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QUASIANALYTIC n-TUPLES OF HILBERT SPACE OPERATORS

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ABSTRACT. The residual and $*$-residual parts of the unitary dilation proved to be especially useful in the study of contractions. A more direct approach to these components, originated in B. Sz.-Nagy, *Acta Sci. Math. (Szeged)* **11**(1947), 152–157, leads to the concept of unitary asymptote, and opens the way for generalizations to more general settings. In this paper a systematic study of unitary asymptotes of commuting n-tuples of general Hilbert space operators is initiated. Special emphasis is put on the study of the quasianalyticity property, which constitutes homogeneous behaviour in localization, and plays a crucial role in the quest for proper hyperinvariant subspaces.

KEYWORDS: *Unitary asymptote, quasianalytic operators, commuting n-tuples of operators, residual sets.*

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INTRODUCTION

The theory of Hilbert space operators contains many beautiful results, yet some of its fundamental questions, like the hyperinvariant subspace problem, are still open; see, e.g., the monographs [4], [6], [7], and [27]. Beside the structure of single operators, multivariable operator theory also came into the focus of researchers' interest; see, e.g., [5], [10], and [25]. The spectral analysis of unitary operators makes possible to explore their structure. Hence, it has been reasonable to relate more general operators to unitaries. Such connection has been made for contractions by Sz.-Nagy's dilation theorem; the resulting theory is presented in [30]. The residual and $*$-residual parts of the unitary dilation proved to be especially useful in the study of contractions.

A more direct approach to these components, originated in [29], leads to the concept of unitary asymptote, and opens the way for generalizations to more general settings. The fundamental properties of unitary asymptotes have been summarized for contractions in Chapter IX of [30], for power bounded operators in [14], for commuting n-tuples of power bounded operators in [1], for representations of abelian semigroups in [15] and [20]; see also the references therein.

In the present paper we initiate a systematic study of unitary asymptotes for commuting n-tuples of general Hilbert space operators, following the categorical approach applied in [2] and Chapter IX of [30]. We note that most of the results are new even in the single variable case, since general operators are considered. In the quest for proper hyperinvariant subspaces of asymptotically non-vanishing contractions, the crucial class is formed by the quasianalytic contractions, which show homogeneous behaviour in localization. For their study see, e.g., [16], [17], [21], [22], and [23]. These investigations were extended to polynomially bounded operators in [18] and [19]; see also [11].

In this paper we provide a systematic study of quasianalyticity in the setting of commuting n-tuples of general Hilbert space operators.

Our paper is organized in the following way.

In Section 1 the concepts of unitary intertwining pairs and unitary asymptotes are introduced. Fundamental properties, like minimality, norm-control and uniqueness are examined. Necessary condition is given for the existence of the unitary asymptote in terms of the spectral radius. Adjoints and restrictions to invariant subspaces are considered.

In Section 2 it is shown that the Lower Orbit Condition (LOC) is a sufficient but not necessary condition of the existence of unitary asymptotes. Application of invariant means yields that (LOC) holds in the power bounded case. A classification is made relying on the annihilating subspace.

Section 3 deals with quasianalyticity and hyperinvariant subspaces. The commutant mapping relates the commutant of the n-tuple

$$\mathbf{T} = (T_1, \dots, T_n)$$

to the commutant of the corresponding n-tuple

$$\mathbf{U} = (U_1, \dots, U_n)$$

of unitaries. We obtain that hyperinvariant subspaces of $\mathbf{U}$ induce hyperinvariant subspaces of $\mathbf{T}$. The basic facts of the spectral analysis of $\mathbf{U}$ are recalled; in particular, the spectral subspaces are hyperinvariant. Quasianalyticity of $\mathbf{T}$ means that the localizations of the spectral measure E of $\mathbf{U}$ at non-zero vectors, corresponding to vectors in the space of $\mathbf{T}$, are equivalent. If this homogeneity breaks, then $\mathbf{T}$ has proper hyperinvariant subspaces.

In Section 4 the local residual sets are introduced. Exploiting the lattice structure of the Borel sets on $\mathbb{T}^n$, the quasianalytic spectral set $\pi(\mathbf{T})$ is defined. Quasianalyticity is related to the cyclicity property of the commutant of $\mathbf{U}$. The absolutely continuous (a.c.) case is also studied. The a.c. global residual set $\omega_{\mathrm{a}}(\mathbf{T})$ is the measurable support of the spectral measure E, and quasianalyticity is equivalent to the coincidence $\pi_{\mathrm{a}}(\mathbf{T}) = \omega_{\mathrm{a}}(\mathbf{T})$.

We shall use the following notation: $\mathbb{N}, \mathbb{Z}_+, \mathbb{Z}, \mathbb{R}_+, \mathbb{R}$ and $\mathbb{C}$ stand for the set of positive integers, non-negative integers, integers, non-negative real numbers, real numbers and complex numbers, respectively. For any $n \in \mathbb{N}$,

$$\mathbb{N}_n := \{k \in \mathbb{N} : k \leqslant n\}, \quad \text{and} \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}.$$

1. EXISTENCE OF UNITARY ASYMPTOTES

Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n-tuple of (bounded, linear) operators on the (complex, separable) Hilbert space $\mathcal{H}$ ($n \in \mathbb{N}$). Giving any other commuting n-tuple $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \dots, \widetilde{T}_n)$ of operators on the Hilbert space $\widetilde{\mathcal{H}}$, the *intertwining set* $\mathcal{I}(\mathbf{T}, \widetilde{\mathbf{T}})$ consists of those (bounded, linear) transformations $X : \mathcal{H} \to \widetilde{\mathcal{H}}$, which satisfy the conditions $XT_i = \widetilde{T}_i X$ ($i \in \mathbb{N}_n$). For any $X \in \mathcal{I}(\mathbf{T}, \widetilde{\mathbf{T}})$ and $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$, $X\mathbf{T}^{\mathbf{k}} = \widetilde{\mathbf{T}}^{\mathbf{k}} X$ holds, where $\mathbf{T}^{\mathbf{k}} := T_1^{k_1} \cdots T_n^{k_n}$. Hence $Xp(\mathbf{T}) = p(\widetilde{\mathbf{T}})X$ is true for every polynomial $p(\mathbf{z}) = \sum_{\mathbf{k}} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$ in the variable $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$, where $\mathbf{z}^{\mathbf{k}} := z_1^{k_1} \cdots z_n^{k_n}$. It is clear that $\mathcal{I}(\mathbf{T}, \widetilde{\mathbf{T}})$ is a Banach space with the operator norm. The *commutant* $\{\mathbf{T}\}' := \mathcal{I}(\mathbf{T}, \mathbf{T})$ of $\mathbf{T}$ is a Banach subalgebra of the operator algebra $\mathcal{L}(\mathcal{H})$.

Let $\mathbf{U} = (U_1, \dots, U_n)$ be a commuting n-tuple of unitary operators on the Hilbert space $\mathcal{K}$. If $X \in \mathcal{I}(\mathbf{T}, \mathbf{U})$, then $(X, \mathbf{U})$ is called a *unitary intertwining pair* of $\mathbf{T}$. The subspace $\mathcal{K}_0 = \bigvee\{\mathbf{U}^{\mathbf{k}} X\mathcal{H} : \mathbf{k} \in \mathbb{Z}^n\}$ is the smallest one, reducing $\mathbf{U}$ and containing the range of X. The restriction $\mathbf{U}_0 = (U_{10}, \dots, U_{n0})$, where $U_{i0} = U_i|_{\mathcal{K}_0}$ ($i \in \mathbb{N}_n$), is a commuting n-tuple of unitaries on $\mathcal{K}_0$. Setting $X_0 : \mathcal{H} \to \mathcal{K}_0, h \mapsto Xh$, the unitary intertwining pair $(X_0, \mathbf{U}_0)$ of $\mathbf{T}$ is called the *minimal part* of $(X, \mathbf{U})$. We say that $(X, \mathbf{U})$ is *minimal*, if $\mathcal{K}_0 = \mathcal{K}$.

The unitary intertwining pairs $(X, \mathbf{U})$ and $(\widetilde{X}, \widetilde{\mathbf{U}})$ of $\mathbf{T}$ are called *similar*, in notation: $(X, \mathbf{U}) \approx (\widetilde{X}, \widetilde{\mathbf{U}})$, if there exists an invertible $Z \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$ satisfying the condition $\widetilde{X} = ZX$. These pairs are called *equivalent*, in notation: $(X, \mathbf{U}) \simeq (\widetilde{X}, \widetilde{\mathbf{U}})$, if there exists a unitary $Z \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$ such that $\widetilde{X} = ZX$.

The unitary intertwining pair $(X, \mathbf{U})$ of $\mathbf{T}$ is a *unitary asymptote* of $\mathbf{T}$, if it is universal in the sense that for any other unitary intertwining pair $(X', \mathbf{U}')$ of $\mathbf{T}$ there exists a unique $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$ such that $X' = Y'X$. It readily follows that every unitary asymptote is minimal.

If $\mathbf{U}'$ is a commuting n-tuple of unitaries on $\mathcal{K}'$ and $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$, then $(X' = Y'X, \mathbf{U}')$ is obviously a unitary intertwining pair of $\mathbf{T}$ and $\|X'\| \leqslant \|Y'\| \cdot \|X\|$. We say that the unitary asymptote $(X, \mathbf{U})$ of $\mathbf{T}$ has *norm-control* with $\kappa \in \mathbb{R}_+$, if $\|Y'\| \leqslant \kappa \|Y'X\|$ holds for every unitary intertwining pair $(Y', \mathbf{U}')$ of $\mathbf{U}$. The smallest possible κ is called the *optimal norm-control*, and it is denoted by $\kappa_{\mathrm{op}} = \kappa_{\mathrm{op}}(X, \mathbf{U})$.

Relying on polar decompositions a routine computation yields that the minimal unitary intertwining pairs of $\mathbf{T}$ can be obtained, up to equivalence, by the aid of operators in the commutant of $\mathbf{U}$.

