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# A NEW NECESSARY CONDITION FOR THE HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. A well known result of C. Cowen states that, for a symbol $\varphi \in L^\infty, \varphi \equiv \overline{f} + g$ $(f, g \in H^2)$, the Toeplitz operator $T_\varphi$ acting on the Hardy space of the unit circle is hyponormal if and only if $f = c + T_{\overline{h}}g$, for some $c \in \mathbb{C}$, $h \in H^\infty$, $\|h\|_\infty \leqslant 1$. In this note we consider possible versions of this result in the *Bergman* space case. Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+$, $m < n$ and $p < q$. By studying the asymptotic behavior of the action of $T_\varphi$ on a particular sequence of vectors, we obtain a sharp inequality involving the above mentioned data. This inequality improves a number of existing results, and it is intended to be a precursor of basic necessary conditions for joint hyponormality of tuples of Toeplitz operators acting on Bergman spaces in one or several complex variables.

KEYWORDS: *Hyponormality, Toeplitz operators, Bergman space, commutators.*

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## NOTATION AND PRELIMINARIES

A bounded operator acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$ is said to be normal if $T^*T = TT^*$; quasinormal if $T$ commutes with $T^*T$; subnormal if $T = N|_{\mathcal{H}}$, where $N$ is normal on a Hilbert space $\mathcal{K}$ which contains $\mathcal{H}$ and $N\mathcal{H} \subseteq \mathcal{H}$; hyponormal if $T^*T \geqslant TT^*$; and 2-hyponormal if $(T, T^2)$ is (jointly) hyponormal, that is

$$\begin{pmatrix} [T^*, T] & [T^{*2}, T] \\ [T^*, T^2] & [T^{*2}, T^2] \end{pmatrix} \geqslant 0.$$

Clearly,

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{2-hyponormal} \Rightarrow \text{hyponormal}.$$

In this paper we focus primarily on the cases $H^2(\mathbb{T})$ and $A^2(\mathbb{D})$, the Hardy space on the unit circle $\mathbb{T}$ and the Bergman space on the unit disk $\mathbb{D}$, respectively. For these Hilbert spaces, we look at Toeplitz operators, that is, the operators obtained by compressing multiplication operators on the respective $L^2$-spaces to the above mentioned Hilbert spaces. We consider possible versions, in the Bergman space context, of C. Cowen's characterization of hyponormality for Toeplitz operators on Hardy space of the unit circle. Concretely, we consider Toeplitz operators on the Bergman space of the unit disk, with symbols of the form

$$\varphi \equiv \alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $m, n, p, q \in \mathbb{Z}_+$, $m < n$ and $p < q$. By letting $T_\varphi$ act on vectors of the form

$$z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

we study the asymptotic behavior of a suitable matrix of inner products, as $k \to \infty$. As a result, we obtain a sharp inequality involving the above mentioned data. We begin with a brief survey of the known results in the Hardy space context.

## 1. THE HARDY SPACE CASE

Let $L^2(\mathbb{T})$ denote the space of square integrable functions with respect to the Lebesgue measure on the unit circle, and let $H^2(\mathbb{T})$ denote the subspace consisting of functions with vanishing negative Fourier coefficients; equivalently, $H^2(\mathbb{T})$ is the $L^2(\mathbb{T})$-closure of the space of analytic polynomials. We also let $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ denote the corresponding Banach spaces of essentially bounded functions on $\mathbb{T}$. The orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$ will be denoted by $P$.

Given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator with *symbol* $\varphi$ acting on the Hardy space is $T_\varphi : H^2(\mathbb{T}) \to H^2(\mathbb{T})$, given by $T_\varphi f := P(\varphi f)$ $(f \in H^2(\mathbb{T}))$. $T_\varphi$ is said to be *analytic* if $\varphi \in H^\infty(\mathbb{T})$.

P.R. Halmos's problem 5 ([8]) asks whether every subnormal Toeplitz operator is either normal or analytic. In 1984, C. Cowen and J. Long answered this question in the negative [4]. Along the way, C. Cowen obtained a characterization of hyponormality for Toeplitz operators, as follows [3]: if $\varphi \in L^\infty$, $\varphi = \overline{f} + g$ $(f, g \in H^2)$, then $T_\varphi$ is hyponormal $\Leftrightarrow f = c + T_{\overline{h}} g$, for some $c \in \mathbb{C}$, $h \in H^\infty$, and $\|h\|_\infty \leqslant 1$. T. Nakazi and K. Takahashi [11] later found an alternative description: for $\varphi \in L^\infty$, let $\mathcal{E}(\varphi) := \{k \in H^\infty : \|k\|_\infty \leqslant 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\}$; then $T_\varphi$ is hyponormal $\Leftrightarrow \mathcal{E}(\varphi) \neq \emptyset$. (For a generalization of Cowen's result, see [7].) In this note we take a first step toward finding suitable generalizations of these results to the case of Toeplitz operators on the Bergman space over the unit disk. We also wish to pursue appropriate generalizations of the results on joint hyponormality of pairs of Toeplitz operators on the Hardy space, obtained in [6] and [5].

At present, there is no known characterization of subnormality of Toeplitz operators in the unit circle in terms of the symbol. However, we do know that every 2-hyponormal Toeplitz operator with a *trigonometric* symbol is subnormal [6]. Thus, a suitable intermediate goal is to find a characterization of 2-hyponormality in terms of the symbol, perhaps using as a starting point either Cowen's or Nakazi–Takahashi's characterizations of hyponormality.

