_\alpha a_j)p_i^{(\alpha)})) < \varepsilon \cdot q_\alpha, \tag{3.4}$$

for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k$, where τ_α is the center-valued trace on $q_\alpha\mathcal{M}$. Let

$$p_i = \sum_\alpha p_i^{(\alpha)}, \quad \text{for all } 1 \leqslant i \leqslant n.$$

Then it is not hard to see that $p_1, \ldots, p_n$ are orthogonal equivalent projections in $\mathcal{M}$ with sum I. By Remark 3.10 and inequality (3.4), we know

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \varepsilon I, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

(iii) $\Rightarrow$ (iv) It is obvious.

(iv) $\Rightarrow$ (v) Assume that there exists a positive integer $n_0 \geqslant 2$ satisfying that *for any $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist n_0 orthogonal equivalent projections $p_1, p_2, \ldots, p_{n_0}$ in $\mathcal{M}$ with sum I satisfying*

$$\tau((p_i a_j - a_j p_i)^*(p_i a_j - a_j p_i)) < \frac{\varepsilon}{n_0^2} I, \quad \forall\, 1 \leqslant i \leqslant n_0, 1 \leqslant j \leqslant k.$$

Let $\lambda = \mathrm{e}^{2\pi \mathrm{i}/n_0}$ be the n_0-th root of unit. Let

$$u = p_1 + \lambda p_2 + \cdots + \lambda^{n_0 - 1} p_{n_0}.$$

Since $p_1, \ldots, p_{n_0}$ are orthogonal equivalent projections in $\mathcal{M}$, we know $\tau(u) = 0$. A quick computation shows that

$$\tau((u a_j - a_j u)^*(u a_j - a_j u)) < \varepsilon I.$$

(v) $\Rightarrow$ (ii) Assume that (v) holds. From Lemma 3.3, there is a family of orthogonal central projections $\{q_\alpha : \alpha \in \Omega\}$ in $\mathcal{M}$ with sum I such that $q_\alpha\mathcal{M}$ is countably decomposable for each $\alpha \in \Omega$.

Next we will show that $q_\alpha\mathcal{M}$ has Property Γ for each α in Ω. Let $x_1, \ldots, x_k$ be elements in $q_\alpha\mathcal{M}$. By the assumption (v), for any $\varepsilon > 0$, there exists a unitary u in $\mathcal{M}$ such that (a) $\tau(u) = 0$ and (b) $\tau((u a_j - a_j u)^*(u a_j - a_j u)) < \varepsilon I$, for all $1 \leqslant j \leqslant k$. Since $q_\alpha\mathcal{M}$ is a countably decomposable type II_1 von Neumann subalgebra, there exists a faithful normal tracial state ρ on $q_\alpha\mathcal{M}$. We can naturally extend ρ on $q_\alpha\mathcal{M}$ to a normal tracial state $\widetilde{\rho}$ on $\mathcal{M}$ by defining $\widetilde{\rho}(x) = \rho(q_\alpha x)$ for all x in $\mathcal{M}$. It is not hard to see that $q_\alpha u$ is a unitary in $q_\alpha\mathcal{M}$ and $\tau_\alpha(q_\alpha u) = \tau(q_\alpha u) = q_\alpha \tau(u) = 0$, where τ_α is a center-valued trace on $q_\alpha\mathcal{M}$. Moreover, by the Dixmier approximation theorem, we have

$$\widetilde{\rho}(x) = \widetilde{\rho}(\tau(x)), \quad \forall\, x \in \mathcal{M}.$$

Hence

$$\begin{aligned} \rho(((q_\alpha u)a_j - a_j(q_\alpha u))^*((q_\alpha u)a_j - a_j(q_\alpha u))) &= \widetilde{\rho}((u a_j - a_j u)^*(u a_j - a_j u)) \\ &= \widetilde{\rho}(\tau((u a_j - a_j u)^*(u a_j - a_j u))) \leqslant \varepsilon, \end{aligned}$$

for all $1 \leqslant i \leqslant k$. By Proposition 3.5 in [14], we conclude that $q_\alpha\mathcal{M}$ has Property Γ.

