J. OPERATOR THEORY
70:2(2018), 419–462
doi: 10.7900/jot.2017may12.2161

THE MINIMAL IDEAL IN MULTIPLIER ALGEBRAS

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Communicated by Marius Dădârlat

ABSTRACT. When $\mathcal{A}$ is a simple, σ-unital, non-unital, non-elementary C^*-algebra, let $I_{\min}$ denote the intersection of the ideals of $\mathcal{M}(\mathcal{A})$ that properly contain $\mathcal{A}$. $I_{\min}$ coincides with the ideal defined by Lin. We prove that $I_{\min} \neq \mathcal{A}$ for several categories of C^*-algebras. If $I_{\min} \neq \mathcal{A}$, then $I_{\min}/\mathcal{A}$ is purely infinite and simple. If $\mathcal{A}$ has strict comparison of positive elements by traces then $I_{\min} = I_{\mathrm{cont}}$, the closure of the linear span of the elements $A \in \mathcal{M}(\mathcal{A})_+$ such that the evaluation map $\widehat{A}(\tau) = \tau(A)$ is continuous. In particular, $I_{\min} \neq I_{\mathrm{cont}}$ for certain Villadsen's AH-algebras.

KEYWORDS: *Multiplier algebras, minimal ideals, strict comparison, Villadsen AH-algebras.*

MSC (2010): Primary 46L05; Secondary 46L35, 46L45.

INTRODUCTION

The ideal structure of the multiplier algebra of a simple, σ-unital non-unital non-elementary, C^*-algebra $\mathcal{A}$ has received over the years a lot of attention. In this paper we will focus on the study of the smallest (closed) ideal properly containing $\mathcal{A}$. Lin ([16], Lemma 2) gave a constructive proof of the existence of such a smallest ideal for AF-algebras in terms of the tracial simplex of the algebra (see Subsection 2.2).

Then Lin and Zhang [20], proved that every simple, separable, non-unital, non-elementary C^*-algebra with property (SP) and with an approximate identity of projections (such algebras do not need to have real rank zero) contains an ℓ^1-sequence of projections (see Definition 2.8 for a generalization). Furthermore, all the principal ideals generated by projections associated to such sequences coincide with the minimal ideal properly containing $\mathcal{A}$.

In [17] Lin defined for every simple σ-unital C^*-algebra an ideal $\mathcal{I}$ in terms of an approximate identity of positive elements and proved that $\mathcal{I}$ is contained in any ideal properly containing $\mathcal{A}$. If $\mathcal{A}$ is separable, then $A \neq \mathcal{I}$.

For simple C^*-algebras with real rank zero, stable rank one, and weakly unperforated K_0, (equivalently, strictly unperforated monoid $V(\mathcal{A})$ of Murray–von Neumann equivalence classes of projections in $\mathcal{A} \otimes \mathcal{K}$) Perera proved that there is a lattice isomorphism between the ideals of $\mathcal{M}(\mathcal{A})$ and the order ideals of $V(\mathcal{A}) \sqcup W_\sigma^d(S_u)$ (see Theorems 2.1 and 3.9 of [24] and notations therein) and then proved ([24], Proposition 4.1) that $V(\mathcal{A}) \sqcup \mathrm{Aff}_{++}(S_u)$ is the smallest order ideal properly containing $V(\mathcal{A})$, thus obtaining the smallest ideal properly containing $\mathcal{A}$. Here $\mathrm{Aff}_{++}(S_u)$ is the space of strictly positive *continuous* affine functions on the state space S_u (see also Subsection 2.2 and Section 4). This ideal, denoted by $L(\mathcal{A})$, plays an important role in the study by Perera [24] and Kucerovsky and Perera [14] of the ideal structure of the multiplier algebra and the characterization for the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$ to be purely infinite.

The goal of this paper is to clarify the relations between the various constructions of the minimal ideal and to further investigate its properties. Throughout the paper, except in Section 6, $\mathcal{A}$ will denote a simple, σ-unital, non-unital and non-elementary C^*-algebra.

We revisit Lin's definition ([17], Lemma 2.1) of a nonclosed left ideal of $\mathcal{M}(\mathcal{A})$ defined in terms of an approximate unit $\{e_n\}$ of positive elements, which we denote by $L(K_0(\{e_n\}))$ and by $I_{\min}$ its norm closure (Lin denoted them by I_0 and I respectively). Properties of $I_{\min}$ are obtained using a bidiagonal decomposition result (Theorem 3.1) which is in the line of the tri-diagonal decomposition of elements in $\mathcal{M}(\mathcal{A})$, first introduced by Elliot in proof of Theorem 3.1 in [8]. More background on bi-diagonal and tri-diagonal decompositions is presented before Theorem 3.1. As a consequence of the proof, one also sees that $I_{\min}$ does not depend on the approximate identity chosen, which was already obtained in [17].

In Remark 2.9 of [17] Lin proved that $I_{\min}$ is contained in the intersection $\mathcal{J}_0$ of all the ideals properly containing $\mathcal{A}$. In Theorem 3.7 we prove that $I_{\min} = \mathcal{J}_0$ and in Theorem 3.8 we show that if $I_{\min}/\mathcal{A}$ is nonzero (and necessarily simple), then it is purely infinite. Furthermore, $\mathcal{A} \neq I_{\min}$ if and only if there exists a *thin* sequence of positive elements for $\mathcal{A}$ (Definition 2.8, Theorem 2.14). This notion can be seen as a generalization of the notion of ℓ^1 sequence of projections introduced for the (SP) case in [20], thus providing a bridge between the approaches in [20] and [17].

If $\mathcal{A}$ is separable, or if $\mathcal{A}$ has the (SP) property and the dimension semigroup of Murray–von Neumann equivalence classes of projections is countable, then a thin sequence exists, and hence $\mathcal{A} \neq I_{\min}$. This includes the case of type II_1 factors.

Except when $\mathcal{A} = \mathcal{K}$, we do not have examples when $\mathcal{A} = I_{\min}$. A natural test case is the nonseparable simple C^*-algebra with both a nonzero finite and an infinite projection studied by Rørdam in [27]. But it still yields $\mathcal{A} \neq I_{\min}$ (see last paragraph of Section 4).

In the case when $\mathcal{A}$ has a nonempty tracial simplex $\mathcal{T}(\mathcal{A})$, another natural ideal inspired by the approaches in [16] and [24] is I_{cont}, the ideal generated by positive elements with continuous evaluation function over $\mathcal{T}(\mathcal{A})$ (Definition 4.1). We show that $\mathcal{A} \subsetneq I_{\text{cont}}$ (Proposition 4.4). If in addition, $\mathcal{A}$ has strict comparison of positive elements by traces, then $I_{\min} = I_{\text{cont}}$, and hence, $\mathcal{A} \neq I_{\min}$ (Theorem 4.6). This result can be seen as a generalization of Perera's construction [24] of the minimal ideal in the case that all quasitraces of $\mathcal{A}$ are traces (e.g., $\mathcal{A}$ is exact), while the weak unperforation of the K_0 group is equivalent to strict comparison by quasitraces, and hence, to strict comparison by traces.

What happens when there is no strict comparison by traces? In the case of the AH-algebras without slow dimension growth studied by Villadsen, which are known to have perforation, we prove that $I_{\min} \neq I_{\text{cont}}$ (Theorem 6.8). In addition, we show that if $\mathcal{A}$ has flat dimension growth, every positive element not in I_{cont} must be full (Theorem 6.10), and hence, I_{cont} contains every other proper ideal of $\mathcal{M}(\mathcal{A})$. If however the dimension growth is very fast, then this is no longer true (Proposition 6.12).

Finally, we prove that if $\mathcal{A}$ has strict comparison of positive elements, then so does $I_{\min}$. This result extends our previous result obtained when $\mathcal{A}$ is separable and has real rank zero ([11], Proposition 3.1). The methods used are inspired by the techniques used in Theorem 6.6 of [12] to prove that $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements if so does $\mathcal{A}$ and $\mathcal{A}$ has quasicontinuous scale in the sense of [14].

1. PRELIMINARIES

1.1. CUNTZ SUBEQUIVALENCE. Cuntz subequivalence in a C^*-algebra $\mathcal{B}$ is denoted by $\preceq$, that is, if $a, b \in \mathcal{B}_+$, then $a \preceq b$ if there is a sequence of elements $x_n \in \mathcal{B}$ such that $\|x_n b x_n^* - a\| \to 0$. If $a \preceq b$ and $b \preceq a$, then a is said to be equivalent to b ($a \sim b$). It is well known that for projections subequivalence in this sense coincides with Murray–von Neumann subequivalence.

We will use the following notation:

$$f_\varepsilon(t) := \begin{cases} 0 & \text{for } t \in [0, \varepsilon], \\ \frac{t-\varepsilon}{\varepsilon} & \text{for } t \in (\varepsilon, 2\varepsilon], \\ 1 & \text{for } t \in (2\varepsilon, \infty). \end{cases} \tag{1.1}$$

For ease of reference we list here the following well known facts (see for instance [5], [26]).

LEMMA 1.1. *Let $\mathcal{B}$ be a C^*-algebra, $a, b \in \mathcal{B}_+$, $x \in \mathcal{B}$, $\delta > 0$. Then*
(i) $xax^* \preceq a$;
(ii) $xx^* \sim x^*x$;
(iii) *if* $a \leqslant b$ *then* $a \preceq b$;
(iv) *if* $\|a - b\| < \delta$, *then* $(a - \delta)_+ \preceq b$;

(v) *if* $a \preceq b$, *then there is* $r \in \mathcal{B}$ *and* $\delta' > 0$ *such that* $(a-\delta)_+ = r(b-\delta')_+ r^*$; *there is also* $r' \in \mathcal{B}$ *such that* $(a-\delta)_+ = r' b r'^*$;
(vi) *if* $a \preceq a'$ *and* $b \preceq b'$, *then* $a+b \preceq a' \oplus b'$;
(vii) *if* $ab = 0$, *then* $(a+b-\delta)_+ = (a-\delta)_+ + (b-\delta)_+$;
(viii) ([12], *Lemma* 2.3) *if* $a \leqslant b$, *then* $(a-\delta)_+ \preceq (b-\delta)_+$;
(ix) ([12], *Lemma* 2.4.(iii)) $(a+b-\delta_1-\delta_2)_+ \preceq (a-\delta_1)_+ + (b-\delta_2)_+$ *for* $\delta_1, \delta_2 \geqslant 0$.

LEMMA 1.2. *Let* $\mathcal{B}$ *be a* C^**-algebra,* $a, b \in \mathcal{B}_+$, *and* $\|a-b\| < \delta$. *Then for all* $\varepsilon \geqslant 0$, $(a-\varepsilon-\delta)_+ \preceq (b-\varepsilon)_+$.

Proof. Since $\|a-(b-\varepsilon)_+\| \leqslant \|a-b\| + \|b-(b-\varepsilon)_+\| < \varepsilon+\delta$, the conclusion follows from Lemma 1.1(iv). ∎

LEMMA 1.3. *Let* $\mathcal{B}$ *be a* C^**-algebra and* $a \in \mathcal{B}_+$. *For every* $\varepsilon > 0$ *there is* $y \in \mathcal{B}$ *such that* $\|a - a^{1/2} y a y^* a^{1/2}\| < \varepsilon$ *and* $\|y a y^*\| = 1$.

Proof. Choose $g_\varepsilon(t) := \sqrt{\frac{f_\varepsilon(t)}{t}}$ and set $y = g_\varepsilon(a)$. Then $yay^* = f_\varepsilon(a)$ and

$$a - a^{1/2} y a y^* a^{1/2} = a(1 - f_\varepsilon(a)),$$

hence both conditions are satisfied. ∎

We need the following results for which we have no handy references. A related result is Lemma 2.3 of [17].

LEMMA 1.4. *Let* $\mathcal{B}$ *be a simple* C^**-algebra and* $0 \neq a, b \in \mathcal{B}_+$. *Then there is* $0 \neq c \in \mathcal{B}_+$ *such that* $c \preceq a$ *and* $c \leqslant b$.

Proof. Since $\mathcal{B}$ is simple, there are elements $x_k, y_k \in \mathcal{B}$ such that

$$\Big\| \sum_{k=1}^{n} x_k a y_k - b \Big\| < \frac{\|b\|}{2}.$$

Then $\sum_{k=1}^{n} x_k a y_k b \neq 0$, and hence, there is some k such that $x_k a y_k b \neq 0$. Then also

$$c := (x_k a y_k b)^* (x_k a y_k b) \neq 0, \quad d := (x_k a y_k b)(x_k a y_k b)^* \neq 0.$$

First notice that

$$d \leqslant \|b\|^2 \|y_k\|^2 \|a\| x_k a x_k^* \preceq a,$$

whence $d \preceq a$. Since $c \sim d$ by Lemma 1.1(ii), it follows that $c \preceq a$. On the other hand,

$$c \leqslant \|a\|^2 \|x_k\|^2 \|y_k\|^2 \|b\| b,$$

hence $c \leqslant b$, by scaling if necessary c, which preserves the relation $c \preceq a$. ∎

For the convenience of the readers, we give the proof of the following well known results.

LEMMA 1.5. *Let $\mathcal{B}$ be a simple, non-elementary C^*-algebra. Then for every element $0 \neq a \in \mathcal{B}_+$ there is an infinite sequence of mutually orthogonal elements $0 \neq a_k \in \mathcal{B}_+$ such that $\sum_{k=1}^{n} a_k \leqslant a$ for all n.*

Proof. Choose $\delta > 0$ such that $(a-\delta)_+ \neq 0$. Then $\mathrm{her}((a-\delta)_+)$ contains a positive element b with infinite spectrum (e.g., 1.11.45 of [18]; in fact it contains an element with spectrum $[0,1]$ by p. 67 of [2]). Since $b \leqslant \frac{\|b\|}{\delta} a$, to simplify notations, assume that $b \leqslant a$. Now choose by compactness a converging sequence of distinct elements $t_k \in \sigma(b)$, and by passing to a subsequence assume that the sequence $\{t_k\}$ is monotone and that $\{|t_k - t_{j+1}|\}$ is also monotone. Let $\varepsilon_k := \frac{1}{5}|t_k - t_{k+1}|$. Then the intervals $[t_k - 2\varepsilon_k, t_k + 2\varepsilon_k]$ are disjoint. Let g_k be the continuous function with

$$g_k(t) := \begin{cases} 0 & t \in [0, t_k - 2\varepsilon_k] \cup [t_k + 2\varepsilon_k, \infty), \\ t_k - \varepsilon_k & t \in [t_k - \varepsilon_k, t_k + \varepsilon_k], \\ \text{linear} & t \in [t_k - 2\varepsilon_k, t_k - \varepsilon_k], \\ \text{linear} & t \in [t_k + \varepsilon_k, t_k + 2\varepsilon_k]. \end{cases}$$

Let $a_k := \frac{g_k(b)}{2^j}$. Then $0 \neq a_k \leqslant \frac{b}{2^j} \leqslant \frac{a}{2^j}$ and $a_i a_k = 0$ for $i \neq j$. Thus we conclude that $\sum_{j=1}^{\infty} a_j \leqslant a$. ∎

LEMMA 1.6. *Let $\mathcal{B}$ be a C^*-algebra and let $a, b, c \in \mathcal{B}_+$ and $x \in \mathcal{B}$. Then*
(i) *$xax^* \sim xa^2x^*$;*
(ii) *$b^{1/2}ab^{1/2} \sim bab$;*
(iii) *if $b \leqslant c$, then $bab \preceq cac$.*

Proof. (i) First we see that $xa^2x^* \leqslant \|a\| xax^*$ and hence $xa^2x^* \preceq xax^*$. For every $\delta > 0$, $0 \leqslant (a-\delta)_+ \leqslant \frac{1}{4\delta}a^2$ and hence $x(a-\delta)_+x^* \preceq xa^2x^*$. Thus

$$xax^* = \lim_{\delta \to 0} x(a-\delta)_+x^* \preceq xa^2x^*.$$

(ii) We have:

$$\begin{aligned} b^{1/2}ab^{1/2} &\sim a^{1/2}ba^{1/2} \quad \text{(by Lemma 1.1(ii))} \\ &\sim a^{1/2}b^2a^{1/2} \quad \text{(by (i))} \\ &\sim bab \quad \text{(by Lemma 1.1(ii))}. \end{aligned}$$

(iii) We have:

$$\begin{aligned} bab &\sim b^{1/2}ab^{1/2} \quad \text{(by (ii))} \\ &\sim a^{1/2}ba^{1/2} \quad \text{(by Lemma 1.1(ii))} \\ &\preceq a^{1/2}ca^{1/2} \quad \text{(by Lemma 1.1(iii), since } a^{1/2}ba^{1/2} \leqslant a^{1/2}ca^{1/2}) \\ &\sim cac \quad \text{(by the same two equivalences above).} \end{aligned}$$ ∎

1.2. THE TRACIAL SIMPLEX AND STRICT COMPARISON. Given a simple σ-unital (possibly unital) C^*-algebra $\mathcal{A}$ and a nonzero positive element e in the Pedersen ideal $\mathrm{Ped}(\mathcal{A})$ of $\mathcal{A}$, denote by $\mathcal{T}(\mathcal{A})$ the collection of the (norm) lower semicontinuous densely defined tracial weights τ on $\mathcal{A}_+$, that are normalized on e. Explicitly, a trace τ is an additive and homogeneous map from $\mathcal{A}_+$ into $[0,\infty]$ (a weight), satisfies the trace condition $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{A}$, the cone $\{x \in \mathcal{A}_+ : \tau(x) < \infty\}$ is dense in $\mathcal{A}_+$ (τ is also called densely finite, or semifinite), satisfies the lower semicontinuity condition $\tau(x) \leqslant \underline{\lim}\, \tau(x_n)$ for $x, x_n \in \mathcal{A}_+$ and $\|x_n - x\| \to 0$, and $\tau(e) = 1$ (τ is normalized on e). We will assume henceforth that $\mathcal{T}(\mathcal{A}) \neq \emptyset$, and hence $\mathcal{A}$ is stably finite.

When equipped with the topology of pointwise convergence on $\mathrm{Ped}(\mathcal{A})$, $\mathcal{T}(\mathcal{A})$ is a Choquet simplex (e.g., see Proposition 3.4 of [30] and [9]). The collection of the extreme points of $\mathcal{T}(\mathcal{A})$ is denoted by $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$ and is called the *extremal boundary* of $\mathcal{T}(\mathcal{A})$. For simplicity's sake we call the elements of $\mathcal{T}(\mathcal{A})$ (respectively, $\partial_{\mathrm{e}}(\mathcal{T}(\mathcal{A}))$) traces (respectively, extremal traces.) Tracial simplexes $\mathcal{T}(\mathcal{A})$ arising from different nonzero positive elements in $\mathrm{Ped}(\mathcal{A})$ are homeomorphic; so we will not specify which element e is used. A trace τ on $\mathcal{A}$ is naturally extended to the trace $\tau \otimes \mathrm{Tr}$ on $\mathcal{A} \otimes \mathcal{K}$, and so we can identify $\mathcal{T}(\mathcal{A} \otimes \mathcal{K})$ with $\mathcal{T}(\mathcal{A})$. For more details, see [9], [30], and also [10] and [12].

