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COBURN–SIMONENKO THEOREM AND INVERTIBILITY OF TOEPLITZ OPERATORS ON THE SPACE OF REAL ANALYTIC FUNCTIONS

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ABSTRACT. We study Toeplitz operators on the space of real analytic functions. We prove that either such an operator or its adjoint is injective. This is an analog of the classical Coburn–Simonenko theorem. We also show that a Toeplitz operator on the space of real analytic functions is invertible if and only if it is a Fredholm operator of index zero.

KEYWORDS: *Toeplitz operator, space of real analytic functions, Fredholm operator, index, Cauchy transform.*

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1. INTRODUCTION

Toeplitz operators and their generalizations consitute one of the most important classes of operators. The classical theory of Toeplitz operators concerns the Hardy spaces $H^p(\mathbb{T})$. A Toeplitz operator on $H^p(\mathbb{T})$ is a bounded linear operator $T : H^p(\mathbb{T}) \to H^p(\mathbb{T})$ such that

$$(T\chi_j, \chi_k) = \int\limits_{\mathbb{T}} T(\chi_j)\overline{\chi}_k \mathrm{d}m = a_{k-j}, \quad j,k \in \mathbb{N}_0$$

for some sequence $\{a_n\}_{n\in\mathbb{Z}}$. We write here χ_n to denote the function $z \mapsto z^n$ restricted to $\mathbb{T}$. The systematic study of Toeplitz operators started with a result of Brown and Halmos [5] according to which an operator $T : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a Toeplitz operator if and only if there exists a function $\phi \in L^\infty$ such that

$$T = T_\phi := PM_\phi. \tag{1.1}$$

The symbol M_ϕ stands for the operator of multiplication by ϕ, which maps the space L^p into itself, and P is the Riesz projection, which on trigonometric polynomials is defined by the condition

$$P : \sum_{n=-N}^{N} c_n \chi_n \mapsto \sum_{n=0}^{N} c_n \chi_n.$$

A famous theorem of M. Riesz states that P is bounded in L^p norm if $1 < p < \infty$. The theory of Toeplitz operators on the Hardy spaces is now well-established. Books [4] and [20] are excellent in-depth monographs concerning this subject. The important formula (1.1) is easy to generalize to other function spaces such as the Bergman spaces [24] and the Fock spaces [25]. One simply substitutes the Riesz projection by other appropriate projections. Other generalizations were also considered [22].

In [8] we considered another prominent space of analysis, namely the space $\mathcal{A}(\mathbb{R})$ of real analytic functions on the real line. Let us recall that a function f is real analytic on $\mathbb{R}$ if locally near any point $x_0 \in \mathbb{R}$ the function f is the sum of a convergent power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n. \tag{1.2}$$

Naturally, the function f need not be real-valued. Thus in this paper we actually study convergent power series and operations on them. Such a power series converges also for complex numbers z such that $|z - x_0| < \rho$, where ρ is the radius of convergence of (1.2). This means that any function $f \in \mathcal{A}(\mathbb{R})$ is holomorphic in some neighbourhood of the real line $\mathbb{R}$. In other words we investigate operators on functions which are holomorphic on some neighbourhood of $\mathbb{R}$. However compared with functions in the Hardy or the Bergman spaces the domain of $f \in \mathcal{A}(\mathbb{R})$ depends on f with the sole condition that it contains the real line. Obviously such functions as $\exp x$ or $\sin x$ are real analytic. Naturally, they are also entire but it is easy to construct a function which is holomorphic on a neighbourhood of $\mathbb{R}$ which is as small as desired. We emphasize that real analytic functions are basic in PDE's. Such fundamental results as the Cauchy–Kovalevskaya theorem or Holmgren's uniqueness theorem are formulated for just this class of functions (cf. [21]).

What makes the study of $\mathcal{A}(\mathbb{R})$ difficult and interesting is that this space is neither a Hilbert space nor Banach, it is not even metrizable. Interestingly, it was shown by Domański and Vogt [14] that this space has no Schauder basis. It is therefore hardly clear what a Toeplitz operator on $\mathcal{A}(\mathbb{R})$ is. The next definition makes this concept precise.

DEFINITION 1.1. We say that a continuous linear operator

$$T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$$

is a *Toeplitz operator* if there exist complex numbers $\ldots, a_{-1}, a_0, a_1, \ldots \in \mathbb{C}$ such that for each $n \in \mathbb{N}_0$ locally near 0

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots . \tag{1.3}$$

Condition (1.3) means that on monomials a Toeplitz operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is defined by the following Toeplitz matrix

$$M = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \ldots \\ a_1 & a_0 & a_{-1} & \ldots \\ a_2 & a_1 & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ddots \end{pmatrix}. \tag{1.4}$$

This matrix determines T completely since polynomials are dense in $\mathcal{A}(\mathbb{R})$ ([7], p. 12). One should however be careful, since as we have already stated, polynomials do not form a Schauder basis in $\mathcal{A}(\mathbb{R})$.

In [8] we developed a theory of such operators. It is a remarkable and rather surprising fact that this theory is similar to the classical Hardy space case. One of the main results in [8] says that an operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Toeplitz operator if and only if there exists a function $F \in \mathcal{X}(\mathbb{R})$ such that

$$T = T_F := \mathcal{C}M_F. \tag{1.5}$$

The symbol $\mathcal{C}$ stands for the Cauchy transform considered on an appropriate function space (we will make this statement precise in Section 3). The operator M_F is the operator of multiplication by $F \in \mathcal{X}(\mathbb{R})$. The latter space serves as the symbol space. It is defined in the following way:

$$\mathcal{X}(\mathbb{R}) := \mathrm{ind}_{K,U} H(U \setminus K), \tag{1.6}$$

i.e. as the inductive limit of the Frechét spaces $H(U \setminus K)$, which consist of all functions holomorphic in $U \setminus K$. The sets U in (1.6) run through all open neighbourhoods of $\mathbb{R}$ and K runs through all compact subsets of the real line. Note the rather surprising similarity between formulas (1.1) and (1.5). Operators of the form (1.5) appear in natural problems of complex analysis. We shall provide examples which motivated our study in the next section.

In the classical theory of Toeplitz operators the Coburn–Simonenko theorem says that either T_ϕ is injective or it has dense image. Equivalently, either T_ϕ is injective or T_ϕ' is injective. The first main result of this paper is an analog of this result for the space of real analytic functions.