(iii) $\Rightarrow$ (i) Assume that (iii) is true. We assume that $\mathcal{M}$ acts on a Hilbert space H. Let $x_1, \ldots, x_k$ be a family of elements in $\mathcal{M}$. From (iii), for any positive integer n, there exists a family of projections $\{p_{ir} : 1 \leqslant i \leqslant n, r \geqslant 1\}$ in $\mathcal{M}$ such that:

(1) for each $r \geqslant 1$, $p_{1,r}, \ldots, p_{n,r}$ are orthogonal equivalent projections in $\mathcal{M}$ with sum I;

(2) moreover,

$$\lim_{r\to\infty} \|\tau((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r}))\| = 0, \quad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

Thus, for any normal tracial state ρ on $\mathcal{M}$, we have

$$\begin{aligned}(3.5)\quad \lim_{r\to\infty} &\|\rho((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r}))\| \\ &= \lim_{r\to\infty} \|\rho(\tau((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r})))\| \\ &\leqslant \lim_{r\to\infty} \|\tau((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r}))\| = 0.\end{aligned}$$

Let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be the collection of all normal tracial states on $\mathcal{M}$. For each $\lambda \in \Lambda$, let $(\pi_\lambda, H_\lambda, \widehat{I}_\lambda)$ be the GNS representation, obtained from ρ_λ, of $\mathcal{M}$ on the Hilbert space $H_\lambda = L^2(\mathcal{M}, \rho_\lambda)$ with a cyclic vector $\widehat{I}_\lambda$ in H_λ. We also let $K = \sum_{\lambda\in\Lambda} H_\lambda$ be the direct sum of Hilbert spaces $\{H_\lambda\}$ and $\pi = \sum_{\lambda\in\Lambda} \pi_\lambda : \mathcal{M} \to B(K)$ be the direct sum of $\{\pi_\lambda\}$. Thus π is a $*$-representation of $\mathcal{M}$ on K defined by

$$\pi(x)((\xi_\lambda)) = (\pi_\lambda(x)\xi_\lambda), \quad \forall\, (\xi_\lambda) \in \sum_{\lambda\in\Lambda} H_\lambda = K.$$

It is not hard to see that π is a normal $*$-representation and $\pi(\mathcal{M})$ is also a von Neumann algebra. By Lemma 3.2(i), π is a $*$-isomorphism from $\mathcal{M}$ onto $\pi(\mathcal{M})$.

We claim that, for all $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k$, $(p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r}) \to 0$ in ultraweak operator topology (or in $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$ topology).

Actually, the claim is equivalent to the statement that

$$\pi((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r})) \to 0 \quad \text{in ultraweak topology.}$$

Note that

$$\{(p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r}) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k, r \in \mathbb{N}\}$$

is a bounded subset in $\mathcal{M}$. It will be enough if we are able to show that

$$\pi((p_{i,r}a_j - a_jp_{i,r})^*(p_{i,r}a_j - a_jp_{i,r})) \to 0 \quad \text{in weak operator topology,}$$

or

$$(3.6)\qquad \pi(p_{i,r}a_j - a_jp_{i,r}) \to 0 \quad \text{in strong operator topology.}$$

By the construction of π, (3.6) follows directly from (3.5).

From the claim in the preceding paragraph, by the definition of Property Γ, we know that $\mathcal{M}$ has property Γ.

(iii) $\Rightarrow$ (vi) From the Dixmier approximation theorem, for any normal tracial state ρ on $\mathcal{M}$, we have

$$\rho(x) = \rho(\tau(x)) \quad \forall\, x \in \mathcal{M}.$$

Now (vi) follows easily from (iii).

(vi) $\Rightarrow$ (vii) It is obvious.

(vii) $\Rightarrow$ (viii) It is similar to (iv) $\Rightarrow$ (v).

(viii) $\Rightarrow$ (ii) Assume that (viii) holds. From Lemma 3.3, there is a family of orthogonal central projections $\{q_\alpha : \alpha \in \Omega\}$ in $\mathcal{M}$ with sum I such that $q_\alpha\mathcal{M}$ is countably decomposable for each $\alpha \in \Omega$. We need to show that $q_\alpha\mathcal{M}$ has Property Γ for each α in Ω.