PROPOSITION 1.1. *Assume that* $(X, \mathbf{U})$ *is a unitary asymptote of* $\mathbf{T}$. *Let* $(X', \mathbf{U}')$ *be a unitary intertwining pair of* $\mathbf{T}$, *where* $X' = Y'X$ *with unique* $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$. *Then the minimal part of* $(X', \mathbf{U}')$ *is equivalent to the minimal part of* $(|Y'|X, \mathbf{U})$, *where* $|Y'| \in \{\mathbf{U}\}'$.

It turns out that norm-control is a general property.

PROPOSITION 1.2. *Assume that* $(X, \mathbf{U})$ *is a unitary asymptote of* $\mathbf{T}$.

(i) *The linear mapping* $\Gamma = \Gamma_{(X,\mathbf{U})} : \{\mathbf{U}\}' \to \mathcal{I}(\mathbf{T}, \mathbf{U}), Y' \mapsto Y'X$ *is invertible.*

(ii) *The unitary asymptote* $(X, \mathbf{U})$ *has optimal norm-control with* $\kappa_{\mathrm{op}} = \|\Gamma^{-1}\|$.

Proof. The mapping Γ is bounded linear, for any unitary intertwining pair of $\mathbf{T}$. It is obvious that Γ is bijective, whenever $(X, \mathbf{U})$ is a unitary asymptote of $\mathbf{T}$. In this case Γ^{-1} is also bounded by the open mapping theorem.

Let $(X', \mathbf{U}')$ be a unitary intertwining pair of $\mathbf{T}$, $X' = Y'X$ with $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$. Since $|Y'| \in \{\mathbf{U}\}'$, it follows that $\|Y'\| = \||Y'|\| \leqslant \|\Gamma^{-1}\| \cdot \||Y'|X\|$. In view of Proposition 1.1 we know that $\||Y'|X\| = \|X'\|$, and so $\|Y'\| \leqslant \|\Gamma^{-1}\| \cdot \|X'\|$. It is clear that $\|\Gamma^{-1}\|$ is the optimal norm-control. ∎

The question of the uniqueness of unitary asymptotes can be settled easily. The straightforward proof of the following statement is left to the reader,

PROPOSITION 1.3. *Suppose that* $(X, \mathbf{U})$ *is a unitary asymptote of* $\mathbf{T}$, *and* $(\widetilde{X}, \widetilde{\mathbf{U}})$ *is a unitary intertwining pair of* $\mathbf{T}$.

(i) *Then* $(\widetilde{X}, \widetilde{\mathbf{U}})$ *is a unitary asymptote of* $\mathbf{T}$ *if and only if* $(\widetilde{X}, \widetilde{\mathbf{U}}) \approx (X, \mathbf{U})$.

(ii) *If* $Z \in \mathcal{I}(\widetilde{\mathbf{U}}, \mathbf{U})$ *is invertible with* $Z\widetilde{X} = X$, *then*

$$\kappa_{\mathrm{op}}(\widetilde{X}, \widetilde{\mathbf{U}}) \leqslant \|Z\| \cdot \kappa_{\mathrm{op}}(X, \mathbf{U}) \leqslant \|Z\| \cdot \|Z^{-1}\| \cdot \kappa_{\mathrm{op}}(\widetilde{X}, \widetilde{\mathbf{U}}).$$

REMARK 1.4. Let $(X, \mathbf{U})$ be a unitary asymptote of $\mathbf{T}$. Clearly, $\kappa_{\mathrm{op}}(X, \mathbf{U}) = 0$ holds exactly when $X = 0$. Let us assume that $X \neq 0$. Given any $0 \neq c \in \mathbb{C}$, and applying Proposition 1.3 with $\widetilde{\mathbf{U}} = \mathbf{U}$ and $Z = cI$, we obtain that $(\widetilde{X} = (1/c)X, \mathbf{U})$ is also a unitary asymptote of $\mathbf{T}$ and $\kappa_{\mathrm{op}}(\widetilde{X}, \mathbf{U}) = |c|\kappa_{\mathrm{op}}(X, \mathbf{U})$. Hence the products $\|\widetilde{X}\| \cdot \kappa_{\mathrm{op}}(\widetilde{X}, \mathbf{U}) = \|X\| \cdot \kappa_{\mathrm{op}}(X, \mathbf{U})$ are independent of the particular choice of c; we call this common value the *absolute optimal norm-control* of $(X, \mathbf{U})$, and denote it by $\kappa_{\mathrm{aop}}(X, \mathbf{U})$. Clearly, $\kappa_{\mathrm{op}}(X, \mathbf{U}) = \|X\|^{-1}\kappa_{\mathrm{aop}}(X, \mathbf{U})$. Notice also that $\|\Gamma_{(X,\mathbf{U})}\| \leqslant \|X\|$ yields that $\kappa_{\mathrm{op}}(X, \mathbf{U}) = \|\Gamma^{-1}_{(X,\mathbf{U})}\| \geqslant 1/\|X\|$, whence $\kappa_{\mathrm{aop}}(X, \mathbf{U}) \geqslant 1$ follows.

REMARK 1.5. If $(X, \mathbf{U})$ and $(\widetilde{X}, \widetilde{\mathbf{U}})$ are unitary asymptotes of $\mathbf{T}$ with norm-control 1, and $\|X\| \leqslant 1, \|\widetilde{X}\| \leqslant 1$, then an easy computation shows that $(X, \mathbf{U}) \simeq (\widetilde{X}, \widetilde{\mathbf{U}})$; furthermore $\|X\| = 1$, provided $X \neq 0$.

Similarity preserves the existence of unitary asymptotes.

PROPOSITION 1.6. *If* $(\widetilde{X}, \widetilde{\mathbf{U}})$ *is a unitary asymptote of the commuting* n*-tuple* $\widetilde{\mathbf{T}}$ *with norm-control* κ *and* $Z \in \mathcal{I}(\mathbf{T}, \widetilde{\mathbf{T}})$ *is invertible, then* $(\widetilde{X}Z, \widetilde{\mathbf{U}})$ *is a unitary asymptote* *of* $\mathbf{T}$ *with norm-control* $\kappa\|Z^{-1}\|$. *Furthermore, we have* $\kappa_{\mathrm{aop}}(\widetilde{X}Z, \widetilde{\mathbf{U}}) \leqslant \|Z\| \cdot \|Z^{-1}\| \cdot \kappa_{\mathrm{aop}}(\widetilde{X}, \widetilde{\mathbf{U}})$.

Proof. Let $(X', \mathbf{U}')$ be a unitary intertwining pair of $\mathbf{T}$. Then $(X'Z^{-1}, \mathbf{U}')$ is a unitary intertwining pair of $\widetilde{\mathbf{T}}$, and so there is a unique $Y' \in \mathcal{I}(\widetilde{\mathbf{U}}, \mathbf{U}')$ such that $X'Z^{-1} = Y'\widetilde{X}$. The last equality is equivalent to $X' = Y'(\widetilde{X}Z)$. Furthermore, $\|Y'\| \leqslant \kappa\|X'Z^{-1}\| \leqslant \kappa\|Z^{-1}\|\|X'\|$. ∎

As an application, we are able to provide simple examples for unitary asymptotes.

COROLLARY 1.7. *Let* $\mathbf{W}$ *be a commuting n-tuple of unitaries on a non-zero space, and let* $Z \in \mathcal{I}(\mathbf{T}, \mathbf{W})$ *be invertible. Then*

(i) $(I, \mathbf{W})$ *is a unitary asymptote of* $\mathbf{W}$ *with optimal norm-control* 1;
(ii) $(Z, \mathbf{W})$ *is a unitary asymptote of* $\mathbf{T}$ *with norm-control* $\|Z^{-1}\|$.

The following example shows that the norm-control with $\|Z^{-1}\|$ is not optimal (even not comparable to the optimal), in general. For simplicity, we deal with the single operator case; i.e., $n = 1$ is assumed.

EXAMPLE 1.8. Given any $\lambda \in \mathbb{T} \setminus \{1\}$, let us consider the operators

$$T = \begin{bmatrix} \lambda & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix},$$

acting on the Hilbert space $\mathbb{C}^2$. It can be easily checked that

$$Z = \begin{bmatrix} 1 & 0 \\ \frac{1}{1-\lambda} & 1 \end{bmatrix} \in \mathcal{I}(T, W) \ \text{ is invertible with } \ Z^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{-1}{1-\lambda} & 1 \end{bmatrix}.$$

Thus, (Z, W) is a unitary asymptote of T with norm-control $\|Z^{-1}\|$. It is clear that $\|Z^{-1}\| \geqslant 1/|1-\lambda|$ can be made as large as we wish, approaching λ to 1. On the other hand, in view of Proposition 1.2, the optimal norm-control for (Z, W) is $\kappa_{\mathrm{op}} = \|\Gamma^{-1}\|$, where $\Gamma : \{W\}' \to \mathcal{I}(T, W), Y' \mapsto Y'Z$. The general form of Y' is

$$Y' = \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{bmatrix}, \quad \text{whence } Y'Z = \begin{bmatrix} \eta_1 & 0 \\ \frac{\eta_2}{1-\lambda} & \eta_2 \end{bmatrix}.$$

Obviously,

$$\begin{aligned} \|\Gamma^{-1}\| &= \sup\left\{ \frac{\|Y'\|}{\|Y'Z\|} : 0 \neq Y' \in \{W\}' \right\} \\ &= (\inf\{\|Y'Z\| : Y' \in \{W\}', \|Y'\| = 1\})^{-1}. \end{aligned}$$

If $1 = \|Y'\| = \max\{|\eta_1|, |\eta_2|\}$, then $\|Y'Z\| \geqslant \max\{|\eta_1|, |\eta_2|\} = 1$; moreover, $\|Y'Z\| = 1$ when $\eta_1 = 1$ and $\eta_2 = 0$. Thus, $\kappa_{\mathrm{op}} = \|\Gamma^{-1}\| = 1$.

Necessary conditions for the existence of non-zero unitary intertwining pairs can be given in terms of the spectral radius. We say that the commuting n-tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is invertible, if each T_i is invertible ($i \in \mathbb{N}_n$); then $\mathbf{T}^{-1} :=$

$(T_1^{-1}, \dots, T_n^{-1})$ is also a commuting n-tuple. The proof of the following statement is left to the reader.