For Toeplitz operators with trigonometric symbols, the following result gives a flavor of what is known about hyponormality.

PROPOSITION 1.1 ([14]). *Suppose*

$$\varphi(z) \equiv \sum_{k=0}^{n} a_k z^k + \overline{\sum_{k=0}^{n} b_k z^k},$$

*with* $a_n \neq 0$. *Let*

$$\begin{pmatrix} \bar{c}_0 \\ \bar{c}_1 \\ \vdots \\ \bar{c}_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ a_2 & a_3 & \cdots & a_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

*Then* $T_\varphi$ *is hyponormal if and only if* $|\Phi_k(c_0, \dots, c_k)| \leqslant 1$ $(0 \leqslant k \leqslant n-1)$, *where* $\Phi_k$ *denotes the Schur function introduced in* [13].

## 2. THE BERGMAN SPACE CASE

By analogy with the case of the unit circle, let $L^\infty \equiv L^\infty(\mathbb{D})$, $H^\infty \equiv H^\infty(\mathbb{D})$, $L^2 \equiv L^2(\mathbb{D})$ and $A^2 \equiv A^2(\mathbb{D})$ denote the relevant spaces in the case of the unit disk $\mathbb{D}$. Similarly, let $P : L^2 \to A^2$ denote the orthogonal projection onto the Bergman space. For $\varphi \in L^\infty$, the Toeplitz operator on the Bergman space with symbol $\varphi$ is

$$T_\varphi : A^2(\mathbb{D}) \to A^2(\mathbb{D}),$$

given by

$$T_\varphi f := P(\varphi f) \quad (f \in A^2).$$

$T_\varphi$ is said to be *analytic* if $\varphi \in H^\infty$.

### 2.1. A REVEALING EXAMPLE.

Let

$$\varphi \equiv \bar{z}^2 + 2z.$$

On the Hardy space $H^2(\mathbb{T})$, $T_\varphi$ is not hyponormal. However, on the Bergman space $A^2(\mathbb{D})$ $T_\varphi$ is hyponormal, as we now prove. Consider a slight variation of the symbol, that is,

$$\varphi \equiv \bar{z}^2 + \alpha z \quad (\alpha \in \mathbb{C}).$$

Observe that

$$\langle [T_\varphi^*, T_\varphi] f, f\rangle = \langle |\alpha|^2 [T_{\overline{z}}, T_z] + [T_{z^2}, T_{\overline{z}^2}] f, f\rangle$$

so that $T_\varphi$ is hyponormal if and only if

$$|\alpha|^2 \|zf\|^2 + \langle P(\overline{z}^2 f), \overline{z}^2 f\rangle \geqslant |\alpha|^2 \langle P(\overline{z} f), \overline{z} f\rangle + \|z^2 f\|^2 \tag{2.1}$$

for all $f \in A^2(\mathbb{D})$.

A calculation now shows that this happens precisely when $|\alpha| \geqslant 2$, as follows. For, given $f \in A^2(\mathbb{D}), f \equiv \sum\limits_0^\infty b_n z^n$, one can apply Lemma 2.1 beow and obtain

$$\|zf\|^2 = \sum_0^\infty \frac{|b_n|^2}{n+2}, \quad \|P(\overline{z}f)\|^2 = \sum_1^\infty |b_n|^2 \frac{n}{(n+1)^2},$$
$$\|z^2 f\|^2 = \sum_0^\infty \frac{|b_n|^2}{n+3}, \quad \text{and} \quad \|P(\overline{z}^2 f)\|^2 = \sum_2^\infty |b_n|^2 \frac{n-1}{(n+1)^2}.$$

Thus, (2.1) becomes

$$|\alpha|^2 \sum_0^\infty \frac{|b_n|^2}{n+2} + \sum_2^\infty |b_n|^2 \frac{n-1}{(n+1)^2} \geqslant |\alpha|^2 \sum_1^\infty |b_n|^2 \frac{n}{(n+1)^2} + \sum_0^\infty \frac{|b_n|^2}{n+3}. \tag{2.2}$$

In short, equation (2.2) must hold for every sequence $(b_n)$ of coefficients of $f$. Consider first a sequence $(b_n)$ with $b_0 := 1$ and $b_n := 0$ for all $n \geqslant 1$. By (2.1), we have $|\alpha|^2 \geqslant \frac{2}{3}$. Next, take $b_0 := 0, b_1 := 1$ and $b_n := 0$ for all $n \geqslant 2$; then (2.1) yields $|\alpha|^2 \geqslant 3$. Finally, if we fix $k \geqslant 2$ and we use a sequence $(b_n)$ defined as $b_0 := 0, b_1 := 0, \ldots, b_{k-1} := 0, b_k := 1$ and $b_n := 0$ for all $n > k$, then (2.2) becomes

$$\frac{|\alpha|^2}{k+2} + \frac{k-1}{(k+1)^2} \geqslant |\alpha|^2 \frac{k}{(k+1)^2} + \frac{1}{k+3}.$$

This immediately leads to the condition

$$|\alpha|^2 \geqslant 4\frac{k+2}{k+3} \quad (\text{all } k \geqslant 2);$$

that is, $|\alpha|^2 \geqslant 4$. As a result, $T_\varphi$ is hyponormal if and only if $|\alpha| \geqslant 2$. It follows that $T_{\overline{z}^2+2z}$ is hyponormal.