Since each $q_\alpha\mathcal{M}$ is a countably decomposable type II_1 von Neumann algebra, there exists a faithful normal tracial state ρ_α on $q_\alpha\mathcal{M}$. Then the normal tracial state ρ_α on $q_\alpha\mathcal{M}$ can be naturally extended to a normal tracial state $\widetilde{\rho}$ on $\mathcal{M}$ by defining $\widetilde{\rho}(x) = \rho_\alpha(q_\alpha x)$ for all $x \in \mathcal{M}$. Let $\varepsilon > 0$ and $a_1, \dots, a_k$ be elements in $q_\alpha\mathcal{M}$. Since (viii) holds, there exists a unitary u in $\mathcal{M}$ such that:

(a) $\tau(u) = 0$;

(b) $\|ua_j - a_ju\|_{2,\widetilde{\rho}} < \varepsilon$, for all $1 \leqslant j \leqslant k$, where $\|\cdot\|_{2,\widetilde{\rho}}$ is the trace norm induced by $\widetilde{\rho}$ on $\mathcal{M}$.

Now it is not hard to verify that $q_\alpha u$ is a unitary in $q_\alpha\mathcal{M}$ satisfying $\tau_\alpha(q_\alpha u) = \tau(q_\alpha u) = 0$, where τ_α is the unique center-valued trace on $q_\alpha\mathcal{M}$. Moreover,

$$\|(q_\alpha u)a_j - a_j(q_\alpha u)\|_{2,\rho_\alpha} = \|(q_\alpha u)a_j - a_j(q_\alpha u)\|_{2,\widetilde{\rho}} = \|ua_j - a_ju\|_{2,\widetilde{\rho}} < \varepsilon.$$

From Proposition 3.5 in [14], it follows that $q_\alpha\mathcal{M}$ has Property Γ for each α in Ω. This ends the whole proof. ∎

4. SIMILARITY DEGREE OF TYPE II_1 VON NEUMANN ALGEBRAS WITH PROPERTY Γ

Let us recall the definition of Property c^*-Γ for unital C^*-algebras.

DEFINITION 4.1 ([14]). Suppose $\mathcal{A}$ is a unital C^*-algebra. We say $\mathcal{A}$ *has Property c^*-Γ* if it satisfies the following condition:

> If π is a unital $*$-representation of $\mathcal{A}$ on a Hilbert space H such that $\pi(\mathcal{A})''$ is a type II_1 factor, then $\pi(\mathcal{A})''$ has Property Γ.

If $\mathcal{A}$ is a separable unital C^*-algebra with Property c^*-Γ, Theorem 5.3 in [14] gives that the similarity degree of $\mathcal{A}$ is no more than 3. Indeed, it was shown in Theorem 5.3 in [14] that, for any C^*-algebra $\mathcal{B}$, if ϕ is a bounded unital homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then $\|\phi\|_{\mathrm{cb}} \leqslant \|\phi\|^3$.

LEMMA 4.2. *Suppose $\mathcal{M}$ is a type II_1 von Neumann algebra with Property Γ. Let τ be the center-valued trace on $\mathcal{M}$ such that $\tau(a) = a$ for any $a \in \mathcal{Z}_\mathcal{M}$, where $\mathcal{Z}_\mathcal{M}$ is the center of $\mathcal{M}$. Suppose F is a finite subset of $\mathcal{M}$. Then there exists a separable unital C^*-subalgebra $\mathcal{A}$ with Property c^*-Γ satisfying $F \subseteq \mathcal{A} \subseteq \mathcal{M}$.*

Proof. Let $F_1 = F = \{x_1, x_2, \ldots, x_k\}$ be a finite subset of $\mathcal{M}$. Since $\mathcal{M}$ has Property Γ, by Theorem 3.11, there exists a 2×2 system of matrix units $\{e_{11}^{(1)}, e_{12}^{(1)}, e_{21}^{(1)}, e_{22}^{(1)}\}$ such that:

(i$_1$) $e_{11}^{(1)} + e_{22}^{(1)} = I$, where I is the identity of $\mathcal{M}$;

(ii$_1$) $\tau((e_{ii}^{(1)}x - xe_{ii}^{(1)})^*(e_{ii}^{(1)}x - xe_{ii}^{(1)})) \leqslant (1/2)I$, for each $x \in F_1$.

