J. OPERATOR THEORY
81:2(2019), 371–405
doi: 10.7900/jot.2018feb20.2213



# $C^*$-ALGEBRAS FOR PARTIAL PRODUCT SYSTEMS OVER $\mathbb{N}$

RALF MEYER and DEVARSHI MUKHERJEE

*Communicated by Marius Dădârlat*

ABSTRACT. We define partial product systems over $\mathbb{N}$. They generalise product systems over $\mathbb{N}$ and Fell bundles over $\mathbb{Z}$. We define Toeplitz $C^*$-algebras and relative Cuntz–Pimsner algebras for them and show that the section $C^*$-algebra of a Fell bundle over $\mathbb{Z}$ is a relative Cuntz–Pimsner algebra. We describe the gauge-invariant ideals in the Toeplitz $C^*$-algebra.

KEYWORDS: *Product system, Fell bundle, $C^*$-correspondence, Cuntz–Pimsner algebra, Toeplitz algebra.*

MSC (2010): 46L55.

## 1. INTRODUCTION

The Cuntz–Pimsner algebras introduced by Pimsner in [15] were generalised, among others, by Muhly and Solel [14] and by Katsura [9]. Fowler [8] generalised Pimsner's construction to product systems. A self-correspondence $\mathcal{E}$ of a $C^*$-algebra $A$ generates a product system over $\mathbb{N}$ by taking $\mathcal{E}_n := \mathcal{E}^{\otimes_A n}$ with the obvious multiplication maps $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \to \mathcal{E}_{n+m}$ for $n, m \in \mathbb{N}$. Any product system over $\mathbb{N}$ is isomorphic to one that is built from a $C^*$-correspondence like this. Another source of product systems over $\mathbb{N}$ are Fell bundles over $\mathbb{Z}$ (see [7]). They consist of Hilbert bimodules $(\mathcal{E}_n)_{n\in\mathbb{Z}}$ with involutions $\mathcal{E}_n \cong \mathcal{E}^*_{-n}$, $x \mapsto x^*$, and multiplication maps $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \to \mathcal{E}_{n+m}$, now for all $n, m \in \mathbb{Z}$. Due to the involutions, the Hilbert bimodules $\mathcal{E}_n$ for $n \in \mathbb{N}$ with the multiplication maps $\mu_{n,m}$ for $n, m \in \mathbb{N}$ suffice to recover the entire Fell bundle. This data gives a product system over $\mathbb{N}$ if and only if the maps $\mu_{n,m}$ are surjective and hence unitary for all $n, m \in \mathbb{N}$. Then the Fell bundle is called *semi-saturated*. In general, the multiplication maps are only isometries of Hilbert bimodules, possibly without adjoint.

The construction of a (relative) Cuntz–Pimsner algebra of a product system splits into two steps. The first builds a Fell bundle over $\mathbb{Z}$, the second takes the section $C^*$-algebra of that Fell bundle. This viewpoint is used in [13] to interpret relative Cuntz–Pismner algebras in bicategorical terms. So there is a close and important link between product systems over $\mathbb{N}$ and Fell bundles over $\mathbb{Z}$. This article describes a common generalisation for both, which we call *partial product systems*.

A partial product system over $\mathbb{N}$ consists of a $C^*$-algebra $A$ with $A,A$-correspondences $\mathcal{E}_n$ for all $n \in \mathbb{N}$ and isometries $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$ for all $n, m \in \mathbb{N}$, subject to several conditions. The obvious conditions are that the multiplication maps $\mu_{n,m}$ be associative, that $\mathcal{E}_0 = A$ and that $\mu_{0,n} : A \otimes_A \mathcal{E}_n \hookrightarrow \mathcal{E}_n$ and $\mu_{n,0} : \mathcal{E}_n \otimes_A A \hookrightarrow \mathcal{E}_n$ be induced by the $A$-bimodule structure on $\mathcal{E}_n$ for each $n \in \mathbb{N}$. Then we speak of a *weak partial product system*. Weak partial product systems on von Neumann algebras have already been used in the study of $E_0$-semigroups, where they are called "superproduct systems" (see [5], [12]).

For a partial product system, we impose two more conditions to get a well-behaved theory. Their role is similar to the compact alignment condition for product systems over quasi-lattice orders. The correspondences $\mathcal{E}_n$ in a weak partial product systems over $\mathbb{N}$ are much more independent than in an ordinary product system, and so the freeness of the monoid $\mathbb{N}$ no longer helps. This makes compact alignment and Nica covariance relevant already over $\mathbb{N}$. Our Toeplitz algebra is, in fact, an analogue of the Nica–Toeplitz algebra.

Our first goal is to define the Toeplitz $C^*$-algebra of a partial product system. We define partial product systems so that, on the one hand, this $C^*$-algebra has a universal property for suitable representations of the partial product system and, on the other hand, is generated concretely by an analogue of the Fock representation. The definition of a representation has some obvious data and conditions and a non-obvious condition needed to make the Toeplitz $C^*$-algebra well-behaved. We first discuss our definition of a representation. Then we discuss the Fock representation. Only then can we formulate the remaining two conditions on partial product systems. They say simply that the Fock representation exists and is a representation.

DEFINITION 1.1. Let $B$ be a $C^*$-algebra. A *weak representation* of a weak partial product system $(A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ in $B$ consists of linear maps $\omega_n : \mathcal{E}_n \to B$ for all $n \in \mathbb{N}$, such that:

(i) $\omega_n(x) \cdot \omega_m(y) = \omega_{n+m}(\mu_{n,m}(x \otimes y))$ for all $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$;

(ii) $\omega_n(x)^*\omega_n(y) = \omega_0(\langle x \,|\, y\rangle)$ for all $n \in \mathbb{N}$, $x, y \in \mathcal{E}_n$.

A weak representation is a *representation* if, in addition,

(iii) $\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{m-n}(\mathcal{E}_{m-n}) \cdot B$ for all $n, m \in \mathbb{N}$ with $m > n > 0$;

(iv) $\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{n-m}(\mathcal{E}_{n-m})^* \cdot B$ for all $n, m \in \mathbb{N}$ with $0 < m < n$.

By convention, $X \cdot Y$ for two subspaces in a $C^*$-algebra $B$ always denotes the closed linear span of the products $x \cdot y$ for $x \in X$, $y \in Y$.

Conditions (i) and (ii) for $n = m = 0$ hold if and only if $\omega_0$ is a $*$-homomorphism. If $n \in \mathbb{N}$, then (i) for $(n, 0)$ and $(0, n)$ and (ii) for $n$ say that $\omega_n$ is a (Toeplitz) representation of the $C^*$-correspondence $\mathcal{E}_n$.

The definition of a partial product system uses the *Fock representation*. This should be a representation in $\mathbb{B}(\mathcal{F})$, where $\mathcal{F}$ is the Hilbert $A$-module direct sum $\bigoplus_{n\in\mathbb{N}} \mathcal{E}_n$. For $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, define

$$S_{n,m}(x) : \mathcal{E}_m \to \mathcal{E}_{n+m}, \quad y \mapsto \mu_{n,m}(x \otimes y).$$

This map is linear and $\|S_{n,m}(x)\| \leqslant \|x\|$. The first assumption for a partial product system asks $S_{n,m}(x)$ to be adjointable for all $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$. Then the operators $S_{n,m}(x)$ for $m \in \mathbb{N}$ combine to an operator $S_n(x) \in \mathbb{B}(\mathcal{F})$ and the maps $S_n$ form a weak representation. The second assumption for a partial product system is that they even form a representation, which we call the *Fock representation*.

DEFINITION 1.2. A *partial product system* is a weak partial product system for which the Fock representation exists and is a representation.

To understand this condition better, we reformulate (iii) and (iv) in Definition 1.1 in case $S_n(x)$ is adjointable for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$. Then (iii) and (iv) are equivalent to the first two cases in the following equation:

$$\omega_n(x)^*\omega_m(y) = \begin{cases} \omega_{m-n}(S_n(x)^*y) & \text{if } m > n, \\ \omega_{n-m}(S_m(y)^*x)^* & \text{if } n > m, \\ \omega_0(\langle y \,|\, x\rangle)^* & \text{if } n = m; \end{cases} \tag{1.1}$$

here $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. The cases $n < m$ and $n > m$ in (1.1) are equivalent to each other by taking adjoints, and the case $n = m$ is condition (ii) in Definition 1.1. So a weak partial product system is a partial product system if and only if the operators $S_n(x)$ on $\mathcal{F}$ are adjointable for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$ and satisfy

$$S_n(x)^*S_m(y) = S_{m-n}(S_n(x)^*y) \tag{1.2}$$

for all $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$ with $m > n$. And a weak representation of a partial product system is a representation if and only if $\omega_n(x)^*\omega_m(y) = \omega_{m-n}(S_n(x)^*y)$ for all $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$ with $m > n$. The formulation of the extra conditions in Definition 1.1 is inspired by a similar treatment of the Cuntz–Pimsner covariance condition of a proper product system in [3] and makes sense without the adjointability of $S_n(x)$. The reformulation in (1.1) guarantees that there is a universal $C^*$-algebra for representations of our partial product system.

We describe an example of a weak partial product system where (1.2) fails. Its definition uses correspondences based on graphs. A graph with vertex and edge sets $V$ and $E$ gives a $C^*$-correspondence $C^*(E)$ over $C_0(V)$. To define a weak partial product system using graphs, we need a common vertex set $V$, graphs $\Gamma_n = (V, E_n, r_n, s_n)$ for all $n \in \mathbb{N}$, and associative, injective multiplication maps $\mu_{n,m} : E_n \times_{s,r} E_m \hookrightarrow E_{n,m}$, where $E_0 = V$ and $r_0, s_0$ are the identity map, and $\mu_{0,n}$ and $\mu_{n,0}$ are the canonical maps. We study when $\mu_{n,m}$ induces an isometry $C^*(E_n) \otimes_{C_0(V)} C^*(E_m) \hookrightarrow C^*(E_{n+m})$ and when these isometries form a partial product system. These conditions are rather restrictive. As it turns out, the category with object set $V$ and arrow set $\bigsqcup_{n\in\mathbb{N}} E_n$ must be the path category of an ordinary graph. The only variation is that the grading is not the standard one, that is, elements of $E_n$ need not be paths of length $n$.

The next theorem generalises an important feature of Pimsner's Toeplitz algebras. Our definitions above are arranged so as to make it true.

THEOREM 1.3. *Let $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ be a partial product system. Let $(S_n)_{n\in\mathbb{N}}$ be its Fock representation. The closed linear span of $S_n(\mathcal{E}_n)S_m(\mathcal{E}_m)^*$ for $m, n \in \mathbb{N}$ is a $C^*$-subalgebra $\mathcal{T}$ of $\mathbb{B}(\mathcal{F})$, and the maps $S_n$ form a representation $\overline{\omega}_n$ of $\mathcal{E}$ in $\mathcal{T}$. This representation in $\mathcal{T}$ is universal: for any representation $(\omega_n : \mathcal{E}_n \to B)_{n\in\mathbb{N}}$ of $\mathcal{E}$ there is a unique $*$-homomorphism $\varrho : \mathcal{T} \to B$ with $\omega_n = \varrho \circ \overline{\omega}_n$ for all $n \in \mathbb{N}$.*

We describe a gauge action of the circle group $\mathbb{T}$ on $\mathcal{T}$ and prove a gauge-equivariant uniqueness theorem for $\mathcal{T}$. This allows us to prove Theorem 1.3. We also describe the fixed-point subalgebra of the gauge action explicitly as an inductive limit $C^*$-algebra. For $m, n \in \mathbb{N}$, there is a unique linear map

$$\Theta_{m,n} : \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to \mathcal{T}$$

with

$$\Theta_{m,n}(|x\rangle\langle y|) = \overline{\omega}_n(x)\overline{\omega}_m(y)^*$$

for all $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. These maps are injective and their images are linearly independent subspaces of $\mathcal{T}$. Their direct sum over all $m, n \in \mathbb{N}$ is a dense $*$-subalgebra in $\mathcal{T}$ because of (1.1).

We classify gauge-invariant ideals $H \triangleleft \mathcal{T}$ in the Toeplitz $C^*$-algebra by two ideals in $A$, namely, the kernel $A \cap H$ and the covariance ideal, consisting of all $a \in A$ that are equal modulo $H$ to an element in the closed linear span of $\mathbb{K}(\mathcal{E}_i)$ for $i \geqslant 1$. Theorem 5.3 describes $H$ through its kernel and covariance ideal. The kernel $I$ is always an invariant ideal, and any invariant ideal may occur. We do not know, in general, which covariance ideals are possible. We also define what it means for a representation to be covariant on an ideal. If an ideal $J$ is the covariance ideal of some representation, then there is a universal $C^*$-algebra $\mathcal{O}(\mathcal{E}, J)$ for representations of $\mathcal{E}$ that are covariant on $J$. We characterise when the canonical representation of $\mathcal{E}$ in $\mathcal{O}(\mathcal{E}, J)$ is faithful: this happens if and only if $J \subseteq K^\perp$ for $K := \bigcap_{n=1}^{\infty} \ker(\vartheta_0^n)$, where $\vartheta_0^n : A \to \mathbb{B}(\mathcal{E}_n)$ is the left action in the correspondence $\mathcal{E}_n$. This allows us to define an analogue of Katsura's $C^*$-algebra for partial product systems. We also define an analogue of Pimsner's $C^*$-algebra as the quotient of $\mathcal{T}$ by the closure of the finite block matrices in the Fock representation. For global product systems, we show that the Katsura and Pimsner $C^*$-algebras defined here agree with those previously constructed. We show that the Katsura $C^*$-algebra of a Fell bundle over $\mathbb{Z}$, restricted to a partial product system over $\mathbb{N}$, is the section $C^*$-algebra of the Fell bundle. Using this, we characterise when a partial product system over $\mathbb{N}$ is the restriction of a Fell bundle over $\mathbb{Z}$. For the partial product systems built from graphs, we show that the Katsura algebra is the graph $C^*$-algebra.

## 2. WEAK PARTIAL PRODUCT SYSTEMS AND REPRESENTATIONS

In this section, we recall some basic notions and study general properties of weak partial product systems and their weak representations. We examine when a weak Fock representation exists, that is, when the operators $S_n(x)$ on the Fock module mentioned in the introduction are adjointable. We illustrate our theory with $C^*$-correspondences built from graphs.