### 2.2. A KEY DIFFERENCE BETWEEN THE HARDY AND BERGMAN CASES.

LEMMA 2.1. *For $u, v \geqslant 0$, we have*

$$P(\overline{z}^u z^v) = \begin{cases} 0 & v < u, \\ \frac{(v-u+1)}{v+1} z^{v-u} & v \geqslant u. \end{cases}$$

*Proof.*

$$\begin{aligned}
P(\overline{z}^u z^v) &= \sum_{j=0}^{\infty} \left\langle \overline{z}^u z^v, \frac{z^j}{\|z^j\|} \right\rangle \frac{z^j}{\|z^j\|} \\
&= \sum_{j=0}^{\infty} \frac{\langle \overline{z}^u z^v, z^j \rangle z^j}{\|z^j\|^2} = \sum_{j=0}^{\infty} (j+1) \langle z^v, z^{u+j} \rangle z^j \\
&= \begin{cases} 0 & v < u, \\ \frac{v-u+1}{v+1} z^{v-u} & v \geqslant u. \end{cases} \qquad \blacksquare
\end{aligned}$$

COROLLARY 2.2. *For $v \geqslant u$ and $t \geqslant w$, we have*

$$\langle P(\overline{z}^u z^v), P(\overline{z}^w z^t) \rangle = \left\langle \frac{v-u+1}{v+1} z^{v-u}, \frac{t-w+1}{t+1} z^{t-w} \right\rangle = \frac{(t-w+1)}{(v+1)(t+1)} \delta_{u+t,v+w}.$$

2.3. SOME KNOWN RESULTS. In this subsection, we briefly summarize a number of partial results relating to the Bergman space case.

(•) (H. Sadraoui [12]). If $\varphi \equiv \overline{g} + f$, the following are equivalent:

(i) $T_\varphi$ is hyponormal on $A^2(\mathbb{D})$;

(ii) $H_{\overline{g}}^* H_{\overline{g}} \leqslant H_{\overline{f}}^* H_{\overline{f}}$;

(iii) $H_{\overline{g}} = C H_{\overline{f}}$, where $C$ is a contraction on $A^2(\mathbb{D})$.

(•) (I.S. Hwang [9]). Let $\varphi \equiv a_{-m}\overline{z}^m + a_{-N}\overline{z}^N + a_m z^m + a_N z^N$ $(0 < m < N)$ satisfying $a_m \overline{a}_N = \overline{a}_{-m} a_{-N}$, then $T_\varphi$ is hyponormal if and only if

$$\frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geqslant \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) \quad \text{if } |a_{-N}| \leqslant |a_N|,$$
$$N^2(|a_{-N}|^2 - |a_N|^2) \leqslant m^2(|a_m|^2 - |a_{-m}|^2) \quad \text{if } |a_N| \leqslant |a_{-N}|.$$

The last condition is not sufficient.

(•) (P. Ahern and Ž. Čučković [1]). Let $\varphi \equiv \overline{g} + f \in L^\infty(\mathbb{D})$, and assume that $T_\varphi$ is hyponormal. Then

$$Bu \geqslant u,$$

where $B$ denotes the Berezin transform and $u := |f|^2 - |g|^2$.

(•) (P. Ahern and Ž. Čučković [1]). Let $\varphi \equiv \overline{g} + f \in L^\infty(\mathbb{D})$, and assume that $T_\varphi$ is hyponormal. Then

$$\limsup_{z \to \zeta} (|f'(z)|^2 - |g'(z)|^2) \geqslant 0$$

for all $\zeta \in \mathbb{T}$. In particular, if $f'$ and $g'$ are continuous at $\zeta \in \mathbb{T}$ then $|f'(\zeta)| \geqslant |g'(\zeta)|$.

(•) (Y. Lu and Y. Shi [10]). The authors study the weighted Bergman space case, and prove the following result (cf. Theorem 2.4(ii) of [10]): let $\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^m + \delta \overline{z}^n$, with $n > m$. Then $m^2(|\beta|^2 - |\gamma|^2) + n^2(|\alpha|^2 - |\delta|^2) \geqslant mn|\overline{\alpha}\beta - \overline{\gamma}\delta|$.

2.4. HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE. The self-commutator of $T_\varphi$ is

$$C := [T_\varphi^*, T_\varphi].$$

We seek necessary and sufficient conditions on the symbol $\varphi$ to ensure that $C \geqslant 0$.

The next result gives a flavor of the type of calculations we face when trying to decipher the hyponormality of a Toeplitz operator on the Bergman space. Although the calculation therein will be superseded by the calculations in the following section, it serves both as a preliminary example and as motivation for the organization of our work.