From (ii$_1$), by the Dixmier approximation theorem, there exist a positive integer n_1, a family of unitaries $v_1^{(1)}, v_2^{(1)}, \ldots, v_{n_1}^{(1)}$ in $\mathcal{M}$ such that:

(iii$_1$) for each $1 \leqslant i \leqslant 2$ and each $x \in F_1$, there is an element y in the convex hull of $\{(v_t^{(1)})^*(e_{ii}^{(1)}x - xe_{ii}^{(1)})^*(e_{ii}^{(1)}x - xe_{ii}^{(1)})v_t^{(1)} : 1 \leqslant t \leqslant n_1\}$ with $\|y\| < 1$.

Let $F_2 = F_1 \cup \{e_{11}^{(1)}, e_{12}^{(1)}, e_{21}^{(1)}, e_{22}^{(1)}\} \cup \{v_1^{(1)}, \ldots, v_{n_1}^{(1)}\}$.

Assume that $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$ have been constructed for some $m \geqslant 2$. Since $\mathcal{M}$ has Property Γ, again by Theorem 3.11, there exists a 2×2 system of matrix units $\{e_{11}^{(m)}, e_{12}^{(m)}, e_{21}^{(m)}, e_{22}^{(m)}\}$ such that:

(i$_m$) $e_{11}^{(m)} + e_{22}^{(m)} = I$, where I is the identity of $\mathcal{M}$;

(ii$_m$) $\tau((e_{ii}^{(m)}x - xe_{ii}^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})) \leqslant (1/(m+1))I$, for each $x \in F_m$.

From (ii$_m$), by the Dixmier approximation theorem, there exist a positive integer n_m, a family of unitaries $v_1^{(m)}, v_2^{(m)}, \ldots, v_{n_m}^{(m)}$ in $\mathcal{M}$ such that:

(iii$_m$) for each $1 \leqslant i \leqslant 2$ and each $x \in F_m$, there is an element y in the convex hull of $\{(v_t^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})v_t^{(m)} : 1 \leqslant t \leqslant n_m\}$ with $\|y\| < 1/m$.

Let $F_{m+1} = F_m \cup \{e_{11}^{(m)}, e_{12}^{(m)}, e_{21}^{(m)}, e_{22}^{(m)}\} \cup \{v_1^{(m)}, v_2^{(m)}, \ldots, v_{n_m}^{(m)}\}$.

Continuing this process, we are able to obtain a sequence $\{F_m\}$, a sequence of system of units $\{e_{11}^{(m)}, e_{12}^{(m)}, e_{21}^{(m)}, e_{22}^{(m)}\}$ such that:

(0) $\{x_1, \ldots, x_k\} = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m \subseteq \cdots$;

(1) for each $m \geqslant 1$, $\{e_{11}^{(m)}, e_{12}^{(m)}, e_{21}^{(m)}, e_{22}^{(m)}\}$ is a system of units such that $e_{11}^{(m)} + e_{22}^{(m)} = I$;

(2) $\tau((e_{ii}^{(m)}x - xe_{ii}^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})) \leqslant (1/(m+1))I$, for each $x \in F_m$;

(3) for each $i = 1, 2$ and each $x \in F_m$, there is an element

$$y \in \operatorname{conv}\{v^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})v : v \text{ is a unitary in } F_{m+1}\}$$

satisfying $\|y\| < \frac{1}{m}$.

