J. OPERATOR THEORY
79:2(2018), 403–418
doi: 10.7900/jot.2017may02.2159

THE JACOBSON RADICAL OF CERTAIN SEMICROSSED PRODUCTS

JASPAR WIART

Communicated by Marius Dădârlat

ABSTRACT. We study the Jacobson radical of the semicrossed product $\mathcal{A} \times_\alpha P$ when $\mathcal{A}$ is a simple C^*-algebra and P is either a semigroup contained in an abelian group or a free semigroup. A full characterization is obtained for a large subset of these semicrossed products and we apply our results to a number of examples.

KEYWORDS: *Semicrossed product, endomorphism, dynamical system, C^*-algebra, finite index conditional expectation, Jacobson radical, semi-simplicity, purely infinite.*

MSC (2010): Primary 47L65; Secondary 46L40.

INTRODUCTION

A C^*-dynamical system is a triple $(\mathcal{A}, \alpha, P)$ consisting of a C^*-algebra $\mathcal{A}$, a semigroup P, and an action α of P on $\mathcal{A}$ by $*$-endomorphisms. The semicrossed product $\mathcal{A} \times_\alpha P$ of $\mathcal{A}$ by P is a universal operator algebra associated to a C^*-dynamical system. In this paper we characterize the Jacobson radical of several classes of semicrossed products of simple C^*-algebras by either semigroups contained in an abelian group or free semigroups.

A full characterization of the Jacobson radical when $\mathcal{A} = C_0(X)$ is a commutative C^*-algebra and $P = \mathbb{Z}_+^n$ was achieved in [4]. In the case $n = 1$ the C^*-dynamical system $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ becomes a topological dynamical system (X, ϕ), where ϕ is a continuous surjection, and the Jacobson radical is generated in a certain way by functions that vanish on the recurrent points of (X, ϕ). For $n \geqslant 2$ their characterization uses a variation on recurrence. When $\mathcal{A}$ is simple, the notion of recurrent points does not seem to arise. However some form of recurrence will likely be needed in the non-simple case.

Our main results show that if $(\mathcal{A}, \alpha, P)$ is a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra, P is either a semigroup contained in an abelian group or a free semigroup, and either

(i) $\mathcal{A}$ is purely infinite (Theorem 2.7), or

(ii) there exists a a faithful conditional expectation $E_s : \alpha_s(1)\mathcal{A}\alpha_s(1) \to \alpha_s(\mathcal{A})$ for each $s \in P$ (Theorem 2.11),

then the Jacobson radical of $\mathcal{A} \times_\alpha P$ is generated by monomials $a \otimes e_s$ where $a \in \mathcal{A}(1 - \alpha_s(1))$ (equivalently monomials such that $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$). These theorems yield a number of corollaries including the case where each α_s is an automorphism (Corollary 2.12) and the case where the range of each α_s is hereditary (Corollary 2.16). We also apply our results to several examples including some standard $*$-endomorphisms of the Cuntz algebra and various shifts on the CAR algebra.

One obstruction to characterizing the Jacobson radical in the non-unital case is that it is not clear that for fixed $s \in P$ the set $\{a \in \mathcal{A} : a \otimes e_s \in \mathrm{rad}(\mathcal{A} \times_\alpha P)\}$ is not all of $\mathcal{A}$. However in two cases we are able to say that the above set is either $\{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$ or all of $\mathcal{A}$. In Proposition 3.2 we show that this holds when $(\mathcal{A}, \alpha, P)$ is an automorphic C^*-dynamical system where $\mathcal{A}$ is simple and P is either contained in an abelian group or a free semigroup. Because $\alpha_s(\mathcal{A}) = \mathcal{A}$, in this case we have that the set is either zero or all or $\mathcal{A}$. With the additional assumption that $P = \mathbb{Z}_+$ we get that the radical of $\mathcal{A} \times_\alpha \mathbb{Z}_+$ is either zero or the ideal generated by $\mathcal{A} \otimes e_1$ (Corollary 3.3). In Corollary 3.5 we see that the above also holds when $(\mathcal{A}, \alpha, P)$ is a C^*-dynamical system where $\mathcal{A}$ is a simple separable C^*-algebra, P is contained in an abelian group, and the range of each α_s is hereditary. These results agree with the unital case because the condition $a\alpha_s(\mathcal{A}) = \{0\}$ is the same as $a \in \mathcal{A}(1 - \alpha_s(1))$. As a final example we apply our results to the action obtained by conjugating the compact operators by the unilateral shift.

1. PRELIMINARIES

A *semigroup* is a set P that is closed under an associative binary operation with identity e. We will restrict ourselves to two classes, namely semigroups that are contained in abelian groups and free semigroups (which are also contained in groups). Such semigroups satisfy left and right cancellation, that is the equalities $st = sr$ and $ts = rs$ both imply $t = r$ for all $s, t, r \in P$.

The *free semigroup* $\mathbb{F}_I^+$ *over the generating set* I is the set of (finite) words with alphabet I with multiplication defined by concatenation. The empty word e is the identity. The map $\ell : \mathbb{F}_I^+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the semigroup of non-negative integers under addition, taking a word $w = i_1 i_2 \cdots i_k$ to k, the *length* of w, is a semigroup homomorphism.

An *action* of a semigroup P on a C^*-algebra $\mathcal{A}$ is a semigroup homomorphism $\alpha : P \to \mathrm{End}(\mathcal{A})$ from P into the group of $*$-endomorphisms of $\mathcal{A}$. Because we will be assuming that our C^*-algebras are simple, each α_s will be an injective

$*$-endomorphism. A *C^*-dynamical system* is a triple $(\mathcal{A}, \alpha, P)$ consisting of a C^*-algebra $\mathcal{A}$, a semigroup P, and an action $\alpha : P \to \mathrm{End}(\mathcal{A})$ of P on $\mathcal{A}$. A *covariant pair* (π, T) for $(\mathcal{A}, \alpha, P)$ is a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ and a contractive representation $T : P \to \mathcal{B}(\mathcal{H})$ of P that together satisfy the covariance relation

$$T_s\pi(a) = \pi\alpha_s(a)T_s \quad \text{for all } a \in \mathcal{A} \text{ and } s \in P.$$

To construct a universal operator algebra with respect to the covariant pairs of $(\mathcal{A}, \alpha, P)$ we begin with the algebra $c_{00}(\mathcal{A}, \alpha, P)$ which is just the vector space $\mathcal{A} \otimes c_{00}(P)$ with multiplication given by the rule

$$(a \otimes e_s)(b \otimes e_t) = (a\alpha_s(b)) \otimes e_{st} \quad \text{for all } s, t \in P \text{ and } a, b \in \mathcal{A}.$$

Each covariant pair gives rise to a representation $\pi \times T : c_{00}(\mathcal{A}, \alpha, P) \to \mathcal{B}(\mathcal{H})$ defined by $(\pi \times T)(a \otimes e_s) = \pi(a)T_s$, which together can be used to construct a family of matrix norms. For each $n \geqslant 1$ we define a norm on $M_n(c_{00}(\mathcal{A}, \alpha, P))$ by

$$\Big\| \sum_{s\in P} A_s \otimes e_s \Big\| = \sup \Big\{ \Big\| \sum_{s\in P} (I_n \otimes T_s)\pi^{(n)}(A_s) \Big\|_{\mathcal{B}(\mathcal{H}^{(n)})} : (\pi, T) \text{ a covariant pair} \Big\}$$

where $A_s \in M_n(\mathcal{A})$ and $A_s = 0$ except finitely often. The *semicrossed product* $\mathcal{A} \times_\alpha P$ *of* $\mathcal{A}$ *by* P is the operator algebra completion of $c_{00}(\mathcal{A}, \alpha, P)$ (or a quotient of it) with respect to the family of matrix norms given above.