THEOREM 1.2. *Assume that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with non-zero $F \in \mathcal{X}(\mathbb{R})$ is a Toeplitz operator on the space of real analytic functions. Then either* $\ker T_F = \{0\}$ *or* $\ker T_F' = \{0\}$.

The Coburn–Simonenko theorem for Hardy spaces (see [6], [23]) implies that a Toeplitz operator $T_\phi : H^p(\mathbb{T}) \to H^p(\mathbb{T}), 1 < p < \infty$ is invertible if and only if T_F is a Fredholm operator of index zero. We also prove such a characterization of Toeplitz operators on $\mathcal{A}(\mathbb{R})$, although our argument is more direct and does not use Theorem 1.2. The reason is that a general Fredholm theory of operators on $\mathcal{A}(\mathbb{R})$ does not seem to be developed. Specifically, our second main result is the following theorem.

THEOREM 1.3. *Assume that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with $F \in \mathcal{X}(\mathbb{R})$ is a Toeplitz operator on the space of real analytic functions. The operator T_F is invertible if and only if T_F is a Fredholm operator of index zero.*

By a Fredholm operator on $\mathcal{A}(\mathbb{R})$ we obviously mean a continuous operator T such that both $\ker T$ and $\mathcal{A}(\mathbb{R})/\operatorname{im} T$ are finite dimensional. Such an operator has necessarily closed range ([8], Proposition 5.1).

The proof of Theorem 1.2 is rather straightforward, although we use a rather deep fact concerning the Cauchy transform on the Hardy spaces on smooth Jordan curves. In order to prove Theorem 1.3 we use our characterization of Fredholm–Toeplitz operators on $\mathcal{A}(\mathbb{R})$ (see Theorem 2 of [8]).

The main motivation for considering Toeplitz operators on $\mathcal{A}(\mathbb{R})$ came with the results of Domański and Langenbruch. The authors in [9], [10], [11], [12] considered operators on $\mathcal{A}(\mathbb{R})$ which are simply diagonal on functions $z \mapsto z^n$ and built a surprisingly rich theory for such operators. It was therefore natural to make one step further and to consider operators defined by Toeplitz matrices. It is important to realize that not much is known about continuous operators on locally convex spaces of holomorphic or differentiable functions with the exception of differential and convolution operators. These theories are much less developed than their Banach or Hilbert space counterparts. We believe that this makes our results interesting.

2. EXAMPLES OF TOEPLITZ OPERATORS

In [8] we proved the following theorem.

THEOREM 2.1. *The following assertions are equivalent:*

(i) *$T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is a Toeplitz operator, i.e. T is a continuous linear operator such that locally near zero*

$$T(x^n)(\xi) = a_{-n} + a_{-n+1}\xi + a_{-n+2}\xi^2 + \cdots \tag{2.1}$$

for some complex numbers $a_n, n \in \mathbb{Z}$.

(ii) *There exists a function $F \in \mathcal{X}(\mathbb{R})$ such that*

$$T = \mathcal{C}M_F, \tag{2.2}$$

where $M_F : \mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ is the multiplication operator $M_F : f \mapsto Ff$ and $\mathcal{C}$ is the Cauchy projection

$$\mathcal{C} : \mathcal{X}(\mathbb{R}) \to \mathcal{A}(\mathbb{R}) \subset \mathcal{X}(\mathbb{R}).$$

Then (2.1) *holds with*

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma F(\zeta)\zeta^{-n-1}\mathrm{d}\zeta, \tag{2.3}$$

where γ is a closed simple curve in $U \setminus K$ surrounding K with index 1 *and $F \in H(U \setminus K)$.*

(iii) *There exists $G \in \mathcal{A}(\mathbb{R})$ and $\Phi \in \mathcal{A}(\mathbb{R})'$ such that*

$$(Tf)(z) = G(z)f(z) + \Big\langle \frac{f(z) - f(\cdot)}{z - \cdot}, \Phi \Big\rangle.$$

Then close to 0*,*

$$G(z) = \sum_{n=0}^{\infty} c_n z^n$$

and (2.1) *holds with $a_n = c_n, n \in \mathbb{N}_0$, and $a_{-n}, n \in \mathbb{N}$, the sequence of moments of Φ, i.e.*

$$a_{-n-1} = \langle z^n, \Phi\rangle, \quad n = 0, 1, 2, \ldots.$$

Theorem 2.1 solves completely the problem of describing the continuous operators on $\mathcal{A}(\mathbb{R})$ which are defined by Toeplitz matrices (1.4). It is also easy to give examples of such operators. It follows from Theorem 2.1 that a Toeplitz operator $T : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is necessarily of the form

$$(T_F f)(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{F(\zeta)f(\zeta)}{\zeta - z}\mathrm{d}\zeta,$$

where F is holomorphic in some set $U \setminus K$, U is an open complex neighbourhood of $\mathbb{R}$ and K is a compact subset of $\mathbb{R}$. The simple closed curve γ surrounds K and z and is contained in the domains of both f and F.

This implies that any operator of multiplication on $\mathcal{A}(\mathbb{R})$ by a function F in $\mathcal{A}(\mathbb{R})$, i.e. by a function holomorphic on some complex neighbourhood of $\mathbb{R}$, is a Toeplitz operator. The matrix of such an operator is lower triangular and it is a consequence of (2.3) that the coefficients $(a_n), n \in \mathbb{N}_0$ are just the Taylor coefficients of F at 0. More interesting examples come with functions in $H_0(\mathbb{C}_\infty \setminus K)$ for some $K \subset\subset \mathbb{R}$, i.e. with elements of $\mathcal{A}(\mathbb{R})'$ (cf. Theorem 3.3 of [8], for explanation). For instance, if $F(z) = 1/(z - a)^2$ with $a \in \mathbb{R}$ then the matrix of T_F in the sense of Definition 1.1 is upper triangular of the form

$$M = \begin{pmatrix} 0 & 0 & 1 & 2a & 3a^2 & \ldots \\ 0 & 0 & 0 & 1 & 2a & \ldots \\ 0 & 0 & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ddots \end{pmatrix},$$

since obviously

$$\frac{1}{(z-a)^2} = \sum_{n=0}^{\infty} \frac{(n+1)a^n}{z^{n+2}},$$

when $|z| > |a|$. We proved ([8], Theorem 3.3) that the symbol space $\mathcal{X}(\mathbb{R})$ decomposes into the direct sum of $\mathcal{A}(\mathbb{R})$ and the dual space $\mathcal{A}(\mathbb{R})'$. Basically, the operators which correspond to symbols in $\mathcal{A}(\mathbb{R})$ have lower triangular matrices and symbols in $\mathcal{A}(\mathbb{R})'$ determine operators with upper triangular matrices.