PROPOSITION 1.9. *Let us assume that there is a unitary intertwining pair* $(X, \mathbf{U})$ *of* $\mathbf{T}$*, with* $X \neq 0$.

(i) *Then* $r(\mathbf{T}^{\mathbf{k}}) \geqslant 1$ *holds, for all* $\mathbf{k} \in \mathbb{Z}_+^n$.

(ii) *If* $\mathbf{T}$ *is invertible, then* $r(\mathbf{T}^{\mathbf{k}}) = 1$ *is true, for all* $\mathbf{k} \in \mathbb{Z}^n$.

We say that $\mathbf{T}$ is of 0-*type*, if $X = 0$ whenever $(X, \mathbf{U})$ is a unitary intertwining pair of $\mathbf{T}$. In this case $(0, \mathbf{0})$ is a degenerate unitary asymptote of $\mathbf{T}$, where $\mathbf{0} = (0, \dots, 0)$ acts on the zero space $\{0\}$. As a consequence, we obtain further examples for unitary asymptotes.

COROLLARY 1.10. *If* $\mathbf{T}$ *is invertible and* $r(\mathbf{T}^{\mathbf{k}}) \neq 1$ *for some* $\mathbf{k} \in \mathbb{Z}^n$*, then* $(0, \mathbf{0})$ *is a unitary asymptote of* $\mathbf{T}$.

We turn to orthogonal sums. If $\mathbf{T}_j = (T_{j1}, \dots, T_{jn})$ is a commuting n-tuple of operators on $\mathcal{H}_j$, for $j = 1, 2$, then $\mathbf{T}_1 \oplus \mathbf{T}_2 := (T_{11} \oplus T_{21}, \dots, T_{1n} \oplus T_{2n})$ is a commuting n-tuple of operators on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$.

PROPOSITION 1.11. *For* $j = 1, 2$*, let* $\mathbf{T}_j = (T_{j1}, \dots, T_{jn})$ *be a commuting* n*-tuple of operators on the Hilbert space* $\mathcal{H}_j$*, and let us assume that* $(X_j, \mathbf{U}_j)$ *is a unitary asymptote of* $\mathbf{T}_j$ *with norm-control* κ_j*. Then* $(X = X_1 \oplus X_2, \mathbf{U} = \mathbf{U}_1 \oplus \mathbf{U}_2)$ *will be a unitary asymptote of* $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$ *with norm-control* $\kappa = \sqrt{2}\max(\kappa_1, \kappa_2)$.

Proof. It is clear that $(X, \mathbf{U})$ is a minimal unitary intertwining pair of $\mathbf{T}$. Suppose that $(X', \mathbf{U}')$ is also a unitary intertwining pair of $\mathbf{T}$. For $j = 1, 2$, the restriction $X'_j := X'|_{\mathcal{H}_j}$ belongs to $\mathcal{I}(\mathbf{T}_j, \mathbf{U}')$, and so there is a unique $Y'_j \in \mathcal{I}(\mathbf{U}_j, \mathbf{U}')$ such that $X'_j = Y'_j X_j$. Then $Y' = [Y'_1 \;\; Y'_2]$ is the unique transformation in $\mathcal{I}(\mathbf{U}, \mathbf{U}')$ satisfying $Y'X = X'$. Furthermore, we have

$$\|Y'\| \leqslant \sqrt{2}\max(\|Y'_1\|, \|Y'_2\|) \leqslant \sqrt{2}\max(\kappa_1\|X'_1\|, \kappa_2\|X'\|) \leqslant \sqrt{2}\max(\kappa_1, \kappa_2)\|X'\|. \quad \blacksquare$$

The next statement deals with the opposite direction.

PROPOSITION 1.12. *Let us assume that* $(X, \mathbf{U})$ *is a unitary asymptote of* $\mathbf{T} = \mathbf{T}_1 \oplus \mathbf{T}_2$*, acting on* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$*, with norm-control* κ*. For* $j = 1, 2$*, let us consider the reducing subspace* $\mathcal{K}_j = \bigvee\{\mathbf{U}^{\mathbf{k}} X \mathcal{H}_j : \mathbf{k} \in \mathbb{Z}^n\}$ *of* $\mathbf{U}$*, and the restrictions* $\mathbf{U}_j := \mathbf{U}|_{\mathcal{K}_j}$ *and* $X_j := X|_{\mathcal{H}_j} \in \mathcal{I}(\mathbf{T}_j, \mathbf{U}_j)$*. Then* $(X_j, \mathbf{U}_j)$ *will be a unitary asymptote of* $\mathbf{T}_j$ $(j = 1, 2)$ *with norm-control* κ*; furthermore,* $(X, \mathbf{U}) \approx (X_1 \oplus X_2, \mathbf{U}_1 \oplus \mathbf{U}_2)$.

Proof. It is clear that $(X_j, \mathbf{U}_j)$ is a minimal unitary intertwining pair of $\mathbf{T}_j$ $(j = 1, 2)$. Suppose that $(X'_j, \mathbf{U}'_j)$ is also a unitary intertwining pair of $\mathbf{T}_j$. Then $(\widetilde{X}_j, \mathbf{U}'_j)$ will be a unitary intertwining pair of $\mathbf{T}$ with $\widetilde{X}_j h := X'_j P_j h$ $(h \in \mathcal{H})$, where $P_j \in \mathcal{L}(\mathcal{H})$ denotes the orthogonal projection onto $\mathcal{H}_j$. Hence there exists unique $\widetilde{Y}_j \in \mathcal{I}(\mathbf{U}, \mathbf{U}'_j)$ such that $\widetilde{Y}_j X = \widetilde{X}_j$. We infer that $Y'_j := \widetilde{Y}_j|_{\mathcal{K}_j} \in \mathcal{I}(\mathbf{U}_j, \mathbf{U}'_j)$ and

$Y_j'X_j = \widetilde{Y}X|_{\mathcal{H}_j} = \widetilde{X}_j|_{\mathcal{H}_j} = X_j'$. Assuming that $Y_j'' \in \mathcal{I}(\mathbf{U}_j, \mathbf{U}_j')$ and $Y_j''X_j = X_j'$, the relations

$$Y_j''\mathbf{U}_j^{\mathbf{k}}X_j = \mathbf{U}_j'^{\mathbf{k}}Y_j''X_j = \mathbf{U}_j'^{\mathbf{k}}Y_j'X_j = Y_j'\mathbf{U}_j^{\mathbf{k}}X_j \quad (\mathbf{k} \in \mathbb{Z}^n)$$

yield that $Y_j'' = Y_j'$. Finally, $\|Y_j'\| \leqslant \|\widetilde{Y}_j\| \leqslant \kappa\|\widetilde{X}_j\| = \kappa\|X_j'\|$, and so $(X_j, \mathbf{U}_j)$ is a unitary asymptote of $\mathbf{T}_j$ with norm-control κ.

Similarity is an immediate consequence of Propositions 1.11 and 1.3. ∎

Concluding this section we provide an example of an operator, which does not have a unitary asymptote (and so it is necessarily not of 0-type). Furthermore, we shall see that, taking restrictions to invariant subspaces or taking adjoints, the existence of unitary asymptotes can be lost.

EXAMPLE 1.13. Let $\{e_j\}_{j\in\mathbb{Z}}$ be an orthonormal basis in the Hilbert space $\widetilde{\mathcal{E}}$, and let $\widetilde{S} \in \mathcal{L}(\widetilde{\mathcal{E}})$ be the bilateral shift defined by $\widetilde{S}e_j = e_{j+1}$ $(j \in \mathbb{Z})$. Let us consider the operator $\widetilde{R} = 2\widetilde{S}$ and its restriction $R = \widetilde{R}|_{\mathcal{E}}$ to the invariant subspace $\mathcal{E} = \bigvee\{e_j\}_{j\in\mathbb{Z}_+}$. It is evident that $(\widetilde{X}, \widetilde{S})$ is a unitary intertwining pair of R, where the injective transformation $\widetilde{X}$ is defined by $\widetilde{X}e_j = 2^{-j}e_j$ $(j \in \mathbb{Z}_+)$. Thus, R is far from being of 0-type.

Let us assume that there exists a unitary asymptote (X, U) of R. For any $\lambda \in \mathbb{T}$, $(X_\lambda, \lambda I_{\mathbb{C}})$ is a unitary intertwining pair of R, where $X_\lambda h = \sum_{j=0}^{\infty} 2^{-j}\lambda^j\langle h, e_j\rangle$ $(h \in \mathcal{E})$. Hence there is a unique $Y_\lambda \in \mathcal{I}(U, \lambda I_{\mathbb{C}})$ such that $X_\lambda = Y_\lambda X$. We may easily infer that λ is an eigenvalue of U. Since the eigenspaces corresponding to distinct eigenvalues are orthogonal to each other, it follows that $\sum_{\lambda\in\mathbb{T}} \bigoplus \ker(U - \lambda I)$ is a subspace of the domain $\mathcal{K}$ of U, which means that the Hilbert space $\mathcal{K}$ is not separable. But this is impossible, because $\mathcal{E}$ is separable and (X, U) is minimal. Therefore, *R does not have a unitary asymptote.*

The operator $\widetilde{R}$ is invertible and $r(\widetilde{R}) = 2$, hence $\widetilde{R}$ is of 0-type by Corollary 1.10. Let W be a unitary operator on a non-zero Hilbert space $\mathcal{F}$. Then $(0 \oplus I, 0 \oplus W)$ *is a unitary asymptote of* $\widetilde{T} = \widetilde{R} \oplus W$ (see Proposition 1.11). The subspace $\mathcal{H} = \mathcal{E} \oplus \mathcal{F}$ is invariant for $\widetilde{T}$; but the *restriction* $T := \widetilde{T}|_{\mathcal{H}} = R \oplus W$ *does not have a unitary asymptote*, since R fails to have a unitary asymptote (see Proposition 1.12).

If (X, U) is a unitary intertwining pair of R^*, then $0 = \|X(R^*)^{j+1}e_j\| = \|U^{j+1}Xe_j\| = \|Xe_j\|$ $(j \in \mathbb{Z}_+)$; hence $X = 0$, and so R^* is of 0-type. We conclude that the *adjoint* $T^* = R^* \oplus W^*$ *has a unitary asymptote:* $(0 \oplus I, 0 \oplus W^*)$.