Recall also that as remarked in 5.3 of [10], by the work of F. Combes ([4], Proposition 4.1, Proposition 4.4) and Ortega, Rørdam, and Thiel ([22], Proposition 5.2) every $\tau \in \mathcal{T}(\mathcal{A})$ has a unique extension, (which we will still denote by τ) to a lower semicontinuous (i.e., normal) tracial weight (trace for short) on the enveloping von Neumann algebra $\mathcal{A}^{**}$, and hence to a trace on the multiplier algebra $\mathcal{M}(\mathcal{A})$.

DEFINITION 1.7. Given a convex compact space K,

(i) $\mathrm{Aff}(K)$ denotes the *Banach space of the continuous real-valued affine functions* on K with the uniform norm;

(ii) $\mathrm{LAff}(K)$ denotes the *collection of the lower semicontinuous affine functions* on K with values in $\mathbb{R} \cup \{+\infty\}$;

(iii) $\mathrm{Aff}(K)_{++}$ (respectively, $\mathrm{LAff}(K)_{++}$) denotes the *cone of the strictly positive functions* (i.e., $f(x) > 0$ for all $x \in K$) in $\mathrm{Aff}(K)$ (respectively, in $\mathrm{LAff}(K)$).

For every $A \in \mathcal{M}(\mathcal{A})_+$, denote by $\widehat{A}$ the evaluation map

$$\mathcal{T}(\mathcal{A}) \ni \tau \to \widehat{A}(\tau) := \tau(A) \in [0,\infty], \tag{1.2}$$

and denote by $\widehat{[A]}$ the dimension map

$$\mathcal{T}(\mathcal{A}) \ni \tau \to \widehat{[A]}(\tau) := d_\tau(A) \in [0,\infty] \tag{1.3}$$

where

$$d_\tau(A) := \lim_n \tau(A^{1/n})$$

is the dimension function.

Then it is well known that $\widehat{A} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ and $\widehat{[A]} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_{++}$ for every $A \neq 0$. By definition of the topology on $\mathcal{T}(\mathcal{A})$, if $a \in \mathrm{Ped}(\mathcal{A})$, then $\widehat{a} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$.

As shown in Remark 5.3 of [22],

$$d_\tau(A) = \tau(R_A) \quad \text{where } R_A \in \mathcal{A}^{**} \text{ is the range projection of } A. \tag{1.4}$$

We will also use frequently the following well known facts. If $A, B \in \mathcal{M}(\mathcal{A})_+$, and $\tau \in \mathcal{T}(\mathcal{A})$ then

$$A \leqslant B \ \Rightarrow\ \widehat{A}(\tau) \leqslant \widehat{B}(\tau), \tag{1.5}$$

$$A \preceq B \ \Rightarrow\ d_\tau(A) \leqslant d_\tau(B), \tag{1.6}$$

$$AB = 0 \ \Rightarrow\ d_\tau(A+B) = d_\tau(A) + d_\tau(A), \tag{1.7}$$

$$\tau(A) \leqslant \|A\| d_\tau(A), \tag{1.8}$$

$$d_\tau((A-\delta)_+) < \frac{1}{\delta}\tau(A) \quad \forall\, \delta > 0. \tag{1.9}$$

We will use the following notions of strict comparison.

DEFINITION 1.8. Let $\mathcal{A}$ be a simple C^*-algebra with $\mathcal{T}(\mathcal{A}) \neq \emptyset$. Then we say that

(i) $\mathcal{A}$ has strict comparison of positive elements by traces if $a \preceq b$ for $a, b \in \mathcal{A}_+$ such that $d_\tau(a) < d_\tau(b)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(b) < \infty$.

(ii) $\mathcal{M}(\mathcal{A})$ has strict comparison of positive elements by traces if $A \preceq B$ whenever $A, B \in \mathcal{M}(\mathcal{A})_+$, A belongs to the ideal $I(B)$ generated by B, and $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$.

Notice that strict comparison is often defined in terms of 2-quasitraces. In Theorem 2.9 of [10] we proved that if a unital simple C^*-algebra of real rank zero and stable rank one has strict comparison of positive elements by traces (equivalently, of projections, due to real rank zero) then all 2-quasitraces are traces. Recently it was shown that in a simple stable C^*-algebra with strict comparison of positive elements by traces all 2-quasitraces are traces ([21], Theorem 3.6).

Notice also that if $\mathcal{A}$ is not unital and hence $\mathcal{M}(\mathcal{A})$ is not simple, $A \preceq B$ still implies that $A \in I(B)$, but this condition does not follow in general from the comparison condition. Indeed if there is an element $B \in \mathcal{A}_+$ with $d_\tau(B) = \infty$ for all $\tau \in \mathcal{T}(\mathcal{A})$ (and this is certainly the case when $\mathcal{A}$ is stable) then the condition $d_\tau(A) < d_\tau(B)$ for all those $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$ is trivially satisfied for every $A \in \mathcal{M}(\mathcal{A})_+$ and yet $A \not\preceq B$.

1.3. CONES AND IDEALS IN C^*-ALGEBRAS.

Let $\mathcal{B}$ be a C^*-algebra and let $K \subset \mathcal{B}_+$. Set

$$L(K) := \{x \in \mathcal{B} : x^*x \in K\}, \tag{1.10}$$

$$L(K)^*L(K) := \Big\{ \sum_{j=1}^{n} x_j^* y_j : x_j, y_j \in L(K),\ n \in \mathbb{N} \Big\}. \tag{1.11}$$

DEFINITION 1.9. Let $\mathcal{B}$ be a C^*-algebra and $K \subset \mathcal{B}_+$.

(i) K is a *cone* if $x + y \in K$ and $tx \in K$ whenever $x, y \in K$ and $0 \leqslant t \in \mathbb{R}$; K is *hereditary* if $x \in K$ whenever $0 \leqslant x \leqslant y \in K$.

(ii) A subalgebra $\mathcal{C} \subset \mathcal{B}$ is hereditary if the cone $\mathcal{C}_+$ is hereditary.

(iii) A cone K is

(a) *invariant* if $axa^* \in K$ whenever $x \in K$ and $a \in \mathcal{B}$;

(b) *strongly invariant* if $x^*x \in K$ whenever $x \in \mathcal{B}$ and $xx^* \in K$;

(c) *weakly invariant* if $axa^* \in \overline{K}$ whenever $x \in K$ and $a \in \mathcal{B}$.

Hereditary cones are also called *order ideals*. It is well known and immediate to see that if K is a hereditary cone, then $L(K)$ is a left ideal of $\mathcal{B}$, $L(K)^*L(K)$ and $L(K)^* \cap L(K)$ are $*$-subalgebras of $\mathcal{B}$, and $L(K)^*L(K) \subset L(K)^* \cap L(K)$. Furthermore, if K is a hereditary cone, then

(1.12) $L(K)$ is two-sided if and only if K is invariant.

(1.13) $L(K) = L(K)^*$ if and only if K is strongly invariant.

THEOREM 1.10. *Let $\mathcal{B}$ be a C^*-algebra and $K \subset \mathcal{B}_+$ be a hereditary cone. Then*

(i) *the norm closure $\overline{K}$ of K is a hereditary cone* ([7], *Theorem* 2.5*);*

(ii) $L(K)^*L(K) = \operatorname{span} K$ *(the collection of complex linear combinations of K) and* $(L(K)^*L(K))_+ = K$ *(*[29], *Proposition* 3.21*);*

(iii) *if K is closed, then $L(K)^*L(K) = L(K)^* \cap L(K)$ and the mappings $\mathcal{B} \to \mathcal{B}_+$, $K \to L(K)$, and $L \to L^* \cap L$ define bijective, order preserving correspondences between the sets of hereditary C^*-subalgebras of $\mathcal{B}$, closed hereditary cones of $\mathcal{B}_+$, and closed left ideals of $\mathcal{B}$ (*[7], *Theorem* 2.4 *and* [23], *Theorem* 1.5.2*).*

We collect here some properties of hereditary cones in C^*-algebras that we will use in this paper.

LEMMA 1.11. *Let $\mathcal{B}$ be a C^*-algebra and $K \subset \mathcal{B}_+$ be a cone.*

(i) *The (norm) closure $\overline{K}$ of K is a cone.*

(ii) *If K is weakly invariant, then $\overline{K}$ is invariant.*

(iii) *If K is invariant, then*

$$\overline{K} = \{x \in \mathcal{B}_+ : (x - \delta)_+ \in K\ \forall\, \delta > 0\}.$$

(iv) *If K is closed and invariant, then it is hereditary and strongly invariant.*

Proof. (i) Obvious.

(ii) Let $x \in \overline{K}$, $a \in \mathcal{B}$ and let $\{x_n\}$ be a sequence in K converging (in norm) to x. Since $ax_na^* \in \overline{K}$ for every n, it follows that $axa^* = \lim_n ax_na^* \in \overline{K}$, that is, $\overline{K}$ is invariant.

(iii) Let

$$K' := \{x \in \mathcal{B}_+ : (x - \delta)_+ \in K\ \forall\, \delta > 0\}.$$

Since $\lim\limits_{\delta\to 0}(x-\delta)_+ = x$ for all $x \in \mathcal{B}_+$, it follows that $K' \subset \overline{K}$. Conversely, let $x \in \overline{K}$, $\delta > 0$, and choose $y \in K$ such that $\|x-y\| < \frac{\delta}{2}$. Then $(x-\delta)_+ = ryr^* \in K$ for some $r \in \mathcal{B}$ by Lemma 1.1(iv) and (v). Thus $x \in K'$, which proves that $K' = \overline{K}$.

(iv) Let $x \leqslant y$, with $x \in \mathcal{B}_+$ and $y \in K$. By Lemma 1.1(iii) and (v) for every $\delta > 0$ there is an $r \in \mathcal{B}$ such that $(x-\delta)_+ = ryr^* \in K$ (because K is invariant). Thus $x = \lim\limits_{\delta\to 0}(x-\delta)_+ \in K$ (because K is closed), which proves that K is hereditary.

Now let $x^*x \in K$ and $x = v|x|$ be the polar decomposition of x. By Lemma 2.1 of [1], $v|x|^{1/n} \in \mathcal{B}$ for every $n \in \mathbb{N}$, hence $(v|x|^{1/n})x^*x(v|x|^{1/n})^* \in K$. Since $|x|^{1/n}|x| \to |x|$ in norm, it follows that also

$$xx^* = vx^*xv^* = \lim_n (v|x|^{1/n})x^*x(v|x|^{1/n})^* \in K,$$

which proves that K is strongly invariant. ∎

In the course of the proof of (iv) we have shown that

(1.14) if K is invariant and $0 \leqslant x \preceq y \in K$ then $(x-\delta)_+ \in K$ for all $\delta > 0$.

From Example 2.5 and Corollary 3.3, we will see that the condition in (iii) that K is invariant cannot be replaced by condition that K is weakly invariant.

COROLLARY 1.12. *Let $\mathcal{B}$ be a C^*-algebra and $K \subset \mathcal{B}_+$ a weakly invariant hereditary cone in a C^*-algebra $\mathcal{B}$; then $\overline{L(K)} = L(\overline{K})$, $\overline{L(K)}$ is a two-sided ideal, and $\overline{L(K)}_+ = \overline{K}$.*

Proof. By Lemma 1.11(i), (ii), and (iv), $\overline{K}$ is a strongly invariant hereditary cone. By (1.13), $L(\overline{K}) = L(\overline{K})^*$ and by Theorem 1.10(i) and (ii), $\operatorname{span}\overline{K} = L(\overline{K})$. Since K is hereditary, $L(K)$ is a left ideal, and hence, so is $\overline{L(K)}$. Moreover, $K \subset L(K)$, hence $\operatorname{span}\overline{K} \subset \overline{L(K)}$, and hence $L(\overline{K}) \subset \overline{L(K)}$. On the other hand, $L(K) \subset L(\overline{K})$, and hence $\overline{L(K)} \subset L(\overline{K})$, and thus $\overline{L(K)} = L(\overline{K})$. ∎

1.4. APPROXIMATE IDENTITIES. When $\mathcal{B}$ is a σ-unital C^*-algebra, and $\{e_n\}$ is an approximate identity, we will always assume that

(1.15) $\{e_n\}$ is strictly increasing ($0 \leqslant e_n \lneqq e_{n+1}$) and that $e_{n+1}e_n = e_n \ \forall\, n$.

It is also convenient to define $e_0 = 0$. Notice that $e_n \in \operatorname{Ped}(\mathcal{A})$ and $\|e_n\| = 1$ for all $n \geqslant 1$.

Notice that

$$(e_{m+1} - e_{n-1})(e_m - e_n) = e_m - e_n \quad \forall\, m > n, \tag{1.16}$$

and hence,

$$e_m - e_n \leqslant R_{e_m - e_n} \leqslant e_{m+1} - e_{n-1} \quad \forall\, m > n. \tag{1.17}$$

REMARK 1.13. We can always pass from an approximate identity satisfying the above conditions to a subsequence $\{f_n\}$ satisfying the following two stronger conditions assumed in [17]:

(i) Let $g_n := f_n - f_{n-1}$ ($f_0 := 0$), then $\|g_n\| = 1$ for all n and $g_n g_m = 0$ for $|m-n| \geqslant 2$.

(ii) There are $a_n \in \mathcal{B}_+$ with $\|a_n\| = 1$ such that $a_n \leqslant g_n$, $a_n g_n = g_n a_n = a_n$ and $a_n g_m = 0$ for $n \neq m$.

Proof. Let $f_n := e_{5n}$. Clearly, $(f_n - f_{n-1})(f_m - f_{m-1}) = 0$ for $|m-n| \geqslant 2$. Set $a_n := e_{5n-1} - e_{5n-4}$. Then $f_n - f_{n-1} \geqslant a_n$ by the monotonicity of e_n and

$$(f_n - f_{n-1})a_n = a_n(f_n - f_{n-1}) = a_n$$

by (1.16). Furthermore, $\|a_n\| = 1$ since by (1.17), $a_n \geqslant R_{e_{5n-2}-e_{5n-3}} \neq 0$; in particular, $\|f_n - f_{n-1}\| = 1$. ∎

2. THE MINIMAL IDEAL AND ITS HEREDITARY CONE

DEFINITION 2.1 ([16], Lemma 2.1). Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with an approximate identity $\{e_n\}$. Then we define the following set of positive elements in $\mathcal{M}(\mathcal{A})$:

$$K_{\mathrm{o}}(\{e_n\}) := \{X \in \mathcal{M}(\mathcal{A})_+ : \forall\, 0 \neq a \in \mathcal{A}_+ \ \exists\, N \in \mathbb{N} \\ \ni\ m > n \geqslant N \Rightarrow (e_m - e_n)X(e_m - e_n) \preceq a\}.$$

REMARK 2.2. (i) By Lemma 1.6(iii),

$$K_{\mathrm{o}}(\{e_n\}) := \{X \in \mathcal{M}(\mathcal{A})_+ : \forall\, 0 \neq a \in \mathcal{A}_+ \ \exists\, N \in \mathbb{N} \\ \ni\ m > N \Rightarrow (e_m - e_N)X(e_m - e_N) \preceq a\}.$$

This equivalent formulation will also be used in the paper.

(ii) If $\mathcal{A}$ has the (SP) property, (i.e., every nonzero hereditary subalgebra of $\mathcal{A}$ contains a nonzero projection), then for every $0 \neq a \in \mathcal{A}_+$ there is a projection $0 \neq p \preceq a$. Thus in the defining property of $K_{\mathrm{o}}(\{e_n\})$ we can replace "for all nonzero elements $a \in \mathcal{A}_+$" with "for all nonzero projections $p \in \mathcal{A}$".

LEMMA 2.3. *We have:*

(i) $X \in K_{\mathrm{o}}(\{e_n\})$ *if and only if* $X^{1/2} \in K_{\mathrm{o}}(\{e_n\})$;

(ii) $K_{\mathrm{o}}(\{e_n\})$ *is a hereditary cone of* $\mathcal{M}(\mathcal{A})$ *if and only if* $\mathcal{A}$ *is non-elementary.*

Proof. (i) Immediate from the definition and Lemma 1.6(i).

(ii) It is also immediate to verify that $K_{\mathrm{o}}(\{e_n\})$ is always hereditary and that if $X \in K_{\mathrm{o}}(\{e_n\})$ then $tX \in K_{\mathrm{o}}(\{e_n\})$ for every $t \geqslant 0$. Assume first that $\mathcal{A}$ is non-elementary and that $X, Y \in K_{\mathrm{o}}(\{e_n\})$. Let $0 \neq a \in \mathcal{A}_+$, then by Lemma 1.5 we can find two elements $0 \neq a', a'' \in \mathcal{A}_+$ with $a'a'' = 0$ and $a' + a'' \leqslant a$. Let N' (respectively, N'') be such that for all $m > n \geqslant N'$ (respectively, $m > n \geqslant N''$), we have $(e_m - e_n)X(e_m - e_n) \preceq a'$ (respectively, $(e_m - e_n)Y(e_m - e_n) \preceq a''$). Hence, for all $m > n \geqslant N := \max(N', N'')$ we have by Lemma 1.1(vi)

$$\begin{aligned}(e_m - e_n)(X+Y)(e_m - e_n) &= (e_m - e_n)X(e_m - e_n) + (e_m - e_n)Y(e_m - e_n)\\ &\preceq a' + a'' \leqslant a.\end{aligned}$$

Thus $K_o(\{e_n\})$ is a cone.

Assume now that $\mathcal{A} = \mathcal{K}$, and hence $\mathcal{M}(\mathcal{A}) = B(\mathcal{H})$, and let $\{e_n\}$ be an increasing sequence of rank n projections. Then it is easy to verify that

$$K_o(\{e_n\}) = \{x \in B(\mathcal{H})_+ : \exists n \ni \operatorname{rank}(1-e_n)x(1-e_n) \leqslant 1\}.$$

Let $\{\eta_n\}$ be an orthonormal basis of $\mathcal{H}$ such that $\operatorname{span}\{\eta_1, \dots, \eta_n\} = R_{e_n}$, and let $\xi := \sum\limits_{j=1}^{\infty} \frac{1}{2^j}\eta_{2j}$ and $\xi' := \sum\limits_{j=1}^{\infty} \frac{1}{2^j}\eta_{2j+1}$. Then both $\xi \otimes \xi$ and $\xi' \otimes \xi'$ belong to $K_o(\{e_n\})$ since they have rank one, but $(1-e_n)(\xi \otimes \xi + \xi' \otimes \xi')(1-e_n)$ has rank two for every n, and hence $\xi \otimes \xi + \xi' \otimes \xi' \notin K_o(\{e_n\})$. ∎

COROLLARY 2.4. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with an approximate identity $\{e_n\}$. Then $L(K_o(\{e_n\})$ is a left ideal and*

$$L(K_o(\{e_n\}))_+ = K_o(\{e_n\}).$$

That $L(K_o(\{e_n\})$ is a left ideal, is an immediate consequence of the fact that $K_o(\{e_n\})$ is a hereditary cone. The equality $K_o(\{e_n\}) = L(K_o(\{e_n\}))_+$ was suggested by H. Lin (private communications). $L(K_o(\{e_n\}))$ was denoted by I_o in [17] where Lin was primarily interested in the continuous case scale where $L(K_o(\{e_n\}))$ is two-sided. However, the following example shows that $K_o(\{e_n\})$ is in general not invariant, i.e., the ideal $L(K_o(\{e_n\}))$ is not two-sided.