It is clear from the definition that $\mathcal{A} \times_\alpha P$ has the universal property that each covariant pair (π, T) gives rise to a completely contractive representation, which we also denote by $\pi \times T$, on $\mathcal{A} \times_\alpha P$ that extends the representation on $c_{00}(\mathcal{A}, \alpha, P)$.

REMARK 1.1. (i) When P is an abelian semigroup we could define covariant pairs using the left covariance relation

$$\pi(a)T_s = T_s\pi\alpha_s(a) \quad \text{for all } a \in \mathcal{A} \text{ and } s \in P.$$

We would then complete the algebra $c_{00}(P, \alpha, \mathcal{A})$, which has the vector space structure of $c_{00}(P) \otimes \mathcal{A}$ with multiplication defined by

$$(e_s \otimes a)(e_t \otimes b) = e_{s+t} \otimes (\alpha_t(a)b) \quad \text{for all } s, t \in P \text{ and } a, b \in \mathcal{A}$$

(where associativity comes from P being abelian), with respect to the family of matrix norms obtained from the left covariant pairs to get the left semicrossed product. This is arguably the better choice when P is abelian and is used more often, see [3]. With minor changes reflecting that multiplication in the left semicrossed product is dual to that of the right semicrossed product, our statements and proofs can be reformulated to handle the left semicrossed product.

(ii) When P is not abelian, the right covariance relation is superior because it yields an associative algebra, (see discussion in Section 5.1 of [3]). To avoid giving two nearly identical proofs, it is convenient for us to use the right covariance relation in the commutative case as well.

(iii) For both the left and right semicrossed products we could impose restrictions on the covariant pairs used in the supremum definition of the matrix norms.

For example taking the supremum over covariant pairs (π, T) for which T is a co-isometric representation of P gives us the co-isometric semicrossed product $\mathcal{A} \times_\alpha^{\mathrm{co}} P$. The isometric and unitary semicrossed products are defined similarly, (see [3] for details). Our results would remain the same in all cases.

The next example shows that for C^*-dynamical systems (A, α, P) where P has the right cancellation property, covariant pairs always exist and $\mathcal{A} \times_\alpha P$ contains a faithful copy of $\mathcal{A}$.

EXAMPLE 1.2 (The orbit representation). Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathcal{A}$. Let $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \ell^2(P)$ and define $\widetilde{\pi} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}})$ and $T : P \to \mathcal{B}(\widetilde{\mathcal{H}})$ by

$$\begin{aligned}
\widetilde{\pi}(a)(\xi \otimes \delta_t) &= (\pi\alpha_t(a)\xi) \otimes \delta_t, \text{ and} \\
T_s(\xi \otimes \delta_t) &= \begin{cases} \xi \otimes \delta_r & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that T_s is well-defined because right cancellation holds in P. We claim that $(\widetilde{\pi}, T)$ is a covariant pair for $(\mathcal{A}, \alpha, P)$. It is clear that $\widetilde{\pi}$ is a (faithful) representation of $\mathcal{A}$ and that T_s is a co-isometry, and is therefore contractive, for each $s \in P$. We verify that T is a semigroup homomorphism

$$\begin{aligned}
T_{s_1} T_{s_2}(\xi \otimes \delta_t) &= \begin{cases} T_{s_1}(\xi \otimes \delta_{r_2}) & \text{if } t = r_2 s_2 \text{ for some } r_2 \in P, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \xi \otimes \delta_{r_1} & \text{if } t = r_1 s_1 s_2 \text{ for some } r_1 \in P, \\ 0 & \text{otherwise,} \end{cases} \\
&= T_{s_1 s_2}(\xi \otimes \delta_t)
\end{aligned}$$

and that the covariance relation is satisfied:

$$\begin{aligned}
T_s \widetilde{\pi}(a)(\xi \otimes \delta_t) &= \begin{cases} (\pi\alpha_{rs}(a)\xi) \otimes \delta_r & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \widetilde{\pi}\alpha_s(a)(\xi \otimes \delta_r) & \text{if } t = rs \text{ for some } r \in P, \\ 0 & \text{otherwise,} \end{cases} \\
&= \widetilde{\pi}\alpha_s(a) T_s(\xi \otimes \delta_t).
\end{aligned}$$

2. THE UNITAL CASE

Recall that the *(Jacobson) radical* $\mathrm{rad}(\mathcal{B})$ of a Banach algebra $\mathcal{B}$ is the intersection of the kernels of all irreducible representations. An element $x \in \mathcal{B}$ is said to be *quasi-nilpotent* if its spectral radius is zero. It is well known ([1], Proposition 25.1) that the radical of a Banach algebra can be characterized using quasi-nilpotence in the following way

$$\begin{aligned}\operatorname{rad}(\mathcal{B}) &= \Big\{x \in \mathcal{B} : \lim_{n\to\infty} \|(xy)^n\|^{1/n} = 0 \text{ for all } y \in \mathcal{B}\Big\} \\ &= \Big\{x \in \mathcal{B} : \lim_{n\to\infty} \|(yx)^n\|^{1/n} = 0 \text{ for all } y \in \mathcal{B}\Big\}.\end{aligned}$$

In particular, taking $y = x$ shows that every $x \in \operatorname{rad}(\mathcal{B})$ is quasi-nilpotent. The fact that the radical is an automorphism invariant ideal follows immediately from this characterization.

Our main results show that under certain assumptions on a C^*-dynamical system $(\mathcal{A}, \alpha, P)$, the radical of the semicrossed product $\mathcal{A} \times_\alpha P$ is generated by the monomials $a \otimes e_s$ satisfying $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$.

When P is abelian there are conditional expectations from $\mathcal{A} \times_\alpha P$ onto the monomials that leave the radical invariant. This tells us that the radical is generated by its monomials. It is therefore natural to consider the set

$$\mathcal{J}_s = \{a \in \mathcal{A} : a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)\}$$

consisting of the coefficients of the s-monomials in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$. These sets turn out to be $\mathcal{A}$–$\alpha_s(\mathcal{A})$-bimodules, and those bimodules are well behaved in simple C^*-algebras. For example $\mathcal{J}_s \cap \alpha_s(\mathcal{A})$ is an ideal of $\alpha_s(\mathcal{A})$, and when $\mathcal{A}$ is simple the intersection must be either $\{0\}$ or all of $\alpha_s(\mathcal{A})$. The case when P is a free semigroup is more complicated because we do not have such conditional expectations. To get around this we will need to define the s-Fourier coefficient of an element in $\mathcal{A} \times_\alpha P$.