Operators T_F, which we call Toeplitz operators, appear in a natural way in interpolation theory. We briefly describe the examples which motivated our interest in this class of operators. In our presentation we follow [16]. Assume that

$$x_1, \ldots, x_m$$

are real numbers and

$$\alpha_1, \ldots, \alpha_m$$

is a set of positive integers such that $\alpha_1 + \cdots + \alpha_m = n$. One looks for a polynomial $P(z)$ satisfying the conditions

$$P(x_k) = f(x_k), \ldots, P^{(\alpha_k - 1)}(x_k) = f^{(\alpha_k - 1)}(x_k), \quad k = 1, 2, \ldots, m. \tag{2.4}$$

The function f is holomorphic in a neighbourhood U of the real line. The following classical result solves the problem.

THEOREM 2.2 ([16], Theorem 2.8). *Let γ be a closed rectifiable Jordan curve contained in U, such that all the interpolation points $x_1, \ldots, x_m$ belong to $I(\gamma)$ and let*

$$\omega(z) = (z - x_1)^{\alpha_1}(z - x_2)^{\alpha_2} \cdots (z - x_m)^{\alpha_m}.$$

Then the integral

$$P(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} d\zeta$$

is a polynomial of degree less than n satisfying conditions (2.4).

Observe that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{\omega(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{\omega(\zeta)} \frac{d\zeta}{\zeta - z}.$$

Thus a solution to the interpolation problem is given by the operator

$$I - T_\omega T_{1/\omega}.$$

Consider now the difference

$$R(z) = f(z) - P(z).$$

We have

$$\begin{aligned} R(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\omega(\zeta)} \frac{\omega(\zeta) - \omega(z)}{\zeta - z} d\zeta \\ &= \frac{\omega(z)}{2\pi i} \int_\gamma \frac{f(\zeta)}{\omega(\zeta)} \frac{d\zeta}{\zeta - z} = T_\omega T_{1/\omega} f(z). \end{aligned}$$

Thus not only the solution to the interpolation problem but also the remainder term is given by the composition of Toeplitz operators. In other words, convergence properties of P when m tends to infinity are controlled by our Toeplitz operators. We remark that the formulas for the interpolation polynomial and the remainder term are known as *Hermite's formulas*. Special cases are Taylor's interpolation polynomials and Lagrange's interpolation polynomials. When $\alpha_1 = \cdots = \alpha_m = p$ one obtains Jacobi's interpolation polynomial

$$\mathcal{J}_{mp-1}(z) = \sum_{k=0}^{p-1} Q_k(z)[q(z)]^k,$$

where $q(z) = (z - x_1) \cdots (z - x_m)$. The functions $Q_k(z)$ are given by a combination of compositions of Toeplitz operators acting on f. The rather delicate problem of convergence of $\mathcal{J}_{mp-1}$ when m tends to ∞ is again governed by Toeplitz operators.

While these examples in our opinion justify our interest in operators of the form T_F on functions holomorphic in some neighbourhood of $\mathbb{R}$, they do not explain why Theorem 1.3 is important. Here we provide another argument. This is again based on [16]. By the first-order divided difference of f with respect to two points x_1 and x_2 we mean the quantity

$$\Delta^{(1)}[f(z); x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then the k-th-order divided difference of $f(z)$ with respect to the $k+1$ points $x_1, \ldots, x_{k+1}$ is defined inductively by the formula

$$\Delta^{(k)}[f(z); x_1, \ldots, x_{k+1}] = \frac{\Delta^{(k-1)}[f(z); x_2, \ldots, x_{k+1}] - \Delta^{(k-1)}[f(z); x_1, \ldots, x_k]}{x_{k+1} - x_1}.$$

Consider now the function $\Delta^{(k)}[f(z); x_1, \ldots, x_k, z]$ as a function of z in some complex neighbourhood of $\mathbb{R}$. Such functions with $k = 1$ appear in one of the proofs of Cauchy's integral formula. A priori, such a function is not defined when z is equal to one of the points $x_1, \ldots, x_k$. But it is easy to observe that

$$\Delta^{(k)}[f(z); x_1, \ldots, x_k, z]$$

can be made to be holomorphic in some neighbourhood of $\mathbb{R}$. Hence it is an element of $\mathcal{A}(\mathbb{R})$. It is also an easy matter to show that

$$\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{\omega_k(\zeta)} \frac{\mathrm{d}\zeta}{\zeta - z} = \Delta^{(k)}[f(z); x_1, \ldots, x_k, z], \tag{2.5}$$

with

$$\omega_k(\zeta) = (\zeta - x_1) \cdots (\zeta - x_k),$$

where γ is a simple curve which surrounds the points $x_1, \ldots, x_k$. Let now

$$g_1, \ldots, g_k$$

be functions in $\mathcal{A}(\mathbb{R})$ and consider the equation

$$\sum_{j=1}^{k} \Delta^{(j)}[g_j(z)f(z); x_1, \ldots, x_j, z] = h(z) \tag{2.6}$$

with a given function $h \in \mathcal{A}(\mathbb{R})$. It is legitimate to call such an equation a divided difference equation and to think about it as a finite-difference analog of a differential equation. The functions $g_1, \ldots, g_k$ and h are locally sums of convergent power series. Thus equation (2.6) can be transformed into a system of equations involving Taylor coefficients of these functions. It is however rather difficult to solve such a system and it is hardly clear why the solution, which is given locally, defines a holomorphic function on $\mathbb{R}$. It follows from (2.5) that equation (2.6) is just an equation of the form

$$T_F f = h$$

for some function F holomorphic in an open neighbourhood of the real line except for a finite number of real poles, i.e. $F \in \mathcal{X}(\mathbb{R})$. Theorem 1.3 says now that if T_F is a Fredholm operator and the index of T_F is equal to 0, then equation (2.6) is uniquely solvable for every function h holomorphic in some complex neighbourhood of $\mathbb{R}$ with f defined in a possibly different neighbourhood of $\mathbb{R}$. Thus there must exist an open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that F does not vanish in $U \setminus K$ and the sum of the number of zeros minus the sum of the number of poles of F in $\mathbb{R}$ must be equal to 0 — this is the winding number of F (cf. Theorem 5.2 below).