EXAMPLE 2.5. Let $\mathcal{A}_o$ be a simple, unital, finite, non-elementary C^*-algebra and let $\mathcal{A} := \mathcal{A}_o \otimes \mathcal{K}$. Let $\{e_{ij}\}$ be the standard matrix units in $\mathcal{K}$, then $e_n := 1 \otimes \sum\limits_{k=1}^{n} e_{kk}$ is an increasing approximate identity of projections of $\mathcal{A}$. Let

$$V := 1 \otimes \sum_{k=1}^{\infty} 2^{-k/2} e_{1,k}.$$

Then $VV^* = e_1 = 1 \otimes e_{11} \in K_o(\{e_n\})$, i.e., $V^* \in L(K_o(\{e_n\}))$. Let

$$P := V^*V = 1 \otimes \sum_{h,k=1}^{\infty} 2^{-(h+k)/2} e_{h,k}.$$

For every $n > 1$ and $0 \neq a \in (\mathcal{A}_o)_+$ with $a \not\sim 1$ we have

$$(e_n - e_{n-1})P(e_n - e_{n-1}) = 1 \otimes 2^{-n} e_{n,n} \sim 1 \otimes e_{11} \not\preceq a \otimes e_{11}.$$

Thus $P \notin K_o(\{e_n\})$, i.e., $V \notin L(K_o(\{e_n\}))$. This example shows that the cone $K_o(\{e_n\})$ is not invariant, and, equivalently, that $L(K_o(\{e_n\}))$ is not a two-sided ideal. It also shows that $K_o(\{e_n\})$ does not satisfy the conclusion of Lemma 1.11(iii) since $P \in \overline{K_o(\{e_n\})}$ and yet $\frac{1}{2}P = (P - \frac{1}{2})_+ \notin K_o(\{e_n\})$. Furthermore, if we choose an approximate identity $f_n = 1 \otimes \sum_{k=1}^{n} f_{kk}$ with $f_{1,1} = \sum_{h,k=1}^{\infty} 2^{-(h+k)/2} e_{h,k}$, we see that $P \in K_o(\{f_n\})$, which shows that $K_o(\{e_n\}) \neq K_o(\{f_n\})$.

We notice that the proof of Lin's main results (e.g., Corollary 3.3 of [19]) did not use I_o or $\bar{I}_o$. In Corollary 3.3 we will see that $\overline{K_o(\{e_n\})}$ is always weakly invariant, and hence, $\overline{K_o(\{e_n\})}$ is strongly invariant and that $\overline{K_o(\{e_n\})}$ does not depend on the approximate identity $\{e_n\}$. Meanwhile, the next lemma shows that refinements of an approximate identity do not change the cone K_o.

LEMMA 2.6. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with an approximate identity $\{e_n\}$. Then $K_o(\{e_n\}) = K_o(\{e_{n_k}\})$ for any strictly increasing sequence n_k of integers.*

Proof. Let $X \in K_o(\{e_{n_k}\})$ and $0 \neq a \in \mathcal{A}_+$. Then there is an $L \in \mathbb{N}$ such that if $k > L$ then $(e_{n_k} - e_{n_L})X(e_{n_k} - e_{n_L}) \preceq a$. Let $m > n_L$ and choose k such that $n_k \geqslant m$. Then $e_m - e_{n_L} \leqslant e_{n_k} - e_{n_L}$, and hence, by Lemma 1.6(iii)

$$(e_m - e_{n_L})X(e_m - e_{n_L}) \preceq (e_{n_k} - e_{n_L})X(e_{n_k} - e_{n_L}) \preceq a.$$

Thus $X \in K_o(\{e_n\})$. The opposite inclusion is obvious. ∎

Given any approximate identity $\{e_n\}$ of $\mathcal{A}$, it is clear that $e_n a e_n \in K_o(\{e_n\})$ for every $a \in \mathcal{A}_+$ and $n \in \mathbb{N}$. Since $e_n a e_n \to a$, it follows that

$$\mathcal{A}_+ \subset \overline{K_o(\{e_n\})}. \tag{2.1}$$

The inclusion $\mathcal{A}_+ \subset K_o(\{e_n\})$ is however equivalent to the condition that $\mathcal{A}$ has continuous scale. Recall that $\mathcal{A}$ is said to have continuous scale if for some (and hence, for every) approximate identity $\{e_n\}$ and for every $0 \neq a \in \mathcal{A}_+$ there is an $N \in \mathbb{N}$ such that $e_m - e_n \preceq a$ for all $m > n \geqslant N$.

LEMMA 2.7. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, and non-elementary C^*-algebra with an approximate identity $\{e_n\}$. The following are equivalent:*

(i) *$\mathcal{A}$ has continuous scale;*
(ii) *$K_o(\{e_n\}) = \mathcal{M}(\mathcal{A})_+$;*
(iii) *$\overline{K_o(\{e_n\})} = \mathcal{M}(\mathcal{A})_+$;*
(iv) *$\mathcal{A}_+ \subset K_o(\{e_n\})$.*

Proof. (i) $\Rightarrow$ (ii) For every $x \in \mathcal{M}(\mathcal{A})_+$ and every $m > n$ we have

$$(e_m - e_n)x(e_m - e_n) \leqslant \|x\|(e_m - e_n) \preceq e_m - e_n.$$

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are obvious.

(iii) $\Rightarrow$ (ii) Since $1 \in \overline{K_o(\{e_n\})}$, there is an $x \in K_o(\{e_n\})$ such that $\|x - 1\| < 1$. Thus x is invertible, and hence, $\rho 1 \leqslant x$ for some scalar $\rho > 0$. Since $K_o(\{e_n\})$ is a hereditary cone, it follows that $1 \in K_o(\{e_n\})$, hence $\mathcal{M}(\mathcal{A})_+ \subset K_o(\{e_n\})$ and thus (ii) holds.

(iv) $\Rightarrow$ (i) Let $b := \sum_{k=1}^{\infty} \frac{1}{k}(e_{k+1}-e_k)$ where the convergence is in norm, and hence, $b \in \mathcal{A}_+ \subset K_o$. Then for every $0 \neq a \in \mathcal{A}_+$ there is an $N \in \mathbb{N}$ such that if $m \geqslant n \geqslant N+1$ then $(e_{m+1}-e_{n-1})b(e_{m+1}-e_{n-1}) \preceq a$. But then by (1.16) we also have

$$\begin{aligned} e_m - e_n &\sim \frac{1}{m-1}(e_m - e_n) = \frac{1}{m-1}(e_{m+1} - e_{n-1})(e_m - e_n)(e_{m+1} - e_{n-1}) \\ &= (e_{m+1} - e_{n-1}) \sum_{k=n}^{m-1} \frac{1}{m-1}(e_{k+1} - e_k)(e_{m+1} - e_{n-1}) \\ &\leqslant (e_{m+1} - e_{n-1}) \sum_{k=n}^{m-1} \frac{1}{k}(e_{k+1} - e_k)(e_{m+1} - e_{n-1}) \\ &\leqslant (e_{m+1} - e_{n-1}) b (e_{m+1} - e_{n-1}) \preceq a. \end{aligned}$$

Thus the scale is continuous. ∎

The implication (iii) $\Rightarrow$ (ii) is essentially the "only if" part of Theorem 2.10 in [17]. The following notions have appeared in various forms and various names in the literature (e.g., [20], and 4.3.11 of [3]) and for ease of reference we present them by the following formal definition.

DEFINITION 2.8. Let $\mathcal{B}$ be a C^*-algebra.

(i) A sequence of elements $0 \neq s_i \in \mathcal{B}_+$ is called *order dense* for $\mathcal{B}$ if for every $0 \neq a \in \mathcal{B}_+$ there is an integer n for which $s_n \preceq a$.

(ii) A sequence of mutually orthogonal elements $0 \neq t_i \in \mathcal{B}_+$ is called *thin* for $\mathcal{B}$ if for every $0 \neq a \in \mathcal{B}_+$ there is an integer N such that $\sum_{i=n}^{m} t_i \preceq a$ for all $m \geqslant n \geqslant N$.

Recall that a thin sequence of projections is called an ℓ^1 sequence in [20]. Clearly, thin sequences are order dense; also if $\{s_i'\}$ is an order dense sequence for $\mathcal{B}$ and $0 \neq s_i \in \mathcal{B}_+$ with $s_i \preceq s_i'$ for every i, then $\{s_i\}$ is also order dense for $\mathcal{B}$. Similarly, let $0 \neq s_i, s_i' \in \mathcal{B}_+$:

(2.2) if $\{s_i'\}$ is thin, $s_i s_j = 0$ for $i \neq j$ and $s_i \preceq s_i'$ $\forall\, i$, then $\{s_i\}$ is thin.

This follows from Lemma 1.1(vi) since $\sum_{i=n}^{m} s_i \preceq \sum_{i=n}^{m} s_i'$ for every $m \geqslant n$. It is also immediate to see that

(2.3) if $\{s_i'\}$ is thin, $s_i = \alpha_i s_i'$ for some $\alpha_i > 0$, then $\{s_i\}$ is thin.

In separable C^*-algebras, it is easy to construct order dense sequences (see also the construction in Lemma 2.4 of [17] and [34] for projections).

PROPOSITION 2.9. *Every separable C^*-algebra has an order dense sequence.*

Proof. Let $\mathcal{B}$ be a separable C^*-algebra and let $\{b_m\}$ be a sequence of positive elements dense in the unit ball of $\mathcal{B}_+$. Let $\{s_n\}$ be an enumeration of the nonzero elements in the collection $\{(b_m - \frac{1}{2})_+ : m \in \mathbb{N}\}$. For every $0 \neq a \in \mathcal{B}_+$ there is an $m \in \mathbb{N}$ such that $\left\| \frac{a}{\|a\|} - b_m \right\| < \frac{1}{2}$. Then $\|b_m\| > \frac{1}{2}$, hence $(b_m - \frac{1}{2})_+ \neq 0$, and thus $(b_m - \frac{1}{2})_+ = s_n$ for some n. Then $s_n \preceq \frac{a}{\|a\|} \sim a$ by Lemma 1.1(iv). Thus $\{s_n\}$ is an order dense sequence. ∎

Another case when order dense sequences are immediate to obtain is the following. For every C^*-algebra $\mathcal{A}$, denote by $D(\mathcal{A})$ the (possibly empty) dimension semigroup of Murray–von Neumann equivalence classes of projections. We say that $D(\mathcal{A})$ is order separable if there is a sequence $\{p_n\}$ of nonzero projections of $\mathcal{A}$ such that for every projection $0 \neq p \in \mathcal{A}$ there is a $p_n \preceq p$. Of course, if $D(\mathcal{A})$ is countable, it is also order separable, but type II_1 von Neumann factors are examples of (non-separable) C^*-algebras with a dimension semigroup $D(\mathcal{A})$ that is order separable but not countable.

PROPOSITION 2.10. *Every C^*-algebra $\mathcal{B}$ with* (SP) *property and with order separable dimension semigroup $D(\mathcal{B})$ has an order dense sequence of projections.*

Proof. By the (SP) property, for every $0 \neq a \in \mathcal{B}_+$ there is a nonzero projection $q \in \mathrm{her}(a)$, and hence, $q \preceq a$. Since $p_n \preceq q$ for some n, we have $p_n \preceq a$. Thus $\{p_n\}$ is order dense for $\mathcal{B}$. ∎

Starting with a thin sequence, we can construct an order dense sequence. For future use in this paper, we will prove a slightly stronger version than needed in this section. When $s, t \in \mathcal{B}_+$ and $n \in \mathbb{N}$, we will denote by ns an n-fold direct sum of s with itself. Then $ns \in M_n(\mathcal{B}_+)$ and the subequivalence relation $ns \preceq t$ is understood to hold in $M_n(\mathcal{B}_+)$. In particular, if $s \preceq t_i$ for $1 \leqslant i \leqslant n$ and t_i are mutually orthogonal, then by Lemma 1.1(vi), $ns \preceq \sum_{i=1}^{n} t_i$.

LEMMA 2.11. *Let $\mathcal{B}$ be a simple non-elementary C^*-algebra. Then for every sequence $\{s_i\}$ of elements $0 \neq s_i \in \mathcal{B}_+$, there is a sequence of mutually orthogonal elements $0 \neq t_i \in \mathcal{B}_+$ such that $n \sum_{i=n}^{m} t_i \preceq s_n$ for every pair of integers $m \geqslant n$.*

Proof. Let $\{a_i\}$ be a sequence of mutually orthogonal elements $0 \neq a_i \in \mathcal{B}_+$ (e.g., see Lemma 1.5). By Lemma 1.4, there are elements $0 \neq s'_i \in \mathcal{B}_+$ with $s'_i \leqslant a_i$ and $s'_i \preceq s_i$. For every i, use Lemma 1.5 to find an infinite sequence $\{s'_{i,j}\}$ of mutually orthogonal nonzero elements $0 \neq s'_{i,j} \in \mathcal{B}$ such that $\sum_{j=1}^{n} s'_{i,j} \leqslant s'_i$ for every n. For every j, set $s_{1,j} = s'_{1,j}$. Applying Lemma 1.4, find an element $0 \neq s_{2,j} \leqslant s'_{2,j}$, such that $s_{2,j} \preceq s_{1,j}$. By iterating the construction, find sequences $\{s_{i,j}\}$ of elements $0 \neq s_{i,j} \leqslant s'_{i,j}$ such that

$$s_{i,j} \preceq s_{i-1,j} \preceq \cdots \preceq s_{1,j} \quad \forall\, i,\, j.$$

Now apply again Lemma 1.5 to find mutually orthogonal elements $0 \neq t_{i,j} \in \mathcal{A}_+$ such that $\sum_{j=1}^{n} t_{i,j} \leqslant s_{i,i}$. By Lemma 1.4 we can assume again that for every i,

$$t_{i,i} \preceq t_{i,i-1} \preceq \cdots \preceq t_{i,1}.$$

Let $t_i := t_{i,i}$. Notice that the sequences $t_i \leqslant s_{i,i} \leqslant s'_i \leqslant a_i$ are mutually orthogonal. Thus for every $n \leqslant i \in \mathbb{N}$

$$nt_i \preceq \sum_{j=1}^{n} t_{i,j} \leqslant s_{i,i},$$

and hence,

$$n \sum_{i=n}^{m} t_i \preceq \sum_{i=n}^{m} s_{i,i} \preceq \sum_{i=n}^{m} s_{n,i} \preceq \sum_{i=n}^{m} s'_{n,i} \leqslant s'_n \preceq s_n. \quad \blacksquare$$

The following consequence is now immediate.

COROLLARY 2.12. *Let $\mathcal{B}$ be a simple non-elementary C^*-algebra. If $\mathcal{B}$ has an order dense sequence $\{s_i\}$, then it has a thin sequence $\{t_i\}$ with $t_i \preceq s_i$ for every i.*

If $\mathcal{A}$ is a simple, σ-unital, non-unital C^*-algebra with an approximate identity $\{e_n\}$, and two sequences of positive integers $\{m_j\}$ and $\{n_j\}$ such that $n_j < m_j < n_{j+1}$ for every j, and $\{d_j\}$ is a sequence of bounded elements $d_j \in \mathcal{A}_+$ for which $d_j \leqslant M_j(e_{m_j} - e_{n_j})$ for some $M_j > 0$, then the elements d_j are mutually orthogonal and the series $\sum_{j=1}^{\infty} d_j$ converges strictly. We will call the sum D of such a sequence *diagonal with respect to* $\{e_n\}$. Furthermore, $D \in \mathcal{A}$ if and only if $\lim_j \|d_j\| = 0$.

LEMMA 2.13. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra and assume that the element $D := \sum_{j=1}^{\infty} d_j$ is diagonal with respect to an approximate identity $\{e_n\}$. Then $D \in K_o(\{e_n\})$ if and only if the sequence $\{d_j\}$ is thin.*

Proof. Let $\{n_j\}$ and $\{m_j\}$ be sequences of positive integers such that $n_j < m_j < n_{j+1}$ for every j and $d_j \leqslant M_j(e_{m_j} - e_{n_j})$ for some $M_j > 0$ and all j. Assume first that the sequence $\{d_j\}$ is thin. Since for every $p \geqslant L \in \mathbb{N}$ we have

$$(1 - e_{n_L}) \sum_{j=1}^{L-1} d_j = 0 \quad \text{and} \quad e_{m_p} \sum_{j=p+1}^{\infty} d_j = 0$$

it follows that

$$(e_{m_p} - e_{n_L})D = (e_{m_p} - e_{n_L}) \sum_{j=L}^{p} d_j.$$

Now let $0 \neq a \in \mathcal{A}_+$ and $L \in \mathbb{N}$ be such that if $p \geqslant L$, then $\sum_{j=L}^{p} d_j \preceq a$. For every $m > N := n_L$ choose p such that $m_p \geqslant m$. Then by Lemma 1.6(iii) and

Lemma 1.1(i), we have

$$(e_m - e_N)D(e_m - e_N) \preceq (e_{m_p} - e_N)D(e_{m_p} - e_N) = (e_{m_p} - e_N)\sum_{j=L}^{p} d_j(e_{m_p} - e_N) \preceq \sum_{j=L}^{p} d_j \preceq a.$$

This proves that $D \in K_o(\{e_n\})$.

Assume now that $D \in K_o(\{e_n\})$ and let $0 \neq a \in \mathcal{A}_+$. Then there is an integer N such that $(e_m - e_N)D(e_m - e_N) \preceq a$ for every $m \geqslant N$. Let L be such that $n_L \geqslant N + 1$ and $p \geqslant L$. Since $\sum_{j=L}^{p} d_j \leqslant M(e_{m_p} - e_{n_L})$ and

$$(e_{m_p+1} - e_{n_L-1})(e_{m_p} - e_{n_L}) = e_{m_p} - e_{n_L},$$

it follows that $(e_{m_p+1} - e_{n_L-1}) \sum_{j=L}^{p} d_j = \sum_{j=L}^{p} d_j$. But then $m_p + 1 > N$, and hence,

$$\begin{aligned}\sum_{j=L}^{p} d_j &= (e_{m_p+1} - e_{n_L-1}) \sum_{j=L}^{p} d_j(e_{m_p+1} - e_{n_L-1}) \\ &\leqslant (e_{m_p+1} - e_{n_L-1})D(e_{m_p+1} - e_{n_L-1}) \preceq (e_{m_p+1} - e_N)D(e_{m_p+1} - e_N) \preceq a.\end{aligned}$$

This proves that $\{d_j\}$ is thin. ∎

THEOREM 2.14. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with an approximate identity $\{e_n\}$. Then the following are equivalent:*

(i) $\mathcal{A}_+ \neq \overline{K_o(\{e_n\})}$;

(ii) $\mathcal{A}$ *has an order dense sequence;*

(iii) $\mathcal{A}$ *has a thin sequence;*

(iv) $\mathcal{A}$ *has a thin sequence d_j such that $D = \sum_{j=1}^{\infty} d_j$ converges strictly to an element $D \in K_o(\{e_n\}) \setminus \mathcal{A}$.*

Proof. As usual, set $K_o = K_o(\{e_n\})$.