Let $A$, $B$, $C$ be $C^*$-algebras. An *$A,B$-correspondence* is a right Hilbert module $\mathcal{E}$ over $B$ with a $*$-homomorphism $\vartheta : A \to \mathbb{B}(\mathcal{E})$ satisfying $\langle \vartheta(a)x \mid y\rangle_B = \langle x \mid \vartheta(a)^* y\rangle_B$ for all $a \in A$, $x, y \in \mathcal{E}$. We often write $ax$ instead of $\vartheta(a)(x)$. *We do not require the left action of $A$ on $\mathcal{E}$ to be nondegenerate, and we allow representations of $C^*$-algebras to be degenerate throughout this article.*

Let $\mathcal{E}$ and $\mathcal{F}$ be an $A,B$- and a $B,C$-correspondence. We equip the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ with the obvious $A,C$-bimodule structure $a(x \otimes y)c = ax \otimes yc$ for $a \in A$, $c \in C$, $x \in \mathcal{E}$, $y \in \mathcal{F}$, and with the $C$-valued inner product

$$\langle x_1 \otimes y_1 \mid x_2 \otimes y_2\rangle_C := \langle y_1 \mid \langle x_1 \mid x_2\rangle_A \cdot y_2\rangle_B \tag{2.1}$$

for $x_1, x_2 \in \mathcal{E}$ and $y_1, y_2 \in \mathcal{F}$. The Hausdorff completion of $\mathcal{E} \odot \mathcal{F}$ for this inner product is an $A,C$-correspondence denoted by $\mathcal{E} \otimes_B \mathcal{F}$ (see [11]). The *identity correspondence* $\mathcal{A}$ on $A$ for a $C^*$-algebra $A$ is $A$ viewed as an $A,A$-correspondence using the obvious bimodule structure and the inner product $\langle a \mid b\rangle_A := a^* b$.

LEMMA 2.1. *Let ${}_A\mathcal{E}_B$, ${}_B\mathcal{F}_C$ and ${}_C\mathcal{G}_D$ be $C^*$-correspondences between the indicated $C^*$-algebras $A, B, C, D$. There are canonical isomorphisms of correspondences*

$$\begin{aligned}
(\mathcal{E} \otimes_B \mathcal{F}) \otimes_C \mathcal{G} &\cong \mathcal{E} \otimes_B (\mathcal{F} \otimes_C \mathcal{G}), & (x \otimes y) \otimes z &\mapsto x \otimes (y \otimes z),\\
\mathcal{A} \otimes_A \mathcal{E} &\cong A \cdot \mathcal{E} \subseteq \mathcal{E}, & a \otimes x &\mapsto ax,\\
\mathcal{E} \otimes_B \mathcal{B} &\cong \mathcal{E}, & x \otimes b &\mapsto xb.
\end{aligned}$$

We usually omit parentheses in tensor products and the associator isomorphism in Lemma 2.1 to reduce the size of our diagrams. They are canonical enough that this cannot cause confusion.

DEFINITION 2.2. A *weak partial product system* over $\mathbb{N}$ consists of

(i) a $C^*$-algebra $A$,
(ii) $A,A$-correspondences $\mathcal{E}_n$ for $n \in \mathbb{N}_{\geqslant 1}$,
(iii) isometric bimodule maps $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$ for $n, m \in \mathbb{N}_{\geqslant 1}$,

such that the following diagrams commute for all $n, m, l \in \mathbb{N}_{\geqslant 1}$ ("associativity"):

$$
\begin{tikzcd}
\mathcal{E}_n \otimes_A \mathcal{E}_m \otimes_A \mathcal{E}_l \arrow[r, hook, "1_{\mathcal{E}_n} \otimes_A \mu_{m,l}"] \arrow[d, hook, "\mu_{n,m} \otimes_A 1_{\mathcal{E}_l}"'] & \mathcal{E}_n \otimes_A \mathcal{E}_{m+l} \arrow[d, hook, "\mu_{n,m+l}"] \\
\mathcal{E}_{n+m} \otimes_A \mathcal{E}_l \arrow[r, hook, "\mu_{n+m,l}"] & \mathcal{E}_{n+m+l}
\end{tikzcd}
\tag{2.2}
$$

(Being isometric means that $\langle \iota(x) \mid \iota(y) \rangle = \langle x \mid y \rangle$ for all $x, y \in \mathcal{E}_n \otimes_A \mathcal{E}_m$.) Let $\mathcal{E}_0 := A$ and let $\mu_{0,m} : \mathcal{A} \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_m$ and $\mu_{m,0} : \mathcal{E}_m \otimes_A \mathcal{A} \hookrightarrow \mathcal{E}_m$ be the canonical isometries from Lemma 2.1. Then the diagram (2.2) commute for all $n, m, l \in \mathbb{N}$.

Let $(A, \mathcal{E}_n, \mu_{n,m})_{n,m \in \mathbb{N}}$ be a weak partial product system. Let $(\omega_n)_{n \in \mathbb{N}}$ be a weak representation of it in a $C^*$-algebra $B$ as in Definition 1.1. That is, $\omega_n : \mathcal{E}_n \to B$ for $n \in \mathbb{N}$ are linear maps satisfying the conditions (i) and (ii) in Definition 1.1. Let $m, n \in \mathbb{N}$. By definition, $\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$ is the closed linear span in $\mathbb{B}(\mathcal{E}_m, \mathcal{E}_n)$ of $|x\rangle\langle y|$ for $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$, where $(|x\rangle\langle y|)(z) := x\langle y \mid z \rangle_A$ for all $z \in \mathcal{E}_m$. There is a unique map $\Theta_{m,n} : \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to B$ with

$$\Theta_{m,n}(|x\rangle\langle y|) = \omega_n(x)\omega_m(y)^* \tag{2.3}$$

for all $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$; this follows from Lemma 3.2 of [15] applied to the representation $\omega_m \oplus \omega_n : \mathcal{E}_m \oplus \mathcal{E}_n \to \mathbb{M}_2(B)$ of $\mathcal{E}_m \oplus \mathcal{E}_n$, by viewing $\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$ as an off-diagonal corner in $\mathbb{K}(\mathcal{E}_m \oplus \mathcal{E}_n)$. These maps are compatible with the multiplication maps and adjoints, that is,

$$\Theta_{m,n}(S) \cdot \Theta_{l,m}(T) = \Theta_{l,n}(S \cdot T), \quad \Theta_{n,m}(S^*) = \Theta_{m,n}(S)^* \tag{2.4}$$

for all $S \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$, $T \in \mathbb{K}(\mathcal{E}_l, \mathcal{E}_m)$; this follows from the case of rank-one operators, which in turn follows easily from the conditions in Definition 1.1:

$$\omega_n(x)\omega_m(y)^*\omega_m(z)\omega_l(w)^* = \omega_n(x)\omega_0(\langle y \mid z \rangle)\omega_l(w)^* = \omega_n(x \cdot \langle y \mid z \rangle)\omega_l(w)^*.$$

In particular,

$$\Theta_{n,0}(x) = \omega_n(x), \quad \Theta_{0,n}(x^*) = \omega_n(x)^*$$

for all $x \in \mathcal{E}_n \cong \mathbb{K}(A, \mathcal{E}_n)$. So (2.4) implies

$$\Theta_{m,n}(T)\omega_m(x) = \omega_n(T(x)) \tag{2.5}$$

for all $m, n \in \mathbb{N}$, $T \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$, $x \in \mathcal{E}_m$.

The maps $\Theta_{m,n}$ map $\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$ into the $C^*$-subalgebra of $B$ that is generated by $\omega_n(\mathcal{E}_n)$. The closed linear span of $\Theta_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n))$ for $m, n \in \mathbb{N}$ need not be an algebra. We will impose more relations to arrange for this later.

DEFINITION 2.3. An ideal $J \triangleleft A$ is *invariant* with respect to a weak partial product system $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m \in \mathbb{N}}$ if $J \cdot \mathcal{E}_n \subseteq \mathcal{E}_n \cdot J$ for all $n \in \mathbb{N}$.

LEMMA 2.4. *Let $(\omega_n)_{n \in \mathbb{N}}$ be a weak representation of a weak partial product system $\mathcal{E}$ in a $C^*$-algebra $B$. The ideal $I := \ker \omega_0$ is invariant and*

$$\ker \omega_n = \mathcal{E}_n \cdot I \subseteq \mathcal{E}_n, \quad \ker \Theta_{m,n} = \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n \cdot I) \subseteq \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n).$$

*Proof.* Let $x \in \mathcal{E}_n$. We have $\omega_n(x) = 0$ if and only if $0 = \omega_n(x)^*\omega_n(x) = \omega_0(\langle x \,|\, x\rangle)$, if and only if $\langle x \,|\, x\rangle \in I$. The latter is equivalent to $x \in \mathcal{E}_n \cdot I$. So $\ker \omega_n = \mathcal{E}_n \cdot I$. This implies that $I$ is invariant because $I \cdot \mathcal{E}_n \subseteq \ker \omega_n$. Let $T \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$. Then $T \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n \cdot I)$ if and only if $T(x) \in \mathcal{E}_n \cdot I$ for all $x \in \mathcal{E}_m$. By the first part, this is equivalent to $\omega_n(T(x)) = \Theta_{m,n}(T)\omega_m(x) = 0$ for all $x \in \mathcal{E}_m$. This follows if $\Theta_{m,n}(T) = 0$. Conversely, if $\Theta_{m,n}(T)\omega_m(x) = 0$ for all $x \in \mathcal{E}_m$, then $\Theta_{m,n}(T) \cdot \Theta_{n,m}(S) = 0$ for all $S \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m)$. Taking $S = T^*$, this implies $\Theta_{m,n}(T) = 0$ because $\Theta_{n,m}(T^*) = \Theta_{m,n}(T)^*$. ∎

Next we seek an analogue of the Fock representation. We want this to exist because it is used by Pimsner [15] to define the Toeplitz $C^*$-algebra. The Fock representation should be a (weak) representation on the Hilbert $A$-module $\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{E}_n$, which we call the *Fock module* of $\mathcal{E}$. Fix $n \in \mathbb{N}$ and $x \in \mathcal{E}_n$. In the Fock representation, $x$ should act on the summand $\mathcal{E}_m$ by the operator

$$S_{n,m}(x) : \mathcal{E}_m \to \mathcal{E}_{n+m}, \quad y \mapsto x \cdot y := \mu_{n,m}(x \otimes y).$$

More precisely, $S_{n,m}(x)$ is the composite of the isometry $\mu_{n,m}$ with the creation operator $\mathcal{E}_m \to \mathcal{E}_n \otimes_A \mathcal{E}_m$, $y \mapsto x \otimes y$, which is adjointable. So the operator above is a well defined bounded linear map. It is always adjointable for $n = 0$, but not for $n > 0$. Sufficient conditions for this are the following:

(1) if the isometries $\mu_{n,m}$ are adjointable;

(2) if the correspondence $\mathcal{E}_n$ is proper, that is, $A$ acts by compact operators on $\mathcal{E}_n$: then the creation operator $\mathcal{E}_m \to \mathcal{E}_n \otimes_A \mathcal{E}_m$ is compact, and then so is $S_{n,m}(x)$ because $\mathbb{K}(\mathcal{F}) \subseteq \mathbb{K}(\mathcal{E})$ if $\mathcal{F} \subseteq \mathcal{E}$ is a Hilbert submodule in a Hilbert module;

(3) if $(\mathcal{E}_n)_{n\in\mathbb{N}}$ comes from a Fell bundle over $\mathbb{Z}$: then the left multiplication map $\mathcal{E}_{n+m} \to \mathcal{E}_m$, $y \mapsto x^* \cdot y$, for $x^* \in \mathcal{E}_{-n}$ is adjoint to $S_{n,m}(x)$.

Condition (1) contains product systems in the usual sense, where each $\mu_{n,m}$ is unitary and hence adjointable.

The adjointability of the operators $S_{n,m}(x)$ is one of the requirements for a partial product system. In other words, we require that for all $x \in \mathcal{E}_n$, $t \in \mathcal{E}_{n+m}$, there is $z \in \mathcal{E}_m$, necessarily unique, with $\langle t \,|\, x \cdot y\rangle_A = \langle z \,|\, y\rangle_A$ for all $y \in \mathcal{E}_m$. If the maps $S_{n,m}(x)$ are adjointable, then so is the *creation operator* $S_n(x) := \sum_{m\in\mathbb{N}} S_{n,m}(x)$ on the Fock module $\mathcal{F}$; we call the adjoint $S_n(x)^*$ an *annihilation operator*. It is easy to see that the maps $S_n(x)$ form a weak representation of our weak partial product system in $\mathbb{B}(\mathcal{F})$. That is, $S_n(x)S_m(y) = S_{n+m}(x \cdot y)$ for $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$, $n, m \in \mathbb{N}_{\geqslant 1}$, and $S_n(x)^*S_n(y) = S_0(\langle x \,|\, y\rangle_A)$ for $x, y \in \mathcal{E}_n$, $n, m \in \mathbb{N}_{\geqslant 1}$, $n \in \mathbb{N}_{\geqslant 1}$.

2.1. CORRESPONDENCES ASSOCIATED TO GRAPHS. A (directed) graph is given by countable discrete sets $V$ and $E$ of vertices and edges and maps $(r, s) : E \rightrightarrows V$ sending an edge to its range and source. It yields a $C_0(V), C_0(V)$-correspondence by completing $C_c(E)$ in the $C_0(V)$-valued inner product

$$\langle x \mid y\rangle(v) := \sum_{e\in s^{-1}(v)} \overline{x(e)}y(e)$$

for $x, y \in C_c(E)$ and $v \in V$ or, equivalently,

$$\langle \delta_x \mid \delta_y\rangle_A = \begin{cases} \delta_{s_n(x)} & \text{if } x = y, \\ 0 & \text{otherwise;} \end{cases}$$

the $C_0(V)$-bimodule structure is given by

$$(x \cdot a)(e) := x(e) \cdot a(s(e)), \quad (a \cdot x)(e) := a(r(e)) \cdot x(e)$$

for $e \in E$, $a \in C_0(V)$, $x \in C_c(E)$. This is a $C^*$-correspondence, and its Cuntz–Pimsner algebra (as modified by Katsura) is the graph $C^*$-algebra of our graph (see [16]).

Now consider two graphs with the same vertex set $V$, with sets of edges $E_1$ and $E_2$ and range and source maps $(r_i, s_i) : E_i \rightrightarrows V$ for $i = 1, 2$. As above, we build two $C_0(V), C_0(V)$-correspondences $\mathcal{E}_1$ and $\mathcal{E}_2$. The composite correspondence $\mathcal{E}_1 \otimes_{C_0(V)} \mathcal{E}_2$ is associated to the graph with edge set $E := E_1 \times_{s,r} E_2$ and $r, s : E \rightrightarrows V$ defined by $r(f, g) := r_1(f)$ and $s(f, g) := s_2(g)$.