3. PRELIMINARIES

3.1. THE SPACE OF REAL ANALYTIC FUNCTIONS AND TOEPLITZ OPERATORS. We refer the reader to [7] for a nice introduction to real analytic functions. Here we only recall facts which are of importance for us. Some information will be repeated from the Introduction. A function $f : \mathbb{R} \to \mathbb{C}$ is real analytic at $x_0 \in \mathbb{R}$ if there exists $r > 0$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{3.1}$$

for $x \in (x_0 - r, x_0 + r)$. A function f is real analytic on $\mathbb{R}$ if it is real analytic at each point $x \in \mathbb{R}$. We use the symbol $\mathcal{A}(\mathbb{R})$ to denote the space of all real analytic functions on $\mathbb{R}$. Naturally, if the sequence (3.1) converges for real $|x - x_0| < r$, it converges also for complex numbers z such that $|z - x_0| < r$. In other words, a real analytic function $f : \mathbb{R} \to \mathbb{C}$ defines a function holomorphic in some open neighbourhood U of the real line. Therefore, we have

$$\mathcal{A}(\mathbb{R}) = \bigcup_{U \supset \mathbb{R}} H(U), \tag{3.2}$$

as sets, where the union is over all open neighbourhoods of $\mathbb{R}$. The symbol $H(U)$ stands for the space of all functions holomorphic in U with the topology of uniform convergence on compact subsets of U. Relation (3.2) is used to topologize the space $\mathcal{A}(\mathbb{R})$. Namely, one equips $\mathcal{A}(\mathbb{R})$ with the strongest locally convex topology which makes all the inclusions $H(U) \hookrightarrow \mathcal{A}(\mathbb{R})$ continuous, where $U \supset \mathbb{R}$ is open. Obviously, this is the inductive topology of the system $\{H(U) \hookrightarrow \mathcal{A}(\mathbb{R})\}_{U\supset\mathbb{R}}$. Such a topology exists. We remark here that this is one of two equivalent ways to equip the space of real analytic functions with a topology. The other is the projective topology induced by the relation

$$\mathcal{A}(\mathbb{R}) = \bigcap_K H(K),$$

where $H(K)$ is the space of germs of holomorphic functions on K and K runs over all compact subsets of $\mathbb{R}$. The topology on $H(K)$ is the topology of the inductive system $H(V) \hookrightarrow H(K)$, where V are all open neighbourhoods of K.

It is a very important result of Martineau [17] that these two topologies on $\mathcal{A}(\mathbb{R})$ coincide. We refer the reader to [18] for background on locally convex spaces, their projective and injective limits.

Every continuous linear functional ξ on $H(K)$, $K \subset\subset \mathbb{C}$, corresponds to a holomorphic function $f_\xi \in H_0(\mathbb{C}_\infty \setminus K)$ (cf. Theorem 1.3.5 in [2]). $H_0(\mathbb{C}_\infty \setminus K)$ stands for the space of all holomorphic functions on $\mathbb{C}_\infty \setminus K$ which vanish at infinity. Naturally, $\mathbb{C}_\infty$ is the Riemann sphere. The (Köthe–Grothendieck) duality between $H(K)$ and $H_0(\mathbb{C}_\infty \setminus K)$ is given by

$$H(K) \times H_0(\mathbb{C}_\infty \setminus K) \ni (g, f) \mapsto \langle g, f\rangle = \frac{1}{2\pi\mathrm{i}} \int_\gamma g(z)f(z)\mathrm{d}z, \tag{3.3}$$

where γ is a finite union of closed curves contained in $U \setminus K$ if $g \in H(U)$, U an open neighbourhood of K, such that $\operatorname{Ind}_\gamma(z) = 1$ for any $z \in K$. This readily implies that

$$\mathcal{A}(\mathbb{R})' \cong (\operatorname{proj}_K H(K))' \cong \operatorname{ind}_K H(K)' \cong \operatorname{ind}_K H_0(\mathbb{C}_\infty \setminus K). \tag{3.4}$$

We can now define Toeplitz operators on $\mathcal{A}(\mathbb{R})$. We start with the definition of the symbol space:

$$\mathcal{X}(\mathbb{R}) := \operatorname{ind}_{U,K} H(U \setminus K),$$

where the sets U run through all open neighbourhoods of $\mathbb{R}$ and K through all compact subsets of $\mathbb{R}$. We showed in [8] that the corresponding inductive topology exists. Assume that $F \in \mathcal{X}(\mathbb{R})$, i.e. that $F \in H(U \setminus K)$, where U is an open set with $\mathbb{R} \subset U$ and K is a compact set such that $K \subset \mathbb{R}$. We assign to F an operator T_F which is a Toeplitz operator in the sense of Definition 1.1. For simplicity we may assume that U is connected and simply connected and K is connected and contains 0. Let $f \in \mathcal{A}(\mathbb{R})$. Thus $f \in H(V)$ for some open neighbourhood of $\mathbb{R}$.

Let $z \in U \cap V$. We put

$$T_F f(z) := \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \frac{F(\zeta) f(\zeta)}{\zeta - z} \mathrm{d}\zeta, \tag{3.5}$$

where γ is a smooth Jordan curve such that $\mathrm{Ind}_{\gamma}(z) = 1$ and $\mathrm{Ind}_{\gamma}(\zeta) = 1$ for $\zeta \in K$. Obviously for any $z \in U \cap V$ we can find such a curve. It follows from Cauchy's theorem that the definition does not depend on γ. We showed in Theorem 1 of [8] that T_F is a continuous operator on $\mathcal{A}(\mathbb{R})$ and every Toeplitz operator on this space in the sense of Definition 1.1 is of this form.

3.2. THE HARDY SPACES AND THE CAUCHY TRANSFORM. Observe that the operator T_F is defined by means of the Cauchy transform $\mathcal{C} = \mathcal{C}_{\gamma}$. We now recall basic information concerning this operator when acting on the Hardy spaces on smooth Jordan curves following the beautiful exposition in [1].