(i) $\Rightarrow$ (ii) $\mathcal{A}_+ \neq \overline{K}_o$ if and only if there is an element $X \in K_o \setminus A$. Then for every k, $(1 - e_k)X(1 - e_k) \neq 0$, hence there is some integer $m_k > k$ such that

$$s_k := (e_{m_k} - e_k)X(e_{m_k} - e_k) \neq 0.$$

By the defining property of K_o, for every $0 \neq a \in \mathcal{A}_+$ there is an integer N such that $s_N \preceq a$.

(ii) $\Rightarrow$ (iii) by Corollary 2.12.

(iii) $\Rightarrow$ (iv) Assume that $\{t_j\}$ is a thin sequence for $\mathcal{A}_+$. By Lemma 1.4, for every j we can find $0 \neq \widetilde{d}_j \in \mathcal{A}_+$ such that $\widetilde{d}_j \preceq t_j$ and $\widetilde{d}_j \leqslant e_{2j} - e_{2j-1}$. Let $d_j := \frac{\widetilde{d}_j}{\|\widetilde{d}_j\|}$ and $D := \sum_{j=1}^{\infty} d_j$. The sequence $\{d_j\}$ is mutually orthogonal and thin by (2.2) and (2.3), and by construction, D is diagonal with respect to $\{e_n\}$. Then by Lemma 2.13, $D \in K_o \setminus \mathcal{A}$.

(iv) $\Rightarrow$ (i) Obvious. ∎

Immediate consequences of Theorem 2.14, Proposition 2.9 and Proposition 2.10, and Lemma 2.7 are the following ((i) was obtained in Lemma 2.4 of [17]).

COROLLARY 2.15. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with an approximate identity $\{e_n\}$. Then $\mathcal{A}_+ \neq \overline{K}_{\mathrm{o}}$ in any of the following cases:*

(i) *$\mathcal{A}$ is separable;*

(ii) *the Cuntz semigroup is order separable;*

(iii) *$\mathcal{A}$ has the* (SP) *property and its dimension semigroup $D(\mathcal{A})$ of Murray–von Neumann equivalence classes of projections is order separable;*

(iv) *$\mathcal{A}$ has a continuous scale.*

We will see in Section 4 that another case when $\mathcal{A}_+ \neq \overline{K}_{\mathrm{o}}$ is when $\mathcal{A}$ has strict comparison of positive elements by traces (see Proposition 4.4 and Theorem 4.6).

3. THE MINIMAL IDEAL

We proceed now to prove that for every approximate identity $\{e_n\}$, as usual, satisfying (1.15), $K_{\mathrm{o}}(\{e_n\})$ is weakly invariant and to obtain properties of $\overline{L(K_{\mathrm{o}}(\{e_n\}))}$. In order to do that, we need first to strengthen a result obtained in Theorem 4.2 of [12]. Diagonal series have proven to be very valuable in working with multiplier algebras, starting with [8] and then [15], [25], [34] among many other. It is well known that a Weyl–von Neumann decomposition of selfadjoint elements into the sum of a diagonal series plus an element in $\mathcal{A}$ of arbitrarily small norm is possible only under additional conditions on $K_1(\mathcal{A})$ (e.g., if $\mathcal{A}$ has real rank zero, the Weyl–von Neumann theorem holds precisely when $\mathcal{M}(\mathcal{A})$ has real rank zero [35]). However a decomposition into a tridiagonal series plus remainder was obtained and used in [15] and [34]. A refinement of that construction, but with fewer hypotheses on $\mathcal{A}$, was obtained in [12] where we proved that if $\mathcal{A}$ is σ-unital, then every positive element $T \in \mathcal{M}(\mathcal{A})_+$ can be decomposed into the sum of a selfadjoint element in $\mathcal{A}$ of arbitrarily small norm and a *bidiagonal* series. A bidiagonal series $D := \sum\limits_{k=1}^{\infty} d_k$ is a strictly converging series with summands $d_k \in \mathcal{A}_+$ such that $d_k d_{k'} = 0$ for $|k-k'| > 1$. In particular, $D = D_{\mathrm{e}} + D_{\mathrm{o}}$, where $D_{\mathrm{e}} := \sum\limits_{k=1}^{\infty} d_{2k}$ and $D_{\mathrm{o}} := \sum\limits_{k=1}^{\infty} d_{2k-1}$ are diagonal series.

If $T \in K_{\mathrm{o}}(\{e_n\})$, the original proof in [12] can be modified to show that the bidiagonal series can be chosen in $\overline{K_{\mathrm{o}}(\{e_n\})}$. Also in order to obtain some further enhancements that will be needed later in this paper, and for the readers' convenience, we will present here a self-contained proof.

THEOREM 3.1. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with approximate identities $\{e_n\}$ and $\{f_m\}$, and let $X^*X \in K_o(\{e_n\})$ for some $X \in \mathcal{M}(\mathcal{A})$. Then for every $\varepsilon > 0$, there exist an element $t = t^* \in \mathcal{A}$ with $\|t\| < \varepsilon$, and a bidiagonal series $D := \sum_{k=1}^{\infty} d_k$ such that $XX^* = D + t$ and $D \in K_o(\{f_m\})$.*

Proof. Without loss of generality, assume that $\|X\| = 1$ and assume also that $XX^* \notin \mathcal{A}$ as the conclusion is trivial when $XX^* \in \mathcal{A}$ (e.g., see (2.1)). By Theorem 2.14, there exists a thin sequence $\{t_k\}$. By the definition of $K_o(\{e_n\})$ there is an increasing sequence $\{N_k\}$, such that

$$(e_m - e_{N_k})X^*X(e_m - e_{N_k}) \preceq t_{k+1} \quad \forall m > N_k.$$

Since $K_o(\{e_n\}) = K_o(\{e_{N_k}\})$ by Lemma 2.6, to simplify notations assume that

$$(e_m - e_n)X^*X(e_m - e_n) \preceq t_{n+1} \quad \forall m > n. \tag{3.1}$$

Fix $\varepsilon > 0$ and construct two sequences $\{m_k\}$ and $\{n_k\}$ of strictly increasing integers as follows. Set $m_0 = n_0 = n_{-1} = 0$, $n_1 = 1$ and $e_0 = f_0 = 0$. Since $\{f_m\}$ is an approximate identity, we can find $m_1 > 0$ such that

$$\|e_{n_1}X^*(1 - f_{m_1})\| < \frac{\varepsilon^2}{4^3}.$$

Then choose $n_2 > n_1 = 1$ such that

$$\|(1 - e_{n_2})X^*f_{m_1}\| < \frac{\varepsilon^2}{4^5}.$$

By iterating, construct strictly increasing sequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$\|e_{n_k}X^*(1 - f_{m_k})\| < \frac{\varepsilon^2}{4^{k+2}} \quad \text{for } k \geqslant 1,$$
$$\|(1 - e_{n_{k-1}})X^*f_{m_{k-2}}\| < \frac{\varepsilon^2}{4^{k+2}} \quad \text{for } k \geqslant 3.$$

When A, B, C are bounded operators, $\|C\| \leqslant 1$, and $0 \leqslant A \leqslant B$, then

$$\|A^{1/2}C\|^2 = \|C^*AC\| \leqslant \|C^*BC\| \leqslant \|BC\|.$$

Using the fact that $\|X\| = 1$ and $\|f_{m_k}\| = 1$ for all k, we can apply this inequality to $A := (e_{n_k} - e_{n_{k-1}})$ and

$$B := e_{n_k} \quad \text{and} \quad C := X^*(1 - f_{m_k})$$

and also to

$$B := 1 - e_{n_{k-1}} \quad \text{and} \quad C := X^*f_{m_{k-2}}.$$

Thus we obtain

$$\|(e_{n_k} - e_{n_{k-1}})^{1/2}X^*(1 - f_{m_k})\| \leqslant \frac{\varepsilon}{2^{k+2}} \quad \text{for } k \geqslant 1,$$
$$\|(e_{n_k} - e_{n_{k-1}})^{1/2}X^*f_{m_{k-2}}\| \leqslant \frac{\varepsilon}{2^{k+2}} \quad \text{for } k \geqslant 3.$$

By the triangle inequality,

$$\|(e_{n_k}-e_{n_{k-1}})^{1/2}X^* - (e_{n_k}-e_{n_{k-1}})^{1/2}X^*(f_{m_k}-f_{m_{k-2}})\|$$
$$= \|(e_{n_k}-e_{n_{k-1}})^{1/2}X^*(1-f_{m_k}) + (e_{n_k}-e_{n_{k-1}})^{1/2}X^* f_{m_{k-2}}\| < \frac{\varepsilon}{2^{k+1}}.$$

From the inequality $\|A^*A - B^*B\| \leqslant (\|A\| + \|B\|)(\|A-B\|)$ and the fact that $\|X\| = 1$ and $\|e_{n_k}\| = \|f_{m_k}\| = 1$, we thus have

$$\|X(e_{n_k}-e_{n_{k-1}})X^* - (f_{m_k}-f_{m_{k-2}})X(e_{n_k}-e_{n_{k-1}})X^*(f_{m_k}-f_{m_{k-2}})\| \leqslant \frac{\varepsilon}{2^k}. \tag{3.2}$$

Set

$$c_k := (f_{m_k}-f_{m_{k-2}})X(e_{n_k}-e_{n_{k-1}})X^*(f_{m_k}-f_{m_{k-2}}), \quad D := \sum_{k=1}^{\infty} c_k.$$

Since f_m is an approximate identity for $\mathcal{A}$ and the sequence

$$c_k \leqslant \|X\|^2(f_{m_k}-f_{m_{k-2}})^2 \leqslant f_{m_k}-f_{m_{k-2}}$$

is uniformly bounded, it is clear that the series converges strictly. Furthermore,

$$XX^* = \sum_{k=1}^{\infty} X(e_{n_k}-e_{n_{k-1}})X^*$$

where the series also converges strictly. Set

$$t := XX^* - D = \sum_{k=1}^{\infty} (X(e_{n_k}-e_{n_{k-1}})X^* - c_k).$$

It follows from (3.2) that this series converges in norm, hence $t = t^* \in \mathcal{A}$. Moreover

$$\|t\| \leqslant \sum_{k=1}^{\infty} \|X(e_{n_k}-e_{n_{k-1}})X^* - c_k\| < \varepsilon.$$

Thus we have the decomposition $XX^* = D + t$. We need to verify that D is a bidiagonal series and that $D \in K_{\mathrm{o}}(\{f_m\})$. We will use now (3.1), which is a consequence of $X^*X \in K_{\mathrm{o}}(\{e_n\}) \setminus \mathcal{A}$. For every $k > 1$

$$\begin{aligned} c_k &\preceq X(e_{n_k}-e_{n_{k-1}})X^* \quad \text{(by Lemma 1.1(i))} \\ &\sim (e_{n_k}-e_{n_{k-1}})^{1/2}X^*X(e_{n_k}-e_{n_{k-1}})^{1/2} \quad \text{(by Lemma 1.1(ii))} \\ &\sim (e_{n_k}-e_{n_{k-1}})X^*X(e_{n_k}-e_{n_{k-1}}) \quad \text{(by Lemma 1.6(ii))} \\ &\preceq t_k \quad \text{(by (3.1))}. \end{aligned}$$

Set $d_k := c_{2k} + c_{2k-1}$. By Lemma 1.1(vi),

$$d_k \preceq t_{2k} + t_{2k-1}. \tag{3.3}$$

Furthermore,

$$d_k \leqslant 2\|X\|^2(f_{m_{2k}} - f_{m_{2k-3}}) \tag{3.4}$$

whence we see that D is bidiagonal. In particular, the even and odd sequences

$$d_{2k} \leqslant 2\|X\|^2(f_{m_{4k}} - f_{m_{4k-3}}), \quad d_{2k+1} \leqslant 2\|X\|^2(f_{m_{4k+2}} - f_{m_{4k-1}}),$$

are both mutually orthogonal, satisfy the intertwining condition of Lemma 2.13, and are thin by (3.3) and (2.2) since

$$d_{2k} \preceq t_{4k} + t_{4k-1}, \quad d_{2k+1} \preceq t_{4k+2} + t_{4k+1}$$

and both sequences $\{t_{4k} + t_{4k-1}\}$ and $\{t_{4k+2} + t_{4k+1}\}$ are thin. But then their sums

$$D_{\mathrm{e}} := \sum_{k=1}^{\infty} d_{2k} \quad \text{and} \quad D_{\mathrm{o}} := \sum_{k=1}^{\infty} d_{2k-1}$$

are both in $K_{\mathrm{o}}(\{f_m\})$, and hence, $D = D_{\mathrm{e}} + D_{\mathrm{o}} \in K_{\mathrm{o}}(\{f_m\})$, which concludes the proof. ∎

REMARK 3.2. If in Theorem 3.1 we start with an element $B \in \mathcal{M}(\mathcal{A})_+$ and drop the hypothesis that $B \in K_{\mathrm{o}}(\{e_n\})$, the same proof yields the decomposition $B = D + t$ where D is a bidiagonal series. Furthermore, if $\{f_m\}$ is an approximate identity, then we can choose D to be the sum $D = D_{\mathrm{e}} + D_{\mathrm{o}}$ of two diagonal series with respect to $\{f_m\}$. In fact to obtain this result we only need to require that $\mathcal{A}$ is σ-unital (see Theorem 4.2 of [12]).

COROLLARY 3.3. *Let $\mathcal{A}$ be simple, σ-unital, non-unital, non-elementary and let $\{e_n\}$, $\{f_m\}$ be two approximate identities for $\mathcal{A}$. Then*

(i) $\overline{K_{\mathrm{o}}(\{e_n\})} = \overline{K_{\mathrm{o}}(\{f_m\})}$;

(ii) $K_{\mathrm{o}}(\{e_n\})$ *is weakly invariant, hence* $\overline{K_{\mathrm{o}}(\{e_n\})}$ *is hereditary and strongly invariant;*

(iii) $\overline{L(K_{\mathrm{o}}(\{e_n\}))}$ *is a two-sided ideal and*

$$\overline{L(K_{\mathrm{o}}(\{e_n\}))} = L(\overline{K_{\mathrm{o}}(\{e_n\}}) = \mathrm{span}\{\overline{K_{\mathrm{o}}(\{e_n\})}\}.$$

Proof. (i) If $T \in K_{\mathrm{o}}(\{e_n\})$, then by applying Theorem 3.1 to $X := T^{1/2}$ we see that $T \in \overline{K_{\mathrm{o}}(\{f_m\})}$, that is, $K_{\mathrm{o}}(\{e_n\}) \subset \overline{K_{\mathrm{o}}(\{f_m\})}$. Thus $\overline{K_{\mathrm{o}}(\{e_n\})} \subset \overline{K_{\mathrm{o}}(\{f_m\})}$. By reversing the role of the approximate identities we obtain equality.

(ii) If $X \in K_{\mathrm{o}}(\{e_n\})$ and $A \in \mathcal{M}(\mathcal{A})$ then

$$(X^{1/2}A^*)(X^{1/2}A^*)^* = X^{1/2}A^*AX^{1/2} \leqslant \|A\|^2 X \in K_{\mathrm{o}}(\{e_n\}),$$

hence by Theorem 3.1,

$$AXA^* = (X^{1/2}A^*)^*(X^{1/2}A^*) \in \overline{K_{\mathrm{o}}(\{e_n\})}.$$

Thus $K_{\mathrm{o}}(\{e_n\})$ is weakly invariant, and hence, by Lemma 1.11(ii) and (iv), we obtain that $\overline{K_{\mathrm{o}}(\{e_n\})}$ is hereditary and strongly invariant.

(iii) Follows immediately from Corollary 1.12 and Theorem 1.10. ∎

The independence of $\overline{L(K_o(\{e_n\}))}$ on the approximate identity was obtained in Remark 2.9 of [17]. From now on, we will denote

$$I_{\min} := \overline{L(K_o(\{e_n\}))}. \tag{3.5}$$

The following result sheds additional light on the relation between $I_{\min}$ and $L(K_o(\{e_n\}))$.

PROPOSITION 3.4. *Let $\mathcal{A}$ be simple, σ-unital, non-unital, non-elementary and let $\{e_n\}$ be an approximate identity for $\mathcal{A}$. Then $I_{\min} = \mathcal{A} + L(K_o(\{e_n\}))$.*

Proof. The inclusion $\mathcal{A} + L(K_o(\{e_n\})) \subset I_{\min}$ is obvious, and to prove equality it is enough to verify that if $D \in (I_{\min})_+ = \overline{K_o(\{e_n\})}$, then it follows that $D \in \mathcal{A} + L(K_o(\{e_n\}))$.

Without loss of generality, $\|D\| \leqslant 1$ and by Remark 3.2 we can assume that D is diagonal with respect to $\{e_n\}$. By further decomposing if necessary $D = \sum_{j=1}^{\infty} d_j$ into a sum of at most three diagonal series, we can assume that there is a sequence m_k such that $(e_{m_k} - e_{m_{k-1}})d_k = d_k$ for all k. To simplify notations, assume that

$$(e_k - e_{k-1})d_k = d_k \quad \forall k \tag{3.6}$$

(setting $e_0 = 0$). By Theorem 2.14, $\mathcal{A}$ has a thin sequence $\{t_j\}$. For every k find $b_k \in K_o(\{e_n\})$ such that $\|D - b_k\| < \frac{1}{k}$ and an integer n_k such that

$$(e_m - e_{n_k})b_k(e_m - e_{n_k}) \preceq t_k \quad \forall m > n_k.$$

Since

$$\|(e_m - e_{n_k})D(e_m - e_{n_k}) - (e_m - e_{n_k})b_k(e_m - e_{n_k})\| \leqslant \|D - b_k\| < \frac{1}{k}$$

it follows from Lemma 1.1(iv) that for all $m > n_k$,

$$\Big((e_m - e_{n_k})D(e_m - e_{n_k}) - \frac{1}{k}\Big)_+ \preceq (e_m - e_{n_k})b_k(e_m - e_{n_k}) \preceq t_k.$$

By (3.6),

$$\Big((e_m - e_{n_k})D(e_m - e_{n_k}) - \frac{1}{k}\Big)_+ = \Big(\sum_{j=n_k}^{m} d_j - \frac{1}{k}\Big)_+ = \sum_{j=n_k}^{m} \Big(d_j - \frac{1}{k}\Big)_+.$$

Set $\delta_j := \frac{1}{k}$ for $n_k \leqslant j < n_{k+1}$. Thus for all $k \in \mathbb{N}$, $\sum_{j=n_k}^{n_{k+1}-1} (d_j - \delta_j)_+ \preceq t_k$. Then for every $0 \neq a \in \mathcal{A}_+$ there is a $K \in \mathbb{N}$ such that $\sum_{j=K}^{k} t_j \preceq a$ for all $k > K$. For all $m > n_K$, choose $n_H \geqslant m$. Then

$$(e_m - e_{n_K})\Big(\sum_{j=1}^{\infty}(d_j - \delta_j)_+\Big)(e_m - e_{n_K}) = \sum_{j=n_K}^{m} (d_j - \delta_j)_+ \leqslant \sum_{j=n_K}^{n_H} (d_j - \delta_j)_+$$

$$\leqslant \sum_{k=K}^{H-1} \sum_{j=n_k}^{n_{k+1}} (d_j - \delta_j)_+ \preceq \sum_{k=K}^{H-1} t_k \preceq a$$

which proves that

$$\sum_{j=1}^{\infty} (d_j - \delta_j)_+ \in K_o(\{e_n\}) \subset L(K_o(\{e_n\})).$$

Finally,

$$D - \sum_{j=1}^{\infty} (d_j - \delta_j)_+ = \sum_{j=1}^{\infty} (d_j - (d_j - \delta_j)_+) \in \mathcal{A}_+$$

since $0 \leqslant d_j - (d_j - \delta_j)_+ \leqslant \delta_j(e_{j+1} - e_j)$. ∎

We proceed now to justify the notation $I_{\min}$. The natural "minimal ideal" is the intersection $\mathcal{J}_o$ of all ideals (not necessarily proper) properly containing $\mathcal{A}$, in symbols

$$\mathcal{J}_o := \bigcap \{\mathcal{J} \lhd \mathcal{M}(\mathcal{A}), \mathcal{A} \subsetneq \mathcal{J}\}. \tag{3.7}$$

Obviously $\mathcal{A} \subset \mathcal{J}_o$, but we do not know whether $\mathcal{A} \neq \mathcal{J}_o$ holds in general. However, we will prove now that $I_{\min} = \mathcal{J}_o$ (see Theorem 3.7). A key tool in that proof, and used also throughout this paper, is the following result obtained in [12].