Let $\mathcal{A}$ be the unital C^*-algebra generated by $\bigcup_{m \in \mathbb{N}} F_m$. Then $\mathcal{A}$ is separable and it follows from the preceding construction that

(4) for $i = 1, 2$ and any $x \in \mathcal{A}$, there exists a sequence of elements $\{y_m\}_{m \geqslant 1}$ in $\mathcal{A}$ such that, for $m \geqslant 1$, each y_m is in the convex hull of $\{v^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})^*(e_{ii}^{(m)}x - xe_{ii}^{(m)})v : v \text{ is a unitary in } \mathcal{A}\}$ and $\lim_{m \to \infty} \|y_m\| = 0$.

Now we are going to show this C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$ has Property c^*-Γ. Suppose π is a unital $*$-representation of $\mathcal{A}$ on a Hilbert space H such that $\pi(\mathcal{A})''$ is a type II_1 factor. Notice that for each $m \in \mathbb{N}$, $\{e_{11}^{(m)}, e_{12}^{(m)}, e_{21}^{(m)}, e_{22}^{(m)}\}$ is a 2×2 system of matrix units in $\mathcal{A}$. It follows that $\{\pi(e_{11}^{(m)}), \pi(e_{12}^{(m)}), \pi(e_{21}^{(m)}), \pi(e_{22}^{(m)})\}$ is also a system of matrix units. Hence $\pi(e_{11}^{(m)}), \pi(e_{22}^{(m)})$ are orthogonal equivalent projections in $\pi(\mathcal{A})''$ with sum I.

It follows from condition (4) that

(4′) for $i = 1, 2$ and any $x \in \pi(\mathcal{A})$, there exists a sequence of elements $\{y_m\}_{m \geqslant 1}$ in $\pi(\mathcal{A})$ such that, for $m \geqslant 1$, each y_m is in the convex hull of

$$\{v^*(\pi(e_{ii}^{(m)})x - x\pi(e_{ii}^{(m)}))^*(\pi(e_{ii}^{(m)})x - x\pi(e_{ii}^{(m)}))v : v \text{ is a unitary in } \pi(\mathcal{A})\}$$

and $\lim\limits_{m\to\infty} \|y_m\| = 0$.

Let ρ be the unique trace on $\pi(\mathcal{A})''$. Since ρ is tracial, condition (4′) implies that, for any $x \in \pi(\mathcal{A})$,

$$\begin{aligned} &\lim_{m\to\infty} \rho((\pi(e_{ii}^{(m)})\pi(x) - \pi(x)\pi(e_{ii}^{(m)}))^*(\pi(e_{ii}^{(m)})\pi(x) - \pi(x)\pi(e_{ii}^{(m)}))) \\ &\qquad = \lim_{m\to\infty} \rho(y_m) = 0. \end{aligned} \tag{4.1}$$

By Kaplansky density theorem, it follows from (4.1) that

$$\lim_{m\to\infty} \rho((\pi(e_{ii}^{(m)})a - a\pi(e_{ii}^{(m)}))^*(\pi(e_{ii}^{(m)})a - a\pi(e_{ii}^{(m)}))) = 0$$

for any $a \in \pi(\mathcal{A})''$. Note that a type II_1 factor is always countably decomposable. By Proposition 3.5 in [14], $\pi(\mathcal{A})''$ has Property Γ, whence we conclude that $\mathcal{A}$ has Property c^*-Γ.

The proof is completed. ∎

It was shown in [2] that the similarity degree of a type II_1 factor with Property Γ is 3. The following theorem gives a generalization.

THEOREM 4.3. *If $\mathcal{M}$ is a type II_1 von Neumann algebra with Property Γ, then the similarity degree $d(\mathcal{M}) = 3$.*

Proof. Since $\mathcal{M}$ is a von Neumann algebra of type II_1, by Corollary 1.9 in [19], it is not nuclear. It follows from Theorem 1 in [11] that $d(\mathcal{M}) \geqslant 3$. In the following we show that $d(\mathcal{M})$ is no more than 3.