Let γ be a C^{∞} smooth Jordan curve. Whenever we speak of a closed curve we mean exactly such a curve. By Jordan's theorem, $\mathbb{C} \setminus \gamma$ consists of precisely two components $I(\gamma)$ and $E(\gamma)$, where $E(\gamma)$ is unbounded. For a closed curve we denote by $\widehat{\gamma}$ the closure $\overline{I(\gamma)}$.

Let γ be a closed curve and denote $\Omega := I(\gamma)$. If k is a positive integer, $C^k(\overline{\Omega})$ denotes the space of continuous complex-valued functions on $\overline{\Omega}$ whose partial derivatives up to and including order k exist and are continuous on Ω and extend continuously to $\overline{\Omega}$. The space $C^{\infty}(\overline{\Omega})$ is the set of functions in $C^k(\overline{\Omega})$ for all k, $A^{\infty}(\Omega)$ denotes the space of holomorphic functions on Ω that are in $C^{\infty}(\overline{\Omega})$. The symbol $A^{\infty}(\mathrm{b}\Omega)$ stands for the set of functions on $\mathrm{b}\Omega$ which are the boundary values of functions in $A^{\infty}(\Omega)$.

For u and v in $C^{\infty}(\mathrm{b}\Omega)$, the L^2 inner product on $\mathrm{b}\Omega$ of u and v is defined via $\langle u, v\rangle_{\mathrm{b}} = \int_{\mathrm{b}\Omega} u\bar{v} \mathrm{d}s$. By $\mathrm{d}s$ we mean the differential element of the arc length on $\mathrm{b}\Omega$. The space $L^2(\mathrm{b}\Omega)$ is defined to be the Hilbert space obtained by completing the space $C^{\infty}(\mathrm{b}\Omega)$ with respect to this inner product. The Hardy space $H^2(\mathrm{b}\Omega)$ is defined to be the closure in $L^2(\mathrm{b}\Omega)$ of $A^{\infty}(\mathrm{b}\Omega)$. When $\Omega = I(\gamma)$ we write $H^2(\gamma)$.

Let u be a C^{∞} function defined on $\mathrm{b}\Omega$. The Cauchy transform of u is a holomorphic function $\mathcal{C}_{\mathrm{b}\Omega} u$ on Ω given by

$$(\mathcal{C}_{\mathrm{b}\Omega} u)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{b}\Omega} \frac{u(\zeta)}{\zeta - z} \mathrm{d}\zeta. \tag{3.6}$$

Following [1] we now gather fundamental properties of the Cauchy transform.

THEOREM 3.1 ([1], Theorem 3.1). *The Cauchy transform maps $C^{\infty}(\mathrm{b}\Omega)$ into $A^{\infty}(\Omega)$.*

This theorem allows us to treat the Cauchy transform as an operator which maps the space $C^{\infty}(\mathrm{b}\Omega)$ into $C^{\infty}(\overline{\Omega})$, or even as an operator from $C^{\infty}(\mathrm{b}\Omega)$ into itself. Notice however that when considered on the boundary $\mathrm{b}\Omega$ the operator

$\mathcal{C}_{\mathrm{b}\Omega}$ is not given by the integral representation (3.6) due to the singularity in the denominator.

THEOREM 3.2 ([1], Theorem 4.1). *The Cauchy transform extends to a bounded operator from $L^2(\mathrm{b}\Omega)$ into $H^2(\mathrm{b}\Omega)$.*

We will denote the extension of the Cauchy transform to the operator from $L^2(\mathrm{b}\Omega)$ to $H^2(\mathrm{b}\Omega)$ by the same symbol $\mathcal{C}_{\mathrm{b}\Omega}$. Thus for any $\phi \in L^\infty$ we can consider Toeplitz operators

$$H^2(\mathrm{b}\Omega) \ni f \mapsto T_{\phi,\mathrm{b}\Omega} f := \mathcal{C}_{\mathrm{b}\Omega}(\phi \cdot f) \in H^2(\mathrm{b}\Omega).$$

As before, when $\Omega = I(\gamma)$ we write $T_{\phi,\gamma}$ and $\mathcal{C}_\gamma$.

The following fact is a consequence of the Coburn–Simonenko theorem for Toeplitz operators on smooth Jordan curves.

THEOREM 3.3. *Assume that γ is a smooth Jordan curve and let $\phi \in L^\infty(\gamma)$. The operator $T_{\phi,\gamma}$ is invertible if and only if it is a Fredholm operator of index 0.*

The Coburn–Simonenko theorem itself is proved in [3] for Carleson curves (Theorem 6.17 therein). Obviously this covers the case of smooth Jordan curves.

A crucial tool in our study is the following theorem.

THEOREM 3.4 ([1], Theorem 3.4). *Suppose $u \in C^\infty(\mathrm{b}\Omega)$. If M is a positive integer, there is a function $\Psi \in C^\infty(\overline{\Omega})$ which vanishes to order M on the boundary such that the boundary values of $\mathcal{C}_{\mathrm{b}\Omega} u$ are expressed via*

$$(\mathcal{C}_{\mathrm{b}\Omega} u)(z) = u(z) - \frac{1}{2\pi \mathrm{i}} \int\int_{\Omega} \frac{\Psi(\zeta)}{\zeta - z} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta},$$

for $z \in \mathrm{b}\Omega$.

Recall that a function Ψ vanishes to order $M \in \mathbb{N}_0$ if Ψ with all its derivatives up to and including order M vanish on $\mathrm{b}\Omega$. Such a function Ψ can be viewed as a function in $C^M(\mathbb{C})$ via extension by zero.