First the case when P is contained in an abelian group G. Given a covariant pair (π, T) of $(\mathcal{A}, \alpha, P)$ and a member $\widehat{g}$ of the dual $\widehat{G}$ we get another covariant pair $(\pi, \widehat{g}T)$ by setting

$$\widehat{g}T_s = \langle \widehat{g}, s\rangle T_s.$$

By applying the universal property of the semicrossed product $\mathcal{A} \times_\alpha P$, we can construct a continuous action $\gamma : \widehat{G} \to \operatorname{Aut}(\mathcal{A} \times_\alpha P)$ of $\widehat{G}$ on $\mathcal{A} \times_\alpha P$ by automorphisms defined on the generators to be $\gamma_{\widehat{g}}(a_t \otimes e_t) = \langle \widehat{g}, t\rangle a_t \otimes e_t$. This action yields a conditional expectation $F_s : \mathcal{A} \times_\alpha P \to \mathcal{A} \otimes e_s$ given by the formula

$$F_s(x) = \int_{\widehat{G}} \overline{\langle \widehat{g}, s\rangle} \gamma_{\widehat{g}}(x) \mathrm{d}\mu = a_s \otimes e_s,$$

where μ is the Haar measure, $x \in \mathcal{A} \times_\alpha P$, and $a_s \in \mathcal{A}$ is called the *s-Fourier coefficient of* x. We note that on the monomials this formula becomes

$$F_s(a \otimes e_t) = \begin{cases} a \otimes e_t & \text{if } t = s, \\ 0 & \text{otherwise.}\end{cases}$$

Since $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is an automorphism invariant ideal, $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ implies $F_s(x) \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$.

Now let P be a free semigroup and fix $s \in P$. By a similar argument as in the abelian case, we get a continuous action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{A} \times_\alpha P)$ of the dual of $\mathbb{Z}$ on $\mathcal{A} \times_\alpha P$ by automorphisms defined on the generators by $\gamma_z(a_t \otimes e_t) = z^{\ell(t)} a_t \otimes e_t$. This action gives us a conditional expectation $F_{\ell(s)} : \mathcal{A} \times_\alpha P \to \mathcal{A} \times_\alpha P$, similar to the one above, defined by the formula

$$F_{\ell(s)}(x) = \int_{\mathbb{T}} \overline{z}^{\ell(s)} \gamma_z(x) \mathrm{d}m(z),$$

where m is normalized Lebesgue measure. On the monomials this formula becomes

$$F_{\ell(s)}(a \otimes e_t) = \begin{cases} a \otimes e_t & \text{if } \ell(t) = \ell(s), \\ 0 & \text{otherwise.} \end{cases}$$

As above $F_{\ell(s)}(x) \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ whenever $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$.

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation. As in Example 1.2, we let $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \ell^2(P)$, $\widetilde{\pi} : \mathcal{A} \to \mathcal{B}(\widetilde{\mathcal{H}})$, and $T : P \to \mathcal{B}(\widetilde{\mathcal{H}})$ to be the orbit representation. Recall that T_t is the co-isometry defined by the formula

$$T_t(\xi \otimes \delta_r) = \begin{cases} \xi \otimes \delta_{r_1} & \text{if } r = r_1 t \text{ for some } r_1 \in P, \\ 0 & \text{otherwise,} \end{cases}$$

and that T_t^* is the isometry $T_t^*(\xi \otimes \delta_r) = \xi \otimes \delta_{rt}$. Observe that the isometries corresponding to words of the same length $\{T_t^* : \ell(t) = \ell(s)\}$ have orthogonal ranges. It now follows that if $y = \sum b_t \otimes e_t \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}$, then $(\widetilde{\pi} \times T)(y) T_s^* = \widetilde{\pi}(b_s)$. We define the *$s$-Fourier coefficient of x* to be the unique $a_s \in \mathcal{A}$ that satisfies $(\widetilde{\pi} \times T) \circ F_{\ell(s)}(x) T_s^* = \widetilde{\pi}(a_s)$.

Together these few paragraphs prove the following lemma.

LEMMA 2.1. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system. If $x \in \mathcal{A} \times_\alpha P$ and $s \in P$, then $\|x\| \geqslant \|a_s\|$, where $a_s \in \mathcal{A}$ is the s-Fourier coefficient of x.*

DEFINITION 2.2. Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system. For each $s \in P$ define $\mathcal{J}_s \subseteq \mathcal{A}$ to be the set of s-Fourier coefficients of the elements in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$.

It turns out that the above sets are very well behaved. The following lemma shows $\mathcal{J}_s$ is an $\mathcal{A}$–$\alpha_s(\mathcal{A})$-bimodule. In particular each $\mathcal{J}_s$ is a left ideal in $\mathcal{A}$ and $\mathcal{J}_s \cap \alpha_s(\mathcal{A})$ is a two-sided ideal in $\alpha_s(\mathcal{A})$.

LEMMA 2.3. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system. For all $s \in P$, the set $\mathcal{J}_s$ is an $\mathcal{A}$–$\alpha_s(\mathcal{A})$-bimodule.*

Proof. Let $x \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ with s-Fourier coefficient $a_s \in \mathcal{J}_s$. The case when P is abelian is easy because $F_s(x) = a_s \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. We simply use the fact that the radical is an ideal to get

$$(b \otimes e_e)(a \otimes e_s)(c \otimes e_e) = ba_s\alpha_s(c) \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$$

for all $b, c \in \mathcal{A}$, whence $ba_s\alpha_s(c) \in \mathcal{J}_s$.

Now let P be a free semigroup. For all $a, b, c \in \mathcal{A}$ and $t \in P$ with $\ell(t) = \ell(s)$ we have

$$(b \otimes e_e)(a \otimes e_t)(c \otimes e_e) = ba\alpha_s(c) \otimes e_t \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}.$$

It now follows that if $y \in \operatorname{span}\{a \otimes e_t : \ell(t) = \ell(s)\}$, then

$$F_{\ell(s)}((b \otimes e_e)y(c \otimes e_e)) = (b \otimes e_e)y(c \otimes e_e).$$

Passing to limits gives us that

$$F_{\ell(s)}((b \otimes e_e)F_{\ell(s)}(x)(c \otimes e_e)) = (b \otimes e_e)F_{\ell(s)}(x)(c \otimes e_e),$$

which is in $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ because the radical is invariant under $F_{\ell(s)}$. Now the calculation

$$\begin{aligned}(\widetilde{\pi} \times T) \circ F_{\ell(s)}((b \otimes e_e)F_{\ell(s)}(x)(c \otimes e_e))T_s^* &= (\widetilde{\pi} \times T)((b \otimes e_e)F_{\ell(s)}(x)(c \otimes e_e))T_s^* \\ &= \widetilde{\pi}(b)((\widetilde{\pi} \times T) \circ F_{\ell(s)}(x)T_s^*)\widetilde{\pi}\alpha_s(c) \\ &= \widetilde{\pi}(b)\widetilde{\pi}(a_s)\widetilde{\pi}\alpha_s(c) = \widetilde{\pi}(ba_s\alpha_s(c)),\end{aligned}$$

which uses the covariance relation $(T_s\widetilde{\pi}(c)^*)^* = (\widetilde{\pi}\alpha_s(c)^*T_s)^*$ in the second line, shows us that the s-Fourier coefficient of the product $(b \otimes e_e)F_{\ell(s)}(x)(c \otimes e_e)$ is $ba_s\alpha_s(c) \in \mathcal{J}_s$. ∎

The following lemma says that if $a \in \mathcal{J}_s$, then $a \otimes e_s$ is quasi-nilpotent.