PROPOSITION 2.3. *Assume* $k, \ell \geqslant \max\{a, b\}$. *Then*

$$\begin{aligned}\langle [T_{\overline{z}^a}, T_{z^b}](z^k+cz^\ell), z^k+cz^\ell\rangle = a^2\Big[\frac{1}{(k+1)^2(k+1+a)} + c^2\frac{1}{(\ell+1)^2(\ell+1+a)}\Big]\delta_{a,b}\\ + ac\Big[\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)}\delta_{a+k,b+\ell}\\ + \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)}\delta_{a+\ell,b+k}\Big].\end{aligned}$$

*Proof.* Keeping in mind that $k, \ell \geqslant \max\{a, b\}$, we calculate the action of the commutator on the binomial $z^k + cz^\ell$:

$$\begin{aligned}&\langle [T_{\overline{z}^a}, T_{z^b}](z^k + cz^\ell), z^k + cz^\ell\rangle\\
&= \langle T_{\overline{z}^a} T_{z^b}(z^k + cz^\ell), z^k + cz^\ell\rangle - \langle T_{z^b} T_{\overline{z}^a}(z^k + cz^\ell), z^k + cz^\ell\rangle\\
&= \langle z^{b+k} + cz^{b+\ell}, z^{a+k} + cz^{a+\ell}\rangle - \langle P(\overline{z}^a z^k + c\overline{z}^a z^\ell), P(\overline{z}^b z^k + c\overline{z}^b z^\ell)\rangle\\
&= \frac{\delta_{a,b}}{a+k+1} + c\frac{\delta_{a+k,b+\ell}}{a+k+1} + c\frac{\delta_{a+\ell,b+k}}{a+\ell+1} + c^2\frac{\delta_{a,b}}{a+\ell+1}\\
&\quad - \frac{(k-b+1)}{(k+1)^2}\delta_{a,b} - c\frac{(k-a+1)}{(k+1)(\ell+1)}\delta_{a+\ell,b+k}\\
&\quad - c\frac{(k-b+1)}{(k+1)(\ell+1)}\delta_{a+k,b+\ell} - c^2\frac{(\ell-b+1)}{(\ell+1)^2}\delta_{a,b}\\
&= \Big[\frac{1}{a+k+1} + c^2\frac{1}{a+\ell+1} - \frac{(k-b+1)}{(k+1)^2} - c^2\frac{(\ell-b+1)}{(\ell+1)^2}\Big]\delta_{a,b}\\
&\quad + c\Big[\frac{1}{a+k+1} - \frac{(k-b+1)}{(k+1)(\ell+1)}\Big]\delta_{a+k,b+\ell}\\
&\quad + c\Big[\frac{1}{a+\ell+1} - \frac{(k-a+1)}{(k+1)(\ell+1)}\Big]\delta_{a+\ell,b+k}\\
&= \Big[\frac{(k+1)(b-a)+ab}{(k+1)^2(k+1+a)} + c^2\frac{(\ell+1)(b-a)+ab}{(\ell+1)^2(\ell+1+a)}\Big]\delta_{a,b}\\
&\quad + c\Big[\frac{1}{a+k+1} - \frac{(\ell-a+1)}{(k+1)(\ell+1)}\Big]\delta_{a+k,b+\ell}\end{aligned}$$

$$
\begin{aligned}
&+ c\Big[\frac{1}{a+\ell+1} - \frac{(k-a+1)}{(k+1)(\ell+1)}\Big]\delta_{a+\ell,b+k} \\
= &\Big[\frac{a^2}{(k+1)^2(k+1+a)} + c^2\frac{a^2}{(\ell+1)^2(\ell+1+a)}\Big]\delta_{a,b} \\
&\text{(if } a \neq b \text{ then this whole expression is 0)} \\
&+ ac\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)}\delta_{a+k,b+\ell} \\
&+ ac\frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)}\delta_{a+\ell,b+k} \\
= &\, a^2\Big[\frac{1}{(k+1)^2(k+1+a)} + c^2\frac{1}{(\ell+1)^2(\ell+1+a)}\Big]\delta_{a,b} \\
&+ ac\Big[\frac{k-\ell+a}{(a+k+1)(k+1)(\ell+1)}\delta_{a+k,b+\ell} \\
&+ \frac{\ell-k+a}{(a+\ell+1)(k+1)(\ell+1)}\delta_{a+\ell,b+k}\Big],
\end{aligned}
$$

as desired. ∎

COROLLARY 2.4. *Assume $a = b$, $k, \ell \geqslant a$ and $k \neq \ell$. Then*

$$\langle [T_{\overline{z}^a}, T_{z^a}](z^k + cz^\ell), z^k + cz^\ell\rangle = a^2\Big[\frac{1}{(k+1)^2(k+1+a)} + c^2\frac{1}{(\ell+1)^2(\ell+1+a)}\Big].$$

2.5. SELF-COMMUTATORS. We focus on the action of the self-commutator $C$ of certain Toeplitz operators $T_\varphi$ on suitable vectors $f$ in the space $A^2(\mathbb{D})$. The symbol $\varphi$ and the vector $f$ are of the form

$$
\begin{aligned}
&\varphi := \alpha z^n + \beta z^m + \gamma\overline{z}^p + \delta\overline{z}^q \quad (n > m; p < q), \quad \text{and} \\
&f := z^k + cz^\ell + dz^r \quad (k < \ell < r),
\end{aligned}
$$

respectively, with $\ell$ and $r$ to be determined later. We also assume that $n - m = q - p$. Our ultimate goal is to study the asymptotic behavior of this action as $k$ goes to infinity. Thus, we consider the expression $\langle Cf, f\rangle$, given by