4. PROOF OF THEOREM 1.2

We start by determining the adjoint of $T_F, F \in \mathcal{X}(\mathbb{R})$ in the sense of duality (3.3). This is rather elementary, we provide the details for the sake of completenes. Let ξ be a continuous funcional on $\mathcal{A}(\mathbb{R})$. It follows from (3.4) that ξ corresponds to a function $f_\xi \in H_0(\mathbb{C}_\infty \setminus L)$ for some compact set $L \subset \mathbb{R}$. Since $F \in \mathcal{X}(\mathbb{R})$ there exist an open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that $F \in H(U \setminus K)$. Let also $g \in \mathcal{A}(\mathbb{R})$. Thus $g \in H(V)$ for some open set $V \supset \mathbb{R}$. It follows from the definition of the operator T_F that $T_F g \in H(U \cap V)$. Let γ be a finite union of closed curves in $(U \cap V) \setminus L$ such that $\operatorname{Ind}_\gamma(z) = 1$ for $z \in L$. Let Γ be a closed curve in $(U \cap V) \setminus (K \cup L)$ such that $\operatorname{Ind}_\Gamma(z) = 1$ for $z \in \gamma$ and $z \in K$. As a result, we also have $\operatorname{Ind}_\Gamma(z) = 1$ for $z \in K \cup L$. By Fubini's theorem we have

$$\langle T_F g, \xi \rangle = \frac{1}{2\pi \mathrm{i}} \int_\gamma T_F g(z) f_\xi(z) \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_\Gamma F(\zeta) g(\zeta) \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f_\xi(z)}{\zeta - z} \mathrm{d}z \mathrm{d}\zeta.$$

It follows from Cauchy's integral formula that for $\zeta \in E(\gamma)$ and sufficiently large $R > 0$,

$$f_\xi(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{-\gamma} \frac{f_\xi(z)}{z - \zeta} \mathrm{d}z + \frac{1}{2\pi \mathrm{i}} \int_{|z|=R} \frac{f_\xi(z)}{z - \zeta} \mathrm{d}z,$$

where the circle $|z| = R$ is oriented counter-clockwise. Since f_ξ vanishes at ∞, we have

$$f_\xi(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f_\xi(z)}{\zeta - z} \mathrm{d}z$$

and, as a result,

$$\langle T_F g, \xi \rangle = \int_\Gamma F(\zeta) g(\zeta) f_\xi(\zeta) \mathrm{d}\zeta. \tag{4.1}$$

Let us now define an operator $S_F : \operatorname{ind}_C H_0(\mathbb{C}_\infty \setminus C) \to \operatorname{ind}_C H_0(\mathbb{C}_\infty \setminus C)$, where C runs over the compact subsets of $\mathbb{R}$. Assume that $f \in H_0(\mathbb{C}_\infty \setminus C)$ for some compact set $C \subset \mathbb{R}$. As before we have $F \in H(U \setminus K)$. Let δ be a closed curve in $U \setminus (K \cup C)$ such that $\operatorname{Ind}_\delta(z) = 1$ for $z \in K \cup C$. We may assume that $K \cup C$ is a connected set. The operator S_F is for $\zeta \in E(\delta)$ defined by the formula

$$S_F f(\zeta) := \frac{1}{2\pi \mathrm{i}} \int_{-\delta} \frac{F(z) f(z)}{z - \zeta} \mathrm{d}z. \tag{4.2}$$

Observe that for any $\zeta \in \mathbb{C}_\infty \setminus (K \cup C)$ there exists a closed curve δ such that $\zeta \in E(\delta)$ and $\operatorname{Ind}_\delta(z) = 1$ for any $z \in K \cup C$. Naturally, by Cauchy's theorem the definition is correct; that is, it does not depend on the choice of δ. Also, $S_F f \in H_0(\mathbb{C}_\infty \setminus (K \cup C))$. It is easy to check that S_F is continuous as an operator on the space $\operatorname{ind} H_0(\mathbb{C}_\infty \setminus C)$. Indeed, it is elementary that S_F maps $H_0(\mathbb{C}_\infty \setminus C)$ continuously into $H_0(\mathbb{C}_\infty \setminus (K \cup C))$. Thus it is a continuous operator between the inductive limits. Thus we have

$$S_F : \operatorname{ind} H_0(\mathbb{C}_\infty \setminus C) \to \operatorname{ind} H_0(\mathbb{C}_\infty \setminus C).$$

We will show that $S_F = T_F'$. We determine the action of $S_F f_\xi$ on h. Since $f_\xi \in H_0(\mathbb{C}_\infty \setminus L)$ and $F \in H(U \setminus K)$ we have by the definition of the operator S_F that $S_F f_\xi \in H_0(\mathbb{C}_\infty \setminus (K \cup L))$. We again assumed that $K \cup L$ is connected. Also, we have $g \in H(V)$. We therefore choose a closed curve Δ in $V \setminus (K \cup L)$ such that $\operatorname{Ind}_\Delta(z) = 1$ for $z \in K \cup L$ and have

$$\langle g, S_F f_\xi \rangle = \frac{1}{2\pi \mathrm{i}} \int_\Delta g(\zeta) S_F f_\xi(\zeta) \mathrm{d}\zeta.$$

Let δ be a closed curve in $I(\Delta)$ such that $\operatorname{Ind}_\delta(z) = 1$ for $z \in K \cup L$. By the definition of the operator S_F we obtain

$$\langle g, S_F f_\xi \rangle = \frac{1}{2\pi i} \int_\Delta g(\zeta) S_F f_\xi(\zeta) \mathrm{d}\zeta = \frac{1}{2\pi i} \int_\Delta g(\zeta) \frac{1}{2\pi i} \int_{-\delta} \frac{F(z) f_\xi(z)}{z - \zeta} \mathrm{d}z \mathrm{d}\zeta$$
$$= \frac{1}{2\pi i} \int_{-\delta} F(z) f_\xi(z) \frac{1}{2\pi i} \int_\Delta \frac{g(\zeta)}{z - \zeta} \mathrm{d}\zeta \mathrm{d}z.$$

By Cauchy's integral formula

$$\langle g, S_F f_\xi \rangle = \frac{1}{2\pi i} \int_\delta F(z) f_\xi(z) g(z) \mathrm{d}z,$$

since if $z \in \delta$ then $z \in I(\Delta)$. By Cauchy's theorem

$$\frac{1}{2\pi i} \int_\delta F(z) f_\xi(z) g(z) \mathrm{d}z = \frac{1}{2\pi i} \int_\Gamma F(\zeta) f_\xi(\zeta) g(\zeta) \mathrm{d}\zeta.$$

Since g and ξ were arbitrary, we have proved the following fact.

PROPOSITION 4.1. *Assume that $F \in \mathcal{X}(\mathbb{R})$. The operator*

$$S_F : \operatorname{ind}_L H_0(\mathbb{C}_\infty \setminus L) \to \operatorname{ind}_L H_0(\mathbb{C}_\infty \setminus L)$$

defined in (4.2) *is the adjoint of T_F in the sense of duality* (3.3).