Analogous counterexamples can be obtained for commuting n-tuples with $n > 1$, observing that (X, U) is a unitary asymptote of T exactly when the pair $(X, (U, I, \ldots, I))$ is a unitary asymptote of $(T, I, \ldots, I)$.

2. ORBIT CONDITIONS

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on the Hilbert space $\mathcal{H}$. The *orbit-infimum* of $\mathbf{T}$ at the vector $h \in \mathcal{H}$ is defined by $\text{o-inf}(\mathbf{T}, h) := \inf\{\|\mathbf{T}^{\mathbf{k}} h\| : \mathbf{k} \in \mathbb{Z}_+^n\}$. If $(X', \mathbf{U}')$ is any unitary intertwining pair of $\mathbf{T}$, then $\|X'h\| = \|\mathbf{U}'^{\mathbf{k}} X'h\| = \|X'\mathbf{T}^{\mathbf{k}} h\| \leqslant \|X'\| \cdot \|\mathbf{T}^{\mathbf{k}} h\|$ $(h \in \mathcal{H}, \mathbf{k} \in \mathbb{Z}_+^n)$, and so the *Upper Orbit Condition* (UOC) automatically holds:

$$\|X'h\| \leqslant \|X'\| \cdot \text{o-inf}(\mathbf{T}, h) \quad \text{for all } h \in \mathcal{H}.$$

The opposite *Lower Orbit Condition* (LOC) *with bound-control* $\kappa \in \mathbb{R}_+$ is an extra requirement:

$$\text{o-inf}(\mathbf{T}, h) \leqslant \kappa \|X'h\| \quad \text{for all } h \in \mathcal{H}.$$

The latter condition together with minimality are sufficient for being a unitary asymptote.

PROPOSITION 2.1. *Let* $(X, \mathbf{U})$ *be a minimal unitary intertwining pair of* $\mathbf{T}$. *If* (LOC) *holds for* $(X, \mathbf{U})$ *with bound-control* κ, *then* $(X, \mathbf{U})$ *is a unitary asymptote of* $\mathbf{T}$ *with norm-control* κ.

Proof. Let $(X', \mathbf{U}')$ be any unitary intertwining pair of $\mathbf{T}$. Given any $h \in \mathcal{H}$, the (UOC) for $(X', \mathbf{U}')$ yields $\|X'h\| \leqslant \|X'\| \cdot \text{o-inf}(\mathbf{T}, h)$, while the (LOC) for $(X, \mathbf{U})$ results $\text{o-inf}(\mathbf{T}, h) \leqslant \kappa \|Xh\|$. Thus $\|X'h\| \leqslant \|X'\| \cdot \kappa \|Xh\|$ $(h \in \mathcal{H})$, and so there is a unique $Y'_+ \in \mathcal{L}((X\mathcal{H})^-, (X'\mathcal{H})^-)$ such that $Y'_+ Xh = X'h$ $(h \in \mathcal{H})$; in particular, $\|Y'_+\| \leqslant \kappa \|X'\|$. By the minimality of $(X, \mathbf{U})$, there exists a unique $Y' \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $Y'\mathbf{U}^{\mathbf{k}} y = \mathbf{U}'^{\mathbf{k}} Y'_+ y$ holds for all $y \in (X\mathcal{H})^-$ and $\mathbf{k} \in \mathbb{Z}^n$. It can be easily seen that $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$, $Y'X = X'$ and $\|Y'\| = \|Y'_+\| \leqslant \kappa \|X'\|$. ∎

Let us consider the homogeneous set $\mathcal{H}_{00}(\mathbf{T}) := \{h \in \mathcal{H} : \text{o-inf}(\mathbf{T}, h) = 0\}$. If $(X', \mathbf{U}')$ is a unitary intertwining pair of $\mathbf{T}$, then the (UOC) implies that $\mathcal{H}_{00}(\mathbf{T}) \subset \ker X'$. Assuming that $(X, \mathbf{U})$ is a unitary asymptote of $\mathbf{T}$, the relation $X' = Y'X$ yields that $\ker X \subset \ker X'$. In particular, if $(\widetilde{X}, \widetilde{\mathbf{U}})$ is another unitary asymptote of $\mathbf{T}$, then $\ker \widetilde{X} = \ker X$; this common nullspace $\mathcal{H}_0(\mathbf{T})$ is called the *annihilating subspace* of $\mathbf{T}$. Clearly, $\mathcal{H}_0(\mathbf{T})$ contains the set $\mathcal{H}_{00}(\mathbf{T})$. It is immediate that coincidence is ensured by the (LOC).

Let us assume that $\mathbf{T}$ has a unitary asymptote. Relying on the annihilating subspace $\mathcal{H}_0(\mathbf{T})$, we may classify the n-tuples:

(i) $\mathbf{T} \in C_{0\cdot}$, if $\mathcal{H}_0(\mathbf{T}) = \mathcal{H}$; this happens exactly, when $(0, \mathbf{0})$ is a unitary asymptote, that is when $\mathbf{T}$ is *of* 0*-type*;

(ii) $\mathbf{T} \in C_{*\cdot}$, if $\mathcal{H}_0(\mathbf{T}) \neq \mathcal{H}$; then $\mathbf{T}$ is called *asymptotically non-vanishing*;

(iii) $\mathbf{T} \in C_{1\cdot}$, if $\mathcal{H}_0(\mathbf{T}) = \{0\} \neq \mathcal{H}$; then $\mathbf{T}$ is called *asymptotically strongly non-vanishing*.

If the adjoint $\mathbf{T}^*$ also has a unitary asymptote, then $\mathbf{T} \in C_{\cdot j}$ by definition if $\mathbf{T}^* \in C_{j\cdot}$ $(j = 0, *, 1)$. Finally, $C_{ij} = C_{i\cdot} \cap C_{\cdot j}$.

Clearly, $\mathcal{H}_{00}(\mathbf{T}) = \mathcal{H}$ implies that $\mathbf{T} \in C_{0\cdot}$. However, Corollary 1.10 shows that $\mathbf{T} \in C_{0\cdot}$ may happen in many other cases, even when (LOC) fails.

The (LOC) does hold in the power bounded setting. Namely, the technique, originated in [29], provides a canonical way of constructing suitable unitary asymptote. Let us assume that $\mathbf{T}$ is a commuting n-tuple of power bounded operators on $\mathcal{H}$, that is $s := \sup\{\|\mathbf{T}^{\mathbf{k}}\| : \mathbf{k} \in \mathbb{Z}_+\} < \infty$. Let $M : \ell^\infty(\mathbb{Z}_+^n) \to \mathbb{C}$ be an invariant mean; see, e.g., [8] and [26]. Then, there exists a minimal unitary intertwining pair $(X_M, \mathbf{U}_M)$ of $\mathbf{T}$, such that

$$\|X_M h\|^2 = M\text{-}\lim_{\mathbf{k}} \|\mathbf{T}^{\mathbf{k}} h\|^2$$

holds, for every $h \in \mathcal{H}$; see, e.g., [1] and [15]. It follows that $(X_M, \mathbf{U}_M)$ satisfies the (LOC) with bound-control 1. Since $\|X_M\| \leqslant s$, we deduce from Proposition 2.1 that $(X_M, \mathbf{U}_M)$ *is a unitary asymptote of* $\mathbf{T}$ *and* $\kappa_{\mathrm{aop}}(X_M, \mathbf{U}_M) \leqslant s$.

It would be interesting to give examples for a commuting n-tuple $\mathbf{T}$ of operators, such that $\mathbf{T}$ is not power bounded, and $\mathbf{T}$ has a unitary asymptote $(X, \mathbf{U})$ satisfying (LOC). An example like that would provide negative answer to the following question.

QUESTION 2.2. *Is power boundedness of* $\mathbf{T}$ *a necessary condition for the existence of a unitary asymptote* $(X, \mathbf{U})$ *(possibly with injective* X*), satisfying* (LOC)*?*

REMARK 2.3. Let $\mathbf{T}=(T_1, \dots, T_n)$ be a commuting n-tuple of power bounded operators on $\mathcal{H}$. For any $i \in \mathbb{N}_n$, o-inf$(\mathbf{T}, h) \leqslant$ o-inf(T_i, h) holds for every $h \in \mathcal{H}$, and so $\mathcal{H}_0(T_i) = \mathcal{H}_{00}(T_i) \subset \mathcal{H}_{00}(\mathbf{T}) = \mathcal{H}_0(\mathbf{T})$. In particular, if $\mathbf{T}$ is asymptotically (strongly) non-vanishing, then T_i is also asymptotically (strongly) non-vanishing. Taking appropriate analytic Toeplitz operators, it can be seen that the converse implication is not true (see Example 3.9).

The following proposition, inspired by a remark of Maria Gamal', shows that unitary asymptotes, not satisfying (LOC), arise naturally.

PROPOSITION 2.4. *Let* $\mathbf{T}$ *be a commuting n-tuple of invertible operators on* $\mathcal{H}$*, and let* $(X, \mathbf{U})$ *be a unitary asymptote of* $\mathbf{T}$ *with norm-control* κ*.*

(i) *Then* $(X, \mathbf{U}^{-1})$ *is a unitary asymptote of* $\mathbf{T}^{-1}$*, with norm-control* κ*.*

(ii) *If* $\mathbf{T}$ *is power bounded, then the following conditions are equivalent:*

(a) $\mathbf{T}^{-1}$ *has a unitary asymptote satisfying* (LOC)*;*

(b) *the given* $(X, \mathbf{U}^{-1})$ *satisfies* (LOC)*;*

(c) X *is invertible;*

(d) $\mathbf{T}$ *is similar to a commuting n-tuple of unitaries.*

Proof. (i) Let $(X', \mathbf{U}'^{-1})$ be a unitary intertwining pair of $\mathbf{T}^{-1}$. Since $(X', \mathbf{U}')$ is a unitary intertwining pair of $\mathbf{T}$, there is a unique $Y' \in \mathcal{I}(\mathbf{U}, \mathbf{U}')$ such that $X' = Y'X$. Then $Y' \in \mathcal{I}(\mathbf{U}^{-1}, \mathbf{U}'^{-1})$ also holds, whence the statement follows. We note also that $(X\mathcal{H})^-$ is reducing for $\mathbf{U}$, and so X has dense range by minimality.