$$\langle [(T_{\alpha z^n+\beta z^m+\gamma\overline{z}^p+\delta\overline{z}^q})^*, T_{\alpha z^n+\beta z^m+\gamma\overline{z}^p+\delta\overline{z}^q}](z^k + cz^\ell + dz^r), z^k + cz^\ell + dz^r\rangle,$$

for large values of $k$ (and consequently large values of $\ell$ and $r$). It is straightforward to see that $\langle Cf, f\rangle$ is a quadratic form in $c$ and $d$, that is,

$$\langle Cf, f\rangle \equiv A_{00} + 2\mathrm{Re}(A_{10}c) + 2\mathrm{Re}(A_{01}d) + A_{20}c\overline{c} + 2\mathrm{Re}(A_{11}\overline{c}d) + A_{02}d\overline{d}. \tag{2.3}$$

Alternatively, the matricial form of (2.3) is

$$\left\langle \begin{pmatrix} A_{00} & A_{10} & A_{01} \\ \overline{A}_{10} & A_{20} & A_{11} \\ \overline{A}_{01} & \overline{A}_{11} & A_{02} \end{pmatrix} \begin{pmatrix} 1 \\ c \\ d \end{pmatrix}, \begin{pmatrix} 1 \\ c \\ d \end{pmatrix} \right\rangle. \tag{2.4}$$

We now observe that the coefficient $A_{00}$ arises from the action of $C$ on the monomial $z^k$, that is,

$$A_{00} = \langle Cz^k, z^k \rangle \equiv \langle [(T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q})^*, T_{\alpha z^n + \beta z^m + \gamma \bar{z}^p + \delta \bar{z}^q}] z^k, z^k \rangle.$$

Similarly,

$$\begin{aligned} & A_{10} = \langle Cz^\ell, z^k \rangle, \\ (2.5) \qquad & A_{01} = \langle Cz^r, z^k \rangle, \\ & A_{20} = \langle Cz^\ell, z^\ell \rangle, \\ (2.6) \qquad & A_{11} = \langle Cz^r, z^\ell \rangle, \\ & A_{02} = \langle Cz^r, z^r \rangle. \end{aligned}$$

To calculate $A_{00}$ explicitly, we first recall that the algebra of Toeplitz operators with analytic symbols is commutative, and therefore $T_{z^n}$ commutes with $T_{z^m}$, $T_{z^p}$ and $T_{z^q}$.

We also recall that two monomials $z^u$ and $z^v$ are orthogonal whenever $u \neq v$. As a result, the only nonzero contributions to $A_{00}$ must come from the inner products $\langle [T^*_{z^n}, T_{z^n}] z^k, z^k \rangle$, $\langle [T^*_{z^m}, T_{z^m}] z^k, z^k \rangle$, $\langle [T^*_{z^p}, T_{z^p}] z^k, z^k \rangle$ and $\langle [T^*_{z^q}, T_{z^q}] z^k, z^k \rangle$ .

Applying Corollary 2.4 we see that

$$A_{00} = \frac{1}{(k+1)^2} \Big( \frac{|\alpha|^2 n^2}{k+n+1} + \frac{|\beta|^2 m^2}{k+m+1} - \frac{|\gamma|^2 p^2}{k+p+1} - \frac{|\delta|^2 q^2}{k+q+1} \Big).$$

Similarly,

$$\begin{aligned} A_{10} = {} & \bar{\alpha}\beta \Big( \frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)} \Big) \delta_{n+k,m+\ell} \\ & + \alpha\bar{\beta} \Big( \frac{1}{\ell+n+1} - \frac{k-n+1}{(k+1)(\ell+1)} \Big) \delta_{m+k,n+\ell} \\ & - \bar{\gamma}\delta \Big( \frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)} \Big) \delta_{q+k,p+\ell} \\ & - \gamma\bar{\delta} \Big( \frac{1}{\ell+q+1} - \frac{k-q+1}{(k+1)(\ell+1)} \Big) \delta_{p+k,q+\ell}. \qquad (2.7) \end{aligned}$$

Now recall that $m < n$ and $k < \ell$, so that $m + k < n + \ell$, and therefore $\delta_{m+k,n+\ell} = 0$. Also, $p < q$ implies $p + k < q + \ell$, so that $\delta_{p+k,q+\ell} = 0$. As a consequence,

$$\begin{aligned} A_{10} = {} & \bar{\alpha}\beta \Big( \frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)} \Big) \delta_{n+k,m+\ell} \\ & - \bar{\gamma}\delta \Big( \frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)} \Big) \delta_{q+k,p+\ell}. \qquad (2.8) \end{aligned}$$

Consider now $A_{01}$, as described in (2.5). We wish to imitate the calculation for $A_{10}$. Observe that $k < r$, so that the vanishing of the relevant $\delta$'s in (2.7) still holds. Thus, we obtain

$$A_{01} = \overline{\alpha}\beta\Big(\frac{1}{r+m+1} - \frac{k-m+1}{(k+1)(r+1)}\Big)\delta_{n+k,m+r} - \overline{\gamma}\delta\Big(\frac{1}{r+p+1} - \frac{k-p+1}{(k+1)(r+1)}\Big)\delta_{q+k,p+r}. \tag{2.9}$$