We are now ready to prove our first main result.

THEOREM 4.2. *Assume that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with a non-zero $F \in \mathcal{X}(\mathbb{R})$ is a Toeplitz operator on the space of real analytic functions. Then either* $\ker T_F = \{0\}$ *or* $\ker T_F' = \{0\}$.

Proof. Assume that there exists a non-zero $u \in \ker T_F$ and a non-zero $v \in \ker T_F'$. As before $F \in H(U \setminus K)$. Let $u \in H(V)$ and $v \in H_0(\mathbb{C}_\infty \setminus L)$. Let Γ be a closed curve in $(U \cap V) \setminus K$ such that $\operatorname{Ind}_\Gamma(z) = 1$ for $z \in K$. By assumption,

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{F(\zeta) u(\zeta)}{\zeta - z} \mathrm{d}\zeta \tag{4.3}$$

for any $z \in I(\Gamma)$. Since F and u belong to $C^\infty(\widehat{\Gamma})$, we also have that $\mathcal{C}_\Gamma(Fu) \in C^\infty(\widehat{\Gamma})$ by Theorem 3.1. It follows from (4.3) that $\mathcal{C}_\Gamma(Fu) \equiv 0$ when $\mathcal{C}_\Gamma(Fu)$ is considered as a function on Γ (cf. the remarks which follow Theorem 3.1). It follows therefore from Theorem 3.4 that

$$0 = \mathcal{C}_\Gamma(Fu)(z) = (F \cdot u)(z) - \frac{1}{2\pi i} \int\int_{I(\Gamma)} \frac{\Psi(\zeta)}{\zeta - z} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}$$

for $z \in \Gamma$ and a function Ψ which is smooth on $\widehat{\Gamma}$ and vanishes to order M on Γ. In other words, for $z \in \Gamma$,

$$(F \cdot u)(z) = \frac{1}{2\pi \mathrm{i}} \iint\limits_{\widehat{\Gamma}} \frac{\Psi(\zeta)}{\zeta - z} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} =: G(z).$$

The function G is of class C^1 on the whole plane and is holomorphic for $z \in \mathbb{C} \setminus \widehat{\Gamma}$. This means that Fu extends to a function holomorphic in $\mathbb{C} \setminus K$. Indeed, let Γ_0 be a proper subarc of the curve Γ. Let γ be an open arc in $(U \cap V) \setminus \widehat{\Gamma}$ whose closure joins the ends of Γ_0. Consider the curve $\Delta := \gamma \cup \Gamma_0$. We may assume that this is a smooth curve. Both the functions Fu and G are holomorphic in a simply connected set $I(\Delta)$ and C^1 on the closure of this set. Also, they are equal on Γ_0. This implies that Fu and G are equal in $I(\Delta)$. Hence, G extends Fu. Observe that G vanishes at ∞.

Consider now the function $F \cdot u \cdot v$. It follows from the above argument that it belongs to $H_0(\mathbb{C}_\infty \setminus (K \cup L))$, since as we showed, $F \cdot u \in H_0(\mathbb{C}_\infty \setminus K)$ and $v \in H_0(\mathbb{C}_\infty \setminus L)$.

On the other hand, $T_F^{\prime} v = 0$. Let now $\zeta \in U \setminus (K \cup L)$. Choose two closed curves Δ and δ both oriented counter-clockwise such that $\delta \subset I(\Delta)$, $\zeta \in I(\Delta) \setminus \overline{I(\delta)}$ and $\operatorname{Ind}_\delta(z) = 1$ for $z \in K \cup L$. From Cauchy's integral formula,

$$(F \cdot v)(\zeta) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Delta} \frac{F(z)v(z)}{z - \zeta} \mathrm{d}z - \frac{1}{2\pi \mathrm{i}} \int\limits_{\delta} \frac{F(z)v(z)}{z - \zeta} \mathrm{d}z.$$

Since $T_F^{\prime} v = 0$, we also have

$$-\frac{1}{2\pi \mathrm{i}} \int\limits_{\delta} \frac{F(z)v(z)}{z - \zeta} \mathrm{d}z = 0.$$

Thus for any $\zeta \in U \setminus (K \cup L)$,

$$(F \cdot v)(\zeta) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Delta} \frac{F(z)v(z)}{z - \zeta} \mathrm{d}z.$$

The function on the right-hand side is holomorphic in $I(\Delta)$. Thus Fv is holomorphic in $I(\Delta)$, which means that it is holomorphic in U.

Consider again the function $F \cdot u \cdot v$. Since $v \in H(V)$ we have

$$F \cdot u \cdot v \in H(U \cap V),$$

and, also

$$F \cdot u \cdot v \in H(U \cap V) \cap H_0(\mathbb{C}_\infty \setminus (K \cup L)).$$

By Liouville's theorem we must have that $F \cdot u \cdot v \equiv 0$. Since u and v are non-zero, it must hold that $F = 0$. This is a contradiction. ∎

5. PROOF OF THEOREM 1.3

We now prove our second main result.

THEOREM 5.1. *Assume that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ with $F \in \mathcal{X}(\mathbb{R})$ is a Toeplitz operator on the space of real analytic functions. The operator T_F is invertible if and only if T_F is a Fredholm operator of index zero.*

We proved in Theorem 2 of [8] the following result.

THEOREM 5.2. *An operator $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, $F \in \mathcal{X}(\mathbb{R})$ is a Fredholm operator if and only if there exist an open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that $F \in H(U \setminus K)$ does not vanish in $U \setminus K$. In this case*

$$\operatorname{index} T_F = -\operatorname{winding} F.$$

Proof of Theorem 5.1. Obviously if T_F is invertible, then it is a Fredholm operator of index 0.

Assume that

$$T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$$

is a Fredholm operator and has index 0. It follows from Theorem 5.2 that there are an open set $U \supset \mathbb{R}$ and a compact set $K \subset \mathbb{R}$ such that $F \in H(U \setminus K)$ and F does not vanish in $U \setminus K$. For simplicity we assume that K is connected, $0 \in K$ and U is simply connected.