Suppose $\phi : \mathcal{M} \to B(H)$ is a bounded unital homomorphism, where H is a Hilbert space. We will show that $\|\phi\|_{\mathrm{cb}} \leqslant \|\phi\|^3$. In fact we are going to prove that, for any $n \in \mathbb{N}$ and any $x = (x_{ij}) \in M_n(\mathcal{M})$,

$$\|\phi^{(n)}(x)\| \leqslant \|\phi\|^3\|x\|. \tag{4.2}$$

Fix $n \in \mathbb{N}$ and $x = (x_{ij}) \in M_n(\mathcal{M})$. We assume that $\|x\| = 1$. Notice that $F = \{x_{ij} : 1 \leqslant i, j \leqslant n\}$ is a finite subset of $\mathcal{M}$. By Lemma 4.2, there is a separable unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}$ with Property c^*-Γ such that $F \subseteq \mathcal{A}$. Let $\widetilde{\phi}$ be the restriction of ϕ on $\mathcal{A}$. Then $\widetilde{\phi} : \mathcal{A} \to B(H)$ is a bounded unital homomorphism. It was shown in the proof of Theorem 5.3 in [14] that

$$\|\widetilde{\phi}\|_{\text{cb}} \leqslant \|\widetilde{\phi}\|^3. \tag{4.3}$$

Since $F \subseteq \mathcal{A}$, it follows from (4.3) that

$$\|\phi^{(n)}(x)\| = \|\widetilde{\phi}^{(n)}(x)\| \leqslant \|\widetilde{\phi}\|^3 \leqslant \|\phi\|^3.$$

Therefore $d(\mathcal{M}) = 3$ and the proof is completed. ∎

Based on Theorem 4.3, a slight modification of the proof of Theorem 5.2 in [14] gives the next corollary.

COROLLARY 4.4. *Let $\mathcal{M}$ be a von Neumann algebra with the type decomposition*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty,$$

where $\mathcal{M}_1$ is a type I *von Neumann algebra, $\mathcal{M}_{c_1}$ is a type* II_1 *von Neumann algebra, $\mathcal{M}_{c_\infty}$ is a type* II_∞ *von Neumann algebra and $\mathcal{M}_\infty$ is a type* III *von Neumann algebra. Suppose $\mathcal{M}_{c_1}$ is a type* II_1 *von Neumann algebra with Property Γ. If ϕ is a bounded unital representation of $\mathcal{M}$ on a Hilbert space H, which is continuous from $\mathcal{M}$, with the topology $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$, to $B(H)$, with the topology $\sigma(B(H), B(H)_\sharp)$, then ϕ is completely bounded and $\|\phi\|_{\text{cb}} \leqslant \|\phi\|^3$.*

Suppose $\mathcal{A}$ is a unital C^*-algebra. Let $\mathcal{I}$ be some index set and

$$l_\infty(\mathcal{I}, \mathcal{A}) = \Big\{ (x_i)_{i \in J} : \text{ for each } i \in \mathcal{I}, x_i \in \mathcal{A} \text{ and } \sup_{i \in \mathcal{I}} \|x_i\| < \infty \Big\}.$$

It was shown in Corollary 17 of [8] that if $\mathcal{M}$ is a type II_1 factor with Property Γ, then $d(l_\infty(\mathcal{I}, \mathcal{M})) \leqslant 5$ for any index set $\mathcal{I}$. The next corollary gives an exact value of $d(l_\infty(\mathcal{I}, \mathcal{M}))$.

COROLLARY 4.5. *If $\mathcal{M}$ is a type* II_1 *factor with Property Γ, then $d(l_\infty(\mathcal{I}, \mathcal{M})) = 3$ for any index set $\mathcal{I}$.*

Proof. Assume that $\mathcal{M}$ is a type II_1 factor with Property Γ. By Theorem 3.11, for any index set $\mathcal{I}$, $l_\infty(\mathcal{I}, \mathcal{M})$ is a type II_1 von Neumann algebra with Property Γ. Therefore

$$d(l_\infty(\mathcal{I}, \mathcal{M})) = 3. \quad ∎$$

Let C be the CAR-algebra $C = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$ (infinite C^*-tensor product of 2×2 matrix algebras). It was shown in Proposition 21 of [8] that, for any index set $\mathcal{I}$, $d(l_\infty(\mathcal{I}, C)) \leqslant 5$. The next corollary gives an exact value of $d(l_\infty(\mathcal{I}, C))$.