LEMMA 2.4. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where P is either contained in an abelian group or a free semigroup. If $a \in \mathcal{J}_s$, then*

$$\lim_{n\to\infty} \|a\alpha_s(a) \cdots \alpha_{s^{(n-1)}}(a)\|^{1/n} = 0. \tag{2.1}$$

Proof. Let $a \in \mathcal{J}_s$. When P is abelian we may use the conditional expectation F_s to show that $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. The n^{th} power of $a \otimes e_s$ is

$$a\alpha_s(a) \cdots \alpha_{s^{n-1}}(a) \otimes e_{s^n}$$

and we apply Lemma 2.1 to get the limit in the statement as a lower bound on the spectral radius of $a \otimes e_s$, which must be zero.

When P is a free semigroup we use the conditional expectation $F_{\ell(s)}$ to find an element $x \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s)\}$, with s-Fourier coefficient a, in the radical. Observing that for all $b_1, b_2 \in \mathcal{A}$ and $t_1, t_2 \in P$,

$$(b_1 \otimes e_{t_1})(b_2 \otimes e_{t_2}) = b_1\alpha_{t_1}(b_2) \otimes e_{t_1t_2} \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(t_1t_2)\},$$

we see that $y^n \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s^n)\}$ whenever $y \in \operatorname{span}\{b \otimes e_t : \ell(t) = \ell(s)\}$. Passing to limits yields $x^n \in \overline{\operatorname{span}}\{b \otimes e_t : \ell(t) = \ell(s^n)\}$. Proceeding by induction on n, assume that the s^n-Fourier coefficient of x^n is $a\alpha_s(a)\cdots\alpha_{s^{(n-1)}}(a)$. Then the calculation

$$\begin{aligned}(\widetilde{\pi}\times T)\circ F_{\ell(s^{n+1})}(x^n x)T^*_{s^{n+1}} &= (\widetilde{\pi}\times T)(x^n)(\widetilde{\pi}\times T)(x)T^*_s T^*_{s^n}\\ &= (\widetilde{\pi}\times T)(x^n)\widetilde{\pi}(a)T^*_{s^n}\\ &= (\widetilde{\pi}\times T)(x^n)T^*_{s^n}\widetilde{\pi}\alpha_{s^n}(a)\\ &= \widetilde{\pi}(a\alpha_s(a)\cdots\alpha_{s^{(n-1)}}(a))\widetilde{\pi}\alpha_{s^n}(a)\end{aligned}$$

shows that the s^{n+1}-Fourier coefficient of x^{n+1} is $a\alpha_s(a)\cdots\alpha_{s^{(n)}}(a)$. As above we apply Lemma 2.1 to complete the proof. ∎

REMARK 2.5. When P is a finitely generated free semigroup there are only finitely many words of a particular length. This means that for $x\in\mathcal{A}\times_\alpha P$,

$$F_{\ell(s)}(x) = a\otimes e_s + \sum_{t\in P_s} a_t\otimes e_t,$$

where $P_s = \{t\in P\setminus\{s\} : \ell(t)=\ell(s)\}$, is a finite sum and the proofs of Lemmas 2.3 and 2.4 simplify. In both proofs, given $a_s\in\mathcal{J}_s$ we can find a finite sum

$$a_s\otimes e_s + \sum_{t\in P_s} a_t\otimes e_t \in \operatorname{rad}(\mathcal{A}\times_\alpha P).$$

By multiplying the sum on the left and right by $b\otimes e_e$ and $c\otimes e_e$ respectively we get

$$ba_s\alpha_s(c)\otimes e_s + \sum_{t\in P_s} ba_t\alpha_t(c)\otimes e_t \in \operatorname{rad}(\mathcal{A}\times_\alpha P)$$

and Lemma 2.3 follows. To get Lemma 2.4 we could observe that the n^{th} power of the sum can be written as

$$(a\alpha_s(a)\cdots\alpha_{s^{n-1}}(a))\otimes e_{s^n} + \sum_{t\in P_{s^n}} a_t\otimes e_t$$

before applying Lemma 2.1.

Lemma 2.4 gives us a way to show when an $a\in\mathcal{A}$ is not in $\mathcal{J}_s$. The next lemma makes use of the fact that any monomial $a\otimes e_s$ that satisfies $(a\otimes e_s)x=0$ for all $x\in\mathcal{A}\times_\alpha P$ is in $\operatorname{rad}(\mathcal{A}\times_\alpha P)$ to show in particular that $\mathcal{J}_s$ is non-zero whenever α_s is a non-unital $*$-endomorphism of a unital C*-algebra.

LEMMA 2.6. *Let $(\mathcal{A},\alpha,P)$ be a C^*-dynamical system where $\mathcal{A}$ is a unital C^*-algebra and P is either contained in an abelian group or a free semigroup. If $p_s=\alpha_s(1)$, then $a(1-p_s)\otimes e_s\in\operatorname{rad}(\mathcal{A}\times_\alpha P)$ for all $a\in\mathcal{A}$. In particular $\mathcal{A}(1-p_s)\subseteq\mathcal{J}_s$.*

Proof. For all $a\in\mathcal{A}(1-p_s)$ and finite sums $\sum_{t\in P} a_t\otimes e_t\in\mathcal{A}\times_\alpha P$ the product

$$a\otimes e_s\Big(\sum_{t\in P} a_t\otimes e_t\Big) = \sum_{t\in P} a\alpha_s(a_t)\otimes e_{st}$$

is zero because

$$a\alpha_s(a_t) = a(1-p_s)p_s\alpha_s(a_t) = 0.$$

Passing to limits we see that $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$. Since $a \otimes e_s$ satisfies the condition in the quasi-nilpotence characterization of the radical, that element must be in $\mathrm{rad}(\mathcal{A} \times_\alpha P)$. ∎

The obvious question raised by the above lemma is: does $\mathcal{J}_s = \mathcal{A}(1 - p_s)$? Although we cannot give a general answer, in the unital case we can prove this equality for two large sets of examples. The first is when $\mathcal{A}$ is a purely infinite simple unital C^*-algebra.