For any closed curve $\gamma \subset U \setminus K$ such that $\operatorname{Ind}_\gamma(z) = 1$ for $z \in K$ we have the Toeplitz operator

$$T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma).$$

Since F does not vanish in $U \setminus K$, the operator $T_{F,\gamma}$ is also a Fredholm operator ([8], Theorem 6.4; see also Theorem 4.1.2 of [19]). Furthermore, without loss of generality we may assume that

$$\operatorname{index}(T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})) = \operatorname{index}(T_{F,\gamma} : H^2(\gamma) \to H^2(\gamma)).$$

This may require shrinking U. We again refer the reader to Theorem 2 of [8] for explanation.

It follows that for any C^∞ smooth Jordan curve $\gamma \subset U \setminus K$ with $\operatorname{Ind}_\gamma(0) = 1$ the operator $T_{F,\gamma}$ has index 0. Thus the operator $T_{F,\gamma}$ is invertible by Theorem 3.3.

We now show that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is bijective. It is a remarkable fact that we can use the open mapping theorem ([18], Theorem 24.30) in order to conclude that $T_F : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$ is invertible. The space $\mathcal{A}(\mathbb{R})$ is an ultrabornological space and has a web (see Chapter 24 of [18] for explanation). It is a consequence of the so called de Wilde theory that T_F is open. It is here where the projective picture plays its role.

Thus we must show that T_F is bijective. Injectivity is obvious. Indeed, assume that $T_F f = 0$ for some function $f \in H(V)$. This means that $\mathcal{C}_\gamma(Ff) = 0$ for any C^∞ smooth Jordan curve $\gamma \subset (U \cap V) \setminus K$ with $\operatorname{Ind}_\gamma(0) = 1$. For any such a curve we have $f \in H^2(\gamma)$ and $T_{F,\gamma} f = 0$. This implies that $f \equiv 0$, since each operator $T_{F,\gamma}$ is injective.

It remains to show that T_F is surjective. Let now $g \in H(V)$ for some open neighbourhood V of $\mathbb{R}$. We will show that $g = T_F f$ for some $f \in \mathcal{A}(\mathbb{R})$. Choose a sequence of C^∞ smooth Jordan curves $\gamma_n \subset (U \cap V) \setminus K$ such that $\operatorname{Ind}_{\gamma_n}(0) = 1$ and

$$\mathbb{R} \subset \bigcup_{n=1}^{\infty} \widehat{\gamma}_n, \tag{5.1}$$

where by $\widehat{\gamma}_n$ we denoted the closure of the domain bounded by γ_n. As we have already noticed, for each $n \in \mathbb{N}$ the operator T_{F,γ_n} is invertible. In particular, it is surjective. Naturally, $g \in H^2(\gamma_n)$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there exists a function $f_n \in H^2(\gamma_n)$ such that $g = T_{F,\gamma_n} f_n$. We claim that the functions f_n define a function in $\mathcal{A}(\mathbb{R})$. A priori, the functions $f_n \in H^2(\gamma_n)$ are defined only on γ_n. We can however take their Cauchy transforms

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{g(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_n} \frac{T_{F,\gamma_n} f_n(\zeta)}{\zeta - z} = \frac{1}{2\pi i} \int_{\gamma_n} \frac{\mathcal{C}_{\gamma_n}(F \cdot f_n)(\zeta)}{\zeta - z} d\zeta,$$

where $\mathcal{C}_{\gamma_n}$ above is understood as an operator on $L^2(\gamma_n)$. According to Proposition 5.3 of [8],

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{\mathcal{C}_{\gamma_n}(F \cdot f_n)(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_n} \frac{F(\zeta) f_n(\zeta)}{\zeta - z} d\zeta$$

for $z \in I(\gamma_n)$. Fix a number $n \in \mathbb{N}$ and choose a C^∞ smooth Jordan curve $\widetilde{\gamma}$ such that

$$\widetilde{\gamma} \subset (I(\gamma_n) \cap I(\gamma_{n+1})) \setminus K$$

and $\operatorname{Ind}_{\gamma}(0) = 1$. For any z in the domain bounded by $\widetilde{\gamma}$ we have

$$g(z) = \frac{1}{2\pi i} \int_{\gamma_n} \frac{F(\zeta) f_n(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{F(\zeta) f_{n+1}(\zeta)}{\zeta - z} d\zeta.$$

It follows from Cauchy's theorem and a standard limit argument based on Theorem 6.3 of [1] that for $z \in I(\widetilde{\gamma})$

$$\frac{1}{2\pi i} \int_{\gamma_n} \frac{F(\zeta) f_n(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{F(\zeta) f_n(\zeta)}{\zeta - z} d\zeta,$$

$$\frac{1}{2\pi i} \int_{\gamma_{n+1}} \frac{F(\zeta) f_{n+1}(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\widetilde{\gamma}} \frac{F(\zeta) f_{n+1}(\zeta)}{\zeta - z} d\zeta.$$

Also, as we have already noticed ([8], Proposition 5.3),

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}}\frac{F(\zeta)f_n(\zeta)}{\zeta-z}\mathrm{d}\zeta=\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}}\frac{\mathcal{C}_{\widetilde{\gamma}}(Ff_n)(\zeta)}{\zeta-z}\mathrm{d}\zeta,$$

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}}\frac{F(\zeta)f_{n+1}(\zeta)}{\zeta-z}\mathrm{d}\zeta=\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}}\frac{\mathcal{C}_{\widetilde{\gamma}}(Ff_{n+1})(\zeta)}{\zeta-z}\mathrm{d}\zeta.$$

Thus

$$T_{F,\widetilde{\gamma}}(f_n-f_{n+1})=0,$$

since if

$$\frac{1}{2\pi\mathrm{i}}\int\limits_{\widetilde{\gamma}}\frac{h(\zeta)}{\zeta-z}\mathrm{d}\zeta\equiv 0$$

for $h\in H^2(\widetilde{\gamma})$ and all $z\in I(\widetilde{\gamma})$, then $h\equiv 0$. Since $T_{F,\widetilde{\gamma}}$ is injective, it follows that $f_n=f_{n+1}$ as functions in $H^2(\widetilde{\gamma})$. Taking the Cauchy integrals of f_n and f_{n+1} we conclude that these functions are equal in $I(\widetilde{\gamma})$. Hence we may treat f_{n+1} as the extension of f_n and f_n as the extension of f_{n+1}. Since n was arbitrary, it follows from (5.1) that we obtain a function f holomorphic in some open neighbourhood of $\mathbb{R}$. By Cauchy's theorem $g=T_F f$. ∎

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