COROLLARY 4.6. *Let $C = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$ (infinite C^*-tensor product of 2×2 matrix algebras). Then, for any infinite index set $\mathcal{I}$, $d(l_\infty(\mathcal{I}, C)) = 3$.*

Proof. Denote by $\mathcal{A} = l_\infty(\mathcal{I}, C) = \sum_{i \in \mathcal{I}} \bigoplus C_i$, where C_i is a copy of C for each $i \in \mathcal{I}$. Let $\mathcal{R}$ and $\mathcal{R}_i$ be the canonical hyperfinite II_1 factor generated by C and

C_i, respectively. Let τ_i be a trace on $\mathcal{R}_i$. Let $\mathcal{M} = l_\infty(\mathcal{I},\mathcal{R}) = \sum_{i\in\mathcal{I}} \bigoplus \mathcal{R}_i$. We might assume that both $\mathcal{M}$ and $\mathcal{A}$ act naturally on the Hilbert space $\sum_{i\in\mathcal{I}} l^2(\mathcal{R}_i,\tau_i)$. Denote by p_i the projection in $\mathcal{A}$ such that $p_i\mathcal{A} = C_i$. It follows that $\sum_{i\in\mathcal{I}} p_i = I$.

First we will prove the following two claims.

Claim 4.6.1. For any $x_1,\ldots,x_k$ in $\mathcal{A}$ and any $\varepsilon > 0$, there exists a system of matrix units $\{E_{st} : 1 \leqslant s,t \leqslant 2\}$ in $\mathcal{A}$ such that $E_{11} + E_{22} = I$ and

$$\|x_jE_{ss} - E_{ss}x_j\| < \varepsilon, \quad \text{for all } 1 \leqslant s \leqslant 2,\ 1 \leqslant j \leqslant k.$$

Proof of Claim 4.6.1. For each $i \in \mathcal{I}$, note that $p_ix_1,\ldots,p_ix_k$ are in a CAR-algebra C_i. Hence there exists a system of matrix units $\{e_{st}^{(i)} : 1 \leqslant s,t \leqslant 2\}$ in C_i such that $e_{11}^{(i)} + e_{22}^{(i)} = p_i$ and

$$\|x_je_{ss}^{(i)} - e_{ss}^{(i)}x_j\| < \frac{\varepsilon}{2}, \quad \text{for all } 1 \leqslant s \leqslant 2,\ 1 \leqslant j \leqslant k.$$

Let

$$E_{st} = \sum_{i\in\mathcal{I}} e_{st}^{(i)}, \quad \text{for all } 1 \leqslant s,t \leqslant 2.$$

Then $\{E_{st} : 1 \leqslant s,t \leqslant 2\}$ is a system of matrix units in $\mathcal{A}$ such that $E_{11} + E_{22} = I$ and we have the following that finishes the proof of Claim 4.6.1:

$$\|x_jE_{ss} - E_{ss}x_j\| = \sup_i \|x_je_{ss}^{(i)} - e_{ss}^{(i)}x_j\| < \varepsilon, \quad \text{for all } 1 \leqslant s \leqslant 2,\ 1 \leqslant j \leqslant k. \quad \blacksquare$$

Claim 4.6.2. For any $x_1,\ldots,x_k$ in $\mathcal{A}$, there exists a separable C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $\mathcal{B}$ is of Property c^*-Γ and all $x_1,\ldots,x_k$ are in $\mathcal{B}$.

Proof of Claim 4.6.2. Let $F_1 = \{x_1,\ldots,x_k\}$. By Claim 4.6.1, there exists a system of matrix units $\{E_{st}^{(1)} : 1 \leqslant s,t \leqslant 2\}$ in $\mathcal{A}$ such that $E_{11}^{(1)} + E_{22}^{(1)} = I$ and

$$\|xE_{ss}^{(1)} - E_{ss}^{(1)}x\| < 1 \quad \text{for all } 1 \leqslant s \leqslant 2,\ 1 \leqslant j \leqslant k, \text{ and } x \in F_1.$$

Let $F_1 = F_1 \cup \{E_{st}^{(1)} : 1 \leqslant s,t \leqslant 2\}$.