THEOREM 2.7. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a purely infinite simple unital C^*-algebra and P is either a subsemigroup of an abelian group or a free semigroup. Then* $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ *is generated by monomials of the form $a(1 - p_s) \otimes e_s$, where $a \in \mathcal{A}$ and $p_s = \alpha_s(1)$.*

Proof. Fix $s \in P$. The projection p_s decomposes $\mathcal{A}$ as $\mathcal{A}p_s \oplus \mathcal{A}(1 - p_s)$. Since we already know from Lemma 2.6 that $a(1 - p_s) \otimes e_s \in \mathrm{rad}(\mathcal{A} \times_\alpha P)$ for all $a \in \mathcal{A}$, it remains to show that $\mathcal{J}_s \cap \mathcal{A}p_s$ is zero. Suppose that $a \in \mathcal{J}_s \cap \mathcal{A}p_s$ is non-zero. Then since $\mathcal{A}$ is a purely infinite simple unital C^*-algebra there exist $b, c \in \mathcal{A}$ such that $bac = 1$ ([2], Theorem V.5.5). But then $bap_s = ba \in \mathcal{J}_s$ is an element of $\mathcal{J}_s$ that does not satisfy (2.1). Indeed estimating

$$\begin{aligned} \|ba\alpha_s(ba) \cdots \alpha_{s^n}(ba)\| &\geqslant \|ba\alpha_s(ba) \cdots \alpha_{s^{n-1}}(ba)\alpha_{s^n}(bac)\| \|c\|^{-1} \\ &= \|ba\alpha_s(ba) \cdots \alpha_{s^{n-1}}(bap_s)p_{s^n}\| \|c\|^{-1} \\ &= \|ba\alpha_s(ba) \cdots \alpha_{s^{n-1}}(ba)\| \|c\|^{-1}, \end{aligned}$$

we see by induction that

$$\lim_{n\to\infty} \|ba\alpha_s(ba) \cdots \alpha_{s^{n-1}}(ba)\|^{1/n} \geqslant \lim_{n\to\infty} \|c\|^{-1} > 0. \quad ∎$$

EXAMPLE 2.8. Let $\mathcal{O}_n$ be the Cuntz algebra on $2 \leqslant n \leqslant \infty$ generators, that is the universal C^*-algebra generated by isometries $\{s_i\}_{i=1}^n$ satisfying

$$\sum_{i=1}^{n} s_i s_i^* = 1 \quad \text{when } n < \infty, \quad \text{or} \quad \sum_{i=1}^{r} s_i s_i^* \leqslant 1 \quad \text{for all } r \in \mathbb{N} \text{ when } n = \infty.$$

It is well known that $\mathcal{O}_n$ is a purely infinite simple unital C^*-algebra. We associate an isometry $s_w \in \mathcal{O}_n$ to each word $w = i_1 i_2 \cdots i_k \in P$ where P is the free semigroup on the generating set $\{1, \ldots, n\}$ when n is finite, or $\mathbb{N}$ when n is infinite, with the convention that $s_e = 1$. Observing that $s_{w_1}s_{w_2} = s_{w_1w_2}$, we get an action of P on $\mathcal{O}_n$ by setting $\alpha_w(a) = s_w a s_w^*$ for each $w \in P$. The C^*-dynamical system $(\mathcal{O}_n, \alpha, P)$ satisfies the hypotheses of Theorem 2.7 and we conclude that $\mathrm{rad}(\mathcal{O}_n \times_\alpha P)$ is generated by monomials of the form $a(1 - s_w s_w^*) \otimes e_w$.

The following is an immediate corollary to Theorem 2.7.

COROLLARY 2.9. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a purely infinite simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. If each $*$-endomorphism α_s is unital then $\mathcal{A} \times_\alpha P$ is semi-simple.*

EXAMPLE 2.10. Let $\mathcal{O}_n$ be the Cuntz algebra with $2 \leqslant n < \infty$ generators $\{s_i\}_{i=1}^n$. Define a unital $*$-endomorphism $\alpha : \mathcal{O}_n \to \mathcal{O}_n$ by

$$\alpha(a) = \sum_{i=1}^{n} s_i a s_i^*.$$

Setting $\alpha_n = \alpha^n$ we get a unital action of $\mathbb{Z}_+$ on $\mathcal{O}_n$. Since $(\mathcal{O}_n, \alpha, \mathbb{Z}_+)$ satisfies the hypotheses of the above corollary we conclude that $\mathcal{O}_n \times_\alpha \mathbb{Z}_+$ is semi-simple.

Our first theorem assumed a restriction on the C^*-algebra. Our second theorem will instead impose a restriction on the action of P on $\mathcal{A}$.

THEOREM 2.11. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. Suppose that for all $s \in P$ there exists a faithful conditional expectation $E_s : p_s\mathcal{A}p_s \to \alpha_s(\mathcal{A})$ where $p_s = \alpha_s(1)$. Then $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $a(1 - p_s) \otimes e_s$, where $a \in \mathcal{A}$.*

Proof. Fix $s \in P$. The projection p_s decomposes $\mathcal{A}$ as $\mathcal{A}p_s \oplus \mathcal{A}(1 - p_s)$. By Lemma 2.6 we already know that $a(1 - p_s) \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ for all $a \in \mathcal{A}$. Since every $a \in \mathcal{J}_s \cap \mathcal{A}p_s$ satisfies $a^*a \in \mathcal{J}_s \cap p_s\mathcal{A}p_s$, it suffices to show that $\mathcal{J}_s \cap p_s\mathcal{A}p_s$ is zero.

Because E_s is a conditional expectation it is an $\alpha_s(\mathcal{A})$-bimodule map. Using the bimodule property of Lemma 2.3 we observe that, since $\mathcal{J}_s \cap p_s\mathcal{A}p_s$ is an $\alpha_s(\mathcal{A})$-bimodule, it must be mapped to an $\alpha_s(\mathcal{A})$-bimodule in $\alpha_s(\mathcal{A})$. It follows that $E_s(\mathcal{J}_s \cap p_s\mathcal{A}p_s)$ is a two-sided ideal in $\alpha_s(\mathcal{A})$ which is non-zero because E_s is faithful. By simplicity $E_s(\mathcal{J}_s \cap p_s\mathcal{A}p_s) = \alpha_s(\mathcal{A})$, and we can find $a \in \mathcal{J}_s \cap p_s\mathcal{A}p_s$ such that $E_s(a) = p_s$. But then

$$\begin{aligned}\|a\alpha_s(a)\cdots\alpha_{s^n}(a)\| &\geqslant \|\alpha_s^{-1}E_s(a\alpha_s(a)\cdots\alpha_{s^n}(a))\| = \|\alpha_s^{-1}(E_s(a)\alpha_s(a)\cdots\alpha_{s^n}(a))\| \\ &= \|\alpha_s^{-1}(p_s\alpha_s(a)\cdots\alpha_{s^n}(a))\| = \|a\alpha_s(a)\cdots\alpha_{s^{n-1}}(a)\|,\end{aligned}$$

and we see by induction that $\lim\limits_{n\to\infty} \|a\alpha_s(a)\cdots\alpha_{s^{n-1}}(a)\|^{1/n} \geqslant 1$. This contradicts (2.1) and we conclude that $\mathcal{J}_s \cap \mathcal{A}p_s$ is zero. ∎

The following corollary is immediate.