(ii) Suppose that $\mathbf{T}$ is power bounded: $s := \sup\{\|\mathbf{T}^{\mathbf{k}}\| : \mathbf{k} \in \mathbb{Z}_+^n\} < \infty$. (Then the existence of the unitary asymptote is guaranteed.) If $\mathbf{T}^{-1}$ has a unitary asymptote satisfying (LOC), then $(X, \mathbf{U}^{-1})$ satisfies (LOC) too by similarity (see Proposition 1.3).

For every $h \in \mathcal{H}$, we have o-inf$(\mathbf{T}^{-1}, h) \geqslant \|h\|/s$. Indeed, if o-inf$(\mathbf{T}^{-1}, h) < \|h\|/s$, then $\|\mathbf{T}^{-\mathbf{k}}h\| < \|h\|/s$ holds for some $\mathbf{k} \in \mathbb{Z}_+^n$, and so we infer that $\|h\| = \|\mathbf{T}^{\mathbf{k}}\mathbf{T}^{-\mathbf{k}}h\| \leqslant s\|\mathbf{T}^{-\mathbf{k}}h\| < \|h\|$, what is a contradiction. Therefore, if (LOC) holds for $(X, \mathbf{U}^{-1})$ with a bound-control $\kappa > 0$, then the inequalities

$$\frac{1}{s}\|h\| \leqslant \text{o-inf}(\mathbf{T}^{-1}, h) \leqslant \kappa\|Xh\| \quad (h \in \mathcal{H})$$

imply that X is invertible, and so $\mathbf{T}$ is similar to the unitary n-tuple $\mathbf{U}$.

Finally, if $\mathbf{T}$ is similar to a unitary n-tuple, then $\mathbf{T}^{-1}$ is power bounded, and so it has a unitary asymptote satisfying (LOC). ∎

Now we give a specific example for an invertible $\mathbf{T} \in C_{1\cdot}$, where (LOC) fails for $\mathbf{T}^{-1}$.

EXAMPLE 2.5. Let us consider the Hilbert space $\ell^2(\mathbb{Z}^n, \beta)$ consisting of the sequences $\xi : \mathbb{Z}^n \to \mathbb{C}$ satisfying $\|\xi\|_\beta^2 := \sum_{\mathbf{k}\in\mathbb{Z}^n} |\xi(\mathbf{k})|^2\beta(\mathbf{k})^2 < \infty$, where $\log_2 \beta(\mathbf{k}) = -\sum_{i=1}^{n} \min(0, k_i)$ for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$. The commuting invertible contractions $\mathbf{T} = (T_1, \dots, T_n)$ are defined on $\ell^2(\mathbb{Z}^n, \beta)$ by $(T_i\xi)(\mathbf{k}) = \xi(\mathbf{k} - \mathbf{e}^{(i)})$, where $\mathbf{e}^{(i)} = (\delta_{i1}, \dots, \delta_{in})$ $(i \in \mathbb{N}_n)$. The commuting unitaries $\mathbf{U} = (U_1, \dots, U_n)$ are defined on $\ell^2(\mathbb{Z}^n)$ by the same formula: $(U_i\xi)(\mathbf{k}) = \xi(\mathbf{k} - \mathbf{e}^{(i)})$. The transformation $X : \ell^2(\mathbb{Z}^n, \beta) \to \ell^2(\mathbb{Z}^n)$, $\xi \mapsto \xi$ intertwines $\mathbf{T}$ with $\mathbf{U}$, and $\|X\xi\| = \lim_{\mathbf{k}} \|\mathbf{T}^{\mathbf{k}}\xi\|_\beta = \|\xi\| \leqslant \|\xi\|_\beta$ holds, for every $\xi \in \ell^2(\mathbb{Z}^n, \beta)$. We obtain that $(X, \mathbf{U})$ is a unitary asymptote of $\mathbf{T}$, satisfying (LOC) with bound-control 1. Since the injective X is not invertible, we infer by Proposition 2.4 that (LOC) fails for the unitary asymptote $(X, \mathbf{U}^{-1})$ of $\mathbf{T}^{-1} \in C_{1\cdot}$.

3. QUASIANALYTIC n-TUPLES OF OPERATORS

Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n-tuple of operators on the Hilbert space $\mathcal{H}$, and let $\mathbf{U} = (U_1, \dots, U_n)$ be a commuting n-tuple of unitaries on $\mathcal{K}$. *Let us assume that $(X, \mathbf{U})$ is a unitary asymptote of $\mathbf{T}$ with norm-control κ, where $X \neq 0$, and so $\mathbf{T} \in C_{*\cdot}$.*

Given any $C \in \{\mathbf{T}\}'$, $(XC, \mathbf{U})$ is a unitary intertwining pair of $\mathbf{T}$. Hence, there exists a unique $D \in \mathcal{I}(\mathbf{U}, \mathbf{U}) = \{\mathbf{U}\}'$ such that $XC = DX$. The transformation

$$\gamma = \gamma_X : \{\mathbf{T}\}' \to \{\mathbf{U}\}', \quad C \mapsto D$$

is called the *commutant mapping* associated with the unitary asymptote $(X, \mathbf{U})$. Obviously, $\gamma(\mathbf{T}^{\mathbf{k}}) = \mathbf{U}^{\mathbf{k}}$ holds for all $\mathbf{k} \in \mathbb{Z}_+^n$. This mapping has been studied in many particular settings; see, e.g., [14], [17], [19] and [21], [30]. The novelty in the next statement is the boundedness of γ, what is an immediate consequence of the norm-control.

PROPOSITION 3.1. *The commutant mapping γ is a bounded algebra-homomorphism, with $\|\gamma\| \leqslant \kappa\|X\|$; taking κ optimal, we obtain $\|\gamma\| \leqslant \kappa_{\mathrm{aop}}(X, \mathbf{U})$.*

REMARK 3.2. Assuming that $(\widetilde{X}, \widetilde{\mathbf{U}})$ is another unitary asymptote of $\mathbf{T}$, let $Z \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$ be the invertible transformation satisfying $ZX = \widetilde{X}$ (see Proposition 1.3). Given any $C \in \{\mathbf{T}\}'$, let us consider the operators $D = \gamma_X(C) \in \{\mathbf{U}\}'$ and $\widetilde{D} = \gamma_{\widetilde{X}}(C) \in \{\widetilde{\mathbf{U}}\}'$. Since $(X, \mathbf{U})$ is a unitary asymptote of $\mathbf{T}$, $\widetilde{D}ZX = \widetilde{D}\widetilde{X} = \widetilde{X}C = ZXC = ZDX \in \mathcal{I}(\mathbf{T}, \widetilde{\mathbf{U}})$ and $\widetilde{D}Z, ZD \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$, it follows that $\widetilde{D}Z = ZD$. Observing that $(Z, \widetilde{\mathbf{U}})$ is a unitary asymptote of $\mathbf{U}$ (see Corollary 1.7), we can see that $\widetilde{D} = ZDZ^{-1} = \gamma_Z(D)$. Therefore, we obtain that $\gamma_{\widetilde{X}} = \gamma_Z \circ \gamma_X$.

The *hyperinvariant subspace lattice* $\operatorname{Hlat}\mathbf{T}$ of $\mathbf{T}$ consists of those subspaces $\mathcal{M}$ of $\mathcal{H}$, which are invariant for the commutant $\{\mathbf{T}\}'$, that is $C\mathcal{M} \subset \mathcal{M}$ holds for all $C \in \{\mathbf{T}\}'$. The relation $\{\mathbf{T}\}' \subset \{T_i\}'$ readily implies that $\operatorname{Hlat}\mathbf{T} \supset \operatorname{Hlat} T_i$ is true, for every $i \in \mathbb{N}_n$. To illustrate this connection we give an example.

EXAMPLE 3.3. Let S be the unilateral shift on the Hardy–Hilbert space H^2; that is $Sf = \chi f$ $(f \in H^2)$, where $\chi(z) = z$ is the identical function. Let us consider the commuting pair $\mathbf{T} = (T_1, T_2)$ of the operators $T_1 = S \oplus S$ and $T_2 = I \oplus 0$ acting on the Hilbert space $\mathcal{H} = H^2 \oplus H^2$.

It is well-known that the commutant of S consists of the analytic Toeplitz operators on H^2 (see, e.g., Section 147 in [12]). Hence the invariant subspaces are also hyperinvariant for S, and by Beurling's theorem they are of the form ϑH^2, where $\vartheta \in H^\infty$ is an inner function ($|\vartheta(\zeta)| = 1$ for a.e. $\zeta \in \mathbb{T}$) or $\vartheta = 0$. Since $C = [C_{ij}]_2 \in \{T_1\}'$ exactly when $C_{ij} \in \{S\}'$ for all $1 \leqslant i, j \leqslant 2$, it can be easily seen that $\operatorname{Hlat} T_1 = \{\vartheta(H^2 \oplus H^2) : \vartheta \in H^\infty \text{ inner or } \vartheta = 0\}$. Obviously, $\{T_2\}' = \{A \oplus B : A, B \in \mathcal{L}(H^2)\}$ and $\operatorname{Hlat} T_2 = \{\{0\}, H^2 \oplus \{0\}, \{0\} \oplus H^2, \mathcal{H}\}$. We conclude that $\{\mathbf{T}\}' = \{T_1\}' \cap \{T_2\}' = \{T_\varphi \oplus T_\psi : \varphi, \psi \in H^\infty\}$, and so $\operatorname{Hlat}\mathbf{T} = \{\vartheta H^2 \oplus \eta H^2 : \vartheta, \eta \in H^\infty \text{ inner or } 0\}$.

Since $\{\mathbf{U}\}'$ is a C^*-algebra, the subspaces in $\operatorname{Hlat}\mathbf{U}$ are reducing. Furthermore, spectral theory provides many subspaces in $\operatorname{Hlat}\mathbf{U}$, as we shall see soon. The hyperinvariant subspaces of $\mathbf{U}$ induce hyperinvariant subspaces of $\mathbf{T}$.