Assume that $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_m$ have been constructed for some $m \geqslant 2$. By Claim 4.6.1, we know there exists a system of matrix units $\{E_{st}^{(m)} : 1 \leqslant s,t \leqslant 2\}$ in $\mathcal{A}$ such that $E_{11}^{(m)} + E_{22}^{(m)} = I$ and

$$\|xE_{ss}^{(m)} - E_{ss}^{(m)}x\| < \frac{1}{m} \quad \text{for all } 1 \leqslant s \leqslant 2,\ 1 \leqslant j \leqslant k, \text{ and } x \in F_m.$$

Let $F_{m+1} = F_m \cup \{E_{st}^{(m)} : 1 \leqslant s,t \leqslant 2\}$.

Using similar arguments as in Lemma 4.2, we are able to obtain an increasing sequence of subsets $\{F_m\}$ of $\mathcal{A}$ such that (a) the C^*-subalgebra $\mathcal{B}$ generated by $\{F_m\}$ in $\mathcal{A}$ is of Property c^*-Γ; and (b) all $x_1,\ldots,x_k$ are in $\mathcal{B}$. This ends the proof of Claim 4.6.2. $\blacksquare$

We continue the proof of the corollary. From Claim 4.6.2, using similar arguments as in Theorem 4.3, we conclude that $d(\mathcal{A}) \leqslant 3$.

Next we will show that $d(\mathcal{A}) \geqslant 3$. Since $\mathcal{I}$ is an infinite set, let $\mathcal{I}_0$ be a countable infinite subset of $\mathcal{I}$. Then $\Big(\sum_{i\in\mathcal{I}\setminus\mathcal{I}_0} p_i\Big)\mathcal{A}$ is a closed two sided ideal of $\mathcal{A}$. Moreover, $\Big(\sum_{i\in I_0} p_i\Big)\mathcal{A} \cong \mathcal{A}/\Big(\sum_{i\in\mathcal{I}\setminus\mathcal{I}_0} p_i\Big)\mathcal{A}$. By Remark 6 in [10], we know that $d(\mathcal{A}) \geqslant d\Big(\Big(\sum_{i\in I_0} p_i\Big)\mathcal{A}\Big)$. In order to show that $d(\mathcal{A}) \geqslant 3$, it suffices to show that $d\Big(\Big(\sum_{i\in I_0} p_i\Big)\mathcal{A}\Big) \geqslant 3$. By replacing $\mathcal{I}$ by $\mathcal{I}_0$, we can assume that $\mathcal{I} = \mathbb{N}$.

Let ω be a free ultra-filter of $\mathbb{N}$ and

$$\mathcal{J} = \Big\{(x_i) \in \mathcal{M}(= l_\infty(\mathbb{N},\mathcal{R}) = \sum_{i\in\mathcal{I}} \oplus\mathcal{R}_i) : \lim_{i\to\omega} \tau_i(x_i^* x_i) = 0\Big\}$$

be a closed two sided ideal of $\mathcal{M}$. By Theorem 7.1 in [15], $\mathcal{M}/\mathcal{J}$ is a type II_1 factor. By Remark 12 in [8], $d(\mathcal{M}/\mathcal{J}) \geqslant 3$.

Let $q : \mathcal{M} \to \mathcal{M}/\mathcal{J}$ be the quotient map. For any element $(x_i) \in \mathcal{M}$, by Kaplansky's density theorem, there exists an element $(\widetilde{x}_i) \in \mathcal{A}$ such that $q((\widetilde{x}_i)) = q((x_i))$. In other words, $q(\mathcal{A}) = \mathcal{M}/\mathcal{J}$. By Remark 6 in [10], we get that $d(\mathcal{A}) \geqslant d(\mathcal{M}/\mathcal{J})$. Combining with the result from the preceding paragraph, we conclude that $d(\mathcal{A}) \geqslant 3$.

Therefore $d(l_\infty(\mathcal{I}, C)) = d(\mathcal{A}) = 3$, when $\mathcal{I}$ is an infinite set. ∎

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