COROLLARY 2.12. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. If every α_s is an automorphism, then $\mathcal{A} \times_\alpha P$ is semi-simple.*

COROLLARY 2.13. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either a subsemigroup of an abelian group or a free semigroup. Suppose that for all $s \in P$ there exists a faithful conditional expectation $E_s : \mathcal{A} \to \alpha_s(\mathcal{A})$. Then each α_s is unital and $\mathcal{A} \times_\alpha P$ is semi-simple.*

Proof. Fix $s \in P$, let $E_s : \mathcal{A} \to \alpha_s(\mathcal{A})$ be a faithful conditional expectation, and let $p_s = \alpha_s(1)$. Observe that since E_s is an $\alpha_s(\mathcal{A})$-bimodule map, both $E_s(1)$

and $E_s(p_s) = \alpha_s(1)$ are units for $\alpha_s(\mathcal{A})$ and are therefore equal. It follows that $E_s(1 - p_s) = 0$ and since E_s is faithful $p_s = 1$ as desired. The result now follows from Theorem 2.11. ∎

EXAMPLE 2.14 (The shift on the CAR algebra). Let $\mathcal{A} = \bigotimes_{n \geqslant 1} M_2$ be the CAR algebra expressed as a tensor product. Extend the map

$$\alpha : a_1 \otimes a_2 \otimes \cdots \mapsto 1_{M_2} \otimes a_1 \otimes a_2 \otimes \cdots$$

defined on the elementary tensors to get a unital $*$-endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ which we call the shift. By setting $\alpha_n = \alpha^n$ we get a unital action of $\mathbb{Z}_+$ on $\mathcal{A}$. Identifying $\mathcal{A} \cong M_2 \otimes \mathcal{A}$ and $\alpha_1(\mathcal{A}) \cong \mathbb{C}1_{M_2} \otimes \mathcal{A}$ we can define a faithful conditional expectation $E_1 : \mathcal{A} \to \alpha_1(\mathcal{A})$ by

$$E_1(a \otimes b) = \mathrm{tr}(a) 1_{M_2} \otimes b,$$

where $\mathrm{tr} : M_2 \to \mathbb{C}$ is the unique faithful tracial state on M_2. One can easily check that for $n \geqslant 2$

$$E_n = \alpha_{n-1} \underbrace{(E_1 \alpha^{-1}) \cdots (E_1 \alpha^{-1})}_{(n-1)\text{-times}} E_1$$

is a faithful conditional expectation from $\mathcal{A}$ onto $\alpha_n(\mathcal{A})$. Thus $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ satisfies the hypothesis of Corollary 2.13 and we conclude that $\mathcal{A} \times_\alpha \mathbb{Z}_+$ is semi-simple.

We say that a conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is *finite index* if there exists a *quasi-basis*, i.e. a set $\{(u_i, u_i^*)\}_{i=1}^n \subseteq \mathcal{A} \times \mathcal{A}$ such that

$$a = \sum_{i=1}^n u_i E(u_i^* a) = \sum_{i=1}^n E(a u_i) u_i^*.$$

When a quasi-basis exists we define the index of E to be

$$\mathrm{Ind}(E) = \sum_{i=1}^n u_i u_i^*.$$

It is well known that $\mathrm{Ind}(E)$ does not depend on the choice of quasi-basis ([9], Proposition 1.2.8). We call a $*$-endomorphism α *finite index* if there exists a finite index conditional expectation from $\mathcal{A}$ onto the range of α. Such $*$-endomorphisms were considered by Exel in [6].

COROLLARY 2.15. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ there exists a finite index conditional expectation $E_s : p_s \mathcal{A} p_s \to \alpha_s(\mathcal{A})$ where $p_s = \alpha_s(1)$. Then $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $a(1 - p_s) \otimes e_s$, where $a \in \mathcal{A}$. Moreover if each α_s is finite index, then $\mathcal{A} \times_\alpha P$ is semi-simple.*

Proof. Let $E_s : p_s \mathcal{A} p_s \to \alpha_s(\mathcal{A})$ be a finite index conditional expectation. By Proposition 2.6.2 of [9], for all positive $a \in \mathcal{A}$ we have $E_s(a) \geqslant \| \mathrm{Ind}(E_s) \|^{-1} a$. It follows that E_s is faithful and we may apply Theorem 2.11 and Corollary 2.13. ∎

We get another special case of Theorem 2.11 when each $*$-endomorphism has hereditary range, a condition which has been considered before in [5], [8]. Corollary 2.16 will follow from the fact that $\alpha_s(\mathcal{A})$ is hereditary if and only if $\alpha_s(\mathcal{A}) = p_s\mathcal{A}p_s$.

COROLLARY 2.16. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple unital C^*-algebra and P is either contained in an abelian group or a free semigroup. Suppose that the range of α_s is hereditary in $\mathcal{A}$ for all $s \in P$. Then* $\mathrm{rad}(\mathcal{A} \times_\alpha P)$ *is generated by monomials of the form $a(1 - p_s) \otimes e_s$, where $a \in \mathcal{A}$ and $p_s = \alpha_s(1)$.*

Proof. It is clear that $\alpha_s(\mathcal{A}) \subseteq p_s\mathcal{A}p_s$ for each $s \in P$. For the reverse inclusion observe that for $0 \leqslant a \in p_s\mathcal{A}p_s$ we have $0 \leqslant a = p_s a p_s \leqslant \|a\| p_s \in \alpha_s(\mathcal{A})$. Since $\alpha_s(\mathcal{A})$ is hereditary, $a \in \alpha_s(\mathcal{A})$. We may now apply Theorem 2.11 because we have $\alpha_s(\mathcal{A}) = p_s\mathcal{A}p_s$. ∎

EXAMPLE 2.17. Let $\mathcal{A}$ be a simple unital C^*-algebra that contains an isometry s. Define a $*$-endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ by $\alpha(a) = sas^*$. We get an action of $\mathbb{Z}_+$ on $\mathcal{A}$ by setting $\alpha_n = \alpha^n$. The range of α_n is hereditary because $\alpha_n(\mathcal{A}) = p_n\mathcal{A}p_n$, where $p_n = \alpha_n(1) = s^n(s^*)^n$. By Corollary 2.16 we conclude that $\mathrm{rad}(\mathcal{A} \times_\alpha \mathbb{Z}_+)$ is generated by monomials of the form $a(1 - p_n) \otimes e_n$.