PROPOSITION 3.5 ([12], Proposition 4.4). *Let $\mathcal{B}$ be a non-unital C^*-algebra and let $A = \sum_{n=1}^{\infty} A_n$, $B = \sum_{n=1}^{\infty} B_n$ where $A_n, B_n \in \mathcal{M}(\mathcal{B})_+$, $A_n A_m = 0$, $B_n B_m = 0$ for $n \neq m$ and the two series converge in the strict topology, and $A_n \preceq (B_n - \delta)_+$ for some $\delta > 0$ and for all n. Then for every $\varepsilon > 0$ and $0 < \delta' < \delta$ there is an $X \in \mathcal{M}(\mathcal{B})$ such that $(A - \varepsilon)_+ = X(B - \delta')_+ X^*$, and hence, $A \preceq (B - \delta')_+ \leqslant B$.*

If the sum of a positive diagonal series in $\mathcal{M}(\mathcal{B})$ is subequivalent to another strictly converging series in $\mathcal{M}(\mathcal{B})$ (not necessarily diagonal) then we can deduce the following relations between the summands.

PROPOSITION 3.6. *Let $\mathcal{B}$ be a non-unital C^*-algebra, let $A = \sum_{k=1}^{\infty} a_k$ and $B = \sum_{k=1}^{\infty} b_k$ be two strictly converging series with $a_k, b_k \in \mathcal{B}_+$ and with the elements a_k mutually orthogonal. If $A \preceq B$, then for every $\delta > 0$ and $M \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that for every $n \geqslant N$ there is an $m \geqslant M$ such that*

$$\sum_{k=N}^{n} (a_k - \delta)_+ \preceq \sum_{k=M}^{m} b_k.$$

Proof. By Lemma 1.1(v), there is an element $X \in \mathcal{M}(\mathcal{B})$ such that $\big(A - \frac{\delta}{6}\big)_+ = XBX^*$, and hence, by Lemma 1.3 there is a $Y \in \mathcal{M}(\mathcal{B})$ such that

$$\Big\| \Big(A - \frac{\delta}{6}\Big)_+ - \Big(\Big(A - \frac{\delta}{6}\Big)_+\Big)^{1/2} YXBX^*Y^* \Big(\Big(A - \frac{\delta}{6}\Big)_+\Big)^{1/2} \Big\| < \frac{\delta}{6} \tag{3.8}$$

and $\|YXBX^*Y^*\| \leqslant 1$. Because of the mutual orthogonality of a_k, and hence, of $\big(a_k - \frac{\delta}{6}\big)_+$, we have for every n

$$\Big(A - \frac{\delta}{6}\Big)_+ = \sum_{k=1}^{\infty} \Big(a_k - \frac{\delta}{6}\Big)_+ \geqslant \sum_{k=n}^{\infty} \Big(a_k - \frac{\delta}{6}\Big)_+. \tag{3.9}$$

If a, b, c are positive elements in a C^*-algebra $\mathcal{C}$ with $a \leqslant b$ and $\|c\| \leqslant 1$, then

$$\begin{aligned} \|a - a^{1/2}ca^{1/2}\| &= \|a^{1/2}(1-c)a^{1/2}\| = \|(1-c)^{1/2}a(1-c)^{1/2}\| \\ &\leqslant \|(1-c)^{1/2}b(1-c)^{1/2}\| = \|b - b^{1/2}cb^{1/2}\|. \end{aligned} \tag{3.10}$$

Thus from (3.8), (3.9), and (3.10) we have for all n

$$\Big\|\sum_{k=n}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+-\Big(\sum_{k=n}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}YXBX^*Y^*\Big(\sum_{k=n}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}\Big\|<\frac{\delta}{6}. \tag{3.11}$$

Since $YX\sum_{k=1}^{M-1} b_kX^*Y^* \in \mathcal{B}$ and $\sum_{k=n}^{\infty} \big(a_k - \frac{\delta}{6}\big)_+ \to 0$ strictly, we can find an integer N such that

$$\Big\|\Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}YX\sum_{k=1}^{M-1}b_kX^*Y^*\Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}\Big\| < \frac{\delta}{6}. \tag{3.12}$$

As a consequence of (3.11) and (3.12) we thus obtain

$$\Big\|\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+-\Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}YX\sum_{k=M}^{\infty}b_kX^*Y^*\Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}\Big\|<\frac{2\delta}{6}, \tag{3.13}$$

and hence

$$\begin{aligned} \sum_{k=N}^{\infty}\Big(a_k-\frac{3\delta}{6}\Big)_+ &\preceq \Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2}YX\sum_{k=M}^{\infty}b_kX^*Y^*\Big(\sum_{k=N}^{\infty}\Big(a_k-\frac{\delta}{6}\Big)_+\Big)^{1/2} \\ &\preceq \sum_{k=M}^{\infty} b_k. \end{aligned}$$

A fortiori, for every $n \geqslant N$, we have $\sum_{k=N}^{n}\big(a_k - \frac{3\delta}{6}\big)_+ \preceq \sum_{k=M}^{\infty} b_k$. Then again by Lemma 1.1(v), there is a $Z \in \mathcal{M}(\mathcal{B})$ such that

$$\sum_{k=N}^{n}\Big(a_k - \frac{4\delta}{6}\Big)_+ = Z\sum_{k=M}^{\infty} b_kZ^*.$$

Choose $e \in \mathcal{B}$ such that $\Big\|Z\sum_{k=M}^{\infty} b_kZ^* - eZ\sum_{k=M}^{\infty} b_kZ^*e\Big\| < \frac{\delta}{6}$, and then choose $m \geqslant M$ such that $\Big\|eZ\sum_{k=m+1}^{\infty} b_kZ^*e\Big\| < \frac{\delta}{6}$. Then

$$\Big\|\sum_{k=N}^{n}\Big(a_k - \frac{4\delta}{6}\Big)_+ - eZ\sum_{k=M}^{m} b_kZ^*e\Big\| < \frac{2\delta}{6},$$

and hence

$$\sum_{k=N}^{n}(a_k-\delta)_+ \preceq eZ\sum_{k=M}^{m} b_k Z^* e \preceq \sum_{k=M}^{m} b_k. \quad \blacksquare$$

The inclusion $I_{\min} \subset \mathcal{J}_{\mathrm{o}}$ in the following theorem has been obtained in Theorem 2.8 of [17], but for completeness's sake we include its proof.

THEOREM 3.7. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra. Then $I_{\min} = \mathcal{J}_{\mathrm{o}}$.*

Proof. To prove that $I_{\min} \subset \mathcal{J}_{\mathrm{o}}$, it is enough to show that given an approximate identity $\{e_n\}$, an element $D \in K_{\mathrm{o}}(\{e_n\})$ and an element $C \in \mathcal{M}(\mathcal{A})_+ \setminus \mathcal{A}$, then $D \in I(C)$. By Theorem 3.1 and Remark 3.2, $C = C_{\mathrm{e}} + C_{\mathrm{o}} + t$ for some $t = t^* \in \mathcal{A}$ and two positive diagonal series C_{e} and C_{o} (with respect to $\{e_n\}$), at least one of which, say C_{e}, does not belong to $\mathcal{A}$. Then, $I(C_{\mathrm{e}}) \subset I(C_{\mathrm{e}} + C_{\mathrm{o}}) = I(C)$, thus it is enough to prove that $D \in I(C_{\mathrm{e}})$. To simplify notations, assume that $C = \sum_{k=1}^{\infty} c_k$ itself is diagonal with respect to $\{e_n\}$. By Theorem 3.1 and Remark 3.2, we can also assume that the series $D = \sum_{k=1}^{\infty} d_k$ is diagonal with respect to $\{e_n\}$. Since $\lim_{\delta\to 0}(C-\delta)_+ = C \notin \mathcal{A}$, there is some $\delta > 0$ such that $(C-\delta)_+ \notin \mathcal{A}$. Since $(C-\delta)_+ = \sum_{k=1}^{\infty}(c_k-\delta)_+$, we can assume without loss of generality that $(c_k-\delta)_+ \neq 0$ for every k.

By Lemma 2.13, the sequence $\{d_j\}$ is thin, hence for every k there is an integer n_k such that

$$\sum_{j=n_k+1}^{m} d_j \preceq (c_k-\delta)_+ \quad \forall\, m \geqslant n_k,\ k \in \mathbb{N}.$$

Choose the sequence $\{n_k\}$ so to be strictly increasing. Then in particular

$$\sum_{j=n_k+1}^{n_{k+1}} d_j \preceq (c_k-\delta)_+ \quad \forall\, k \in \mathbb{N},$$

and hence, by Proposition 3.5,

$$\sum_{j=n_1+1}^{\infty} d_j \preceq \left(C-\frac{\delta}{2}\right)_+ \leqslant C.$$

Thus $D \in I(C)$, which shows that $I_{\min} \subset \mathcal{J}_{\mathrm{o}}$.

Now to prove that $\mathcal{J}_{\mathrm{o}} = I_{\min}$, we need to consider only the case that $\mathcal{A} \neq \mathcal{J}_{\mathrm{o}}$. We will prove that then $\mathcal{J}_{\mathrm{o}}$ contains a thin sequence, which by Theorem 2.14 implies that $\mathcal{A} \neq I_{\min}$ and hence that $\mathcal{J}_{\mathrm{o}} \subset I_{\min}$. Equality then holds by the first part of the proof.

Choose $D \in (\mathcal{J}_{\mathrm{o}})_+ \setminus \mathcal{A}$ and by invoking Theorem 3.1 and Remark 3.2 as in the first part of the proof, assume that $D := \sum_{k=1}^{\infty} d_k$ is diagonal with respect to $\{e_n\}$.

Let $\delta > 0$ be such that $(D-\delta)_+ \notin \mathcal{A}$. We claim that the sequence $\{(d_k-\delta)_+\}$ is thin. Since $\sum_{k=1}^{\infty}(d_k-\delta)_+ = (D-\delta)_+ \notin \mathcal{A}$, we can assume without loss of generality that $(d_k-\delta)_+ \neq 0$ for all k. Let $0 \neq a \in \mathcal{A}_+$. By Lemma 2.11 applied to the stationary sequence $s_i = a$, there is a sequence of mutually orthogonal elements $0 \neq t_i \in \mathcal{A}_+$ such that $n\sum_{i=n}^{m} t_i \preceq a$ for every pair of integers $m \geqslant n$. By Lemma 1.4 there are elements $0 \neq a_i' \leqslant e_{2i}-e_{2i-1}$ and $a_i' \preceq t_i$ for every i. Let $a_i := \frac{a_i'}{\|a_i'\|}$. Then the series converges strictly to an element $A := \sum_{i=1}^{\infty} a_i \in \mathcal{M}(\mathcal{A}) \setminus \mathcal{A}$ because $a_i \leqslant \frac{1}{\|a_i'\|}(e_{2i}-e_{2i-1})$ and $\|a_i\| = 1$ for every i. Furthermore, for every $m \geqslant M \in \mathbb{N}$ we have

$$M\sum_{i=M}^{m} a_i \sim M\sum_{i=M}^{m} a_i' \preceq M\sum_{i=M}^{m} t_i \preceq a. \tag{3.14}$$

Since $\mathcal{A} \subsetneq I(A)$, it follows that $\mathcal{J}_{\mathrm{o}} \subset I(A)$, and hence there is some M such that $\left(D-\frac{\delta}{2}\right)_+ \preceq MA$. By Proposition 3.6, there is some N such that for every $n \geqslant N$ there is $m \geqslant M$ for which

$$\sum_{k=N}^{n}(d_k-\delta)_+ \preceq \sum_{i=M}^{m} Ma_i \sim M\sum_{i=M}^{m} a_i \preceq a.$$

This proves that the sequence $\{(d_k-\delta)_+\}$ is thin and thus concludes the proof. ∎

In Corollary 3.3 of [19], Lin obtained that if $\mathcal{A}$ is non-unital, σ-unital, non-elementary, and simple, then $\mathcal{A}$ has continuous scale if and only if $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is simple, if and only if $\mathcal{M}(\mathcal{A})/\mathcal{A}$ is simple and purely infinite. Continuing Lin's work we prove the following theorem.

THEOREM 3.8. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra and assume that $I_{\min} \neq \mathcal{A}$. Then $I_{\min}/\mathcal{A}$ is purely infinite simple.*

Proof. By Theorem 3.7, it is trivial to see that $I_{\min}/\mathcal{A}$ is simple. Denote by $\pi : I_{\min} \to I_{\min}/\mathcal{A}$ the canonical quotient map. Choose a positive element $T \in I_{\min} \setminus \mathcal{A}$. Given an approximate identity $\{e_n\}$, by Theorem 3.1 and Remark 3.2 we can find a series $D := \sum_{k=1}^{\infty} d_k$ diagonal with respect to $\{e_n\}$ and with $0 \neq \pi(D) \leqslant \pi(T)$. Choose $\delta > 0$ such that $(D-\delta)_+ \notin \mathcal{A}$. By the diagonality of D, $(D-\delta)_+ = \sum_{k=1}^{\infty}(d_k-\delta)_+$ and assume that $(d_k-\delta)_+ \neq 0$ for every k. Apply Lemma 2.11 to the sequence $\{(d_k-\delta)_+\}$ to find a mutually orthogonal sequence $\{c_k''\}$ of elements $0 \neq c_k'' \in \mathcal{A}_+$ such that $nc_k'' \preceq (d_k-\delta)_+$ for every $n \in \mathbb{N}$ and $k \geqslant n$, where nc_k'' denotes as before the n-fold direct sum of c_k'' with itself. Choose $0 \neq c_k' \leqslant e_{2k}-e_{2k-1}$ with $c_k' \preceq c_k''$ for every k. Define $c_k := \frac{c_k'}{\|c_k'\|}$ and $C := \sum_{k=1}^{\infty} c_k$.

Then the series converge strictly and $C \notin \mathcal{A}$. Moreover,

$$nc_k \preceq nc''_k \preceq (d_k - \delta)_+ \quad \forall\, k \geqslant n.$$

By Proposition 3.5,

$$n \sum_{k=n}^{\infty} c_k \preceq \sum_{k=n}^{\infty} d_k.$$

But then

$$n\pi(C) = n\pi\Big(\sum_{k=n}^{\infty} c_k\Big) \preceq \pi\Big(\sum_{k=n}^{\infty} d_k\Big) = \pi(D) \leqslant \pi(T) \quad \forall\, n \in \mathbb{N}.$$

In particular, $\pi(C) \preceq \pi(T)$, that is, $C \in (I_{\min})_+ \setminus \mathcal{A}$.

On the other hand, $I_{\min}/\mathcal{A} = \mathcal{J}_0/\mathcal{A}$ by Theorem 3.7, and hence it is simple. Thus for every $\varepsilon > 0$ there is an m such that $\pi((T-\varepsilon)_+) \preceq m\pi(C)$, and hence

$$\pi((T-\varepsilon)_+) \oplus \pi((T-\varepsilon)_+) \preceq 2m\pi(C) \preceq \pi(T).$$

Since ε is arbitrary, it follows that $\pi(T) \oplus \pi(T) \preceq \pi(T)$ which proves that $I_{\min}/\mathcal{A}$ is purely infinite. ∎

4. THE MINIMAL IDEAL WHEN $\mathcal{A}$ HAS STRICT COMPARISON

DEFINITION 4.1. Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. Set:

(i) $K_c := \{X \in \mathcal{M}(\mathcal{A})_+ : \widehat{X} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))\}$;

(ii) $I_{\mathrm{cont}} := \overline{L(K_c)}$.

PROPOSITION 4.2. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. Then*

(i) *K_c is a hereditary strongly invariant cone; $L(K_c)$ is a two-sided selfadjoint ideal and hence so is $\overline{L(K_c)} = L(\overline{K}_c)$;*

(ii) *We have:*

$$\begin{aligned}(I_{\mathrm{cont}})_+ = \overline{K}_c &= \{X \in \mathcal{M}(\mathcal{A})_+ : \widehat{X} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))\}^- \\ &= \{X \in \mathcal{M}(\mathcal{A})_+ : \widehat{(X-\delta)}_+ \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))\ \forall\, \delta > 0\};\end{aligned}$$

(ii) *for a projection $P \in \mathcal{M}(\mathcal{A})$, $P \in I_{\mathrm{cont}}$ if and only if $\widehat{P}$ is continuous;*

(iv) $I_{\mathrm{cont}} = \overline{\mathrm{span}\, K_c}$.

Proof. (i) Since the map $\mathcal{M}(\mathcal{A})_+ \ni X \to \widehat{X} \in \mathrm{LAff}(\mathcal{T}(\mathcal{A}))_+$ satisfies the conditions $\widehat{X+Y} = \widehat{X} + \widehat{Y}$ and $\widehat{tX} = t\widehat{X}$ for $X, Y \in \mathcal{M}(\mathcal{A})_+$ and $t \in \mathbb{R}_+$, it is clear that K_c is a cone. Moreover, if $0 \leqslant X \leqslant Y \in K_c$, then

$$\widehat{X} + \widehat{Y-X} = \widehat{Y}.$$

Since $\widehat{Y}$ is affine and continuous and both $\widehat{X}$ and $\widehat{Y-X}$ are affine, lower semicontinuous, and non-negative, it is immediate to verify that both must be continuous.

Thus $X \in K_c$, and hence, K_c is hereditary. Since $\widehat{X^*X} = \widehat{XX^*}$ for all $X \in \mathcal{M}(\mathcal{A})$, K_c is strongly invariant. Therefore, the rest of the conclusions in (i) follow from (1.13), Lemma 1.11, (1.12), and Corollary 1.12.

(ii) By Corollary 1.12 and Theorem 1.10(i) and (ii) we have that

$$(\overline{L(K_c)})_+ = L(\overline{K}_c)_+ = \overline{K}_c$$

which is the first equality in (ii). The second equality is given by Lemma 1.11(iii).

(iii) Since $(P-\delta)_+ = \begin{cases} (1-\delta)P & 0 \leqslant \delta < 1, \\ 0 & \delta \geqslant 1, \end{cases}$ we have by (ii) that $P \in (I_{\mathrm{cont}})_+$ if and only if $\widehat{P} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$.