Now let $r_n, s_n : E_n \rightrightarrows V$ for $n \in \mathbb{N}_{>0}$ be graphs with the same vertex set $V$. Let $\widetilde{\mu}_{n,m} : E_n \times_{s,r} E_m \to E_{n+m}$ be *injective* maps for all $n, m \in \mathbb{N}$, which we write multiplicatively as $x \cdot y := \widetilde{\mu}_{n,m}(x, y)$ for $x \in E_n$, $y \in E_m$ with $s_n(x) = r_m(y)$. Assume that $r_{n+m}(x \cdot y) = r_n(x)$ and $s_{n+m}(x \cdot y) = s_m(y)$ and that these multiplication maps are associative. Let $E_0 = V$ and $r_0 = s_0 = \mathrm{id}_V$ and let the multiplication maps $\widetilde{\mu}_{0,m} : V \times_{s,r} E_m \to E_m$ and $\widetilde{\mu}_{m,0} : E_m \times_{s,r} V \to E_m$ be the obvious maps $(r(x), x) \mapsto x$, $(x, s(x)) \mapsto x$. We are going to build a weak partial product system out of this data. The construction will also show that the assumptions above are necessary to get a weak partial product system.

Let $A := C_0(V)$ and let $\mathcal{E}_n$ be the $A, A$-correspondence associated to the graph $E_n$ as above. The characteristic functions $(\delta_x)_{x\in E_n}$ form a basis in $C_c(E_n)$, which is dense in $\mathcal{E}_n$. So $(\delta_x \otimes \delta_y)_{x\in E_n, y\in E_m}$ is a basis in $C_c(E_n) \odot C_c(E_m)$, which maps to a dense subset in $\mathcal{E}_n \otimes_A \mathcal{E}_m$. For $x \in E_n$, $y \in E_m$, define

$$\mu_{n,m}(\delta_x \otimes \delta_y) := \begin{cases} \delta_{x\cdot y} & \text{if } s_n(x) = r_m(y), \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.5. *The map $\mu_{n,m} : C_c(E_n) \odot C_c(E_m) \to C_c(E_{n+m})$ is isometric for the $C_0(V)$-valued inner product and extends uniquely to an isometric bimodule map $\mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$.*

*Proof.* Let $x, x' \in E_n$, $y, y' \in E_m$. We write $\delta_{a=b}$ for the Kronecker delta to distinguish it more clearly from our characteristic functions. We declare that $\delta_{x\cdot y} = 0$ if $s_n(x) \neq r_m(y)$, so that $\mu_{n,m}(\delta_x \otimes \delta_y) = \delta_{x\cdot y}$ holds for all $x \in E_n$, $y \in E_m$.

On the one hand, we compute

$$\begin{aligned}\langle \mu_{n,m}(\delta_x \otimes \delta_y) \mid \mu_{n,m}(\delta_{x'} \otimes \delta_{y'}) \rangle(v) &= \langle \delta_{x\cdot y} \mid \delta_{x'\cdot y'} \rangle(v) \\ &= \delta_{s(x)=r(y)} \cdot \delta_{s(x')=r(y')} \cdot \delta_{x\cdot x'=y\cdot y'} \cdot \delta_{s(x\cdot y)=v}.\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}\langle \delta_x \otimes \delta_y \mid \delta_{x'} \otimes \delta_{y'} \rangle(v) &= \langle \delta_y \mid \langle \delta_x \mid \delta_{x'} \rangle \delta_{y'} \rangle(v) = \delta_{x=x'} \langle \delta_y \mid \delta_{s(x)} \delta_{y'} \rangle(v) \\ &= \delta_{x=x'} \delta_{s(x)=r(y')} \langle \delta_y \mid \delta_{y'} \rangle(v) = \delta_{x=x'} \delta_{s(x)=r(y')} \delta_{y=y'} \delta_{s(y)=v}.\end{aligned}$$

Since $\widetilde{\mu}_{n,m}$ is injective, $x \cdot x' = y \cdot y'$ if and only if $x = x'$ and $y = y'$. We assumed $s(x \cdot y) = s(y)$ as well. So both expressions above are equal. This shows that $\mu_{n,m}$ preserves the inner products between the elements $\delta_x \otimes \delta_y$ and $\delta_{x'} \otimes \delta_{y'}$ for $(x, y), (x', y') \in E_n \times_{s,r} E_m$. Hence it induces an isometry $\mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$.

The condition $r_{n+m}(x \cdot y) = r_n(x)$ for all $(x, y) \in E_n \times_{s,r} E_m$ implies that $\mu_{n,m}$ is a left module homomorphism. The condition $s_{n+m}(x \cdot y) = s_m(y)$ implies that it is a right module homomorphism. ∎

The associativity condition in (2.2) is clearly equivalent to

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

for all $x \in E_n$, $y \in E_m$, $z \in E_l$, $n, m, l \in \mathbb{N}_{>0}$. Thus we obtain a weak partial product system when we add this to our requirements. Our choices for $E_0$, $\widetilde{\mu}_{0,m}$ and $\widetilde{\mu}_{m,0}$ say that $\mathcal{E}_0 = \mathcal{A}$ and that $\mu_{0,m}$ and $\mu_{m,0}$ are the canonical maps $\mathcal{A} \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_m$ and $\mathcal{E}_m \otimes_A \mathcal{A} \xrightarrow{\sim} \mathcal{E}_m$ as in Lemma 2.1.

The disjoint union $E := \bigsqcup_{n\in\mathbb{N}} E_n$ is a category with object set $V$, using the multiplication maps $\widetilde{\mu}_{n,m}$ for $n, m \in \mathbb{N}$. The decomposition of $E$ as a disjoint union may be encoded by the functor from $E$ to the monoid $(\mathbb{N}, +)$ which maps elements of $E_n$ to $n$. We want to describe a representation $(\omega_n)_{n\in\mathbb{N}}$ of the weak partial product system above through representations of this category. The representation $\omega_n$ of $\mathcal{E}_n$ is given by the operators $T_x := \omega_n(\delta_x)$ for $x \in E_n$. Since $\omega_0$ is a representation of $C_0(V)$, the operators $T_v$ for $v \in E_0 = V$ are orthogonal projections. Conditions (i) and (ii) in Definition 1.1 say, first, that $T_x T_y$ is $T_{x\cdot y}$ if $s(x) = r(y)$, and 0 otherwise and, secondly, that $T_x^* T_y = \delta_{x,y} T_{s(y)}$ if $x, y \in E_n$ for the same $n \in \mathbb{N}$. So each $T_x$ is an isometry from $T_{s(x)}B \subseteq B$ into $T_{r(x)}B \subseteq B$, and the ranges of these isometries for $x \in E_n$ with fixed $n \in \mathbb{N}$ are orthogonal. If $V$ has only one element, then our category $E$ becomes a monoid, and the map $x \mapsto T_x$ is a representation of this monoid by isometries with the extra property that the isometries $T_x$ for $x \in E_n$ with fixed $n \in \mathbb{N}$ have orthogonal ranges.

We now examine the Fock representation. The Fock module $\mathcal{F}$ is the Hilbert $C_0(V)$-module with basis $(\delta_x)_{x\in E}$ with $E = \bigsqcup E_n$ as above, and with the inner product $\langle \delta_x \mid \delta_y \rangle = \delta_{x=y} \delta_{s(x)}$. The creation operator for $x \in E$ acts on $\mathcal{F}$ by $S_{\delta_x}(\delta_y) := \delta_{x\cdot y}$ for $x, y \in E$. Since the multiplication map on $E_n \times_{s,r} E_m$ is injective for each $n, m \in \mathbb{N}$ by assumption, the map $S_{\delta_x}$ is a partial isometry with domain spanned by $\delta_y$ with $r(y) = s(x)$ and image spanned by $\delta_{x\cdot y}$ for all such $y$.

This image is complementable, the complement being spanned by those $\delta_y$ with $y \in E \setminus (x \cdot E)$. So $S_{\delta_x}$ has the adjoint

$$S^*_{\delta_x}(\delta_y) := \begin{cases} z & \text{if } y = x \cdot z \text{ for some } z \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Since the adjointable operators form a Banach space, it follows that $S_\xi$ is adjointable for all $\xi \in \mathcal{E}_n$. Hence the Fock representation exists as a weak representation.

## 3. REPRESENTATIONS AND PARTIAL PRODUCT SYSTEMS

In this section, we restrict attention to weak partial product systems for which the weak Fock representation exists. We show that the extra conditions for a representation, (iii) and (iv) in Definition 1.1, are equivalent to (1.1). We show an example as in Section 2.1 for which these conditions fail for the Fock representation. We define *partial product systems* by requiring that the Fock representation be defined and be a representation. We study the extra conditions needed for representations for global product systems and Fell bundles. Finally, we relate our notion of representation and partial product system to Nica covariance and compact alignment for product systems over quasi-lattice orders.

PROPOSITION 3.1. *Let $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ be a weak partial product system for which the weak Fock representation exists, that is, the creation operators on its Fock module are adjointable. Let $(\omega_n)_{n\in\mathbb{N}}$ be a weak representation of $\mathcal{E}$ in a $C^*$-algebra $B$. Define $\Theta_{m,n} : \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to B$ as in* (2.3). *The following are equivalent:*

(i) *$\omega_n(x)^*\omega_m(y) = \omega_{m-n}(S_n(x)^*y)$ for all $n, m \in \mathbb{N}_{\geqslant 1}$ with $m > n$ and $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$;*

(ii) *$\omega_n(x)^*\omega_m(y) = \omega_{n-m}(S_m(y)^*x)^*$ for all $n, m \in \mathbb{N}_{\geqslant 1}$ with $n > m$ and $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$;*

(iii) *$\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{m-n}(\mathcal{E}_{m-n})$ for all $n, m \in \mathbb{N}_{>0}$ with $m > n$;*

(iv) *$\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \subseteq \omega_{n-m}(\mathcal{E}_{n-m})^*$ for all $n, m \in \mathbb{N}_{>0}$ with $m < n$;*

(v) *$\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \cdot B \subseteq \omega_{m-n}(\mathcal{E}_{m-n}) \cdot B$ for all $n, m \in \mathbb{N}$ with $m > n$;*

(vi) *$\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) \cdot B \subseteq \omega_{n-m}(\mathcal{E}_{n-m})^* \cdot B$ for all $n, m \in \mathbb{N}_{>0}$ with $m < n$;*

(vii) *if $m, n, p, q \in \mathbb{N}$, then*

$$\Theta_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)) \cdot \Theta_{p,q}(\mathbb{K}(\mathcal{E}_p, \mathcal{E}_q)) \subseteq \begin{cases} \Theta_{m-q+p,n}(\mathbb{K}(\mathcal{E}_{m-q+p}, \mathcal{E}_n)) & \text{if } m \geqslant q, \\ \Theta_{p,n+q-m}(\mathbb{K}(\mathcal{E}_p, \mathcal{E}_{n+q-m})) & \text{if } m \leqslant q. \end{cases}$$

*These equivalent conditions characterise when $(\omega_n)_{n\in\mathbb{N}}$ is a representation.*

*Proof.* The condition in (ii) is the adjoint of the condition in (i). So these two conditions are equivalent. Similarly, (iii) and (iv) are equivalent. The implications

$$\text{(i)} \Rightarrow \text{(iii)} \Rightarrow \text{(v)}, \quad \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(vi)}$$

are trivial. We are going to show that (v) implies (i). An analogous argument shows that (vi) implies (ii). This will complete the proof that conditions (i)–(vi) are equivalent. A representation is a weak representation that also satisfies the conditions (v) and (vi). Hence each of our equivalent conditions characterises representations among weak representations.

Let $z \in \mathcal{E}_{m-n}$. Then

$$\begin{aligned}\omega_{m-n}(z)^*\omega_{m-n}(S_n(x)^*(y)) &= \omega_0(\langle z \mid S_n(x)^*(y)\rangle) = \omega_0(\langle S_n(x)(z) \mid y\rangle)\\ &= \omega_0(\langle x \cdot z \mid y\rangle) = \omega_m(x\cdot z)^*\omega_m(y)\\ &= \omega_{m-n}(z)^*\omega_n(x)^*\omega_m(y).\end{aligned}$$

Letting

$$X := \omega_{m-n}(S_n(x)^*(y)) - \omega_n(x)^*\omega_m(y),$$

this becomes $\omega_{m-n}(z)^* \cdot X = 0$ for all $z \in \mathcal{E}_{m-n}$ or $X^* \cdot \omega_{m-n}(\mathcal{E}_{m-n}) \cdot B = 0$. Condition (v) implies $X \cdot B \subseteq \omega_{m-n}(\mathcal{E}_{m-n})B$. Thus $X^* \cdot X \cdot B = 0$. Hence $X = 0$.

Condition (vii) contains (iii) and (iv) as special cases because

$$\omega_n(\mathcal{E}_n) = \Theta_{0,n}(\mathbb{K}(\mathcal{E}_0, \mathcal{E}_n)), \quad \omega_n(\mathcal{E}_n)^* = \Theta_{n,0}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_0)).$$

It remains to prove that (iii) implies (vii). By definition,

$$\Theta_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)) = \omega_n(\mathcal{E}_n)\omega_m(\mathcal{E}_m)^*.$$

Hence

$$\Theta_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)) \cdot \Theta_{p,q}(\mathbb{K}(\mathcal{E}_p, \mathcal{E}_q)) = \omega_n(\mathcal{E}_n) \cdot (\omega_m(\mathcal{E}_m)^*\omega_q(\mathcal{E}_q)) \cdot \omega_p(\mathcal{E}_p)^*.$$

Now we apply (iii) to the two factors in the middle. If $m > q$, then

$$\begin{aligned}\omega_n(\mathcal{E}_n)\omega_m(\mathcal{E}_m)^*\omega_q(\mathcal{E}_q)\omega_p(\mathcal{E}_p)^* &\subseteq \omega_n(\mathcal{E}_n)\omega_{m-q}(\mathcal{E}_{m-q})^*\omega_p(\mathcal{E}_p)^*\\ &\subseteq \omega_n(\mathcal{E}_n)\omega_{m-q+p}(\mathcal{E}_{m-q+p})^*\\ &= \Theta_{m-q+p,n}(\mathbb{K}(\mathcal{E}_{m-q+p}, \mathcal{E}_n)),\end{aligned}$$

both for $q > 0$ and $q = 0$. If $m < q$, then

$$\begin{aligned}\omega_n(\mathcal{E}_n)\omega_m(\mathcal{E}_m)^*\omega_q(\mathcal{E}_q)\omega_p(\mathcal{E}_p)^* &\subseteq \omega_n(\mathcal{E}_n)\omega_{q-m}(\mathcal{E}_{q-m})\omega_p(\mathcal{E}_p)^*\\ &\subseteq \omega_{n+q-m}(\mathcal{E}_{n+q-m})\omega_p(\mathcal{E}_p)^*\\ &= \Theta_{p,n+q-m}(\mathbb{K}(\mathcal{E}_p, \mathcal{E}_{n+q-m})),\end{aligned}$$

both for $m > 0$ and $m = 0$. The case $m = q$ also works. Thus (iii) implies (vii). ∎

Equation (2.4) and condition (vii) in Proposition 3.1 imply that

$$\sum_{m,n=0}^{\infty} \Theta_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)) \subseteq B$$

is a $*$-subalgebra for any representation. For $n = m$ and $p = q$, the condition (vii) in Proposition 3.1 says that

$$\Theta_n(\mathbb{K}(\mathcal{E}_n)) \cdot \Theta_p(\mathbb{K}(\mathcal{E}_p)) \subseteq \Theta_{\max\{n,p\}}(\mathbb{K}(\mathcal{E}_{\max\{n,p\}})); \tag{3.1}$$

here we abbreviated $\Theta_n := \Theta_{n,m}$. So $\sum_{n=0}^{N} \Theta_n(\mathbb{K}(\mathcal{E}_n))$ is a $*$-subalgebra as well for all $N \in \mathbb{N} \cup \{\infty\}$.