PROPOSITION 3.4. *For every $\mathcal{N} \in \operatorname{Hlat}\mathbf{U}$, we have $\mathcal{M} = X^{-1}\mathcal{N} \in \operatorname{Hlat}\mathbf{T}$; in particular, $\mathcal{H}_0(\mathbf{T}) = X^{-1}(\{0\}) \in \operatorname{Hlat}\mathbf{T}$.*

Proof. Given any $C \in \{\mathbf{T}\}'$, let us consider $D = \gamma(C) \in \{\mathbf{U}\}'$. For any $h \in \mathcal{M}$, $Xh \in \mathcal{N}$ implies $DXh \in \mathcal{N}$. Since $XCh = DXh$, it follows that $Ch \in \mathcal{M}$. ∎

Now we turn to the spectral analysis of the commuting n-tuple $\mathbf{U}$ of unitaries. We recall that the spectrum of the abelian C^*-algebra, generated by $\mathbf{U}$, can be identified with a non-empty, compact subset of the n-dimensional torus $\mathbb{T}^n$, called the (joint) *spectrum* of $\mathbf{U}$, and denoted by $\sigma(\mathbf{U})$ (see, e.g., [28]). We note also that $\sigma(\mathbf{U})$ coincides with the (joint) approximate point spectrum of $\mathbf{U}$ (see [24] and [25]). By the spectral theorem (see, e.g., [6]) there exists a unique spectral measure $E_{\mathbf{U}} : \mathcal{B}_n \to \mathcal{P}(\mathcal{K})$ such that $\mathbf{U}^{\mathbf{k}} = \int_{\mathbb{T}^n} \mathbf{z}^{\mathbf{k}} \, \mathrm{d}E_{\mathbf{U}}(\mathbf{z})$ holds for all $\mathbf{k} \in \mathbb{Z}^n$. (Here $\mathcal{B}_n$ denotes the σ-algebra of all Borel subsets of $\mathbb{T}^n$, and $\mathcal{P}(\mathcal{K})$ stands for the set of orthogonal projections in $\mathcal{K}$.) The support of $E_{\mathbf{U}}$ is the spectrum $\sigma(\mathbf{U})$. We note yet that $D \in \{\mathbf{U}\}'$ if and only if D commutes with every spectral projection $E_{\mathbf{U}}(\omega)$ $(\omega \in \mathcal{B}_n)$. Hence, *the spectral subspaces $E_{\mathbf{U}}(\omega)\mathcal{K}$ are all hyperinvariant for* $\mathbf{U}$.

The *localization* of the spectral measure $E = E_{\mathbf{U}}$ at the vector $v \in \mathcal{K}$ is the finite positive Borel measure E_v on $\mathbb{T}^n$, defined by $E_v(\omega) := \langle E(\omega)v, v\rangle$ $(\omega \in \mathcal{B}_n)$. Clearly, $E_v(\mathbb{T}^n) = \|v\|^2$. Let $M_+(\mathbb{T}^n)$ be the set of all finite, positive Borel measures on $\mathbb{T}^n$. Given any $\mu, \nu \in M_+(\mathbb{T}^n)$, the notation $\mu \overset{\mathrm{a}}{\prec} \nu$ means that μ is absolutely continuous with respect to ν, that is $\nu(\omega) = 0$ implies $\mu(\omega) = 0$ $(\omega \in \mathcal{B}_n)$. The measures μ and ν are equivalent, in notation: $\mu \overset{\mathrm{a}}{\sim} \nu$, if $\mu \overset{\mathrm{a}}{\prec} \nu$ and $\nu \overset{\mathrm{a}}{\prec} \mu$, that is $\mu(\omega) = 0$ holds exactly when $\nu(\omega) = 0$.

Let $\widetilde{\mathbf{U}} = (\widetilde{U}_1, \ldots, \widetilde{U}_n)$ be also a commuting n-tuple of unitary operators, acting on the non-zero Hilbert space $\widetilde{\mathcal{K}}$, and let $\widetilde{E} : \mathcal{B}_n \to \mathcal{P}(\widetilde{\mathcal{K}})$ be the spectral measure of $\widetilde{\mathbf{U}}$. Given any $Z \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$, let us consider the unitary intertwining pair $(\widetilde{X}, \widetilde{\mathbf{U}})$ of $\mathbf{T}$, where $\widetilde{X} := ZX$. The following technical lemma relates localizations of E and $\widetilde{E}$.

LEMMA 3.5. *If Z is injective, then $\widetilde{E}_{\widetilde{X}h} = E_{|Z|Xh} \overset{\mathrm{a}}{\sim} E_{Xh}$ hold, for every $h \in \mathcal{H}$.*

Proof. Taking the polar decomposition $Z = W|Z|$, it can be easily seen that $|Z| \in \{\mathbf{U}\}'$ is injective and $W \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}})$ is an isometry. The subspace $\widetilde{\mathcal{K}}_1 = W\mathcal{K}$ is reducing for $\widetilde{\mathbf{U}}$, so the restriction $\widetilde{\mathbf{U}}_1 := \widetilde{\mathbf{U}}|_{\mathcal{K}_1}$ is a commuting n-tuple of unitaries on $\widetilde{\mathcal{K}}_1$. Let $\widetilde{E}_1 : \mathcal{B}_n \to \mathcal{P}(\widetilde{\mathcal{K}}_1)$ be the spectral measure of $\widetilde{\mathbf{U}}_1$. Clearly, $W_1 \in \mathcal{I}(\mathbf{U}, \widetilde{\mathbf{U}}_1)$ is a unitary transformation, where $W_1 v = Wv$ $(v \in \mathcal{K})$. Relying on the uniqueness part of the Riesz representation theorem, it is easy to show that $W_1 E(\omega) W_1^* = \widetilde{E}_1(\omega) = \widetilde{E}(\omega)|_{\widetilde{\mathcal{K}}_1}$ hold for all $\omega \in \mathcal{B}_n$. Thus, for any $h \in \mathcal{H}$, we have

$$\begin{aligned}\widetilde{E}_{\widetilde{X}h}(\omega) &= \langle \widetilde{E}(\omega)\widetilde{X}h, \widetilde{X}h\rangle = \langle \widetilde{E}_1(\omega) W_1|Z|Xh, W_1|Z|Xh\rangle \\ &= \langle W_1 E(\omega)|Z|Xh, W_1|Z|Xh\rangle = \langle E(\omega)|Z|Xh, |Z|Xh\rangle = E_{|Z|Xh}(\omega) \quad (\omega \in \mathcal{B}_n).\end{aligned}$$

Since $|Z| \in \{\mathbf{U}\}'$, it follows that $|Z|E(\omega) = E(\omega)|Z|$, and so $E(\omega)|Z|Xh = |Z|E(\omega)Xh$. Taking into account that $|Z|$ is injective, we infer that $E_{|Z|Xh}(\omega) = 0$ if and only if $E_{Xh}(\omega) = 0$. (Notice that $E_v(\omega) = \|E(\omega)v\|^2$, for all $v \in \mathcal{K}$.) Therefore, we obtain that $\widetilde{E}_{\widetilde{X}h} = E_{|Z|Xh} \overset{\mathrm{a}}{\sim} E_{Xh}$. ∎

We say that the *unitary intertwining pair* $(X', \mathbf{U}')$ of $\mathbf{T}$ is *quasianalytic*, if $X' \neq 0$ and $E'_{X'u} \overset{\mathrm{a}}{\sim} E'_{X'v}$ holds for all $u, v \in \mathcal{H} \setminus \{0\}$, where $E' : \mathcal{B}_n \to \mathcal{P}(\mathcal{K}')$ denotes the spectral measure of $\mathbf{U}'$. The *commuting n-tuple* $\mathbf{T}$ of operators is called *quasianalytic*, if it has a quasianalytic unitary asymptote $(X, \mathbf{U})$. In view of Proposition 1.3 and Lemma 3.5, we can see that if one of the unitary asymptotes is quasianalytic, then so are all of them.

PROPOSITION 3.6. *If* $\mathbf{T}$ *is a quasianalytic commuting n-tuple of operators, then it is asymptotically strongly non-vanishing:* $\mathbf{T} \in C_{1\cdot}$.

Proof. Let $(X, \mathbf{U})$ be a unitary asymptote of $\mathbf{T}$, and let $u \in \mathcal{H}$ be a vector such that $Xu \neq 0$. Then, for every $0 \neq v \in \mathcal{H}$, the relation $E_{Xv} \overset{a}{\sim} E_{Xu} \neq 0$ yields that $E_{Xv} \neq 0$, and so $Xv \neq 0$. Thus, $\mathcal{H}_0(\mathbf{T}) = \ker X = \{0\}$. ∎

Quasianalyticity is a property of homogeneity type. If it is broken, then hyperinvariant subspaces arise.