In short, $A_{01}$ can be obtained from $A_{10}$ by replacing $\ell$ by $r$. In a completely analogous way, we can calculate $A_{11}$, by replacing $\ell$ by $r$ and $k$ by $\ell$ in (2.7):

$$A_{11} = \overline{\alpha}\beta\Big(\frac{1}{r+m+1} - \frac{\ell-m+1}{(\ell+1)(r+1)}\Big)\delta_{n+\ell,m+r} - \overline{\gamma}\delta\Big(\frac{1}{r+p+1} - \frac{\ell-p+1}{(r+1)(\ell+1)}\Big)\delta_{q+\ell,p+r}. \tag{2.10}$$

Also, $A_{20}$ and $A_{02}$ follow the pattern of $A_{00}$:

$$A_{20} = \frac{1}{(\ell+1)^2}\Big(\frac{|\alpha|^2n^2}{\ell+n+1} + \frac{|\beta|^2m^2}{\ell+m+1} - \frac{|\gamma|^2p^2}{\ell+p+1} - \frac{|\delta|^2q^2}{\ell+q+1}\Big), \quad \text{and}$$
$$A_{02} = \frac{1}{(r+1)^2}\Big(\frac{|\alpha|^2n^2}{r+n+1} + \frac{|\beta|^2m^2}{r+m+1} - \frac{|\gamma|^2p^2}{r+p+1} - \frac{|\delta|^2q^2}{r+q+1}\Big).$$

Recall again that $k < \ell < r$. We now make a judicious choice to simplify the forms of $A_{10}$, $A_{11}$ and $A_{01}$. That is, we let $\ell := n+k-m$ and $r := \ell+q-p$. It follows that $n+k = m+\ell < m+r$ and $q+k < q+\ell = p+r$. Therefore, both Kronecker deltas appearing in $A_{01}$ are zero, and thus $A_{01} = 0$. Moreover,

$$A_{10} = \overline{\alpha}\beta\Big(\frac{1}{\ell+m+1} - \frac{k-m+1}{(k+1)(\ell+1)}\Big) - \overline{\gamma}\delta\Big(\frac{1}{\ell+p+1} - \frac{k-p+1}{(k+1)(\ell+1)}\Big)\delta_{q+k,p+\ell} \quad \text{and} \tag{2.11}$$

$$A_{11} = \overline{\alpha}\beta\Big(\frac{1}{r+m+1} - \frac{\ell-m+1}{(r+1)(\ell+1)}\Big)\delta_{n+\ell,m+r} - \overline{\gamma}\delta\Big(\frac{1}{r+p+1} - \frac{\ell-p+1}{(r+1)(\ell+1)}\Big). \tag{2.12}$$

The $3\times 3$ matrix associated with $C$ becomes

$$M := \begin{pmatrix} A_{00} & A_{10} & 0 \\ \overline{A_{10}} & A_{20} & A_{11} \\ 0 & \overline{A_{11}} & A_{02} \end{pmatrix}.$$

We now wish to study the asymptotic behavior of $k^3M$ as $k \to \infty$. Surprisingly, $k^3A_{00}$, $k^3A_{02}$ and $k^3A_{20}$ all have the same limit as $k \to \infty$. Also, $k^3A_{10}$ and $k^3\overline{A_{11}}$ have the same limit. To see this, observe that

$$k^3A_{00} = \frac{k^2}{(k+1)^2}\Big(\frac{k|\alpha|^2n^2}{k+n+1} + \frac{k|\beta|^2m^2}{k+m+1} - \frac{k|\gamma|^2p^2}{k+p+1} - \frac{k|\delta|^2q^2}{k+q+1}\Big),$$

so that

$$a := \lim_{k\to\infty} k^3 A_{00} = |\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2.$$

Then $\lim_{k\to\infty} k^3 A_{20} = \lim_{k\to\infty} k^3 A_{02} = a$. In terms of the remaining entries of $k^3 M$, recall the assumption $n - m = q - p$, and let $g := n - m = q - p$. It follows that $\ell = k + g$ and $r = \ell + g = k + 2g$. By using these values in (2.11), we obtain

$$\begin{aligned} k^3 A_{10} = {} & \overline{\alpha}\beta \frac{k^3 mn}{(k+1)(k+g+1)(k+g+m+1)} \\ & - \overline{\gamma}\delta \frac{k^3 pq}{(k+1)(k+g+1)(k+g+p+1)}, \end{aligned}$$

so that

$$\rho := \lim_{k\to\infty} k^3 A_{10} = \overline{\alpha}\beta mn - \overline{\gamma}\delta pq.$$

The calculation for $A_{11}$ is entirely similar, and one gets $\lim_{k\to\infty} k^3 A_{11} = \rho$.