The conditions (v) and (vi) concern inclusions between certain right ideals (or Hilbert submodules) of $B$. Hence they are similar to nondegeneracy conditions. Thus Proposition 3.1 is similar in spirit to Proposition 2.5 of [2].

A *partial product system* is a weak partial product system for which the weak Fock representation exists and is a representation, that is, satisfies the equivalent conditions in Proposition 3.1. These assumptions will be used in the next section to define and study the Toeplitz algebra of a partial product system.

We now examine the difference between weak representations and representations for several classes of weak partial product systems. First, we examine product systems, then restrictions of Fell bundles over $\mathbb{Z}$. Finally, we study examples coming from graphs as in Section 2.1.

A *product system* is a weak partial product system $(A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ where the multiplication maps $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{n+m}$ for $n, m \in \mathbb{N}_{>0}$ are unitary.

PROPOSITION 3.2. *Any weak representation of a product system is a representation. Product systems are partial product systems.*

*Proof.* Let $(A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ be a product system and let $(\omega_n)_{n\in\mathbb{N}}$ be a weak representation of it in a $C^*$-algebra $B$. Let $n, k \in \mathbb{N}_{>0}$. Then $\omega_n(\mathcal{E}_n) \cdot \omega_k(\mathcal{E}_k) = \omega_{n+k}(\mathcal{E}_{n+k})$, that is, the closed linear span of $\omega_n(\xi) \cdot \omega_k(\eta)$ for $\xi \in \mathcal{E}_n$, $\eta \in \mathcal{E}_k$ is dense in $\omega_{n+k}(\mathcal{E}_{n+k})$. If $m > n$, write $m = n + k$. Then

$$\omega_n(\mathcal{E}_n)^* \cdot \omega_m(\mathcal{E}_m) = \omega_n(\mathcal{E}_n)^* \cdot \omega_n(\mathcal{E}_n) \cdot \omega_k(\mathcal{E}_k) \subseteq \omega_0(A) \cdot \omega_k(\mathcal{E}_k) \subseteq \omega_{m-n}(\mathcal{E}_{m-n}).$$

So any weak representation satisfies the condition (i) in Proposition 3.1. The Fock representation exists as a weak representation because the maps $\mu_{n,m}$ for $n > 0$ are adjointable. It is a representation by the statement already proved. That is, our product system is a partial product system. ∎

A *Fell bundle* over the group $\mathbb{Z}$ is given by Banach spaces $B_n$ for $n \in \mathbb{Z}$ with multiplication maps $B_n \times B_m \to B_{n+m}$ and involutions $B_n \to B_{-n}$ with certain properties (see Definition 16.1 of [7]). These properties are equivalent to the existence of injective maps $B_n \hookrightarrow B$ for some $C^*$-algebra $B$ such that the multiplication maps and involutions in $(B_n)_{n\in\mathbb{N}}$ are the restrictions of the multiplication and involution in $B$. In particular, $B_0$ is a $C^*$-algebra. Each $B_n$ becomes a Hilbert $B_0$-bimodule by $\langle\langle b \mid c\rangle\rangle := bc^*$ and $\langle b \mid c\rangle := b^*c$ for all $b, c \in B_n$. The Fell bundle multiplication maps $B_n \times B_m \to B_{n+m}$ induce isometries $\mu_{n,m} : B_n \otimes_{B_0} B_m \hookrightarrow B_{n+m}$ of $B_0, B_0$-correspondences for all $n, m \in \mathbb{N}$.

PROPOSITION 3.3. *Let $\mathcal{B} = (B_n)_{n\in\mathbb{Z}}$ be a Fell bundle. Its restriction to $\mathbb{N}$ is a partial product system; here each $B_n$ is viewed as a $B_0, B_0$-correspondence, even a Hilbert $B_0$-bimodule, as above.*

*Proof.* The creation operator $S_n(x) : B_m \to B_{n+m}$, $y \mapsto x \cdot y$, for $x \in B_n$ is adjointable with adjoint $S_n(x)^*(y) = x^* \cdot y$. So

$$S_n(x)^* S_m(y) z = x^* \cdot (y \cdot z) = (x^* \cdot y) \cdot z = S_{m-n}(S_n(x)^* y)(z)$$

for $n, m, k \in \mathbb{N}$ with $n \leqslant m$ and $x \in B_n$, $y \in B_m$, $z \in B_k$. Thus the Fock representation $(S_n)_{n\in\mathbb{N}}$ satisfies condition (i) in Proposition 3.1. So it is a representation and we have got a partial product system over $\mathbb{N}$. ∎

DEFINITION 3.4. Let $D$ be a $C^*$-algebra. A *representation* of a Fell bundle $\mathcal{B} = (B_n)_{n\in\mathbb{Z}}$ in $D$ consists of linear maps $\omega_n : B_n \to D$ for $n \in \mathbb{Z}$ such that:
(i) $\omega_0 : B_1 \to D$ is a $*$-homomorphism;
(ii) $\omega_n(b)\omega_m(c) = \omega_{n+m}(bc)$ for all $n, m \in \mathbb{Z}$, $b \in B_n$, $c \in B_m$;
(iii) $\omega_{-n}(b^*) = \omega_n(b)^*$ for all $n \in \mathbb{Z}$, $b \in B_n$.

It is immediate from the definitions that the restriction of a representation of a Fell bundle over $\mathbb{Z}$ to $\mathbb{N}$ is a representation of the partial product system over $\mathbb{N}$. The Fock representation, however, is never the restriction of a Fell bundle representation. So a Fell bundle over $\mathbb{Z}$ has strictly fewer representations than its partial product system restriction to $\mathbb{N}$.

When is a (weak) partial product system over $\mathbb{N}$ the restriction of a Fell bundle over $\mathbb{Z}$? We shall answer this question in Theorem 5.14. In this section, we only study some easy aspects of this question. A necessary condition is that each $\mathcal{E}_n$ be a Hilbert $A$-bimodule. The following example show that this is not yet sufficient, even if we also assume $\mathcal{E}$ to be a partial product system.

EXAMPLE 3.5. Let $A = \mathbb{C} \oplus \mathbb{C}$ and let $\varphi$ be the partial isomorphism on $A$ that maps the first summand identically onto the second summand. This partial isomorphism corresponds to the Hilbert $A$-bimodule $\mathcal{E}_\varphi = \mathbb{C}$ with the $A$-bimodule structure $(a_1, a_2) \cdot x \cdot (b_1, b_2) := a_2 \cdot x \cdot b_1$ and the left and right inner products $\langle\langle x \mid y \rangle\rangle := (0, x\overline{y})$, $\langle x \mid y \rangle := (\overline{x}y, 0)$ for $a_1, a_2, x, y, b_1, b_2 \in \mathbb{C}$. Since the range and source ideals of $\mathcal{E}_\varphi$ are orthogonal, $\mathcal{E}_\varphi \otimes_A \mathcal{E}_\varphi \cong 0$. Let $\mathcal{E}_n := \mathcal{E}_\varphi$ for all $n \in \mathbb{N}_{\geqslant 1}$ and let $\mu_{n,m} : \mathcal{E}_n \otimes_A \mathcal{E}_m \cong 0 \hookrightarrow \mathcal{E}_{n+m}$ be the zero map for all $n, m \in \mathbb{N}_{\geqslant 1}$. We claim that this is a partial product system over $\mathbb{N}$. All its fibres are Hilbert $A$-bimodules. The maps $\mu_{n,m}$ are isometric and associative. The creation operator $S_n(x)$ for $x \in \mathcal{E}_n$ vanishes on $\mathcal{E}_m$ for $m \in \mathbb{N}_{\geqslant 1}$ and maps $(b_1, b_2) \in \mathcal{E}_0 = A$ to $x \cdot b_1 \in \mathcal{E}_n$. Hence $S_n(x)$ is adjointable, and its adjoint vanishes on $\mathcal{E}_m$ for all $m \neq n$. Thus the Fock representation satisfies condition (i) in Proposition 3.1. So we have a partial product system of Hilbert bimodules. It cannot come from a Fell bundle over $\mathbb{Z}$, however. A Fell bundle also has injective multiplication maps

$$\mathcal{E}_n^* \otimes_A \mathcal{E}_m \cong \mathcal{E}_{-n} \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{m-n}.$$

We have $\mathcal{E}_n^* \otimes_A \mathcal{E}_m = \mathcal{E}_\varphi^* \otimes_A \mathcal{E}_\varphi$, which is the Hilbert bimodule that corresponds to the identity map on the first summand in $A$. Since there is no non-zero map from this to $\mathcal{E}_\varphi$, there is no multiplication map $\mathcal{E}_n^* \otimes_A \mathcal{E}_m \hookrightarrow \mathcal{E}_{m-n}$ for $m > n$.

REMARK 3.6. Any product system of Hilbert bimodules is the restriction of a Fell bundle over $\mathbb{Z}$. Indeed, a product system is determined by the correspondence $\mathcal{E}_1$, and so a product system of Hilbert bimodules is given by a single Hilbert bimodule. The Fell bundle generated by it is described in [1]. This does not yet give all Fell bundles over $\mathbb{Z}$, however: we only get those Fell bundles that are semi-saturated, that is, have surjective multiplication maps $\mu_{n,m}$ for all $n, m \geqslant 0$.

EXAMPLE 3.7. There is a weak partial product system of the type introduced in Section 2.1 for which the weak Fock representation exists but is not a representation. Let

$$E_0 = V = \{v_0, v_1, v_2, v_3\}, \quad E_1 = \{a\}, \quad E_2 = \{c, d\}, \quad E_3 = \{b\}, \quad E_4 = \{e\},$$

and $E_n = \emptyset$ for $n \geqslant 5$ with $a : v_1 \to v_3$, $b : v_0 \to v_1$, $c : v_2 \to v_3$, $d : v_0 \to v_2$, and $e = a \cdot b = c \cdot d$. This determines the range and source maps $r_n, s_n : E_n \rightrightarrows V$ and the multiplication maps $\widetilde{\mu}_{n,m} : E_n \times_{s_n, r_m} E_m \hookrightarrow E_{n+m}$ for all $n, m \in \mathbb{N}$. The category $\bigsqcup_{n\in\mathbb{N}} E_n$ is the one that describes commutative squares:

$$\begin{tikzcd} v_0 \arrow[r, "b"] \arrow[d, "d"'] \arrow[dr, "e"] & v_1 \arrow[d, "a"] \\ v_2 \arrow[r, "c"'] & v_3, \end{tikzcd} \qquad e = a \circ b = c \circ d.$$

The resulting $C^*$-algebra is $A = \mathbb{C}[V] = \mathbb{C}^4$. The Fock module over $A$ is the $\mathbb{C}$-vector space with basis

$$E := \{v_0, v_1, v_2, v_3, a, b, c, d, e\}$$

with the $A$-bimodule structure given by the range and source maps and with $\langle x \,|\, y \rangle := \delta_{x=y} s(x)$ for all $x, y \in E$. The Fock representation of the weak partial product system is given on the basis vectors by the following table:

| | $v_0$ | $v_1$ | $v_2$ | $v_3$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|---|---|---|
| $S_1(a)$ | 0 | $a$ | 0 | 0 | 0 | $e$ | 0 | 0 | 0 |
| $S_2(c)$ | 0 | 0 | $c$ | 0 | 0 | 0 | 0 | $e$ | 0 |
| $S_2(d)$ | $d$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_3(b)$ | $b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $S_4(e)$ | $e$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

.

So $S_1(\mathcal{E}_1)\mathcal{F} = \mathbb{C}[a, e]$, $S_2(\mathcal{E}_2)\mathcal{F} = \mathbb{C}[c, d, e]$, and $S_1(\mathcal{E}_1)^*(S_2(\mathcal{E}_2)\mathcal{F}) = \mathbb{C}[b]$. This is not contained in $S_1(\mathcal{E}_1)\mathcal{F}$.

PROPOSITION 3.8. *A weak partial product system of the form built in Section* 2.1 *is a partial product system if and only if* $E = \bigsqcup_{n\in\mathbb{N}} E_n$ *is the path category of some directed graph* $\Gamma$*, but where generators in* $\Gamma$ *may have degrees different from* 1.

*Proof.* We have already seen that the weak Fock representation exists, that is, the operators $S_n(\xi)$ for $\xi \in \mathcal{E}_n$ are adjointable. When is it a representation? This means that $S_n(x)^* S_m(y) z \in S_{m-n}(\mathcal{E}_{m-n})\mathcal{F}$ for $m, n \in \mathbb{N}$ with $m > n$ and all $x \in E_n$, $y \in E_m$, $z \in E_k$. By definition, $S_n(\delta_x)^* S_m(\delta_y)(\delta_z) = S_n(\delta_x)^*(\delta_{y\cdot z})$ is equal to $\langle \delta_x \mid \delta_s \rangle \delta_t = \delta_{x=s}\delta_t$ if $y \cdot z = s \cdot t$ for some $s \in E_n$ and $t \in E_{m+k-n}$, and 0 otherwise. So $S_n(\delta_x)^* S_m(\delta_y)(\delta_z) = \delta_t$ if $y \cdot z = x \cdot t$ and 0 otherwise. We need this to belong to $S_{m-n}(\mathcal{E}_{m-n})\mathcal{F}$, that is, to be either 0 or of the form $\delta_{a\cdot b}$ for some $a \in E_{m-n}$, $b \in E_k$. Thus the weak Fock representation is a representation if and only if $y \cdot z = x \cdot t$ for $x \in E_n$, $y \in E_m$, $z \in E_k$, $t \in E_{m+k-n}$ implies that $t = a \cdot b$ for some $a \in E_{m-n}$, $b \in E_k$. Since we already assumed that the multiplication map $E_n \times_{s,r} E_k \to E_{n+k}$ is injective, the equation $y \cdot z = (x \cdot a) \cdot b$ implies $z = b$ and $y = x \cdot a$. That is, the equation $y \cdot z = x \cdot t$ has only the trivial solutions with $y = x \cdot a$ and $t = a \cdot z$ in $E$.