(iv) Since by (i) and Theorem 1.10, $I_{\mathrm{cont}} = L(\overline{K}_c) = \operatorname{span} \overline{K}_c$ is closed, it is immediate to see that $\operatorname{span} \overline{K}_c = \overline{\operatorname{span} K_c}$. ∎

Notice that if $\mathcal{A} = \mathcal{K}$, then K_c consists of the positive cone of the trace class operators, and hence, $I_{\mathrm{cont}} = \mathcal{K}$.

It is immediate to verify that $\mathcal{A} \subset I_{\mathrm{cont}}$. Indeed $(a-\delta)_+ \in \mathrm{Ped}(\mathcal{A})$ for every $\delta > 0$ and $a \in \mathcal{A}_+$, hence $\widehat{(a-\delta)}_+$ is continuous, that is, $(a-\delta)_+ \in K_c$. Thus $a \in \overline{K}_c \subset I_{\mathrm{cont}}$. To further relate I_{cont} to $\mathcal{A}$ and to $I_{\min}$ we need first the following lemma.

LEMMA 4.3. *Let $\mathcal{A}$ be a simple, non-elementary C^*-algebra with $\mathcal{T}(\mathcal{A}) \neq \emptyset$. Then for every $\varepsilon > 0$ there is an element $0 \neq c \in \mathcal{A}_+$ such that $d_\tau(c) < \varepsilon$ for all $\tau \in \mathcal{T}(\mathcal{A})$. Furthermore, the element c can be chosen in* $\mathrm{Ped}(\mathcal{A})$.

Proof. Let $0 \neq f \in \mathrm{Ped}(\mathcal{A})_+$ and recall that $\widehat{f} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_+$. Choose $\delta > 0$ such that $(f-\delta)_+ \neq 0$, and an integer $M \geqslant \frac{\|\widehat{f}\|}{\varepsilon\delta}$. By Lemma 1.5 we can find nonzero positive mutually orthogonal elements a_j such that $\sum_{j=1}^{M} a_j \preceq (f-\delta)_+$. By Lemma 1.4 choose a nonzero positive element $c \preceq a_j$ for $1 \leqslant j \leqslant M$. By Lemma 1.1(vi) it follows that

$$Mc \preceq \sum_{j=1}^{M} a_j \preceq (f-\delta)_+.$$

Thus for every $\tau \in \mathcal{T}(\mathcal{A})$

$$\begin{aligned} Md_\tau(c) &= d_\tau(Mc) \quad \text{(by (1.7))} \\ &\leqslant d_\tau((f-\delta)_+) \quad \text{(by (1.6))} \\ &\leqslant \frac{1}{\delta}\tau(f) \quad \text{(by (1.9))} \\ &\leqslant \frac{1}{\delta}\|\widehat{f}\|. \end{aligned}$$

Thus $d_\tau(c) < \varepsilon$. Finally, $(c-\delta)_+ \in \mathrm{Ped}(\mathcal{A})$ for every $\delta > 0$. Choose $\delta > 0$ such that $(c-\delta)_+ \neq 0$. Then $d_\tau((c-\delta)_+) \leqslant d_\tau(c) < \varepsilon$ for all $\tau \in \mathcal{T}(\mathcal{A})$. ∎

PROPOSITION 4.4. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$. Then $\mathcal{A} \subsetneq I_{\mathrm{cont}}$.*

Proof. By Lemma 4.3, there is an infinite sequence $\{\widetilde{a}_k\}$ of elements $0 \neq \widetilde{a}_k \in \mathcal{A}_+$ such that $d_\tau(\widetilde{a}_k) \leqslant \frac{1}{2^k}$ for all k and all $\tau \in \mathcal{T}(\mathcal{A})$. By Lemma 1.4 we can find $0 \neq a'_k \leqslant e_{3k} - e_{3k-1}$ with $a'_k \preceq \widetilde{a}_k$ for all k. Let $a_k := \frac{a'_k}{\|a'_k\|}$. Then

$$\tau(a_k) \leqslant d_\tau(a_k) \leqslant d_\tau(a'_k) \leqslant d_\tau(\widetilde{a}_k) \leqslant \frac{1}{2^k} \quad \forall\, k \text{ and } \forall\, \tau \in \mathcal{T}(\mathcal{A}).$$

Furthermore, $a_k \leqslant \frac{1}{\|a'_k\|}(e_{3k+1} - e_{3k-2}) \in \mathrm{Ped}(\mathcal{A})$, hence $\widehat{a}_k \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_+$. Let $A := \sum_{k=1}^{\infty} a_k$. Then the series converges strictly and since it is diagonal (i.e., $a_k a_{k'} = 0$ for $k \neq k'$) and does not converge in norm, $A \notin \mathcal{A}$. On the other hand, $\widehat{A} = \sum_{k=1}^{\infty} \widehat{a}_k$ is continuous since the series is uniformly convergent. Thus $A \in (I_{\mathrm{cont}})_+$. ∎

PROPOSITION 4.5. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with nonempty $\mathcal{T}(\mathcal{A})$. Then $K_{\mathrm{o}}(\{e_n\}) \subset K_{\mathrm{c}}$ for every approximate identity $\{e_n\}$. Consequently, $I_{\min} \subset I_{\mathrm{cont}}$.*

Proof. Let $0 \neq X \in K_{\mathrm{o}}(\{e_n\})$ and $\varepsilon > 0$. By Lemma 4.3 we can find an element $0 \neq c \in \mathcal{A}_+$ such that $d_\tau(c) < \frac{\varepsilon}{\|X\|}$ for every $\tau \in \mathcal{T}(\mathcal{A})$. By the definition of $K_{\mathrm{o}}(\{e_n\})$ there is an $N \in \mathbb{N}$ such that

$$(e_n - e_m)X(e_n - e_m) \preceq c \quad \forall\, n > m \geqslant N.$$

Now $X^{1/2}\widehat{(e_n - e_m)}X^{1/2} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_+$ because $X^{1/2}(e_n - e_m)X^{1/2} \in \mathrm{Ped}(\mathcal{A})$. Moreover,

$$\begin{aligned} X^{1/2}\widehat{(e_n - e_m)}X^{1/2}(\tau) &= (e_n - e_m)^{1/2}\widehat{X(e_n} - e_m)^{1/2}(\tau) \\ &\leqslant \|X\| d_\tau((e_n - e_m)^{1/2}X(e_n - e_m)^{1/2}) \\ &= \|X\| d_\tau((e_n - e_m)X(e_n - e_m)) \leqslant \|X\| d_\tau(c) < \varepsilon. \end{aligned}$$

Thus the series $\widehat{X} = \sum_{n=1}^{\infty} X^{1/2}\widehat{(e_n - e_{n-1})}X^{1/2}$ converges uniformly and hence $X \in K_{\mathrm{c}}$. This proves that $K_{\mathrm{o}}(\{e_n\}) \subset K_{\mathrm{c}}$, and hence, $I_{\min} \subset I_{\mathrm{cont}}$. ∎

In general, $I_{\min}$ may fail to coincide with I_{cont} as we will see in section 6.

THEOREM 4.6. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with strict comparison of positive elements by traces. Then $I_{\min} = I_{\mathrm{cont}}$.*

Proof. By Proposition 4.5 we need to prove that $(I_{\mathrm{cont}})_+ \subset (I_{\min})_+$. As in the proof of Theorem 3.7, it is enough to verify that if $\{e_n\}$ is an approximate identity for $\mathcal{A}$, $D = \sum_{k=1}^{\infty} d_k$ is diagonal with respect to $\{e_n\}$, and $D \in I_{\mathrm{cont}}$, then

$D \in I_{\min}$. Let $\delta > 0$, and by dropping if necessary the zero summands in the series $(D-\delta)_+ = \sum\limits_{k=1}^{\infty} (d_k - \delta)_+$, assume that $(d_k-\delta)_+ \neq 0$ for all k. We claim that the sequence $\{(d_k-\delta)_+\}$ is thin.

Let $0 \neq a \in \mathcal{A}_+$. Recall that the function $d_\tau(a)$ is lower semicontinuous, and hence, $\min\limits_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a) > 0$. By Proposition 4.2, $\widehat{(D-\delta)}_+ \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$ and since $(d_k-\delta)_+ \in \mathrm{Ped}(\mathcal{A})$ for all k, also $\widehat{(d_k-\delta)}_+ \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))$. Since

$$\tau((D-\delta)_+) = \sum_{k=1}^{\infty} \tau((d_k-\delta)_+),$$

by Dini's theorem the series converges uniformly on $\mathcal{T}(\mathcal{A})$ for every $\delta > 0$. In particular, there is an N such that if $j \geqslant i \geqslant N$ and $\tau \in \mathcal{T}(\mathcal{A})$, then

$$\sum_{k=i}^{j} \tau\Big(\Big(d_k - \frac{\delta}{2}\Big)_+\Big) < \frac{\delta}{2} \min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a). \tag{4.1}$$

By (1.9), $d_\tau((d_k-\delta)_+) \leqslant \frac{2}{\delta}\tau\big(\big(d_k - \frac{\delta}{2}\big)_+\big)$, and hence, by (1.7),

$$d_\tau\Big(\sum_{k=i}^{j}(d_k-\delta)_+\Big) = \sum_{k=i}^{j} d_\tau((d_k-\delta)_+) \leqslant \sum_{k=i}^{j} \frac{2}{\delta}\tau\Big(\Big(d_k - \frac{\delta}{2}\Big)_+\Big) < \min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau(a).$$

By the hypothesis of strict comparison of positive elements by traces, we thus have that $\sum\limits_{k=i}^{j} (d_k-\delta)_+ \preceq a$, which proves that the sequence $\{(d_k-\delta)_+\}$ is thin. But then $(D-\delta)_+ \in K_{\mathrm{o}}(\{e_n\})$ by Lemma 2.13. Since δ is arbitrary, it follows that $D \in I_{\min} = \overline{K_{\mathrm{o}}(\{e_n\})}$ which concludes the proof. ∎

As a consequence of this theorem, any counterexample for $I_{\min} \neq \mathcal{A}$, could only be found among non-separable C^*-algebras with no strict comparison of positive elements. Among such algebras is the C^*-algebra $\mathcal{A}$ introduced by Rørdam to provide an example of a simple unital C^*-algebra with both infinite projections and nonzero finite projections ([27], Theorem 5.6). Recall that $\mathcal{A}$ is the C^*-inductive limit $\mathcal{A} = \lim\limits_{n\to\infty} \mathcal{M}(\mathcal{B}_n)$, where all $\mathcal{B}_n$ are separable C^*-algebras. So, while the algebras $\mathcal{M}(\mathcal{B}_n)$ are not separable and hence neither is $\mathcal{A}$, the order dense sequences for $\mathcal{B}_n$ are order dense also for $\mathcal{M}(\mathcal{B}_n)$ and therefore their union is order dense for $\mathcal{A}$. As a consequence $I_{\min} \neq \mathcal{A}$.

5. STRICT COMPARISON IN THE MINIMAL IDEAL

In Theorem 6.6 of [12] we proved that if $\mathcal{A}$ is a σ-unital simple C^*-algebra with strict comparison of positive elements by traces and with quasicontinuous scale (e.g., with finite extremal boundary), then strict comparison of positive elements by traces (see Definition 1.8) holds also in $\mathcal{M}(\mathcal{A})$. In this section we will show that if we restrict our attention to comparison between elements in I_{cont}, then strict comparison holds *without* requiring the scale to be quasicontinuous.

For the first step we list here a slightly modified version of Lemma 6.2 in [12].

LEMMA 5.1. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$ and let $A \in (I_{\text{cont}})_+$, $B \in \mathcal{M}(\mathcal{A})_+$, and assume that $d_\tau(A) < d_\tau(B)$ for every $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ and $\alpha > 0$ such that $d_\tau((A-\varepsilon)_+) + \alpha \leqslant d_\tau((B-\delta)_+)$ for every $\tau \in \mathcal{T}(\mathcal{A})$.*

The proof being essentially the same, we refer the reader to Lemma 6.2 of [12]. The only difference is that here we need to replace the condition used in Lemma 6.2 of [12] that $\widehat{A}:_K$ is continuous for some closed subset K of $\mathcal{T}(\mathcal{A})$, with the condition that $\widehat{\left(A-\frac{\varepsilon}{2}\right)_+}$ is continuous on the whole of $\mathcal{T}(\mathcal{A})$, which follows from Proposition 4.2.

The next lemma extends the results of Lemma 6.4 in [12].

LEMMA 5.2. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital C^*-algebra with nonempty tracial simplex $\mathcal{T}(\mathcal{A})$ and let $B = \sum_{k=1}^{\infty} b_k$ be a strictly converging series with $b_k \in \mathcal{A}_+$ for all k and $b_n b_m = 0$ for $|n-m| \geqslant 2$. Assume that $B \in (I_{\text{cont}})_+$ and that $\delta > 0$. Then*

(i) $d_\tau\Big(\Big(\sum_{k=n}^{\infty} b_k - \delta\Big)_+\Big) \downarrow 0$ *uniformly on $\mathcal{T}(\mathcal{A})$;*

(ii) *for every $\varepsilon > 0$ and $0 < \delta' < \delta$ there is an n such that for all $\tau \in \mathcal{T}(\mathcal{A})$*

$$d_\tau\Big(\Big(\sum_{k=1}^{n} b_k - \delta'\Big)_+\Big) > d_\tau\Big(\Big(\sum_{k=1}^{\infty} b_k - \delta\Big)_+\Big) - \varepsilon.$$

Proof. (i) The sequence $\Big\{d_\tau\Big(\Big(\sum_{n}^{\infty} b_k - \delta\Big)_+\Big)\Big\}$ is monotone decreasing by Lemma 1.1(viii) and (1.6). Moreover, by Lemma 1.1(ix)

$$d_\tau\Big(\Big(\sum_{k=n}^{\infty} b_k - \delta\Big)_+\Big) \leqslant d_\tau\Big(\Big(\sum_{k\geqslant n,\, k \text{ even}} b_k - \frac{\delta}{2}\Big)_+\Big) + d_\tau\Big(\Big(\sum_{k\geqslant n,\, k \text{ odd}} b_k - \frac{\delta}{2}\Big)_+\Big).$$

The series of the even and odd terms separately are diagonal and dominated by B, hence they still belong to I_{cont}. Thus it is enough to assume that $\sum_{k=1}^{\infty} b_k$ itself is diagonal.

Then $\left(B-\frac{\delta}{2}\right)_+ \in K_{\text{c}}$ by Proposition 4.2(ii), hence

$$\widehat{\Big(B-\frac{\delta}{2}\Big)_+} = \sum_{k=1}^{\infty} \widehat{\Big(b_k-\frac{\delta}{2}\Big)_+} \in \operatorname{Aff}(\mathcal{T}(\mathcal{A}))_+ .$$

Since also $\widehat{\left(b_k - \frac{\delta}{2}\right)_+} \in \mathrm{Aff}(\mathcal{T}(\mathcal{A}))_+$ for every k, by Dini's theorem this series converges uniformly. But then

$$d_\tau\Big(\Big(\sum_{k=n}^{\infty} b_k - \delta\Big)_+\Big) = \sum_{k=n}^{\infty} d_\tau((b_k - \delta)_+) \leqslant \frac{2}{\delta}\sum_{k=n}^{\infty} \tau\Big(\Big(b_k - \frac{\delta}{2}\Big)_+\Big) \to 0$$

uniformly on $\mathcal{T}(\mathcal{A})$.

(ii) Again, by Lemma 1.1(ix), for every $0 < \delta' < \delta$ we have

$$d_\tau\Big(\Big(\sum_{k=1}^{\infty} b_k - \delta\Big)_+\Big) \leqslant d_\tau\Big(\Big(\sum_{k=1}^{n} b_k - \delta'\Big)_+\Big) + d_\tau\Big(\Big(\sum_{k=n+1}^{\infty} b_k - (\delta - \delta')\Big)_+\Big).$$

By (i), we can choose n such that $d_\tau\Big(\Big(\sum_{n+1}^{\infty} b_k - (\delta - \delta')\Big)_+\Big) < \varepsilon$ for all τ. ∎

REMARK 5.3. If in the above lemma there is a projection $P \in I_{\mathrm{cont}}$ such that $B \leqslant \|B\|P$, then it is easy to verify that the uniform convergence in (i) holds also for $\delta = 0$ (see also Lemma 6.4 of [12]). Furthermore, (ii) strengthens to the statement that

$$d_\tau\Big(\Big(\sum_{k=1}^{n} b_k - \delta\Big)_+\Big) \to d_\tau\Big(\Big(\sum_{k=1}^{\infty} b_k - \delta\Big)_+\Big) \quad \text{uniformly on } \mathcal{T}(\mathcal{A}).$$

However, these stronger results do not hold in general as it is readily seen by considering $B := \sum_{k=1}^{\infty} \frac{1}{k}(e_{k+1} - e_k)$ for some approximate identity $\{e_n\}$ in a stable algebra $\mathcal{A}$. Indeed then $B \in \mathcal{A} \subset I_{\mathrm{cont}}$, but $d_\tau\Big(\sum_{k=n}^{\infty} b_k\Big) = \infty$ for all n.

We are ready now to prove that strict comparison holds for $I_{\min}$ provided that it holds for $\mathcal{A}$.