It follows that the asymptotic behavior of $k^3 M$ is given by the tridiagonal matrix

$$\begin{pmatrix} a & \rho & 0 \\ \overline{\rho} & a & \rho \\ 0 & \overline{\rho} & a \end{pmatrix}.$$

Now, if instead of using a vector of the form

$$f := z^k + cz^\ell + dz^r \quad (k < \ell < r),$$

with $\ell = k + g$ and $r = \ell + g = k + 2g$ (that is, a vector of the form $f := z^k + cz^{k+g} + dz^{k+2g}$) we were to use a longer vector with similar power structure,

$$f_N := z^k + c_1 z^{k+g} + c_2 z^{k+2g} + \cdots + c_N z^{k+Ng},$$

it is not hard to see that the asymptotic behavior of the associated matrix would still be given by the tridiagonal matrix with $a$ in the diagonal and $\rho$ in the super-diagonal. To see this, one only need to observe that the entries of the matrix $P$ associated with $\langle Cf, f\rangle$ will follow the same pattern as the entries in $M$. For example, when $N = 3$ the $(3,4)$-entry of $P$ will follow the pattern of $A_{10}$ above, with $\ell$ and $k$ replaced by $k + 3g$ and $k + 2g$, respectively. Similarly, the $(2,4)$-entry of $P$ will follow the pattern of $A_{01}$ above, with $r$ and $k$ replaced by $k + 3g$ and $k + g$, respectively. As a result, it is straightforward to see that, like $A_{01}$, the entry $P_{24}$ will be zero. As for $P_{34}$, one gets

$$\begin{aligned} P_{34} = {} & \overline{\alpha}\beta \Big( \frac{1}{k+3g+m+1} - \frac{k+2g-m+1}{(k+2g+1)(k+3g+1)} \Big) \delta_{n+k+2g,m+k+3g} \\ & - \overline{\gamma}\delta \Big( \frac{1}{k+3g+p+1} - \frac{k+2g-p+1}{(k+3g+1)(k+2g+1)} \Big) \delta_{q+k+2g,p+k+3g}. \end{aligned}$$

As before,

$$k^3 P_{34} = \overline{\alpha}\beta \frac{k^3 mn}{(k+2g+1)(k+3g+1)(k+3g+m+1)} \\ - \overline{\gamma}\delta \frac{k^3 pq}{(k+2g+1)(k+3g+1)(k+3g+p+1)},$$

and once again,

$$\lim_{k\to\infty} k^3 P_{34} = \overline{\alpha}\beta mn - \overline{\gamma}\delta pq = \rho.$$

In summary, the hyponormality of $T_\varphi$, detected by the positivity of the self-commutator $C$, leads to the positive semi-definiteness of the tridiagonal matrix $P$ of size $(N+1)\times(N+1)$. Since this must be true for all $N \geqslant 1$, it follows that a necessary condition for the hyponormality of $T_\varphi$ is the positive semi-definiteness of the infinite tridiagonal matrix

$$Q := \begin{pmatrix} a & \rho & 0 & \cdots \\ \overline{\rho} & a & \rho & \cdots \\ 0 & \overline{\rho} & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We now consider the spectral behavior of $Q$ as an operator on $\ell^2(\mathbb{Z}_+)$.

LEMMA 2.5. *For $a \in \mathbb{R}$ and $\rho \in \mathbb{C}$, the spectrum of the infinite tridiagonal matrix $Q$ is $[a-2|\rho|, a+2|\rho|]$.*

*Proof.* This result is well known; we present a proof for the sake of completeness. Observe that $Q$ is the canonical matrix representation of the Toeplitz operator on $H^2(\mathbb{T})$ with symbol $\varphi(z) := a + 2\mathrm{Re}(\overline{\rho}z)$. Since the symbol is harmonic, it follows that the spectrum of $T_\varphi \equiv aI + T_{\overline{\rho}z+\rho\overline{z}}$ is the set $a + 2\mathrm{Re}(\{\overline{\rho}z : z \in \mathbb{D}\}^-) = a + 2[-|\rho|, |\rho|]$, as desired. ∎

As a consequence, if $Q$ is positive (as an operator on $\ell^2(\mathbb{Z}_+)$), then

$$a \geqslant 2|\rho|.$$

2.6. MAIN RESULT.

THEOREM 2.6. *Assume that $T_\varphi$ is hyponormal on $A^2(\mathbb{D})$, with*

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \quad (n > m; p < q).$$

*Assume also that $n - m = q - p$. Then*

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geqslant 2|\overline{\alpha}\beta mn - \overline{\gamma}\delta pq|. \tag{2.13}$$

2.7. A SPECIFIC CASE. When $p = m$ and $q = n$ in

$$\varphi := \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q \quad (n > m; p < q)$$

the inequality

$$|\alpha|^2 n^2 + |\beta|^2 m^2 - |\gamma|^2 p^2 - |\delta|^2 q^2 \geqslant 2|\overline{\alpha}\beta mn - \overline{\gamma}\delta pq|$$

reduces to

$$n^2(|\alpha|^2 - |\delta|^2) + m^2(|\beta|^2 - |\gamma|^2) \geqslant 2mn|\overline{\alpha}\beta - \overline{\gamma}\delta|.$$

This not only generalizes previous estimates, but also sharpens them, since previous results did not include the factor 2 in the right-hand side.

## 3. WHEN IS $T_\varphi$ NORMAL?

We conclude this paper with a description of those symbols $\varphi$ in Theorem 2.6 which produce a normal operator $T_\varphi$. We first recall a result of S. Axler and Ž. Čučković.