We call an element $x \in \bigsqcup_{n=1}^{\infty} E_n$ *irreducible* if the only product decompositions $x = a \cdot b$ with $a, b \in E$ are $x = 1_{r(x)} \cdot x$ and $x = x \cdot 1_{s(x)}$. The irreducible elements in $E$ with the restriction of the range and source maps form a directed graph $\Gamma$. Since $E$ is a category, any path in the graph $\Gamma$ defines an element in $E$. This defines a functor from the path category of $\Gamma$ to $E$. This functor is the identity on objects. It is surjective on arrows because if an arrow is not irreducible, we may write it as a non-trivial product of two strictly shorter arrows, and decomposing these as long as possible will eventually write our arrow as a product of irreducible arrows.

In a path category, the equation $y \cdot z = x \cdot t$ only has the trivial solutions. Conversely, we claim that the functor from the path category of $\Gamma$ to $E$ is injective if the equation $y \cdot z = x \cdot t$ has only the trivial solutions and the multiplication maps $E_n \times_{s,r} E_m \to E_{n+m}$ are injective for all $n, m \in \mathbb{N}$. Indeed, assume that two paths $a_1 \cdots a_k$ and $b_1 \cdots b_l$ in the graph $\Gamma$ are mapped to the same arrow in $E$. Since $y \cdot z = x \cdot t$ has only trivial solutions and $a_k$ and $b_l$ are irreducible, we must have $a_k = b_l$ and $a_1 \cdots a_{k-1} = b_1 \cdots b_{l-1}$. By induction, we conclude that our two paths are equal. So the functor from the path category of $\Gamma$ to $E$ is an isomorphism of categories if $E$ gives a partial product system. The path category of $\Gamma$ differ from $E$ only through the grading, that is, the functor to $\mathbb{N}$. Whereas all generators in a usual path category have length 1, they may belong to $E_n$ for any $n \in \mathbb{N}$. ∎

Roughly speaking, Proposition 3.8 says that the "partial" analogues of graph $C^*$-algebras are not more general than graph $C^*$-algebras. The only thing that is modified is the gauge action because the edges of the graph $\Gamma$ may belong to $\mathcal{E}_n$ and thus have degree $n$ for any $n \in \mathbb{N}$.

3.1. ANALOGY WITH NICA COVARIANCE. We briefly discuss the analogy between the extra condition for a weak representation to be a representation and Nica covariance for representations of product systems over quasi-lattice orders (see Definition 5.1 of [8]). Let $(G, P)$ be a quasi-lattice ordered group; for $p, q \in P$, let $p \vee q \in P \cup \{\infty\}$ be their least upper bound or $\infty$ if $p, q$ have no upper bound in $P$. Let $(A, \mathcal{E}_p, \mu_{p,q})_{p,q\in P}$ be a product system over $P$. Let $(\omega_p)_{p\in P}$ be a Toeplitz representation of the product system in a $C^*$-algebra $B$. The representation is called *Nica covariant* if, for all $p, q \in P$,

$$\omega_p(\mathcal{E}_p)\cdot\omega_p(\mathcal{E}_p)^*\cdot\omega_q(\mathcal{E}_q)\cdot\omega_q(\mathcal{E}_q)^*\subseteq\begin{cases}\omega_{p\vee q}(\mathcal{E}_{p\vee q})\cdot\omega_{p\vee q}(\mathcal{E}_{p\vee q})^* & p\vee q\in P,\\ 0 & p\vee q=\infty.\end{cases} \tag{3.2}$$

Here we use $\Theta_p(\mathbb{K}(\mathcal{E}_p)) = \omega_p(\mathcal{E}_p)\omega_p(\mathcal{E}_p)^*$ by (2.3).

LEMMA 3.9. *A Toeplitz representation $(\omega_p)_{p\in P}$ of a product system over $(G, P)$ satisfies* (3.2) *if and only if*

$$\omega_p(\mathcal{E}_p)^*\omega_q(\mathcal{E}_q)\subseteq\begin{cases}\omega_{p^{-1}(p\vee q)}(\mathcal{E}_{p^{-1}(p\vee q)})\omega_{q^{-1}(p\vee q)}(\mathcal{E}_{q^{-1}(p\vee q)})^* & p\vee q\in P,\\ 0 & p\vee q=\infty.\end{cases} \tag{3.3}$$

*Proof.* Equation (3.3) multiplied on the left by $\omega_p(\mathcal{E}_p)$ and on the right by $\omega_q(\mathcal{E}_q)^*$ becomes (3.2) because $\omega_p(\mathcal{E}_p)\cdot\omega_{p^{-1}(p\vee q)}(\mathcal{E}_{p^{-1}(p\vee q)}) = \omega_{p\vee q}(\mathcal{E}_{p\vee q})$ as in the proof of Proposition 3.2. Conversely, multiply (3.2) on the left by $\omega_p(\mathcal{E}_p)^*$ and on the right by $\omega_q(\mathcal{E}_q)$. We may simplify

$$\begin{aligned}\omega_p(\mathcal{E}_p)^*\cdot\omega_p(\mathcal{E}_p)\cdot\omega_p(\mathcal{E}_p)^* &= \omega_0(\langle\mathcal{E}_p\,|\,\mathcal{E}_p\rangle)\cdot\omega_p(\mathcal{E}_p)^*\\ &= \omega_p(\mathcal{E}_p\cdot\langle\mathcal{E}_p\,|\,\mathcal{E}_p\rangle)^* = \omega_p(\mathcal{E}_p)^*\end{aligned}$$

and similarly for $q$. And

$$\begin{aligned}\omega_p(\mathcal{E}_p)^*\cdot\omega_{p\vee q}(\mathcal{E}_{p\vee q}) &= \omega_p(\mathcal{E}_p)^*\cdot\omega_p(\mathcal{E}_p)\cdot\omega_{p^{-1}(p\vee q)}(\mathcal{E}_{p^{-1}(p\vee q)})\\ &= \omega_0(\langle\mathcal{E}_p\,|\,\mathcal{E}_p\rangle)\cdot\omega_{p^{-1}(p\vee q)}(\mathcal{E}_{p^{-1}(p\vee q)})\\ &= \omega_{p^{-1}(p\vee q)}(\langle\mathcal{E}_p\,|\,\mathcal{E}_p\rangle\cdot\mathcal{E}_{p^{-1}(p\vee q)}) \subseteq \omega_{p^{-1}(p\vee q)}(\mathcal{E}_{p^{-1}(p\vee q)}).\end{aligned}$$

In this way, (3.2) implies (3.3). ∎

Now let $(G, P)$ be $(\mathbb{Z}, \mathbb{N})$. If $p, q \in \mathbb{N}$, then $p^{-1}(p \vee q) = \max\{p, q\} - p$ is 0 if $p \geqslant q$ and $q - p$ if $p \leqslant q$. So (3.3) becomes $\omega_p(\mathcal{E}_p)^*\cdot\omega_q(\mathcal{E}_q) \subseteq \omega_{q-p}(\mathcal{E}_{q-p})$ if $p \leqslant q$ and $\omega_p(\mathcal{E}_p)^*\cdot\omega_q(\mathcal{E}_q) \subseteq \omega_{p-q}(\mathcal{E}_{p-q})^*$ if $p \geqslant q$. Proposition 3.1 shows that each of these conditions is equivalent to the conditions in Definition 1.1. Proposition 3.2 shows that (3.3) holds automatically for weak representations of a global product system. In other words, any Toeplitz representation of a product system over $\mathbb{N}$ is Nica covariant, which is Remark 5.2 of [8].

## 4. THE TOEPLITZ ALGEBRA

In this section, we define the Toeplitz $C^*$-algebra as the universal $C^*$-algebra for representations of a partial product system. We equip the Toeplitz $C^*$-algebra with a gauge action of the circle group $\mathbb{T}$, describe the spectral subspaces, and prove a gauge-equivariant uniqueness theorem. We describe the fixed-point subalgebra of the gauge action as an inductive limit and then show that the Fock representation generates a faithful representation of the Toeplitz $C^*$-algebra.

DEFINITION 4.1. Let $\mathcal{E} = (A, (\mathcal{E}_n)_n, \mu_{n,m})$ be a partial product system. Its Toeplitz algebra is a $C^*$-algebra $\mathcal{T}$ with a representation $(\overline{\omega}_n)_{n\in\mathbb{N}}$ that is universal in the following sense: for any representation $(\omega_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$ in a $C^*$-algebra $B$, there is a unique $*$-homomorphism $\varrho : \mathcal{T} \to B$ with $\varrho \circ \overline{\omega}_n = \omega_n$ for all $n \in \mathbb{N}$.

For a product system in the usual sense, any weak representation is a representation by Proposition 3.2. Hence the universal property of the Toeplitz $C^*$-algebra is the usual one in this case. So our definition of the Toeplitz $C^*$-algebra generalises the usual definition for product systems.

PROPOSITION 4.2. *Any partial product system $(A, \mathcal{E}_n, \mu_{n,m})$ has a Toeplitz algebra. It is the universal $C^*$-algebra generated by the symbols $\overline{\omega}_n(x)$ for $n \in \mathbb{N}$, $x \in \mathcal{E}_n$, subject to the following relations:*

(i) *$\overline{\omega}_0$ is a $*$-homomorphism;*

(ii) *the maps $x \mapsto \overline{\omega}_n(x)$ are linear for all $n \in \mathbb{N}_{\geqslant 1}$;*

(iii) *$\overline{\omega}_n(x)\overline{\omega}_m(y) = \overline{\omega}_{n+m}(\mu_{n,m}(x \otimes y))$ for all $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$, $n, m \in \mathbb{N}$;*

(iv) *$\overline{\omega}_n(x)^*\overline{\omega}_n(y) = \overline{\omega}_0(\langle x \,|\, y\rangle_A)$ for all $x \in \mathcal{E}_n$, $y \in \mathcal{E}_n$ $n \in \mathbb{N}$;*

(v) *$\omega_n(x)^*\omega_m(y) = \omega_{m-n}(S_n(x)^*(y))$ for all $n, m \in \mathbb{N}$ with $m > n$ and $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$.*

*Proof.* Let $\mathfrak{T}$ be the $*$-algebra with the generators and relations as above. Any $C^*$-norm on $\mathfrak{T}$ satisfies

$$\|\overline{\omega}_n(x)\| = \|\overline{\omega}_n(x)^*\overline{\omega}_n(x)\|^{1/2} = \|\overline{\omega}_0(\langle x \,|\, x\rangle_A)\|^{1/2} \leqslant \|\langle x \,|\, x\rangle_A\|^{1/2} = \|x\|_{\mathcal{E}_n},$$

where the inequality uses that $\overline{\omega}_0$ is a $*$-homomorphism. Hence the set of all $C^*$-norms on $\mathfrak{T}$ has a maximum. We claim that the completion of $\mathfrak{T}$ in this maximal $C^*$-norm is the Toeplitz algebra of the partial product system. The relations that define $\mathfrak{T}$ are exactly those that are needed to make $\overline{\omega}_n$ a representation of our partial product system in $\mathcal{T}$. Here we use Proposition 3.1, which shows that (v) implies $\omega_n(x)^*\omega_m(y) = \omega_{n-m}(S_m(y)^*(x))^*$ for $m < n$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. Any representation $(\omega_n)_{n\in\mathbb{N}}$ of the partial product system in a $C^*$-algebra $B$ induces a unique $*$-homomorphism $\mathcal{T} \to B$ mapping $\overline{\omega}_n(x) \mapsto \omega_n(x)$ for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$. ∎

Next we define a gauge action on $\mathcal{T}$. The maps

$$\alpha_n : \mathcal{E}_n \to C(\mathbb{T}, \mathcal{T}), \quad x \mapsto (t \mapsto t^n\overline{\omega}_n(x)),$$

form a representation of the partial product system because $\overline{\omega}_n$ is one. By the universal property of $\mathcal{T}$, there is a unique $*$-homomorphism $\alpha : \mathcal{T} \to C(\mathbb{T}, \mathcal{T})$ with $\alpha(\overline{\omega}_n(x)) = \alpha_n(x)$ for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$. Define $\alpha_t : \mathcal{T} \to \mathcal{T}$ for $t \in \mathbb{T}$ by $\alpha_t := \mathrm{ev}_t \circ \alpha$. Since $\alpha$ is a $*$-homomorphism, each $\alpha_t$ is a $*$-homomorphism and $t \mapsto \alpha_t(x)$ is continuous for each $x \in \mathcal{T}$. Since $\alpha_t(\alpha_s(\overline{\omega}_n(x))) = (ts)^n\overline{\omega}_n(x) =$

$\alpha_{ts}(\overline{\omega}_n(x))$ and $\alpha_1(\overline{\omega}_n(x)) = \overline{\omega}_n(x) = \mathrm{id}_{\mathcal{T}}(\overline{\omega}_n(x))$ for $x \in \mathcal{E}_n$ and $n \in \mathbb{N}$, the uniqueness part of the universal property of $\mathcal{T}$ implies $\alpha_t\alpha_s = \alpha_{ts}$ for all $t, s \in \mathbb{T}$ and $\alpha_1 = \mathrm{id}_{\mathcal{T}}$. Thus each $\alpha_t$ is bijective and $\mathbb{T} \ni t \mapsto \alpha_t$ is a continuous action of $\mathbb{T}$ on $\mathcal{T}$ by $*$-automorphisms.

A circle action on a $C^*$-algebra gives a lot of useful extra structure (see [6]). We now use this for the gauge action on $\mathcal{T}$. We define its spectral subspaces

$$\mathcal{T}_n := \{x \in \mathcal{T} : \alpha_t(x) = t^n x \text{ for all } t \in \mathbb{T}\}$$

for $n \in \mathbb{Z}$. These spectral subspaces form a so-called $\mathbb{Z}$-grading, that is, $\mathcal{T}_n\mathcal{T}_m \subseteq \mathcal{T}_{n+m}$ and $\mathcal{T}_n^* = \mathcal{T}_{-n}$ for all $n, m \in \mathbb{Z}$, and the closed linear span of the subspaces $\mathcal{T}_n$ is dense in $\mathcal{T}$. In particular,

$$\mathcal{T}_0 := \{x \in \mathcal{T} : \alpha_t(x) = x \text{ for all } t \in \mathbb{T}\}$$

is a $C^*$-subalgebra, the *fixed-point subalgebra* of $\alpha$.