THEOREM 5.4. *Let $\mathcal{A}$ be a simple, σ-unital, non-unital, non-elementary C^*-algebra with strict comparison of positive elements by traces, $A, B \in (I_{\min})_+$ and assume that $B \notin \mathcal{A}$. If $d_\tau(A) < d_\tau(B)$ for all $\tau \in \mathcal{T}(\mathcal{A})$ for which $d_\tau(B) < \infty$, then $A \preceq B$.*

Proof. Let $\varepsilon > 0$. By Theorem 4.6, $I_{\min} = I_{\mathrm{cont}}$. Thus by Lemma 5.1 there is a $\delta > 0$ and $\alpha > 0$ such that

$$d_\tau((A - \varepsilon)_+) + \alpha \leqslant d_\tau((B - 4\delta)_+) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A}).$$

By the assumption that $B \notin \mathcal{A}$, we can reduce if necessary δ so to also have $(B - 4\delta)_+ \notin \mathcal{A}$. By Theorem 3.1 and Remark 3.2, $B = \sum_{k=1}^{\infty} b_k + t$ where $\sum_{k=1}^{\infty} b_k$ is a strictly converging bi-diagonal series, $t = t^* \in \mathcal{A}$, and $\|t\| < \delta$. Then by Lemma 1.2

$$(B - 4\delta)_+ \preceq \Big(\sum_{k=1}^{\infty} b_k - 3\delta\Big)_+ \preceq B \tag{5.1}$$

whence by (1.6) for all τ

$$d_\tau((A-\varepsilon)_+) + \alpha \leqslant d_\tau\Big(\Big(\sum_{k=1}^{\infty} b_k - 3\delta\Big)_+\Big).$$

By Lemma 5.2(ii), there is an n_1 such that

$$d_\tau((A-\varepsilon)_+) < d_\tau\Big(\Big(\sum_{k=1}^{n_1} b_k - 2\delta\Big)_+\Big) \quad \forall \tau \in \mathcal{T}(\mathcal{A})\,. \tag{5.2}$$

Since $(B-4\delta)_+ \notin \mathcal{A}$, we have by (5.1) that $\Big(\sum\limits_{k=1}^{\infty} b_k - 2\delta\Big)_+ \notin \mathcal{A}$. But then

$$\forall\, n\ \exists\, m \geqslant n \text{ such that } \Big(\sum_{k=n}^{m} b_k - 2\delta\Big)_+ \neq 0. \tag{5.3}$$

Otherwise if there was an n such that $\Big(\sum\limits_{k=n}^{m} b_k - 2\delta\Big)_+ = 0$ for all m, the strict convergence $\Big(\sum\limits_{k=n}^{m} b_k - 2\delta\Big)_+ \to \Big(\sum\limits_{k=n}^{\infty} b_k - 2\delta\Big)_+$ (see Lemma 3.1 of [12]) would imply that $\Big(\sum\limits_{k=n}^{\infty} b_k - 2\delta\Big)_+ = 0$, and hence, from Lemma 1.1(ix),

$$\Big(\sum_{k=1}^{\infty} b_k - 2\delta\Big)_+ \preceq \sum_{k=1}^{n-1} b_k + \Big(\sum_{k=n}^{\infty} b_k - 2\delta\Big)_+ \in \mathcal{A},$$

a contradiction. Now starting with the integer n_1 just constructed, and by the same argument, define inductively an increasing sequence $\{n_k\}$ of integers $n_k \geqslant n_{k-1}+2$ such that

$$\Big(\sum_{j=n_k+2}^{n_{k+1}} b_j - 2\delta\Big)_+ \neq 0 \quad \forall\, k.$$

Let $d_1 := \sum\limits_{j=1}^{n_1} b_j$ and $d_{k+1} := \sum\limits_{j=n_k+2}^{n_{k+1}} b_j$. By construction, $d_n d_m = 0$ for $n \neq m$ and

$$\sum_{k=1}^{\infty} d_k \leqslant \sum_{k=1}^{\infty} b_k. \tag{5.4}$$

By construction $(d_k - 2\delta)_+ \neq 0$ for all k and the function $d_\tau((d_k-2\delta)_+)$ is lower semicontinuous and strictly positive. Let

$$\beta_k := \min_{\tau \in \mathcal{T}(\mathcal{A})} d_\tau((d_k - 2\delta)_+).$$

By (5.2) we also have

$$d_\tau((A-\varepsilon)_+) < d_\tau((d_1 - 2\delta)_+) \quad \forall\, \tau. \tag{5.5}$$

Now apply Theorem 3.1 and Remark 3.2 to decompose A into the strictly converging sum of a series $\sum\limits_{j=1}^{\infty} a_j$ and a selfadjoint remainder $a \in \mathcal{A}$ with $a_j \in \mathcal{A}_+$ for all j, $a_j a_k = 0$ for $|j-k| \geqslant 2$, and $\|a\| \leqslant \varepsilon$. By Lemma 5.2(i) we can find a strictly increasing sequence of integers $\{m_k\}$ such that

$$d_\tau\Big(\Big(\sum_{j=m_k+1}^{\infty} a_j - 2\varepsilon\Big)_+\Big) < \beta_{k+1} \quad \forall\, \tau \in \mathcal{T}(\mathcal{A})\,.$$

Set $m_o = 0$ and $c_k := \sum\limits_{j=m_{k-1}+1}^{m_k} a_j$. We claim that

$$d_\tau((c_k - 2\varepsilon)_+) < d_\tau((d_k - 2\delta)_+) \quad \forall\, \tau \in \mathcal{T}(\mathcal{A}),\ k \geqslant 1. \tag{5.6}$$

For $k = 1$ we have

$$(c_1 - 2\varepsilon)_+ = \Big(\sum_{j=1}^{m_1} a_j - 2\varepsilon\Big)_+ \preceq \Big(\sum_{j=1}^{\infty} a_j - 2\varepsilon\Big)_+ \preceq (A-\varepsilon)_+.$$

where the first sub-equivalence follows from Lemma 1.1(viii) and the second one from Lemma 1.2. Then by (1.6) and (5.5),

$$d_\tau((c_1 - 2\varepsilon)_+) \leqslant d_\tau((A-\varepsilon)_+) < d_\tau((d_1 - 2\delta)_+),$$

that is, (5.6) holds for $k = 1$. For $k \geqslant 2$, by Lemma 1.1(viii) and (1.6) we have for all $\tau \in \mathcal{T}(\mathcal{A})$ that

$$d_\tau((c_k - 2\varepsilon)_+) \leqslant d_\tau\Big(\Big(\sum_{j=m_{k-1}+1}^{\infty} a_j - 2\varepsilon\Big)_+\Big) < \beta_k,$$

and hence, (5.6) also holds.

By the strict comparison of positive elements in $\mathcal{A}$, it follows that

$$(c_k - 2\varepsilon)_+ \preceq (d_k - 2\delta)_+ \quad \forall\, k \geqslant 1. \tag{5.7}$$

By construction, $\sum\limits_{k=1}^{\infty} a_k = \sum\limits_{k=1}^{\infty} c_k$ with convergence in the strict topology and $c_n c_m = 0$ for $|n-m| \geqslant 2$. Thus $C_e := \sum\limits_{k=1}^{\infty} c_{2k}$ and $C_o := \sum\limits_{k=1}^{\infty} c_{2k-1}$ are two diagonal series also converging strictly and $\sum\limits_{k=1}^{\infty} a_k = C_e + C_o$. Furthermore,

$$(C_e - 2\varepsilon)_+ = \sum_{k=1}^{\infty} (c_{2k} - 2\varepsilon)_+ \text{ and } (C_o - 2\varepsilon)_+ = \sum_{k=1}^{\infty} (c_{2k-1} - 2\varepsilon)_+.$$

By Proposition 3.5 we have

$$(C_e - 3\varepsilon)_+ \prec \Big(\sum_{k=1}^{\infty} d_{2k} - \delta\Big)_+ \quad \text{and} \quad (C_o - 3\varepsilon) \prec \Big(\sum_{k=1}^{\infty} d_{2k-1} - \delta\Big)_+. \tag{5.8}$$

Therefore

$$\begin{aligned}(A - 7\varepsilon)_+ &\preceq (C_e + C_o - 6\varepsilon)_+ \quad \text{(by Lemma 1.2)}\\ &\preceq (C_e - 3\varepsilon)_+ + (C_o - 3\varepsilon)_+ \quad \text{((by Lemma 1.1(ix))}\end{aligned}$$

$$\begin{aligned}
&\preceq \Big(\sum_{k=1}^{\infty} d_{2k} - \delta\Big)_+ \oplus \Big(\sum_{k=1}^{\infty} d_{2k-1} - \delta\Big)_+ \quad \text{(by Lemma 1.1(vi))}\\
&= \Big(\sum_{k=1}^{\infty} d_k - \delta\Big)_+ \quad \text{(by Lemma 1.1(vii))}\\
&\preceq \Big(\sum_{k=1}^{\infty} b_k - \delta\Big)_+ \quad \text{(by (5.4), Lemma 1.1(viii))}\\
&\preceq B \quad \text{(by Lemma 1.1(iv))}.
\end{aligned}$$

Since ε is arbitrary, we conclude that $A \preceq B$. ∎

6. AN EXAMPLE WHERE $I_{\min} \neq I_{\mathrm{cont}}$.

From Theorem 4.6, examples where $I_{\min} \neq I_{\mathrm{cont}}$ can be found only among "pathological" algebras that do not have strict comparison of positive elements. In this section we prove that the algebras constructed by Villadsen in [33] provide such examples. We will largely follow his notations. Let

$$X_0 = \mathbb{D}^{n_0} \quad \text{and} \quad X_i = X_{i-1} \times \mathbb{C}P^{n_i} \quad \text{for } i \in \mathbb{N},$$

that is,

$$X_i = \mathbb{D}^{n_0} \times \mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \times \cdots \times \mathbb{C}P^{n_i}.$$

We will always assume that

$$n_i \geqslant \sigma(i) := \begin{cases} 1 & i = 0, \\ i(i!) & i \geqslant 1, \end{cases} \tag{6.1}$$

and hence,

$$\dim(X_i) = 2\sum_{k=0}^{i} n_k \geqslant 2\sum_{k=0}^{i} \sigma(k) = 2(i+1)!. \tag{6.2}$$

This condition, together with the appropriate connecting maps, will guarantee that the AH-algebra $\mathcal{A}$ constructed in this process will not have *slow dimension growth*, which by Corollary 4.6 of [32] would imply strict comparison of positive elements. We refer the reader to Villadsen's definition ([33], pp. 1092–1093) of the connecting maps

$$\Phi_{i,i+1} : C(X_i) \otimes \mathcal{K} \to C(X_{i+1}) \otimes \mathcal{K} \tag{6.3}$$

and their compositions

$$\Phi_{i,j} = \Phi_{j-1,j} \circ \cdots \circ \Phi_{i,i+1} : C(X_i) \otimes \mathcal{K} \to C(X_j) \otimes \mathcal{K}.$$

Identifying as usual projections with complex vector bundles, given a complex vector bundle η over X_i, $\Phi_{i,i+1}(\eta)$ denotes a complex vector bundle over X_{i+1}.

Denoting by $k\eta$ (respectively, kq) the k-fold direct sum of the vector bundle η (respectively, of the projection q) with itself, we then have

$$\Phi_{i,i+1}(\eta) \cong \eta \times ((i+1)\operatorname{rank}(\eta))\gamma_{n_{i+1}}. \tag{6.4}$$

Here γ_k denotes the universal line bundle over the projective space $\mathbb{C}P^k$ (see (6.9) below for a key property of γ_k). Iterating we have for every $j > i$,

$$\begin{aligned}\Phi_{i,j}(\eta) \cong \eta &\times \frac{\sigma(i+1)}{(i+1)!}\operatorname{rank}(\eta)\gamma_{n_{i+1}} \times \frac{\sigma(i+2)}{(i+1)!}\operatorname{rank}(\eta)\gamma_{n_{i+2}} \times \cdots \\ &\times \frac{\sigma(j-1)}{(i+1)!}\operatorname{rank}(\eta)\gamma_{n_{j-1}} \times \frac{\sigma(j)}{(i+1)!}\operatorname{rank}(\eta)\gamma_{n_j}.\end{aligned} \tag{6.5}$$

In particular, since for every i and j, $\operatorname{rank}(\gamma_i) = 1$, $\sum_{k=0}^{j}\sigma(k) = (j+1)!$, and

$$\operatorname{rank}(\Phi_{i,j}(\eta)) = \operatorname{rank}(\eta)\Big(1 + \sum_{k=i+1}^{j}\frac{\sigma(k)}{(i+1)!}\operatorname{rank}(\gamma_{n_k})\Big),$$

we then have

$$\operatorname{rank}(\Phi_{i,j}(\eta)) = \frac{(j+1)!}{(i+1)!}\operatorname{rank}(\eta) \quad \forall j \geqslant i. \tag{6.6}$$

Let θ be a trivial line bundle over X_0 and set

$$p_i := \Phi_{0,i}(\theta) \quad \text{for } i>0; \quad \mathcal{A}_i = p_i(C(X_i)\otimes\mathcal{K})p_i \quad \text{for } i\geqslant 0; \quad \mathcal{A} = \varinjlim(\mathcal{A}_i, \Phi_{i,i+1}).$$

Here $\Phi_{i,i+1}$ denotes the *restriction* of $\Phi_{i,i+1}$ to $\mathcal{A}_i$. Let $\Phi_{i,\infty} : \mathcal{A}_i \to \mathcal{A}$ denote the unital embedding $\mathcal{A}_i \hookrightarrow \mathcal{A}$. By Villadsen's construction, these maps are injective and we denote by $\Phi_{\infty,i} : \Phi_{i,\infty}(\mathcal{A}_i) \to \mathcal{A}_i$ the inverse map of $\Phi_{i,\infty}$. As usual, we will identify $\mathcal{A}_i$ with their images in $\mathcal{A}$ and focus on the algebraic inductive limit $\bigcup \mathcal{A}_i \subset \mathcal{A}$.

For ease of reference, notice that

$$\operatorname{rank}(p_i) = (i+1)! \quad \forall\, i. \tag{6.7}$$

By [6] (see also a short proof in [33]), $\mathcal{A}$ is a simple, unital, AH-algebra and it has a unique tracial state τ. Villadsen proved that if $n_i = n\sigma(i)$ for a fixed $n \in \mathbb{N}$, then $\mathcal{A}$ has stable rank $n+1$. What interests us here is that by (6.2) and (6.7), $\inf_i \frac{\dim(X_i)}{\operatorname{rank}(p_i)} \geqslant 2$ and hence $\mathcal{A}$ does not have slow dimension growth, the group $K_0(\mathcal{A})$ has perforation, and $\mathcal{A}$ does not have strict comparison of projections by its trace. The same holds for other choices of $n_i \geqslant \sigma(i)$ as readily seen from Villadsen's construction.

We will show that $I_{\min} \neq I_{\text{cont}}$ for the underlying algebra $\mathcal{A}\otimes\mathcal{K}$ and that every element outside I_{cont} is full if $\sup \frac{n_i}{\sigma(i)} < \infty$ ($\mathcal{A}$ has *flat dimension growth*), while this is not the case for an unbounded dimension growth as $n_i = i!\sigma(i)$.

To prove these results, we will focus on *diagonal* projections of $\mathcal{M}(\mathcal{A}\otimes\mathcal{K})$, i.e. projections of the form $S = \bigoplus_{k=1}^{\infty} t_k s_k$ where $t_k \in \mathbb{N}$, s_k is a projection in $\Phi_{k,\infty}(\mathcal{A}_k)$, and $t_k s_k$ is the direct sum of t_k copies of s_k.

To determine if the diagonal projection S is in I_{cont} is easy. Since $\mathcal{A}$ has a unique tracial state τ, and hence, $I_{\mathrm{cont}} = I_\tau$, the projection S is in I_{cont} if and only if $\tau(S) < \infty$, i.e., $\sum_{k=1}^{\infty} t_k \tau(s_k) < \infty$. If $\eta_k = \Phi_{\infty,k}(s_k)$ is the complex vector bundle over X_k corresponding to s_k, i.e., $s_k = \Phi_{k,\infty}(\eta_k)$, then $\tau(s_k) = \frac{\mathrm{rank}(\eta_k)}{\mathrm{rank}(p_k)} = \frac{\mathrm{rank}(\eta_k)}{(k+1)!}$ by (6.6) and hence,

$$\tau(S) = \sum_{k=0}^{\infty} \frac{t_k\,\mathrm{rank}(\eta_k)}{(k+1)!}. \tag{6.8}$$

To construct a diagonal projection $S \notin I_{\min}$ we will make use of algebraic topology tools, more precisely, properties of the Euler classes. For a complex vector bundle η on a compact metric space X, $e(\eta)$ will denote the Euler class in the cohomology ring $H^*(X)$. To simplify notations, we will suppress explicit reference to the base space X. We start by recalling that for the universal line bundles γ_{n_i} used in defining the connecting maps (6.4), we have

$$e(\gamma_{n_i})^n \begin{cases} \neq 0 & n \leqslant n_i, \\ = 0 & n > n_i. \end{cases} \tag{6.9}$$

LEMMA 6.1. *Let η be a vector bundle over X_i and let $j > i$.*
(i) *If $e(\eta) = 0$, then $e(\Phi_{i,j}(\eta)) = 0$.*
(ii) *If $e(\eta) \neq 0$ and $\mathrm{rank}(\eta) \leqslant (i+1)!$, then $e(\Phi_{i,j}(\eta)) \neq 0$.*

Proof. Recall the fact that the Euler class of $\Phi_{i,j}(\eta)$ is the cup product of the Euler classes of its components in the Cartesian product in (6.5) (viewed as vector bundles over X_j via pullbacks of the relevant projection maps). That is,

$$\begin{aligned} e(\Phi_{ij}(\eta)) &= e(\eta) e\Big(\frac{\sigma(i+1)}{(i+1)!}\,\mathrm{rank}(\eta)\gamma_{n_{i+1}}\Big) \cdots e\Big(\frac{\sigma(j)}{(i+1)!}\,\mathrm{rank}(\eta)\gamma_{n_j}\Big) \\ &= e(\eta) e(\gamma_{n_{i+1}})^{\frac{\sigma(i+1)}{(i+1)!}\,\mathrm{rank}(\eta)} \cdots e(\gamma_{n_j})^{\frac{\sigma(j)}{(i+1)!}\,\mathrm{rank}(\eta)}. \end{aligned}$$

Thus if $e(\eta)$ vanishes, so does $e(\Phi_{ij}(\eta))$. By the Kunneth formula, since the cohomology groups considered have no torsion, it follows that $e(\Phi_{ij}(\eta)) \neq 0$ if and only if all the factors in the above decomposition do not vanish. By (6.9), a necessary and sufficient condition for that to happen is that $n_k \geqslant \frac{\sigma(k)}{(i+1)!}\,\mathrm{rank}(\eta)$ for all $i < k \leqslant j$. By the assumption (6.1), a sufficient condition for that to happen is that $\mathrm{rank}(\eta) \leqslant (i+1)!$. ∎

Recall that, in each building block $\mathcal{A}_i \otimes \mathcal{K}$, we are identifying projections with vector bundles over X_i. Thus if a projection p belongs to $\mathcal{A}_i \otimes \mathcal{K}$ for some i

we associate with it the sequence $\{\eta_j\}_j^\infty$ of the vector bundles

$$\eta_j := (\Phi_{\infty,j} \otimes \mathrm{id})(p)$$

over the spaces X_j, and $\eta_j = \Phi_{ij}(\eta_i)$ for $j \geqslant i$. In view of Lemma 6.1, it is convenient to set the following definition.

DEFINITION 6.2. Let $p \in \Big(\bigcup_{j=0}^{\infty} \mathcal{A}_j\Big) \otimes \mathcal{K}$ be a projection. We say that:

(i) $e(p) = 0$ if $e(\eta_i) = 0$ for some i for which $p \in \mathcal{A}_i \otimes \mathcal{K}$ (and hence $e(\eta_j) = 0$ for every $j \geqslant i$);

(ii) $e(p) \neq 0$ if $e(\eta_j) \neq 0$ for every j for which $p \in \mathcal{A}_j \otimes \mathcal{K}$.

In order to verify that $e(p) \neq 0$, by Lemma 6.1 it is sufficient to show that $e(\eta_i) \neq 0$ and that $\operatorname{rank}(\eta_i) \leqslant (i+1)!$ for the smallest i for which $p \in \mathcal{A}_i \otimes \mathcal{K}$.