LEMMA 3.1 ([2]). *Let $\varphi$ be harmonic and bounded on $\mathbb{D}$. Then $T_\varphi$ is normal if and only if there exist a pair of complex numbers $a$ and $b$ such that $(a, b) \neq (0, 0)$ and $F := a\varphi + b\overline{\varphi}$ is constant on $\mathbb{D}$.*

Assume now that $T_\varphi$ is normal. By Lemma 3.1, there exist $a$ and $b$ such that $(a, b) \neq (0, 0)$ and $F := a\varphi + b\overline{\varphi}$ is constant. In what follows, we write a harmonic symbol as $\varphi \equiv f + \overline{g}$, with $f$ and $g$ analytic. A straightforward calculation using $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$, applied to $F$, shows that $(|a|^2 - |b|^2)\frac{\partial f}{\partial z} = 0$. If $f$ is constant, a similar calculation shows that $g$ is also constant, and a fortiori $\varphi$ is constant. Thus, without loss of generality, we can assume that $f$ is not constant, and therefore $|a| = |b| > 0$. If we write $a = |a|\mathrm{e}^{\mathrm{i}\theta}$ and $b = |a|\mathrm{e}^{\mathrm{i}\eta}$, it is not hard to see that $\varphi + \mathrm{e}^{\mathrm{i}(\eta-\theta)}\overline{\varphi}$ is constant on $\mathbb{D}$. Let $\psi := \lambda\varphi$, with $\lambda := \mathrm{e}^{-(\mathrm{i}/2)(\eta-\theta)}$. Let $\lambda := \mathrm{e}^{\mathrm{i}(\eta-\theta)}$, so that $|\lambda| = 1$. We conclude that $\varphi + \lambda\overline{\varphi}$ is constant on $\mathbb{D}$.

THEOREM 3.2. *Let $\varphi \equiv \alpha z^n + \beta z^m + \gamma \overline{z}^p + \delta \overline{z}^q$ (with $n < m, p < q, n - m = q - p$), and let $\lambda := \mathrm{e}^{\mathrm{i}(\eta-\theta)}$, as above. Then $T_\varphi$ is normal if and only if $\varphi$ is of one of exactly three types:*

(i) $\varphi = \alpha z^n - \lambda\overline{\alpha}\overline{z}^n$ *(when $n = p$);*

(ii) $\varphi = \alpha z^n + \beta z^m - \lambda\overline{\beta}\overline{z}^m - \lambda\overline{\alpha}\overline{z}^n$ *(when $m = p$); or*

(iii) $\varphi = \beta z^m - \lambda\overline{\beta}\overline{z}^m$ *(when $m = q$).*

*Proof.* ($\Rightarrow$) Assume that $T_\varphi$ is normal. From the discussion in the paragraph immediately preceding Theorem 3.2, we can always assume that $\varphi + \lambda\overline{\varphi}$ is constant on $\mathbb{D}$, for some $\lambda \in \mathbb{T}$. Since $\varphi$ is clearly nonconstant, we know that $G := \varphi + \lambda\overline{\varphi}$ is a constant trigonometric polynomial, with analytic monomials $z^m$, $z^n$, $z^p$ and $z^q$. Since $\varphi$ is a nonconstant harmonic function, in the above mentioned list of four monomials we must necessarily have at least two identical monomials. Since $m < n$, $p < q$ and $n - m = q - p$, we are led to consider the following three cases.

*Case* 1. $n = p$ (and therefore $m < n = p < q$); here

$$G = \beta z^m + (\alpha + \lambda\overline{\gamma})z^n + \lambda\overline{\delta}z^q + \lambda\overline{\beta z^m + (\alpha + \lambda\overline{\gamma})z^n + \lambda\overline{\delta}z^q},$$

from which it easily follows that $\beta = 0$, $\gamma = -\lambda\overline{\alpha}$ and $\delta = 0$. Then $\varphi = \alpha z^n - \lambda\overline{\alpha z^n}$, as desired.

*Case* 2. $m = p$ (and therefore $m = p < q = n$); here

$$G = (\alpha + \lambda\overline{\delta})z^n + (\beta + \lambda\overline{\gamma})z^m + \lambda\overline{(\alpha + \lambda\overline{\delta})z^n + (\beta + \lambda\overline{\gamma})z^m},$$

so that $\alpha + \lambda\overline{\delta} = 0$ and $\beta + \lambda\overline{\gamma} = 0$. It readily follows that $\delta = -\lambda\overline{\alpha}$ and $\gamma = -\lambda\overline{\beta}$. We then get $\varphi = \alpha z^n + \beta z^m - \lambda\overline{\alpha z^n + \beta z^m}$, as desired.

*Case* 3. $m = q$, which leads to $\varphi = \beta z^m - \lambda\overline{\beta z^m}$.

($\Leftarrow$) For the converse, observe that in each of the three representations we have $\varphi + \lambda\overline{\varphi} = 0$, which implies $T_\varphi^* = -\overline{\lambda}T_\varphi$. Therefore, $T_\varphi^*$ commutes with $T_\varphi$, so $T_\varphi$ is normal.

The proof is now complete. ∎

REMARK 3.3. The form of (i), (ii) and (iii) in Theorem 3.2 is entirely consistent with Theorem 2.6. For instance, consider Case 1: here $\beta = \delta = 0$ and $\gamma = -\overline{\alpha}$, so that both sides of (2.13) are equal to 0.

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ADDED IN PROOFS. Yang Wen (JiangXi University of Sciences and Technology, PRC) has recently brought to our attention that the inequality (2.13) can also be derived by a careful application of the results in [1], using an appropriate split of the symbol $\varphi$.