We define the *spectral projections* $E_n : \mathcal{T} \to \mathcal{T}$ for $n \in \mathbb{Z}$ by

$$E_n(x) := \int_{\mathbb{T}} t^{-n}\alpha_t(x)\,\mathrm{d}t$$

for $x \in \mathcal{T}$, where $\mathrm{d}t$ denotes the normalised Haar measure on the compact group $\mathbb{T}$. Each $E_n$ is norm contractive and is an idempotent operator with image $\mathcal{T}_n$ that vanishes on $\mathcal{T}_m$ for $m \neq n$. In particular, $E_0 : \mathcal{T} \to \mathcal{T}_0$ is a conditional expectation. It is well known to be faithful, that is, if $x \in \mathcal{T}$ satisfies $x \geqslant 0$ and $E_0(x) = 0$, then $x = 0$. This implies the following gauge-invariant uniqueness theorem.

THEOREM 4.3. *Let $B$ be a $C^*$-algebra with an action $\beta$ of $\mathbb{T}$. Let $\varrho : \mathcal{T} \to B$ be a $\mathbb{T}$-equivariant $*$-homomorphism. If $\varrho$ is injective on $\mathcal{T}_0 \subseteq \mathcal{T}$, then it is injective on $\mathcal{T}$.*

*Proof.* Let $\varrho(x) = 0$ for some $x \in \mathcal{T}$. Let $E_0^B$ denote the $0^{\text{th}}$ spectral projection on $B$. Since $\varrho$ is $\mathbb{T}$-equivariant, it intertwines the spectral projections $E_0$ on $\mathcal{T}$ and $E_0^B$ on $B$. So

$$\varrho(E_0(x^*x)) = E_0^B(\varrho(x^*x)) = E_0^B(\varrho(x)^*\varrho(x)) = 0.$$

This implies $E_0(x^*x) = 0$ because $E_0(x^*x) \in \mathcal{T}_0$, and $\varrho|_{\mathcal{T}_0}$ is injective. Then $x = 0$ because $E_0$ is faithful. ∎

Let $\overline{\Theta}_{m,n} : \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to \mathcal{T}$ be the maps defined by the representation $\overline{\omega}_n$ as in (2.3), that is, $\overline{\Theta}_{m,n}(|x\rangle\langle y|) = \overline{\omega}_n(x)\overline{\omega}_m(y)^*$ for $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. The relations in Proposition 4.2 imply that any word in the generators of $\mathfrak{T}$ may be reduced to one of the form $\overline{\omega}_n(x)\overline{\omega}_m(y)^*$ for some $n, m \in \mathbb{N}$, $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. Therefore, the closed linear span of $\overline{\Theta}_{m,n}(\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n))$ for $n, m \in \mathbb{N}$ is a dense subspace in $\mathcal{T}$. It is a $*$-subalgebra by Proposition 3.1.(vii).

LEMMA 4.4. *The spectral subspace $\mathcal{T}_k \subseteq \mathcal{T}$ for $k \in \mathbb{Z}$ is the closed linear span of $\overline{\Theta}_{n,n+k}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_{n+k}))$ for all $n \in \mathbb{N}$ with $n \geqslant -k$.*

*Proof.* If $x \in \mathcal{E}_{n+k}$, $y \in \mathcal{E}_n$, then $\alpha_t(\overline{\omega}_{n+k}(x)\overline{\omega}_n(y)^*) = t^k\overline{\omega}_{n+k}(x)\overline{\omega}_n(y)^*$. Thus $\overline{\omega}_{n+k}(x)\overline{\omega}_n(y)^* \in \mathcal{T}_k$. Now (2.3) implies that $\overline{\Theta}_{n,n+k}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_{n+k}))$ is contained in $\mathcal{T}_k$. We have already observed that the linear span of $\overline{\Theta}_{n,m}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_m))$ is dense in $\mathcal{T}$. If $x \in \mathcal{T}_k$ and $\varepsilon > 0$, then there is a finite linear combination $\sum\limits_{n,m\in\mathbb{N}} \overline{\Theta}_{n,m}(x_{n,m})$ with $x_{n,m} \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m)$ that is $\varepsilon$-close to $x$. Then

$$E_k\Big(\sum_{n,m\in\mathbb{N}} \overline{\Theta}_{n,m}(x_{n,m})\Big) = \sum_{n,n+k\in\mathbb{N}} \overline{\Theta}_{n,n+k}(x_{n,n+k})$$

is still $\varepsilon$-close to $x$. So the closed linear span of $\overline{\Theta}_{n,n+k}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_{n+k}))$ is dense in $\mathcal{T}_k$. ∎

THEOREM 4.5. *The map*

$$\bigoplus_{j=0}^{N} \overline{\Theta}_j : \bigoplus_{j=0}^{N} \mathbb{K}(\mathcal{E}_j) \to \mathcal{T}_0$$

*is injective and its image $\mathcal{T}_{0,N}$ is a $C^*$-subalgebra of $\mathcal{T}_0$. These $C^*$-subalgebras for $N \to \infty$ form an inductive system with colimit $\mathcal{T}_0$.*

*Proof.* The Fock representation $(S_n)_{n\in\mathbb{N}}$ induces a $*$-homomorphism $\varphi : \mathcal{T} \to \mathbb{B}(\mathcal{F})$ with $\varphi(\overline{\omega}_n(x)) = S_n(x)$ for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$. Define the $*$-homomorphisms $\Theta_n : \mathbb{K}(\mathcal{E}_n) \to \mathbb{B}(\mathcal{F})$ as in (2.3). Then $\varphi(\overline{\Theta}_n(x)) = \Theta_n(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{K}(\mathcal{E}_n)$. The operator $\Theta_n(x)$ is block diagonal on $\mathcal{F} := \bigoplus\limits_{j=0}^{\infty} \mathcal{E}_j$. Its first $n-1$ diagonal entries are 0, and the $n^{\text{th}}$ diagonal entry is $x$ acting on $\mathcal{E}_n$. Let $x_j \in \mathbb{K}(\mathcal{E}_j)$ for $j = 0, \dots, N$ satisfy $\sum\limits_{j=0}^{N} \Theta_j(x_j) = 0$. We prove recursively that $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, and so on, by looking at the $j^{\text{th}}$ diagonal entry for $j = 0, \dots, N$. So the composite map

$$\bigoplus_{j=0}^{N} \mathbb{K}(\mathcal{E}_j) \twoheadrightarrow \mathcal{T}_{0,N} \subseteq \mathcal{T} \xrightarrow{\varphi} \mathbb{B}(\mathcal{F})$$

is injective. Even more, the recursive proof above shows that this injective map has a bounded inverse on its image. So its image is closed in $\mathcal{T}_0$. Equation (3.1) and Lemma 4.4 imply that $\mathcal{T}_{0,N}$ is a $*$-subalgebra in $\mathcal{T}_0$. Since it is closed as well, it is a $C^*$-subalgebra. Lemma 4.4 shows that the union $\bigcup\limits_{N=0}^{\infty} \mathcal{T}_{0,N}$ is dense in $\mathcal{T}_0$. Hence $\mathcal{T}_0$ is the inductive limit $C^*$-algebra of the inductive system $(\mathcal{T}_{0,N})_{N\in\mathbb{N}}$. ∎

THEOREM 4.6. *The Fock representation $(S_n)_{n\in\mathbb{N}}$ on the Fock module $\mathcal{F}$ over $A$ induces a faithful representation of $\mathcal{T}$. So $\mathcal{T}$ is isomorphic to the $C^*$-subalgebra of $\mathbb{B}(\mathcal{F})$ generated by $S_n(\mathcal{E}_n)$ for all $n \in \mathbb{N}$.*

*Proof.* Let $\varphi : \mathcal{T} \to \mathbb{B}(\mathcal{F})$ be the representation induced by the Fock representation; this exists because $\mathcal{E}$ is a partial product system. The proof of Theorem 4.5 shows that the restriction of $\varphi$ to $\mathcal{T}_{0,N} \subseteq \mathcal{T}$ is injective for all $N \in \mathbb{N}$. Since $\mathcal{T}_0$ is the inductive limit of these $C^*$-subalgebras by Theorem 4.5, it follows that $\varphi|_{\mathcal{T}_0}$ is injective. The Fock Hilbert module carries an obvious gauge action with spectral subspaces $\mathcal{F}_n = \mathcal{E}_n$. Let $\beta : \mathbb{T} \to \mathbb{B}(\mathcal{F})$ be the induced action. The Fock representation $\varphi$ is $\mathbb{T}$-equivariant because $S_n(x)$ belongs to the $n^{\text{th}}$ spectral subspace of $\mathbb{B}(\mathcal{F})$ for all $n \in \mathbb{N}$, $x \in \mathcal{E}_n$. Hence Theorem 4.3 shows that $\varphi$ is injective. Its image is the $C^*$-subalgebra generated by the operators $S_n(x)$ for $n \in \mathbb{N}$, $x \in \mathcal{E}_n$ because the elements $\overline{\omega}_n(x)$ for $n \in \mathbb{N}$, $x \in \mathcal{E}_n$ generate $\mathcal{T}$. ∎

PROPOSITION 4.7. *The map* $\bigoplus_{m,n\in\mathbb{N}} \overline{\Theta}_{m,n} : \bigoplus_{m,n\in\mathbb{N}} \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to \mathcal{T}$ *is injective.*

*Proof.* It suffices to prove that $\bigoplus_{m,n} \Theta_{m,n} : \bigoplus_{m,n} \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n) \to \mathcal{T} \hookrightarrow \mathbb{B}(\mathcal{F})$ is injective. We describe operators on $\mathcal{F}$ by block matrices. If $x \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$, then the $j,k$-entry of $\Theta_{m,n}(x)$ vanishes unless $j-k = n-m$ and $k \geqslant m$, and the $n,m$-entry is $x : \mathcal{E}_m \to \mathcal{E}_n$. Let $\sum_{m,n\in\mathbb{N}} \Theta_{m,n}(x_{m,n}) = 0$ for some $x_{m,n} \in \mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$ with only finitely many non-zero $x_{n,m}$. Now we examine the $j,k$-entries of $\sum_{m,n\in\mathbb{N}} \Theta_{m,n}(x_{m,n})$ for increasing $j$. For $j = 0$, we see that $x_{m,0} = 0$ for all $m \in \mathbb{N}$. An induction over $j$ shows that $x_{m,j} = 0$ for all $m \in \mathbb{N}$ and all $j \in \mathbb{N}$. ∎

4.1. HEREDITARY RESTRICTIONS AND QUOTIENTS. Throughout this subsection, let $A$ be a $C^*$-algebra and let $\mathcal{E} := (A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ be a partial product system over $A$. Let $\mathcal{T}$ be its Toeplitz $C^*$-algebra. We are going to restrict the partial product system to a hereditary $C^*$-subalgebra $H \subseteq A$ and a quotient $A/I$ for an *invariant* ideal $I$ in $A$. We show that the Toeplitz $C^*$-algebra for the restriction to $A/I$ is a quotient of $\mathcal{T}$.

First let $H \subseteq A$ be a hereditary $C^*$-subalgebra. Define

$$\mathcal{E}_n|_H := H \cdot \mathcal{E}_n \cdot H \subseteq \mathcal{E}_n;$$

the set of all products $a \cdot x \cdot b$ with $a, b \in H$, $x \in \mathcal{E}_n$ is already a closed linear subspace of $\mathcal{E}_n$ by the Cohen–Hewitt factorisation theorem. The multiplication maps in the given partial product system $(\mathcal{E}_n)_{n\in\mathbb{N}}$ restrict to multiplication maps

$$\mu_{n,m}|_H : \mathcal{E}_n|_H \otimes_H \mathcal{E}_m|_H \hookrightarrow \mathcal{E}_{n+m}|_H.$$

LEMMA 4.8. *The Hilbert* $H,H$*-correspondences* $\mathcal{E}_n|_H$ *and the multiplication maps* $\mu_{n,m}|_H$ *form a partial product system* $\mathcal{E}|_H$ *over* $H$.

*Proof.* Let $\mathcal{F}$ be $\bigoplus_{n=0}^{\infty} \mathcal{E}_n$ with the Fock representation $(S_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. The Hilbert submodule $\mathcal{F} \cdot H = \bigoplus_{n=0}^{\infty} \mathcal{E}_n \cdot H$ is invariant for $S_n(x)$ and $S_n(x)^*$ for all $x \in \mathcal{E}_n$. The

Hilbert submodule $H \cdot \mathcal{F} \cdot H = \bigoplus_{n=0}^{\infty} H \cdot \mathcal{E}_n \cdot H$ is still invariant for $S_n(x)$ and $S_n(x)^*$ for all $x \in H \cdot \mathcal{E}_n \cdot H$. We claim that $S'_n(x) = S_n(x)|_{H \cdot \mathcal{F} \cdot H}$ for $n \in \mathbb{N}$, $x \in H \cdot \mathcal{E}_n \cdot H$ defines a representation $(S'_n)_{n\in\mathbb{N}}$ of $(\mathcal{E}_n|_H, \mu_{n,m}|_H)_{n,m\in\mathbb{N}}$ on $H \cdot \mathcal{F} \cdot H$. The operators $S'_n(x)$ for $n \in \mathbb{N}$, $x \in \mathcal{E}_n$ are adjointable with adjoint $S'_n(x)^* = S_n(x)^*|_{H \cdot \mathcal{F} \cdot H}$. The conditions in Definition 1.1 are inherited from the corresponding ones for the representation $(S_n)_{n\in\mathbb{N}}$ of $\mathcal{E}_n$. The representation $S'_n$ is unitarily equivalent to the Fock representation of $(\mathcal{E}_n|_H, \mu_{n,m}|_H)_{n,m\in\mathbb{N}}$. ∎

Let $I \triangleleft A$ be an invariant ideal. Then

$$\mathcal{E}_n|_{A/I} := \mathcal{E}_n / \mathcal{E}_n \cdot I$$

is a Hilbert $A/I, A/I$-module in a canonical way. The left action of $A$ descends to $A/I$ because $I \cdot \mathcal{E}_n \subseteq \mathcal{E}_n \cdot I$. The multiplication map $\mu_{n,m}$ induces a well defined multiplication map

$$\mu_{n,m}|_{A/I} : \mathcal{E}_n|_{A/I} \otimes_{A/I} \mathcal{E}_m|_{A/I} \to \mathcal{E}_{n+m}|_{A/I}.$$