EXAMPLE 2.18 (Non-commuting non-unital shifts on the CAR algebra). Let $\mathcal{A} = \bigotimes\limits_{n\geqslant 1} M_2$ be the CAR algebra and

$$q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We define two non-unital shifts, $\alpha_1 : \mathcal{A} \to \mathcal{A}$ and $\alpha_2 : \mathcal{A} \to \mathcal{A}$, on the elementary tensors by

$$\begin{aligned} \alpha_1 &: a_1 \otimes a_2 \otimes \cdots \mapsto q_1 \otimes a_1 \otimes a_2 \otimes \cdots, \quad \text{and} \\ \alpha_2 &: a_1 \otimes a_2 \otimes \cdots \mapsto q_2 \otimes a_1 \otimes a_2 \otimes \cdots \end{aligned}$$

which extend to $*$-endomorphisms on $\mathcal{A}$. Let $\mathbb{F}_2^+$ be the free semigroup on the generating set $\{1, 2\}$. To get an action α of $\mathbb{F}_2^+$ on $\mathcal{A}$ we set

$$\alpha_w = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_k},$$

where $w = i_1 i_2 \cdots i_k \in \mathbb{F}_2^+$. The range of each α_w is hereditary for each $w \in \mathbb{F}_2^+$ because $\alpha_w(\mathcal{A}) = p_w\mathcal{A}p_w$, where

$$p_w = \alpha_w(1) = q_{i_1} \otimes q_{i_2} \otimes \cdots \otimes q_{i_k} \otimes 1.$$

Thus by Corollary 2.16 we conclude that $\mathrm{rad}(\mathcal{A} \times_\alpha \mathbb{F}_2^+)$ is generated by monomials of the form $a(1 - p_w) \otimes e_w$.

3. THE NON-UNITAL CASE

The main obstruction in obtaining a characterization of the radical in the non-unital simple case is that without a unit it is not obvious that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$ or even $\mathcal{J}_s \neq \mathcal{A}$. Because of this, in this section, we must assume that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$, which in the abelian semigroup case is equivalent to $\mathcal{J}_s \neq \mathcal{A}$. Even with that assumption the proofs of Theorems 2.7 and 2.11 do not generalize. Theorem 2.7 used the fact that for each non-zero element a in a purely infinite simple unital C^*-algebra $\mathcal{A}$ there exist $b, c \in \mathcal{A}$ such that $bac = 1$, and the characterization of the multiplicative domain of the conditional expectation used in the proof of Theorem 2.11 required that the map was unital. We will however be able to obtain non-unital versions of Corollaries 2.12 and 2.16.

LEMMA 3.1. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where P is contained in an abelian group. Let $s \in P$. Then*
(i) *we have $\mathcal{J}_s \subseteq \mathcal{J}_{st}$ for all $t \in P$, and*
(ii) *if $\alpha_s(\mathcal{A}) \subseteq \mathcal{J}_s$, then $\mathcal{J}_s = \mathcal{A}$.*

Proof. (i) Let $a \in \mathcal{J}_s$ and $t \in P$. For all finite sums $\sum\limits_{r\in P} a_r \otimes e_r \in \mathcal{A} \times_\alpha P$,

$$a \otimes e_{st}\Big(\sum_{r\in P} a_r \otimes e_r\Big) = a \otimes e_s\Big(\sum_{r\in P} \alpha_t(a_r) \otimes e_{tr}\Big)$$

is quasi-nilpotent because $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. Passing to limits we see that $(a \otimes e_{st})x$ is quasi-nilpotent for all $x \in \mathcal{A} \times_\alpha P$. It follows that $a \otimes e_{st} \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ and $a \in \mathcal{J}_{st}$.

(ii) Suppose $\alpha_s(\mathcal{A}) \subseteq \mathcal{J}_s$, let $a \in \mathcal{A}$, and let $\sum\limits_{t\in P} a_t \otimes e_t \in \mathcal{A} \times_\alpha P$ be a finite sum. From (i) and the fact that the radical of $\mathcal{A} \times_\alpha P$ is generated by its monomials we have $\sum\limits_{t\in P} \alpha_s(a_t) \otimes e_{st} \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. It follows that

$$a \otimes e_s\Big(\sum_{t\in P} a_t \otimes e_t\Big) = a \otimes e_e\Big(\sum_{t\in P} \alpha_s(a_t) \otimes e_{st}\Big)$$

is quasi-nilpotent. Passing to limits we see $(a \otimes e_s)x$ is quasi-nilpotent for all $x \in \mathcal{A} \times_\alpha P$, whence $a \otimes e_s \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. ∎

PROPOSITION 3.2. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple C^*-algebra and α is an action of either a semigroup contained in an abelian group or a free semigroup on $\mathcal{A}$ by $*$-automorphisms. Then $\mathcal{J}_s$ equals either $\mathcal{A}$ or $\{0\}$ for each $s \in P$.*

Proof. Because $\mathcal{J}_s$ is an $\mathcal{A}$–$\alpha_s(\mathcal{A})$-bimodule and $\alpha_s(\mathcal{A}) = \mathcal{A}$, $\mathcal{J}_s$ is an ideal of the simple C^*-algebra $\mathcal{A}$. It follows that $\mathcal{J}_s$ is either $\mathcal{A}$ or $\{0\}$. ∎

COROLLARY 3.3. *Let $(\mathcal{A}, \alpha, \mathbb{Z}_+)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple C^*-algebra and α is an action of $\mathbb{Z}_+$ on $\mathcal{A}$ by $*$-automorphisms. Then* $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ *is either zero or the ideal generated by $\mathcal{A} \otimes e_1$.*

Proof. If $a \otimes e_0 \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$, then $(a^* \otimes e_0)(a \otimes e_0) = a^* a \otimes e_0$ should be quasi-nilpotent. We apply the C^*-identity to the spectral radius formula

$$\lim_{n\to\infty} \|(a^* a \otimes e_0)^n\|^{1/n} = \lim_{n\to\infty} \|(a^* a)^n\|^{1/n} = \|a\|$$

to show that a must be zero and $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is contained in the ideal generated by $\mathcal{A} \otimes e_1$.

Suppose that $\mathcal{A} \times_\alpha P$ is not semi-simple. Then by the previous proposition there is some $n \geqslant 1$ for which $\mathcal{J}_n = \mathcal{A}$. By Lemma 3.1(i) we have $\mathcal{A} = \mathcal{J}_n \subseteq \mathcal{J}_{n+k}$ for all $k \in \mathbb{Z}_+$. Observe that for all $a \in \mathcal{A}$ and finite sums $\sum\limits_{k=0}^{m} a_k \otimes e_k \in \mathcal{A} \times_\alpha P$ we can write

$$\Big((a \otimes e_1) \sum_{k=0}^{m} a_k \otimes e_k\Big)^n = \sum_{k=n}^{n(m+1)} b_k \otimes e_k \in \operatorname{rad}(\mathcal{A} \times_\alpha P),$$

for some $b_k \in \mathcal{A}$. Passing to limits we see that $((a \otimes e_1)x)^n \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$ for all $x \in \mathcal{A} \times_\alpha P$, therefore $(a \otimes e_1)x$ is quasi-nilpotent and $a \otimes e_1 \in \operatorname{rad}(\mathcal{A} \times_\alpha P)$. ∎

PROPOSITION 3.4. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple C^*-algebra and P is either a semigroup contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ there exists $0 < b_s \in \alpha_s(\mathcal{A})$ such that $\alpha_s(\mathcal{A}) = \overline{b_s \mathcal{A} b_s}$. If*