THEOREM 3.7. *If the asymptotically non-vanishing, commuting n-tuple* $\mathbf{T}$ *of operators is not quasianalytic, then* Hlat $\mathbf{T}$ *is non-trivial.*

Proof. Since $\mathbf{T} \in C_{*\cdot}$, it has a unitary asymptote $(X, \mathbf{U})$, where $\ker X = \mathcal{H}_0(\mathbf{T}) \neq \mathcal{H}$. In view of Proposition 3.4, we may assume that $\mathcal{H}_0(\mathbf{T}) = \{0\}$, when X is injective. Let $E : \mathcal{B}_n \to \mathcal{P}(\mathcal{K})$ be the spectral measure of $\mathbf{U}$. Since $\mathbf{T}$ is not quasianalytic, we can find non-zero vectors $u, v \in \mathcal{H}$ so that $E_{Xu} \overset{a}{\not\prec} E_{Xv}$. Thus, there exists $\omega_1 \in \mathcal{B}_n$ such that $E_{Xv}(\omega_1) = 0$ and $E_{Xu}(\omega_1) > 0$. Setting $\omega_2 = \mathbb{T}^n \setminus \omega_1$, let us consider the decomposition $\mathcal{K} = \mathcal{N}_1 \oplus \mathcal{N}_2$, where $\mathcal{N}_j = E(\omega_j)\mathcal{K} \in$ Hlat $\mathbf{U}$ for $j = 1, 2$. We know from Proposition 3.4 that $\mathcal{M} = X^{-1}\mathcal{N}_2 \in$ Hlat $\mathbf{T}$. The equality $E_{Xv}(\omega_1) = 0$ yields that $Xv \in \mathcal{N}_2$, whence $v \in \mathcal{M}$ follows. On the other hand, $E_{Xu}(\omega_1) > 0$ implies that $Xu \notin \mathcal{N}_2$, and so $u \notin \mathcal{M}$. Consequently, $\mathcal{M}$ is a proper hyperinvariant subspace of $\mathbf{T}$. ∎

COROLLARY 3.8. *Let* $\mathbf{T}$ *be a commuting n-tuple of class* $C_{*\cdot}$. *If there exists an injection* $Y \in \mathcal{I}(\widetilde{\mathbf{U}}, \mathbf{T})$, *where* $\widetilde{\mathbf{U}}$ *is a commuting n-tuple of unitaries on* $\widetilde{\mathcal{K}}$ *and the spectrum* $\sigma(\widetilde{\mathbf{U}})$ *is not a single point of* $\mathbb{T}^n$, *then* $\mathbf{T}$ *is not quasianalytic, and so* Hlat $\mathbf{T}$ *is non-trivial.*

Proof. Let $\widetilde{E}$ denote the spectral measure of $\widetilde{\mathbf{U}}$. Since $\sigma(\widetilde{\mathbf{U}})$ is not a singleton, a Borel set $\omega_1 \in \mathcal{B}_n$ can be given so that $\widetilde{E}(\omega_1) \neq 0$ and $\widetilde{E}(\omega_2) \neq 0$ with $\omega_2 = \mathbb{T}^n \setminus \omega_1$. Choosing non-zero vectors $v_j \in \widetilde{E}(\omega_j)\widetilde{\mathcal{K}}$ $(j = 1, 2)$, the relation $\widetilde{E}(\omega_1)\widetilde{E}(\omega_2) = 0$ yields that the non-zero localizations $\widetilde{E}_{v_1}$ and $\widetilde{E}_{v_2}$ are singular to each other.

Let $(X, \mathbf{U})$ be a unitary asymptote of $\mathbf{T}$, and let E be the spectral measure of $\mathbf{U}$. Suppose that $\mathbf{T}$ is quasianalytic. Then X is injective by Proposition 3.6, and so $Z = XY \in \mathcal{I}(\widetilde{\mathbf{U}}, \mathbf{U})$ is also injective. Since $(I, \widetilde{\mathbf{U}})$ is a unitary asymptote of $\widetilde{\mathbf{U}}$ (see Corollary 1.7), an application of Lemma 3.5 results in that $E_{Zv_j} \overset{a}{\sim} \widetilde{E}_{v_j}$ for $j = 1, 2$. Thus $E_{X(Yv_1)}$ is not equivalent to $E_{X(Yv_2)}$, which means that $\mathbf{T}$ is not quasianalytic. We arrived at a contradiction, and so **T** cannot be quasianalytic. Theorem 3.7 implies that Hlat **T** is non-trivial. ∎

We conclude this section with an example.

EXAMPLE 3.9. For every $i \in \mathbb{N}_n$, let a non-constant function $\varphi_i \in H^\infty$ be given so that $\|\varphi_i\|_\infty = 1$ and $\Omega(\varphi_i) = \{\zeta \in \mathbb{T} : |\varphi_i(\zeta)| = 1\}$ is of positive Lebesgue measure. Let us assume also that $\Omega := \bigcap_{i=1}^{n} \Omega(\varphi_i)$ and $\mathbb{T} \setminus \Omega$ are of positive measure. Let us consider the commuting n-tuple $\mathbf{T} = (T_{\varphi_1}, \ldots, T_{\varphi_n})$ of analytic Toeplitz operators on the Hardy–Hilbert space H^2; that is $T_{\varphi_i} f = \varphi_i f$ for all $f \in H^2$ $(i \in \mathbb{N}_n)$. Let us consider also the commuting n-tuple $\mathbf{U} = (U_{\varphi_1}, \ldots, U_{\varphi_n})$ of unitary multiplication operators on the Hilbert space $L^2(\Omega) = \chi_\Omega L^2(\mathbb{T})$, where χ_Ω stands for the characteristic function of Ω and $L^2(\mathbb{T})$ is defined with respect to the normalized Lebesgue measure m on $\mathbb{T}$. Thus, $U_{\varphi_i} g = \varphi_i g$ for all $g \in L^2(\Omega)$ $(i \in \mathbb{N}_n)$. It is clear that $X \in \mathcal{I}(\mathbf{T}, \mathbf{U})$, where $Xf := \chi_\Omega f$ $(f \in H^2)$. In view of the F. & M. Riesz theorem $f(\zeta) \neq 0$ for a.e. $\zeta \in \mathbb{T}$, whenever $0 \neq f \in H^2$. Hence X is injective. Furthermore, for every $0 \neq f \in H^2$, Xf is cyclic for the unitary operator U_Ω, defined by $U_\Omega g = \chi g$ $(g \in L^2(\Omega))$; that is $\bigvee_{j \in \mathbb{Z}_+} U_\Omega^j Xf = L^2(\Omega)$ (see, e.g., Section 146 in [12] and [3]). In particular, X has dense range, and so $(X, \mathbf{U})$ is a minimal unitary intertwining pair of **T**. Since the (LOC) holds with bound-control $\kappa = 1$, it follows that $(X, \mathbf{U})$ is a unitary asymptote of **T** with norm-control $\kappa = 1$ (see Proposition 2.1). We obtain also that $\mathcal{H}_0(\mathbf{T}) = \ker X = \{0\}$, that is $\mathbf{T} \in C_{1\cdot}$. Taking into account that $U_\Omega \in \{\mathbf{U}\}'$, we conclude that Xf is cyclic for the commutant $\{\mathbf{U}\}' : \bigvee\{DXf : D \in \{\mathbf{U}\}'\} = L^2(\Omega)$, for every $0 \neq f \in H^2$. Therefore, **T** must be quasianalytic; see the proof of Theorem 3.7.

Notice also that **T** is of 0-type when $m(\Omega) = 0$, while $T_i \in C_{1\cdot}$ holds for all $i \in \mathbb{N}_n$. Furthermore, if $m(\Omega) = 1$ then **T** is a quasianalytic n-tuple of commuting isometries.

4. LOCAL AND GLOBAL SPECTRAL INVARIANTS

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be again a commuting n-tuple of operators on the separable Hilbert space $\mathcal{H}$. Let $\mathbf{U} = (U_1, \ldots, U_n)$ be a commuting n-tuple of unitaries on the Hilbert space $\mathcal{K}$, and let us assume that $(X, \mathbf{U})$ is a unitary asymptote of **T** with $X \neq 0$; that is $\mathbf{T} \in C_{*\cdot}$. Let $E : \mathcal{B}_n \to \mathcal{P}(\mathcal{K})$ denote the spectral measure of **U**, where $\mathcal{B}_n$ stands for the σ-algebra of all Borel subsets of $\mathbb{T}^n$.

We say that the Borel sets $\omega_1, \omega_2 \in \mathcal{B}_n$ *are equal E-a.e.*, in notation: $\omega_1 \stackrel{E}{=} \omega_2$, if $E(\omega_1 \triangle \omega_2) = 0$, that is when $E(\omega_1) = E(\omega_2)$. In this way we obtain an equivalence relation on $\mathcal{B}_n$. The Borel set ω_1 is *contained in ω_2 E-a.e.*, in notation: $\omega_1 \overset{E}{\subset} \omega_2$, if $E(\omega_1 \setminus \omega_2) = 0$, that is when $E(\omega_1) \leqslant E(\omega_2)$. This partial ordering makes $\mathcal{B}_n$ (actually, the quotient set $\mathcal{B}_n / \overset{E}{=}$) to be a *complete lattice*. (The separability of $\mathcal{K}$ is exploited here.)

Let us consider the localization of E at a vector $y \in \mathcal{K} : E_y(\omega) = \langle E(\omega)y, y\rangle = \|E(\omega)y\|^2$ $(\omega \in \mathcal{B}_n)$. It is immediate that $E_y(\omega) = 0$ if and only if $E(\omega)y = 0$, and $E_y(\omega) = \|y\|^2$ exactly when $E(\omega)y = y$. Furthermore, $E_y(\omega) = \|y\|^2$ if and only if $E_y(\omega^c) = 0$. The Borel set

$$\omega(\mathbf{U}, y) := \overset{E}{\bigwedge}\{\omega \in \mathcal{B}_n : E_y(\omega) = \|y\|^2\} = \Big(\overset{E}{\bigvee}\{\omega' \in \mathcal{B}_n : E_y(\omega') = 0\}\Big)^c$$

is called the *local residual set* of $\mathbf{U}$ at y; it is determined E-a.e.. Clearly,

$$E(\omega(\mathbf{U}, y))\mathcal{K} = \bigcap\{E(\omega)\mathcal{K} : y \in E(\omega)\mathcal{K}\} \quad \text{and}$$
$$E(\omega(\mathbf{U}, y)^c)\mathcal{K} = \bigvee\{E(\omega')\mathcal{K} : E(\omega')y = 0\}.$$

The next proposition contains further basic properties; the proof is left to the reader.

PROPOSITION 4.1. *Using the previous notation, we have:*

(i) *For every $y \in \mathcal{K}$, $E(\omega(\mathbf{U}, y))\mathcal{K}$ is the smallest spectral subspace containing y.*

(ii) *For any $y_1, y_2 \in \mathcal{K}$, we have $E_{y_1} \overset{a}{\sim} E_{y_2}$ if and only if $\omega(\mathbf{U}, y_1) = \omega(\mathbf{U}, y_2)$.*

(iii) $\overset{E}{\bigvee}\{\omega(\mathbf{U}, y) : y \in \mathcal{K}\} = \mathbb{T}^n$.