PROPOSITION 4.9. *The restriction $\mathcal{E}|_{A/I} := (A/I, \mathcal{E}_n|_{A/I}, \mu_{n,m}|_{A/I})_{n,m\in\mathbb{N}}$ is a partial product system. Its Toeplitz $C^*$-algebra is a quotient of $\mathcal{T}$ by a $\mathbb{T}$-invariant ideal.*

*Proof.* The beginning of the following proof works for any ideal $I \triangleleft A$ and will later be used in this generality. Let $\mathcal{F}$ be $\bigoplus_{n=0}^{\infty} \mathcal{E}_n$ with the Fock representation $(S_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. The Hilbert submodule $\mathcal{F} \cdot I = \bigoplus_{n\in\mathbb{N}} \mathcal{E}_n \cdot I$ is invariant for all adjointable operators on $\mathcal{F}$. Hence there is a unital $*$-homomorphism

$$\pi : \mathbb{B}(\mathcal{F}) \to \mathbb{B}(\mathcal{F}/\mathcal{F}I).$$

We may identify $\mathcal{F}/\mathcal{F}I \cong \mathcal{F} \otimes_A A/I$, and then $\pi(T) = T \otimes 1$ for all $T \in \mathbb{B}(\mathcal{F})$. The Hilbert modules $\mathcal{F}$ and $\mathcal{F}/\mathcal{F}I$ carry obvious $\mathbb{Z}$-gradings, which induce actions of the circle group $\mathbb{T}$ on $\mathbb{B}(\mathcal{F})$ and $\mathbb{B}(\mathcal{F}/\mathcal{F}I)$ that are continuous in the strict topology. The homomorphism $\pi$ is grading-preserving. Composing the Fock representation $(S_n)_{n\in\mathbb{N}}$ with the $*$-homomorphism $\pi$ gives a representation $(S'_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$ on $\mathcal{F}/\mathcal{F}I$. The $C^*$-algebra generated by $S'_n(\mathcal{E}_n)$ is $\pi(\mathcal{T})$. Thus it is a quotient of $\mathcal{T}$. Since $\pi$ is $\mathbb{T}$-equivariant, its kernel is a gauge-invariant ideal in $\mathcal{T}$.

Now assume that $I$ is invariant. Then $S'_n|_{\mathcal{E}_n I} = 0$ because

$$\mathcal{E}_n \cdot I \cdot \mathcal{E}_m \subseteq \mathcal{E}_n \cdot \mathcal{E}_m \cdot I \subseteq \mathcal{E}_{n+m} \cdot I$$

for all $n, m \in \mathbb{N}$. Hence $(S'_n)_{n\in\mathbb{N}}$ descends to a representation of $\mathcal{E}|_{A/I}$. This representation is unitarily equivalent to the Fock representation. Hence $\mathcal{E}|_{A/I}$. is a partial product system. Its Toeplitz $C^*$-algebra is isomorphic to $\pi(\mathcal{T})$ by Theorem 4.6. And this is a quotient of $\mathcal{T}$ by a gauge-invariant ideal. ∎

## 5. GAUGE-INVARIANT IDEALS IN THE TOEPLITZ ALGEBRA

Let $B$ be a $C^*$-algebra with a continuous $\mathbb{T}$-action and let $B_n \subseteq B$ be the homogeneous subspaces. Call an ideal $J \triangleleft B_0$ *invariant* if $J \cdot B_n \subseteq B_n \cdot J$ for all $n \in \mathbb{Z}$; since $B_n^* = B_{-n}$, this is equivalent to $J \cdot B_n = B_n \cdot J$ only for $n \in \mathbb{N}_{>0}$. Given a $\mathbb{T}$-invariant ideal $I \triangleleft B$, its restriction $I \cap B_0$ is an invariant ideal in $B_0$.

Conversely, if $J \triangleleft B_0$ is an invariant ideal, then $J \cdot B = B \cdot J$ is a $\mathbb{T}$-invariant ideal in $B$. It is well known that these two maps are isomorphisms inverse to each other between the lattices of gauge-invariant ideals in $B$ and of invariant ideals in $B_0$.

We are going to apply this general result to the Toeplitz $C^*$-algebra of a partial product system with its canonical $\mathbb{T}$-action. The gauge-invariant ideals of the Toeplitz $C^*$-algebra of an ordinary product system are described completely by Katsura [10]. Like Katsura, we describe gauge-invariant ideals in a Toeplitz $C^*$-algebra by a pair of ideals $I \triangleleft J \triangleleft A$. Here the ideal $I$ is invariant, and any invariant ideal occurs. We do not know, in general, which ideals $J$ are possible. So our result is not as complete as Katsura's result for product systems.

Throughout this section, we fix a $C^*$-algebra $A$ and a partial product system $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$. Let $\mathcal{T}$ be its Toeplitz $C^*$-algebra and let $\mathcal{T}_n \subseteq \mathcal{T}$ for $n \in \mathbb{Z}$ be its homogeneous subspaces for the canonical $\mathbb{T}$-action. Let $\mathcal{F}$ be $\bigoplus_{n=0}^{\infty} \mathcal{E}_n$ with the Fock representation $(S_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$. To simplify notation, we view $\mathbb{K}(\mathcal{E}_m, \mathcal{E}_n)$ for $m, n \in \mathbb{N}$ as subspaces of $\mathcal{T}$, that is, we drop the name $\Theta_{m,n}$ for their canonical embeddings. In particular, we view $\mathcal{E}_n \cong \mathbb{K}(A, \mathcal{E}_n)$ and $A = \mathcal{E}_0$ as subspaces of $\mathcal{T}$.

LEMMA 5.1. *If $H \triangleleft \mathcal{T}$ is an ideal, then $I := H \cap A \triangleleft A$ is an invariant ideal, that is, $I \cdot \mathcal{E}_n \subseteq \mathcal{E}_n \cdot I$ for all $n \in \mathbb{N}$. Conversely, if $I$ is an invariant ideal in $A$, then the kernel $H$ of the canonical homomorphism $\mathcal{T} \to \mathcal{T}(\mathcal{E}|_{A/I})$ is a gauge-invariant ideal with $I = H \cap A$. It is the minimal ideal $H \triangleleft \mathcal{T}$ with $I \subseteq H \cap A$.*

*Proof.* Let $H \triangleleft \mathcal{T}$. There is a canonical representation $(\omega_n)_{n\in\mathbb{N}}$ of $\mathcal{E}$ in $\mathcal{T}/H$, and $I$ is its kernel. Hence $I$ is invariant by Lemma 2.4. Conversely, let $I \triangleleft A$ be an invariant ideal. Then we may restrict our partial product system to the quotient $A/I$ as in Section 4.1. This restriction is a partial product system $\mathcal{E}|_{A/I}$, whose Toeplitz $C^*$-algebra $\mathcal{T}(\mathcal{E}|_{A/I})$ is a quotient of $\mathcal{T}$ by some gauge-invariant ideal $H$ by Proposition 4.9. The map from $A/I$ to the Toeplitz $C^*$-algebra of $\mathcal{E}|_{A/I}$ is injective by Theorem 4.5. So the intersection $H \cap A$ is $I$.

Let $L \triangleleft \mathcal{T}$ be any ideal with $I \subseteq L \cap A$. The canonical representation of $\mathcal{E}$ in $\mathcal{T}$ induces a representation in $\mathcal{T}/L$. This representation kills $I$ and hence also $\mathcal{E}_n \cdot I$ for all $n \in \mathbb{N}$. Therefore, it descends to a representation of $\mathcal{E}|_{A/I}$. Hence the quotient map $\mathcal{T} \twoheadrightarrow \mathcal{T}/L$ factors through the Toeplitz $C^*$-algebra of $\mathcal{E}|_{A/I}$. Thus $H$ above is minimal among the ideals $L \triangleleft \mathcal{T}$ with $I \subseteq L \cap A$. ∎

DEFINITION 5.2. Let $(\omega_n)_{n\in\mathbb{N}}$ be a representation of the partial product system $\mathcal{E}$ in a $C^*$-algebra $B$. Let $\mathcal{E}_{\geqslant n}B \subseteq B$ denote the closed linear span of $\omega_m(\mathcal{E}_m)\cdot B$ for all $m \geqslant n$. The *covariance ideal* of the representation is

$$J(\omega_n) := \{a \in A : \omega_0(a)\cdot B \subseteq \mathcal{E}_{\geqslant 1}B\}.$$

We call $(\omega_n)_{n\in\mathbb{N}}$ *covariant* on an ideal $J \triangleleft A$ if $J \subseteq J(\omega_n)$.

We are going to characterise a gauge-invariant ideal $H$ by the pair $(I, J)$ of ideals in $A$, where $I := H \cap A$ and $J$ is the covariance ideal of the canonical representation of $\mathcal{E}$ in $\mathcal{T}/H$.

For any ideal $J \triangleleft A$, there is a $C^*$-algebra extension

$$\mathbb{B}(\mathcal{F}, \mathcal{F}J) \rightarrowtail \mathbb{B}(\mathcal{F}) \twoheadrightarrow \mathbb{B}(\mathcal{F}/\mathcal{F}J),$$

which is also $\mathbb{T}$-equivariant (see the proof of Proposition 4.9). Hence

$$V(J) := \mathcal{T} \cap \mathbb{B}(\mathcal{F}, \mathcal{F}J)$$

is a gauge-invariant ideal in $\mathcal{T}$ for each ideal $J \triangleleft A$. It consists of those operators in $\mathcal{T}$ that act by zero on $\mathcal{F}/\mathcal{F}J$. Let $L^0(J) \subseteq \mathbb{B}(\mathcal{F}/\mathcal{F}J)$ be the $\mathbb{T}$-invariant $*$-subalgebra of finite block matrices:

$$L^0(J) := \{x \in \mathbb{B}(\mathcal{F}/\mathcal{F}J) : \text{there is } N \in \mathbb{N} \text{ with } x_{n,m} = 0 \text{ for } n > N \text{ or } m > N\}.$$

Here $x_{n,m} \in \mathbb{B}(\mathcal{E}_m/\mathcal{E}_mJ, \mathcal{E}_n/\mathcal{E}_nJ)$ denotes the $n,m$-matrix entry of $x$. Let $L(J) \subseteq \mathcal{T}$ be the preimage of $\overline{L^0(J)} \subseteq \mathbb{B}(\mathcal{F}/\mathcal{F}J)$. This is an ideal because $\omega_n(x)$ is a multiplier of $L^0(J)$ for each $x \in \mathcal{E}_n$, $n \in \mathbb{N}$. The ideal $L(J)$ is also $\mathbb{T}$-invariant.

THEOREM 5.3. *Let $H$ be a gauge-invariant ideal in the Toeplitz $C^*$-algebra of $\mathcal{E}$. Let $I := H \cap A$ and let $J$ be its covariance ideal. Then $H = V(J) \cap L(I)$. In particular, the ideals $I$ and $J$ determine $H$ uniquely.*

The proof will use a couple of lemmas. First we examine the covariance ideal more closely. The right ideal $\mathcal{E}_{\geqslant 1}\mathcal{T}$ is easily seen to be the closure of $\sum_{n\geqslant 0, m\geqslant 1} \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m)$. Its intersection with $\mathcal{T}_0$ is the closure of $\sum_{m\geqslant 1} \mathbb{K}(\mathcal{E}_m)$. So the covariance ideal $J$ for the canonical representation in $\mathcal{T}/H$ is the set of all $a \in A$ that are identified with an element in the closure of $\sum_{m\geqslant 1} \mathbb{K}(\mathcal{E}_m)$ in the quotient $\mathcal{T}/H$. For instance, a Cuntz–Pimsner-like covariance condition for $\mathcal{E}_m$ would identify $a \sim \vartheta_0^m(a)$ for certain elements $a \in A$ with $\vartheta_0^m(a) \in \mathbb{K}(\mathcal{E}_m)$. Here $\vartheta_j^k : \mathbb{K}(\mathcal{E}_j) \to \mathbb{B}(\mathcal{E}_k)$ for $0 \leqslant j \leqslant k$ is the map defined by the Fock representation, that is, $\vartheta_j^k(|x\rangle\langle y|)(z) = S_j(x)S_j(y)^*z$ for all $x, y \in \mathcal{E}_j$, $z \in \mathcal{E}_k$. The covariance ideal allows more complicated relations that identify some elements of $A$ with elements of the closure of $\sum_{m\geqslant 1} \mathbb{K}(\mathcal{E}_m)$.

If $n, k \in \mathbb{N}$, then

$$\omega_n(\mathcal{E}_n)\mathcal{E}_{\geqslant k}B \subseteq \mathcal{E}_{\geqslant n+k}B, \quad \omega_n(\mathcal{E}_n)^*\mathcal{E}_{\geqslant n+k}B \subseteq \mathcal{E}_{\geqslant k}B; \tag{5.1}$$

the second property uses condition (v) in Proposition 3.1. Hence the induced representations $\Theta_{n,m} : \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m) \to B$ satisfy

$$\Theta_{n,m}(\mathbb{K}(\mathcal{E}_n, \mathcal{E}_m))\mathcal{E}_{\geqslant k}B \subseteq \mathcal{E}_{\geqslant \max\{m,k-n+m\}}B. \tag{5.2}$$

LEMMA 5.4. *Let $(\omega_n)_{n\in\mathbb{N}}$ be a representation of the partial product system $\mathcal{E}$ in a $C^*$-algebra $B$ with covariance ideal $J$. Then*

$$\mathbb{K}(\mathcal{E}_n, \mathcal{E}_m J) = \{T \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m) : \Theta_{n,m}(T) \cdot \mathcal{E}_{\geqslant n}B \subseteq \mathcal{E}_{\geqslant m+1}B\}.$$

*Proof.* Let $T \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m)$. We first assume $T \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m J)$. Then $T$ is in the closed linear span of operators of the form $|x\rangle a\langle y|$ with $x \in \mathcal{E}_m$, $a \in J$, $y \in \mathcal{E}_n$. Equation (5.1) implies

$$\Theta_{n,m}(|x\rangle a\langle y|)\mathcal{E}_{\geqslant n}B \subseteq \omega_m(x)\omega_0(a)B \subseteq \omega_m(x)\mathcal{E}_{\geqslant 1}B \subseteq \mathcal{E}_{\geqslant m+1}B.$$

Hence $\Theta_{n,m}(T)(\mathcal{E}_{\geqslant n}B) \subseteq \mathcal{E}_{\geqslant m+1}B$.