(i) *P is abelian and $\mathcal{J}_s \neq \mathcal{A}$, or*

(ii) *P is free and $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$,*

then

$$\mathcal{J}_s = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\} = \{a \in \mathcal{A} : ab_s = 0\}.$$

Proof. The equality

$$\{a \in \mathcal{A} : ab_s = 0\} = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$$

is easy and the containment

$$\mathcal{J}_s \supseteq \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\}$$

is clear because an argument similar to the one in the proof of Lemma 2.6 shows that $a\alpha_s(\mathcal{A}) = \{0\}$ implies $(a \otimes e_s)x = 0$ for all $x \in \mathcal{A} \times_\alpha P$. For the reverse inclusion suppose that $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$, which by Lemma 3.1 is equivalent to $\mathcal{J}_s \neq \mathcal{A}$ in the abelian semigroup case. The bimodule property of $\mathcal{J}_s$ guarantees $0 \leqslant b_s a^* a b_s \in \mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$ for all $a \in \mathcal{J}_s$. It follows that $ab_s = 0$ for all $a \in \mathcal{J}_s$. ∎

COROLLARY 3.5. *Let $(\mathcal{A}, \alpha, P)$ be a C^*-dynamical system where $\mathcal{A}$ is a simple separable C^*-algebra and P is either a semigroup contained in an abelian group or a free semigroup. Suppose that for each $s \in P$ the range of α_s is hereditary in $\mathcal{A}$. If*

(i) *P is abelian and $\mathcal{J}_s \neq \mathcal{A}$, or*

(ii) *P is free and $\mathcal{J}_s \cap \alpha_s(\mathcal{A}) = \{0\}$,* *then*

$$\mathcal{J}_s = \{a \in \mathcal{A} : a\alpha_s(\mathcal{A}) = \{0\}\} = \{a \in \mathcal{A} : ab_s = 0\},$$

where $0 < b_s \in \alpha_s(\mathcal{A})$ *is an element that satisfies* $\alpha_s(\mathcal{A}) = \overline{b_s \mathcal{A} b_s}$.

Proof. Recall that if $\mathcal{B}$ is a separable hereditary C^*-subalgebra of a C^*-algebra $\mathcal{A}$, then there exists $0 \leqslant b \in \mathcal{B}$ such that $\mathcal{B}$ is the closure of $b\mathcal{A}b$ ([7], Theorem 3.2.5). Therefore there exists $0 < b_s \in \alpha_s(\mathcal{A})$ such that $\alpha_s(\mathcal{A})$ is the closure of $b_s\mathcal{A}b_s$ and we can apply the previous proposition. ∎

EXAMPLE 3.6 (The unilateral shift and the compacts). Let $\mathcal{K}$ be the compact operators on $\ell^2(\mathbb{Z}_+) = \overline{\operatorname{span}}\{\xi_i : i \geqslant 0\}$, let $S \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ be the unilateral shift, and let $S_n = S^n$. Since $\mathcal{K}$ is an ideal of $\mathcal{B}(\ell^2(\mathbb{Z}_+))$, we can define an action of $\mathbb{Z}_+$ on $\mathcal{K}$ by setting $\alpha_n(K) = S_n K S_n^*$ for all $K \in \mathcal{K}$. Corollary 3.5 applies because the range $\alpha_n(\mathcal{K}) = S_n S_n^* \mathcal{K} S_n S_n^*$ of each α_n is hereditary. To compute the radical we need only demonstrate $\mathcal{J}_n \neq \mathcal{K}$ for each $n \in \mathbb{Z}_+$.

We claim that $S_n^* P_n \notin \mathcal{J}_n$, where P_n is the orthogonal projection onto $\mathbb{C}\xi_n$. To see this first note that for all $k \geqslant 1$

$$\begin{aligned}\alpha_{(k-1)n}(S_n^*P_n)\alpha_{kn}(S_n^*P_n) &= S_{(k-1)n}(S_n^*P_n)S_{(k-1)n}^* \cdot S_{kn}(S_n^*P_n)S_{kn}^* \\ &= \alpha_{(k-1)n}(S_n^*P_n)S_n^*\end{aligned}$$

and then estimate

$$\begin{aligned}\lim_{k\to\infty} \|S_n^*P_n\alpha_n(S_n^*P_n)\cdots\alpha_{(k-1)n}(S_n^*P_n)\|^{1/k} &= \lim_{k\to\infty} \|S_n^*P_nS_{(k-1)n}^*\|^{1/k} \\ &\geqslant \lim_{k\to\infty} \|S_n^*P_nS_{(k-1)n}^*\xi_{kn}\|^{1/k} = 1,\end{aligned}$$

which by Lemma 2.4 tells us that $S_n^*P_n \notin \mathcal{J}_n$. We conclude by Corollary 3.5 that $\mathcal{J}_n = \{K \in \mathcal{K} : K\alpha_n(\mathcal{K}) = \{0\}\}$. Exploiting the fact that $\mathcal{K} \subseteq \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is an ideal, we can write $\mathcal{J}_n = \mathcal{K}(I - S_nS_n^*)$, where $I \in \mathcal{B}(\ell^2(\mathbb{Z}_+))$ is the identity. It follows that $\operatorname{rad}(\mathcal{A} \times_\alpha P)$ is generated by monomials of the form $K(I - S_nS_n^*) \otimes e_n$, which mirrors the characterization of the radical in Corollary 2.16.

REFERENCES

[1] F.F. BONSALL, J. DUNCAN, *Complete Normed Algebras*, Springer-Verlag, New York-Heidelberg 1973.

[2] K.R. DAVIDSON, *C^*-Algebras by Example*, Amer. Math. Soc., Providence RI 1996.

[3] K.R. DAVIDSON, A.H. FULLER, E.T.A. KAKARIADIS, Semicrossed products of operator algebras by semigroups, *Mem. Amer. Math Soc.*, to appear.

[4] K.R. DAVIDSON, A. KATAVOLOS, A. MANOUSSOS, The Jacobson radical for analytic crossed products, *J. Funct. Anal.* **187**(2001), 129–145.

[5] R. EXEL, A new look at the crossed-product of a C^*-algebra by an endomorphism, *Ergodic Theory Dynam. System* **23**(2003), 1733–1750.

[6] R. EXEL, Crossed-products by finite index endomorphisms and KMS states, *J. Funct. Anal.* **199**(2003), 153–188.

[7] G.J. MURPHY, *C^*-Algebras and Operator Theory*, Academic Press, Inc., Boston 1990.

[8] G.J. MURPHY, Crossed products of C^*-algebras by endomorphisms, *Integral Equations Operator Theory* **24**(1996), 298–319.

[9] Y. WATATANI, Index for C^*-subalgebras, *Mem. Amer. Math. Soc.* **83**(1990), vi+117.

JASPAR WIART, DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, N2L 3G1, CANADA

E-mail address: jwiart@uwaterloo.ca

Received May 2, 2017.