COROLLARY 6.3. *Let $q, r \in \Big(\bigcup_{j=0}^{\infty} \mathcal{A}_j\Big) \otimes \mathcal{K}$ be projections, $q \preceq r$ and $e(q) = 0$. Then $e(r) = 0$.*

Proof. There is an i such that $q, r \in \mathcal{A}_i \otimes \mathcal{K}$, $e((\Phi_{\infty,i} \otimes \mathrm{id})(q)) = 0$, and the subequivalence $q \preceq r$ holds within $\mathcal{A}_i \otimes \mathcal{K}$, i.e., $r = q' \oplus s$ for some projections $q', s \in \mathcal{A}_i \otimes \mathcal{K}$ with $q' = vv^*$ and $q = v^*v$ for some $v \in \mathcal{A}_i \otimes \mathcal{K}$. But then

$$\begin{aligned} e((\Phi_{\infty,i} \otimes \mathrm{id})(r)) &= e((\Phi_{\infty,i} \otimes \mathrm{id})(q'))e((\Phi_{\infty,i} \otimes \mathrm{id})(s)) \\ &= e((\Phi_{\infty,i} \otimes \mathrm{id})(q))e((\Phi_{\infty,i} \otimes \mathrm{id})(s)) = 0. \end{aligned}$$

By Definition 6.2, $e(r) = 0$. ∎

We will construct now two sequences of projections $\{q_i\}$ and $\{r_i\}$ in $\mathcal{A} \otimes \mathcal{K}$ for which $e(q_i) = 0$ and $e(r_i) \neq 0$ for all i.

By the definition of p_i it is immediate to find a trivial complex line bundle $\theta_i \leqslant p_i$ over X_i. Let $q_i := \Phi_{i,\infty}(\theta_i) \otimes e_{ii} \in \mathcal{A} \otimes \mathcal{K}$, so that the projections q_i are mutually orthogonal. Then it is clear that

$$Q := \bigoplus_{i=1}^{\infty} q_i \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \setminus \mathcal{A} \otimes \mathcal{K} \quad \text{and} \quad \tau(Q) = \sum_{i=1}^{\infty} \tau(q_i) = \sum_{i=1}^{\infty} \frac{1}{(i+1)!} < \infty,$$

and hence,

$$Q \in I_{\mathrm{cont}}. \tag{6.10}$$

Furthermore, by construction,

$$e(q_i) = 0 \quad \forall\, i. \tag{6.11}$$

Next, from the definition of p_i and the construction of the maps $\Phi_{i,i+1}$ in (6.3), we see that there is a complex line bundle $\rho_i \in C(X_i) \otimes \mathcal{K}$ with $\rho_i \leqslant p_i$ and $\rho_i \sim \pi_i^{2*}(\gamma_{n_i})$ where π_i^{2*} denotes the pull back map from vector bundles on $\mathbb{C}P^{n_i}$ to those on the space X_i. Thus $e(\rho_i)^k = 0$ if and only if $e(\gamma_{n_i})^k = 0$, i.e., by (6.9), if and only if $k > n_i$. When there is no risk of confusion, we will write γ_{n_i} for ρ_i as well as for the pullbacks to vector bundles over X_i for $j > i$. Set

$$r_i := \Phi_{i,\infty}(\rho_i) \otimes e_{ii} \in \mathcal{A}_i \otimes \mathcal{K} \subset \mathcal{A} \otimes \mathcal{K}\,.$$

By definition, the projections r_i are mutually orthogonal. Set

$$R := \sum_{i=1}^{\infty} r_i.$$

It is then clear that $R \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K}) \setminus \mathcal{A} \otimes \mathcal{K}$,

$$\tau(R) = \sum_{i=1}^{\infty} \tau(r_i) = \sum_{i=1}^{\infty} \frac{1}{(i+1)!} < \infty,$$

and hence

$$R \in I_{\text{cont}}. \tag{6.12}$$

LEMMA 6.4. *$e(nr_i) \neq 0$ for all $n \leqslant \min(n_i, (i+1)!)$. In particular, $e(r_i) \neq 0$ for all i.*

Proof. By (6.9) and the assumption that $n \leqslant n_i$ we have

$$e(n\gamma_{n_i}) = e(\gamma_{n_i})^n \neq 0.$$

Moreover, $\operatorname{rank}(n\gamma_{n_i}) = n \leqslant (i+1)!$, hence $e(nr_i) \neq 0$ by Lemma 6.1 and Definition 6.2. ∎

LEMMA 6.5. *For all integers $j > i$*

(i) $\left(\frac{j!}{i!}\right) r_j \preceq r_i$;

(ii) $\sum_{k=i+1}^{j} r_k \preceq r_i$;

(iii) $\sum_{k=i}^{j} r_k \preceq 2r_i$.

Proof. (i) By (6.4) we have $\Phi_{i,i+1}(\rho_i) \cong \rho_i \times (i+1)\gamma_{n_{i+1}}$, and hence, $(i+1)r_{i+1} \preceq r_i$. Then (i) follows immediately.

(ii) The proof is by induction on $j - i \geqslant 1$. By (i), $r_{i+1} \preceq (i+1)r_{i+1} \preceq r_i$ so the condition holds for $j - i = 1$. Assume condition (ii) holds for some $j - i > 1$ and hence $\sum_{k=i+2}^{j+1} r_k \preceq r_{i+1}$. Then

$$\sum_{k=i+1}^{j+1} r_k \preceq r_{i+1} \oplus \sum_{k=i+2}^{j+1} r_k \preceq 2r_{i+1} \preceq (i+1)r_{i+1} \preceq r_i$$

where the last relation in the chain holds by (i).

(iii) Obvious from (ii). ∎

LEMMA 6.6. *The sequence $\{r_k\}$ is order dense (see Definition 2.8).*

Proof. In view of Lemma 1.1(iv) and the density of $\bigcup_{i=1}^{\infty} \Phi_{i,\infty}(\mathcal{A}_i)$ in $\mathcal{A}$, in order to show that $\{r_k\}$ is order dense in $\mathcal{A} \otimes \mathcal{K}$, it is enough to show that for every $i, h \in \mathbb{N}$ and $0 \neq a \in p_i(C(X_i) \otimes M_h(\mathbb{C}))_+ p_i$ there is some $j > i$ such that $r_j \preceq \Phi_{i,j}(a)$.

To do that we need to examine more closely the construction of the connecting maps $\Phi_{i,i+1}$ and their iterations $\Phi_{i,j}$. We again refer the reader to the definition in [33] and also to [13]. Disregarding the isomorphism between $\mathcal{K} \otimes \mathcal{K}$ and $\mathcal{K}$, we may view $\Phi_{i,i+1}(a)$ to be in the following matrix form:

$$\begin{pmatrix} a \circ \pi_{i+1,i} & & & \\ & a(\pi_{i+1,i}(y_{i+1}^1)) \otimes r_{i+1}^1 & & \\ & & \ddots & \\ & & & a(\pi_{i+1,i}(y_{i+1}^{i+1})) \otimes r_{i+1}^{i+1} \end{pmatrix}$$

where r_{i+1}^k are mutually orthogonal projections all equivalent to r_{i+1}, $\pi_{i+1,i}$ denotes the projection from X_{i+1} onto X_i, and the points $y_j^k \in X_j$ are chosen so that the collection of their projections $\{\pi_{j,i}(y_j^k) : 1 \leqslant k \leqslant j, j \geqslant i\}$ is dense in X_i for every i. Since a is a continuous, there is a $j > i$ and a $1 \leqslant k \leqslant j$ such that $a(\pi_{j,i}(y_j^k)) \neq 0$. But then, $0 \leqslant a(\pi_{j,i}(y_j^k)) \otimes r_j^k \leqslant \Phi_{i,i+1}(a)$. By diagonalizing $a(\pi_{j,i}(y_j^k))$, we can find a $\lambda > 0$ and a rank one projection s such that $\lambda s \otimes r_j^k \leqslant \Phi_{i,j}(a)$, and hence, $r_j \preceq \Phi_{i,j}(a)$. This proves the claim. ∎

COROLLARY 6.7. *The projection R belongs to $I_{\min} \setminus \mathcal{A} \otimes \mathcal{K}$, and hence it generates $I_{\min}$.*

Proof. Let $e_n := 1_{\mathcal{A}} \otimes \sum_{k=1}^{n} e_{kk}$; then $\{e_n\}$ is an approximate identity of $\mathcal{A} \otimes \mathcal{K}$. By Lemma 6.6, the sequence $\{r_k\}$ is order dense, and hence by Lemma 6.5 it is thin. But then $R \in K_o(\{e_n\}) \subset I_{\min}$ by Lemma 2.13. Since $R \notin \mathcal{A} \otimes \mathcal{K}$ and $I_{\min}$ is minimal among the ideals properly containing $\mathcal{A} \otimes \mathcal{K}$, it follows that R generates $I_{\min}$. ∎

THEOREM 6.8. *The projection Q does not belong to $I_{\min}$, and hence $I_{\min} \neq I_{\mathrm{cont}}$.*

Proof. Assume by contradiction that $Q \in I_{\min}$. Then $I_{\min} = I(R)$ by Corollary 6.7 and hence there is an $n \in \mathbb{N}$ such that $Q \leqslant nR$, i.e., $\bigoplus_{k=1}^{\infty} q_k \preceq \bigoplus_{k=1}^{\infty} nr_k$. Choose i such that $n \leqslant \sigma(i-1)$. Then $n \leqslant \min(n_{i-1}, i!)$ by the assumption (6.1) and hence $e(nr_{i-1}) \neq 0$ by Lemma 6.4. On the other hand, by Proposition 3.6 there are $m, j \in \mathbb{N}$, $j \geqslant i$, such that $q_m \preceq \bigoplus_{k=i}^{j} nr_k$. By Lemma 6.5(ii), $q_m \preceq nr_{i-1}$ and since $e(q_m) = 0$, it follows from Corollary 6.3 that $e(nr_{i-1}) = 0$, a contradiction. ∎

REMARK 6.9. (i) A consequence of Lemma 6.6 is the known fact that Villadsen's algebras have the (SP) property (e.g., see the proof of the (SP) property for the Villadsen's type algebras studied in [28]).

(ii) The same argument in the proof of Theorem 6.8 shows that $q_m \not\preceq r_i$ for every $m, i \in \mathbb{N}$ which is an illustration of the well known fact that strict comparison of projections does not hold in $\mathcal{A} \otimes \mathcal{K}$.

Notice that so far we have only assumed that $n_i \geqslant \sigma(i)$. We can obtain more if we assume that $\mathcal{A}$ has *flat dimension growth*, that is $\sup \frac{\dim(X_i)}{\operatorname{rank}(p_i)} < \infty$, (see Definition 1.2 of [31]), which are exactly Villadsen's *finite* stable rank algebras studied in [33].

THEOREM 6.10. *Assume that $\mathcal{A}$ has flat dimension growth, then I_{cont} is the largest proper ideal of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.*

Proof. To prove that I_{cont} contains every proper ideal of $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, it suffices to prove that if $S \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})_+ \setminus I_{\text{cont}}$ then $I(S) = \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$, namely S is full. Assume without loss of generality that $\|S\| = 1$. By Theorem 3.1 and Remark 3.2, $S = D_{\mathrm{e}} + D_{\mathrm{o}} + a$ where $a = a^* \in \mathcal{A} \otimes \mathcal{K} \subset I_{\text{cont}}$ and D_{e} and D_{o} are diagonal series. Then at least one of the two series must also not belong to I_{cont}. To simplify notations, assume that S itself is diagonal, namely $S = \bigoplus_{k=1}^{\infty} s_k$ where $s_k \in (\mathcal{A} \otimes \mathcal{K})_+$ for every k and the series converges strictly. Furthermore, find $\delta > 0$ for which $\tau(S - \delta)_+ = \infty$. Let $M := \sup \frac{\dim(X_i)}{\operatorname{rank}(p_i)}$ and choose an increasing subsequence $\{m_k\}$ such that $\sum_{j=m_k+1}^{m_{k+1}} \tau((s_j - \delta)_+) > \frac{M}{2} + 2$. To simplify notations, assume $m_k = k$, i.e., $\tau((s_k - \delta)_+) > \frac{M}{2} + 2$ for every k. It was proven in Lemma 2.5 of [13] that for every $0 \neq c \in (\mathcal{A} \otimes M_n)_+$ and $\varepsilon > 0$, there is a projection q with $\left|\tau(q) - \lim_{k\to\infty} \tau(c^{1/k})\right| < \varepsilon$ and $q \in \overline{c(\mathcal{A} \otimes M_n)c}$, and hence, $q \preceq c$. While the standard assumption in [13] was that $n_i = \sigma(i)$, no conditions on n_i were used in the proof of that lemma. Moreover, it is routine to extend that lemma to $0 \neq c \in (\mathcal{A} \otimes \mathcal{K})_+$. Thus we can find a sequence $\{q_k\}$ of projections $q_k \preceq (s_k - \delta)_+$, such that for all k

$$\tau(q_k) > d_\tau((s_k - \delta)_+) - \frac{1}{2^k} \geqslant \tau((s_k - \delta)_+)) - \frac{1}{2^k} > \frac{M}{2} + \tau(1_{\mathcal{A}} \otimes e_{kk}).$$

By Definition 2.1 of [31], $M \geqslant \operatorname{drr}(\mathcal{A})$ (we refer the reader to [31] for the definition of the *dimension-rank ratio* of $\mathcal{A}$) and by Theorem 3.10 of [31] it follows that

$$1_{\mathcal{A}} \otimes e_{kk} \preceq q_k \preceq (s_k - \delta)_+ \quad \forall k.$$

Then by Proposition 3.5 we have that

$$1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})} = \bigoplus_{k=1}^{\infty} 1_{\mathcal{A}} \otimes e_{kk} \preceq \bigoplus_{k=1}^{\infty} s_k = S$$

which proves that S is full. ∎

Without the flat dimension growth condition, the conclusion of Theorem 6.10 no longer necessarily holds. To show that, first we need the following refinement of Lemma 6.5.

LEMMA 6.11. *Let* $\eta = \bigoplus_{k=i}^{j} \Phi_{k,j}(t_k\gamma_{n_k})$, *where* $i \leqslant j$ *are integers and* $\{t_k\}$ *is a monotone nondecreasing sequence of integers. For every* $j' \geqslant j$ *we have*

$$\Phi_{j,j'}(\eta) \cong m_i\gamma_{n_i} \times m_{i+1}\gamma_{n_{i+1}} \times \cdots \times m_{j'}\gamma_{n_{j'}}$$

where $m_k \in \mathbb{N}$ *and*

$$m_k \leqslant \begin{cases} t_i & k = i, \\ t_k\Big(1 + e\frac{\sigma(k)}{(i+1)!}\Big) & i+1 \leqslant k \leqslant j, \\ t_j e\frac{\sigma(k)}{(i+1)!} & j+1 \leqslant k \leqslant j'. \end{cases}$$

Proof. From (6.5) we have for every $j' \geqslant j$

$$\begin{array}{lclclclclclcl}
\Phi_{i,j'}(t_i\gamma_{n_i}) & \cong & t_i\gamma_{n_i} & \times & t_i\frac{\sigma(i+1)}{(i+1)!}\gamma_{n_{i+1}} & \times & \cdots & \times & t_i\frac{\sigma(j)}{(i+1)!}\gamma_{n_j} & \times & \cdots & \times & t_i\frac{\sigma(j')}{(i+1)!}\gamma_{n_{j'}} \\
\Phi_{i+1,j'}(t_{i+1}\gamma_{n_{i+1}}) & \cong & & & t_{i+1}\gamma_{n_{i+1}} & \times & \cdots & \times & t_{i+1}\frac{\sigma(j)}{(i+2)!}\gamma_{n_j} & \times & \cdots & \times & t_{i+1}\frac{\sigma(j')}{(i+2)!}\gamma_{n_{j'}} \\
\Phi_{i+2,j'}(t_{i+2}\gamma_{n_{i+2}}) & \cong & & & & & \cdots & \times & t_{i+2}\frac{\sigma(j)}{(i+3)!}\gamma_{n_j} & \times & \cdots & \times & t_{i+2}\frac{\sigma(j')}{(i+3)!}\gamma_{n_{j'}} \\
\vdots & & \vdots & & & & & \vdots & & & & \vdots & \\
\Phi_{j,j'}(t_j\gamma_{n_j}) & \cong & & & & & & & t_j\gamma_{n_j} & \times & \cdots & \times & t_j\frac{\sigma(j')}{(j+1)!}\gamma_{n_{j'}}
\end{array}$$

Recall that if $\rho_1 \cong s_1\alpha \times t_1\beta$ and $\rho_2 \cong s_2\alpha \times t_2\beta$ for some complex vector bundles α and β on spaces X and Y, and integers s_1, s_2, t_1, t_2, then

$$\rho_1 \oplus \rho_2 \cong (s_1 + s_2)\alpha \times (t_1 + t_2)\beta.$$

Thus by summing the integer multipliers of the universal bundles γ_{n_k} we obtain that

$$m_k = \begin{cases} t_i & k = i, \\ t_k + \sigma(k) \sum\limits_{h=i+1}^{k} \frac{t_{h-1}}{h!} & i+1 \leqslant k \leqslant j, \\ \sigma(k) \sum\limits_{h=i+1}^{j} \frac{t_{h-1}}{h!} & j+1 \leqslant k \leqslant j'. \end{cases}$$

By using the Lagrange remainder of the Taylor series for the exponential function, we see that $\sum\limits_{h=i+1}^{k} \frac{1}{h!} \leqslant \frac{e}{(i+1)!}$. This inequality together with the monotonicity of the sequence $\{t_k\}$ establishes the claim. ∎

PROPOSITION 6.12. *Let* $R_\infty := \bigoplus_{k=1}^{\infty} k!r_k$. *Then* $R_\infty \notin I_{\text{cont}}$. *If* $n_k \geqslant k!\sigma(k)$, *then* $Q \notin I(R_\infty)$.

Proof. Clearly $R_\infty \in \mathcal{M}(\mathcal{A}\otimes\mathcal{K})\setminus\mathcal{A}\otimes\mathcal{K}$ is a projection and $R_\infty \notin I_{\text{cont}}$ follows from $\tau(R_\infty) = \sum\limits_{k=1}^{\infty} \frac{k!}{(k+1)!} = \infty$. To show that $Q \notin I(R_\infty)$ we reason as in the proof of Theorem 6.8. For every $n \in \mathbb{N}$, choose i such that $(i+1)! \geqslant 2en$ and let $j \geqslant i$. Let η be the complex vector bundle over X_j corresponding to $\sum\limits_{k=i}^{j} nk!r_k$.

Then $\eta \cong \bigoplus\limits_{k=i}^{j} \Phi_{k,j}(nk!\gamma_{n_k})$, and hence, by Lemma 6.11,

$$\Phi_{j,j'}(\eta) \cong nm_i\gamma_{n_i} \times nm_{i+1}\gamma_{n_{i+1}} \times \cdots \times nm_{j'}\gamma_{n_{j'}}.$$

Since

$$nm_k \leqslant \begin{cases} ni! & \text{for } k=i, \\ nk!\big(1+e\frac{\sigma(k)}{(i+1)!}\big) & \text{for } i<k\leqslant j, \\ \frac{en}{(i+1)!}j!\sigma(k) & \text{for } j+1<k\leqslant j', \end{cases} \qquad \leqslant k!\sigma(k) \leqslant n_k,$$

we see that $e(\Phi_{j,j'}(\eta)) \neq 0$ for every $j' \geqslant j$. Thus $e\Big(n\bigoplus\limits_{k=i}^{j} t_k r_k\Big) \neq 0$ by Definition 6.2. Reasoning as in the proof of Theorem 6.8, we then conclude that $Q \notin I(R_\infty)$. ∎

Acknowledgements. This work was partially supported by the Simons Foundation (grant No 245660 to Victor Kaftal and grant No 281966 to Shuang Zhang).

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Received May 12, 2017; revised September 28, 2017.