For every vector $h \in \mathcal{H}$, $\omega(\mathbf{T}, h) := \omega(\mathbf{U}, Xh)$ is called the *local residual set* of $\mathbf{T}$ at h. This Borel set is determined E-a.e., and it is independent of the particular choice of the unitary asymptote (see Lemma 3.5 and its proof). Since the reducing subspace $\overline{\mathcal{K}} = \bigvee\{E(\omega(\mathbf{T}, h))\mathcal{K} : h \in \mathcal{H}\}$ contains the range of X, we infer that $\overline{\mathcal{K}} = \mathcal{K}$, and so

$$\overset{E}{\bigvee}\{\omega(\mathbf{T}, h) : h \in \mathcal{H}\} = \mathbb{T}^n.$$

The Borel set $\pi(\mathbf{T}) := \overset{E}{\bigwedge}\{\omega(\mathbf{T}, h) : h \in \mathcal{H} \setminus \{0\}\}$ is called the *quasianalytic spectral set* of $\mathbf{T}$. It is also determined E-a.e. and independent of the special choice of $(X, \mathbf{U})$. In view of Proposition 4.1 we obtain the following theorem.

THEOREM 4.2. *The commuting n-tuple $\mathbf{T} \in C_{*\cdot}$ is quasianalytic if and only if $\pi(\mathbf{T}) = \mathbb{T}^n$ holds E-a.e..*

Quasianalyticity can be also characterized in terms of cyclicity. To deduce this theorem we need some auxiliary results concerning the spectral analysis of $\mathbf{U}$. The following two propositions must be known.

The measure $\mu = \mu_{\mathbf{U}} \in M_+(\mathbb{T}^n)$ is called a *scalar spectral measure* of $\mathbf{U}$, if it is equivalent to the spectral measure $E : \mu(\omega) = 0$ if and only if $E(\omega) = 0$ $(\omega \in \mathcal{B}_n)$. The next statement can be proved applying the technique of Chapter IX in [6].

PROPOSITION 4.3. *We have:*

(i) *For any $y \in \mathcal{K}$, the localization E_y is a scalar spectral measure of* $\mathbf{U}$ *if and only if $\omega(\mathbf{U}, y) = \mathbb{T}^n$.*

(ii) *There exists a vector $y \in \mathcal{K}$ such that $\omega(\mathbf{U}, y) = \mathbb{T}^n$.*

Let μ be a scalar spectral measure of $\mathbf{U}$. The mapping $\Phi_{\mathbf{U}} : L^\infty(\mu) \to \mathcal{L}(\mathcal{K})$, $f \mapsto f(\mathbf{U}) := \int_{\mathbb{T}^n} f \, \mathrm{d}E$ is a well-defined, faithful representation, called the *functional calculus* for $\mathbf{U}$, induced by the spectral measure E. Applying basic tools of functional analysis (see Section V.12 in [6], Section 12 in [7] and Section I.3.4 in [9]), the range of $\Phi_{\mathbf{U}}$ can be identified.

PROPOSITION 4.4. *The range of the functional calculus $\Phi_{\mathbf{U}}$ coincides with the double commutant $\{\mathbf{U}\}''$, and so* $\operatorname{Hlat} \mathbf{U} = \{E(\omega)\mathcal{K} : \omega \in \mathcal{B}_n\}$.

Now we can prove the cyclic characterization.

THEOREM 4.5. *For the commuting n-tuple $\mathbf{T} \in C_{*\cdot}$, the following conditions are equivalent:*

(i) $\mathbf{T}$ *is quasianalytic;*

(ii) *the vector Xh is cyclic for the commutant $\{\mathbf{U}\}'$, for every non-zero $h \in \mathcal{H}$;*

(iii) *for every non-zero $h \in \mathcal{H}$, $E(\omega)Xh \neq 0$ whenever $E(\omega) \neq 0$ $(\omega \in \mathcal{B}_n)$.*

Proof. Suppose that, for a non-zero $h \in \mathcal{H}$, the vector Xh is not cyclic for $\{\mathbf{U}\}'$. Then the hyperinvariant subspace $\mathcal{N} = \{DXh : D \in \{\mathbf{U}\}'\}^-$ of $\mathbf{U}$, induced by Xh, is not equal to $\mathcal{K}$. By Proposition 4.4, there exists a Borel set $\omega_h \in \mathcal{B}_n$ such that $E(\omega_h)\mathcal{K} = \mathcal{N}$. Since $\omega(\mathbf{T}, h) = \omega_h$ does not essentially coincide with $\mathbb{T}^n$ with respect to E, we infer that $\mathbf{T}$ is not quasianalytic (see Proposition 4.3 and Theorem 4.2). Thus, (i) implies (ii).

Since the spectral subspaces are hyperinvariant, it is immediate that (ii) yields (iii). Finally, (iii) means that, for every non-zero $h \in \mathcal{H}$, E_{Xh} is a scalar spectral measure. Hence all these localizations are equivalent, and so $\mathbf{T}$ is quasianalytic. ∎

A finer distinction can be made among Borel sets in the absolutely continuous (a.c.) case. Let m_n denote the normalized Lebesgue measure on $\mathbb{T}^n$. (More precisely, we consider its restriction to $\mathcal{B}_n$.) Let us assume that $\mathbf{T} \in C_{*\cdot}$ is an *a.c. commuting n-tuple*, which means that $\mu \overset{\mathrm{a}}{\prec} m_n$, where μ is a scalar spectral measure of $\mathbf{U}$. Taking the Radon–Nikodym derivative $0 \leqslant g_E = \mathrm{d}\mu / \mathrm{d}m_n \in L^1(m_n) = L^1(\mathbb{T}^n)$, the Borel set $\omega_{\mathrm{a}}(\mathbf{U}) := \{\mathbf{z} \in \mathbb{T}^n : g_E(\mathbf{z}) > 0\}$ is called the *a.c. (global) residual set* of $\mathbf{U}$. For every $y \in \mathcal{K}$, $E_y \overset{\mathrm{a}}{\prec} m_n$ also holds. Taking $0 \leqslant g_y = \mathrm{d}E_y / \mathrm{d}m_n \in L^1(\mathbb{T}^n)$, the Borel set $\omega_{\mathrm{a}}(\mathbf{U}, y) := \{\mathbf{z} \in \mathbb{T}^n : g_y(\mathbf{z}) > 0\}$ is called the *a.c. local residual set* of $\mathbf{U}$ at y. It is immediate that $E(\omega_{\mathrm{a}}(\mathbf{U}, y))\mathcal{K}$ is the smallest spectral subspace, containing y. It is clear also that $\omega_{\mathrm{a}}(\mathbf{U}, y) \subset \omega_{\mathrm{a}}(\mathbf{U})$ for all $y \in \mathcal{K}$, and $\omega_{\mathrm{a}}(\mathbf{U}, y) = \omega_{\mathrm{a}}(\mathbf{U})$ for some $y \in \mathcal{K}$ (see Proposition 4.3). We obtain that $\omega_{\mathrm{a}}(\mathbf{U})$ is the *measurable support* of E (with respect to m_n), that is $E(\omega) \neq 0$ holds exactly when $m_n(\omega \cap \omega_{\mathrm{a}}(\mathbf{U})) > 0$ $(\omega \in \mathcal{B}_n)$.

For any $\omega_1, \omega_2 \in \mathcal{B}_n$, the equivalence relation $\omega_1 \stackrel{\mathrm{e}}{=} \omega_2$ holds by definition if $m_n(\omega_1 \triangle \omega_2) = 0$. The relation $\omega_1 \stackrel{\mathrm{e}}{\subset} \omega_2$, defined by $m_n(\omega_1 \setminus \omega_2) = 0$, is a partial ordering on $\mathcal{B}_n$, which makes $\mathcal{B}_n$ (actually, the quotient set $\mathcal{B}_n / \stackrel{\mathrm{e}}{=}$) to be a complete lattice.

For any $h \in \mathcal{H}$, $\omega_{\mathrm{a}}(\mathbf{T}, h) := \omega_{\mathrm{a}}(\mathbf{U}, Xh)$ is called the *a.c. local residual set* of $\mathbf{T}$ at h. It follows by the minimality of $(X, \mathbf{U})$ that the *a.c. (global) residual set* $\omega_{\mathrm{a}}(\mathbf{T}) := \stackrel{\mathrm{e}}{\bigvee}\{\omega_{\mathrm{a}}(\mathbf{T}, h) : h \in \mathcal{H}\}$ of $\mathbf{T}$ coincides with $\omega_{\mathrm{a}}(\mathbf{U})$.

The Borel set $\pi_{\mathrm{a}}(\mathbf{T}) := \stackrel{\mathrm{e}}{\bigwedge}\{\omega_{\mathrm{a}}(\mathbf{T}, h) : 0 \neq h \in \mathcal{H}\}$ is called the *a.c. quasianalytic spectral set* of $\mathbf{T}$. The previous spectral invariants are determined up to sets of zero Lebesgue measure, and they are independent of the special choice of the unitary asymptote. We note also that these definitions are compatible with those given in [19] for single a.c. polynomially bounded operators.

For any $\omega_1, \omega_2 \in \mathcal{B}_n$, $\omega_1 \stackrel{E}{\subset} \omega_2$ holds if and only if $\omega_1 \cap \omega_{\mathrm{a}}(\mathbf{U}) \stackrel{\mathrm{e}}{\subset} \omega_2 \cap \omega_{\mathrm{a}}(\mathbf{U})$. Furthermore, $\omega_{\mathrm{a}}(\mathbf{T}, h) \stackrel{E}{=} \omega(\mathbf{T}, h)$ is true for every $h \in \mathcal{H}$. It follows that $\omega_{\mathrm{a}}(\mathbf{T}) \stackrel{E}{=} \omega(\mathbf{T}) \stackrel{E}{=} \mathbb{T}^n$ and $\pi_{\mathrm{a}}(\mathbf{T}) \stackrel{E}{=} \pi(\mathbf{T})$. Thus, the following statement can be derived from Theorem 4.2.

THEOREM 4.6. *For the a.c. commuting n-tuple* $\mathbf{T} \in C_{*\cdot}$, *the following conditions are equivalent:*

(i) $\mathbf{T}$ *is quasianalytic;*

(ii) $\pi_{\mathrm{a}}(\mathbf{T}) = \omega_{\mathrm{a}}(\mathbf{T})$*;*

(iii) $\omega_{\mathrm{a}}(\mathbf{T}, h) = \omega_{\mathrm{a}}(\mathbf{T})$*, for all non-zero* $h \in \mathcal{H}$*.*

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