Conversely, assume $\Theta_{n,m}(T)(\mathcal{E}_{\geqslant n}B) \subseteq \mathcal{E}_{\geqslant m+1}B$. Let $x \in \mathcal{E}_n$, $y \in \mathcal{E}_m$. Then

$$\begin{aligned}\omega_0(\langle y \mid T(x)\rangle)B &= \omega_m(y)^*\omega_m(T(x))B = \omega_m(y)^*\Theta_{n,m}(T)\omega_n(x)B\\ &\subseteq \omega_m(y)^*\Theta_{n,m}(T)\mathcal{E}_{\geqslant n}B \subseteq \omega_m(y)^*\mathcal{E}_{\geqslant m+1}B \subseteq \mathcal{E}_{\geqslant 1}B.\end{aligned}$$

Thus $\langle y \mid T(x)\rangle \in J$. Since $y$ is arbitrary, this implies $T(x) \in \mathcal{E}_m \cdot J$. Since $x$ is arbitrary, this implies $T \cdot \mathbb{K}(\mathcal{E}_n) \subseteq \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m \cdot J)$. Multiplying $T$ with an approximate unit in $\mathbb{K}(\mathcal{E}_n)$, we get $T \in \mathbb{K}(\mathcal{E}_n, \mathcal{E}_m \cdot J)$. ∎

LEMMA 5.5. *Let $(\omega_n)_{n\in\mathbb{N}}$ be a representation of the partial product system $\mathcal{E}$ in a $C^*$-algebra $B$. Let $J$ be its covariance ideal and let $I := \ker\omega_0$. Let $\omega : \mathcal{T} \to B$ be the associated representation of the Toeplitz $C^*$-algebra $\mathcal{T}$. Let $x_i \in \mathbb{K}(\mathcal{E}_i)$ for $i = 0, \dots, N$ and $X := \sum_{i=1}^{N} x_i \in \mathcal{T}$. The following are equivalent:*

(i) $\omega(X) = 0$ *in $B$;*

(ii) $\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i) \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell \cdot J)$ *if $\ell < N$ and* $\sum_{i=0}^{N} \vartheta_i^\ell(x_i) \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell \cdot I)$ *if $\ell \geqslant N$.*

*Proof.* We prove (i) ⇒ (ii) and assume $\omega(X) = 0$. Let $\ell \in \mathbb{N}$ and $y \in \mathcal{E}_\ell$. Assume first that $\ell \geqslant N$. Then

$$0 = \omega(X)\omega_\ell(y) = \omega_\ell\Big(\sum_{i=0}^{N} \vartheta_i^\ell(x_i)y\Big)$$

by (2.5). This implies $\sum_{i=0}^{N} \vartheta_i^\ell(x_i)y \in \mathcal{E}_\ell \cdot I$ by Lemma 2.4 as asserted in (ii). Now let $\ell < N$. The same computation as above shows that

$$\omega_\ell\Big(\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i)y\Big) + \sum_{i=\ell+1}^{N} \omega(x_i)\omega_\ell(y) = 0.$$

Since $\omega(x_i)B \subseteq \mathcal{E}_{\geqslant \ell+1}B$ for $i \geqslant \ell+1$, this implies

$$\omega_\ell\Big(\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i)y\Big)B \subseteq \mathcal{E}_{\geqslant \ell+1}B.$$

Hence $\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i)y \in \mathbb{K}(\mathcal{E}_0, \mathcal{E}_\ell \cdot J) = \mathcal{E}_\ell \cdot J$ by Lemma 5.4. Since $y \in \mathcal{E}_\ell$ is arbitrary, this implies $\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i) \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell \cdot J)$. This finishes the proof that (i) $\Rightarrow$ (ii).

Now we prove, conversely, that (ii) implies (i). If $X$ satisfies the conditions in (ii), then so does $X^*$. Hence we may replace $X$ by the two self-adjoint elements $X + X^*$ and $\mathrm{i}^{-1}(X - X^*)$. So we may assume without loss of generality that $X$ is self-adjoint. The assumption in (ii) for $\ell \geqslant N$ and Lemma 2.4 imply $\omega(X)\omega_\ell(y) = 0$ for all $y \in \mathcal{E}_\ell$, $\ell \geqslant N$. Thus $\omega(X)$ vanishes on $\mathcal{E}_{\geqslant N}B = 0$. Now let $0 \leqslant \ell < N$ and $y \in \mathcal{E}_\ell$. Then $\sum_{i=0}^{\ell} \vartheta_i^\ell(x_i)y \in \mathcal{E}_\ell J$ by assumption. Hence

$$\begin{aligned}\omega(X)\omega_\ell(y)B &= \sum_{i=0}^{\ell} \omega_\ell(\vartheta_i^\ell(x_i)y)B + \sum_{i=\ell+1}^{N} \omega(x_i)\omega_\ell(y)B \subseteq \omega_\ell(\mathcal{E}_\ell J)B + \mathcal{E}_{\geqslant \ell+1}B \\ &\subseteq \omega_\ell(\mathcal{E}_\ell)\mathcal{E}_{\geqslant 1}B + \mathcal{E}_{\geqslant \ell+1}B \subseteq \mathcal{E}_{\geqslant \ell+1}B.\end{aligned}$$

This implies $\omega(X)\mathcal{E}_{\geqslant \ell}B \subseteq \mathcal{E}_{\geqslant \ell+1}B$ because $\omega(X)\mathcal{E}_{\geqslant \ell+1}B \subseteq \mathcal{E}_{\geqslant \ell+1}B$ for any $X \in \mathcal{T}_0$ by (5.2). Thus $\omega(X)^N B \subseteq \mathcal{E}_{\geqslant N}B$. Hence $\omega(X)^{N+1} = 0$. Then $\omega(X) = 0$ because $X$ is self-adjoint. This finishes the proof that (ii) implies (i). ∎

*Proof of Theorem* 5.3. Both $H$ and $V(J) \cap L(I)$ are gauge-invariant ideals in $\mathcal{T}$. Hence they are equal if and only if their intersections with $\mathcal{T}_0$ are equal. And by Theorem 4.5, these intersections are equal if and only if the intersections with $\mathcal{T}_{0,N}$ are equal for all $N \in \mathbb{N}$. Since these intersections are ideals, it suffices to prove that they have the same positive elements. So let $y \in \mathcal{T}_{0,N}$ be a positive element. Describe $y$ by a block diagonal matrix on $\mathcal{F}$ with entries $y_\ell \in \mathbb{B}(\mathcal{E}_\ell)$ for $\ell \in \mathbb{N}$. Here $y_\ell = \sum_{i=0}^{\ell} \vartheta_i^\ell(x_i)$ in the notation of Lemma 5.5. So Lemma 5.5 says that $y \in H$ if and only if $y_\ell \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell J)$ for $0 \leqslant \ell < N$ and $y_\ell \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell I)$ for $\ell \geqslant N$. Since $I \subseteq J$, this implies $y_\ell \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell J)$ for all $\ell \in \mathbb{N}$. And this is equivalent to $y \in \mathcal{T} \cap \mathbb{B}(\mathcal{F}, \mathcal{F}J) = V(J)$. Let $\overline{y}_\ell$ be the operator on $\mathcal{E}_\ell/\mathcal{E}_\ell I$ induced by $y_\ell$. We have $y_\ell \in \mathbb{B}(\mathcal{E}_\ell, \mathcal{E}_\ell I)$ if and only if $\overline{y}_\ell = 0$. The condition $y \in L(J)$ is equivalent to $\lim_{\ell\to\infty} \|\overline{y}_\ell\| = 0$. This clearly follows if $\overline{y}_\ell = 0$ for $\ell \geqslant N$. We claim the converse implication. Since $y$ is positive, we may rewrite $\lim_{\ell\to\infty} \|\overline{y}_\ell\| = 0$ as follows: for each $\varepsilon > 0$, there is $M \in \mathbb{N}$ such that $(\overline{y}_\ell - \varepsilon)_+ = 0$ for all $\ell \geqslant M$; here $(\overline{y}_\ell - \varepsilon)_+$ means the positive part of $\overline{y}_\ell - \varepsilon$. We may choose $M \geqslant N$. Then $y \in \mathcal{T}_{0,M}$, and Lemma 5.5 and the conditions $(\overline{y}_\ell - \varepsilon)_+ = 0$ for $\ell \geqslant M$ and $y \in V(J)$ imply $(y - \varepsilon)_+ \in H$. Since $(y - \varepsilon)_+ \in \mathcal{T}_{0,N}$, Lemma 5.5 implies $(\overline{y}_\ell - \varepsilon)_+ = 0$ already for $\ell \geqslant N$. Since this holds for all $\varepsilon > 0$, we get $\overline{y}_\ell = 0$ for all $\ell \geqslant N$. ∎

By Theorem 5.3, the lattice of gauge-invariant ideals in $\mathcal{T}$ is isomorphic to the lattice of pairs of ideals $(I, J)$ in $A$ that occur as the kernel and covariance ideal for a gauge-invariant ideal in $\mathcal{T}$. If $(I, J)$ comes from a gauge-invariant ideal $H$, then $H = V(J) \cap L(I)$ by Theorem 5.3. So the question is when $A \cap V(J) \cap L(I) = I$ holds and the covariance ideal of $V(J) \cap L(I) \triangleleft \mathcal{T}$ is $J$.

We already know that $I$ must be invariant and that any invariant ideal may occur. And $I \subseteq J$ is trivial. Given an invariant ideal $I$, we may form the quotient partial product system $\mathcal{E}_n / \mathcal{E}_n I$. Its Toeplitz $C^*$-algebra is isomorphic to a quotient of $\mathcal{T}$ by a gauge-invariant ideal by Proposition 4.9. Its kernel and covariance ideal are $I, I$ by Theorem 4.5. So

$$\mathcal{T}(\mathcal{E}|_{A/I}) \cong \mathcal{T}(\mathcal{E})/(V(I) \cap L(I)) = \mathcal{T}(\mathcal{E})/V(I)$$

by Theorem 5.3. We may replace the original partial product system by $\mathcal{E}|_{A/I}$. This reduces our problem to the case $I = 0$.

DEFINITION 5.6. Let $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m \in \mathbb{N}}$ be a partial product system and let $J \triangleleft A$ be an ideal. Let

$$\mathcal{O}(\mathcal{E}, J) := \mathcal{T}/(V(J) \cap L(0)).$$

This quotient inherits a representation of $\mathcal{E}$ and a $\mathbb{T}$-action from $\mathcal{T}$ because $V(J) \cap L(0)$ is a $\mathbb{T}$-invariant ideal in $\mathcal{T}$. We call $\mathcal{O}(\mathcal{E}, J)$ the *$J$-covariance algebra* if the canonical representation of $\mathcal{E}$ in $\mathcal{O}(\mathcal{E}, J)$ has covariance ideal $J$. Let $K := \bigcap_{n=1}^{\infty} \ker \vartheta_0^n$. We call

$$\mathcal{O}_{\mathrm{Pimsner}}(\mathcal{E}) := \mathcal{O}(\mathcal{E}, A), \quad \mathcal{O}_{\mathrm{Katsura}}(\mathcal{E}) := \mathcal{O}(\mathcal{E}, K^{\perp})$$

the *Pimsner algebra* and the *Katsura algebra* of $\mathcal{E}$, respectively.

The name "covariance algebra for $J$" is justified by the universal property in Proposition 5.7 below. The Pimsner algebra is the quotient $\mathcal{T}/L(0)$ because $V(A) = \mathcal{T}$. Here $L(0)$ is defined as the norm-closure of the finite block matrices in $\mathcal{T}$. This is exactly how Pimsner defines his $C^*$-algebra for a $C^*$-correspondence in [15]. By Theorem 5.3, the Pimsner algebra is the smallest quotient of $\mathcal{T}$ that is defined by a covariance condition: any gauge-invariant quotient that is strictly smaller is of the form $\mathcal{T}/(V(J) \cap L(I))$ with a non-zero invariant ideal $I$.

PROPOSITION 5.7. *Let $J \triangleleft A$ be an ideal such that the canonical representation of $\mathcal{E}$ in $\mathcal{O}(\mathcal{E}, J)$ has covariance ideal $J$. Then $*$-homomorphisms $\mathcal{O}(\mathcal{E}, J) \to B$ for a $C^*$-algebra $B$ are naturally in bijection with representations of the partial product system $\mathcal{E}$ in $B$ that are covariant on $J$.*

*Proof.* Representations of $\mathcal{O}(\mathcal{E}, J)$ are in bijection with representations of $\mathcal{T}$ that kill $V(J) \cap L(0) \triangleleft \mathcal{T}$. The universal property of the Toeplitz $C^*$-algebra gives a bijection between representations of $\mathcal{T}$ and representations of $\mathcal{E}$ in $B$. Let $(\omega_n)_{n \in \mathbb{N}}$ be a representation of $\mathcal{E}$ in $B$ and let $\omega : \mathcal{T} \to B$ be the induced $*$-homomorphism. Let $I = \ker \omega_0 \triangleleft A$ and let $J(\omega_n)$ be the covariance ideal of $(\omega_n)_{n\in\mathbb{N}}$. Then $\ker\omega = V(J(\omega_n)) \cap L(I)$ by Theorem 5.3. If $(\omega_n)$ is covariant on $J$, that is, $J \subseteq J(\omega_n)$, then $V(J) \cap L(0) \subseteq V(J(\omega_n)) \cap L(I)$ and hence $\omega|_{V(J)\cap L(0)} = 0$. Then $\omega$ factors through $\mathcal{O}(\mathcal{E}, J)$. Conversely, assume that $\omega$ factors through $\mathcal{O}(\mathcal{E}, J)$. The covariance ideal for the representation of $\mathcal{E}$ in $\mathcal{O}(\mathcal{E}, J)$ is $J$ by assumption. So $\omega$ is covariant on $J$. ∎

THEOREM 5.8. *The canonical $*$-homomorphism $A \to \mathcal{O}(\mathcal{E}, J)$ is faithful if and only if $J \subseteq K^{\perp}$ for the ideal*

$$K := \bigcap_{\ell=1}^{\infty} \ker(\vartheta_0^{\ell} : A \to \mathbb{B}(\mathcal{E}_{\ell})).$$

*Proof.* Assume first that $J$ is not contained in $K^{\perp}$. Then $J \cap K \neq 0$ and we may pick a non-zero element $a \in J \cap K$. In the Fock representation, $a \in A$ acts by the block diagonal operator with entries $\vartheta_0^{\ell}(a) \in \mathbb{B}(\mathcal{E}_{\ell})$ for $\ell \in \mathbb{N}$. By assumption, this belongs to $J$ for $\ell = 0$ and vanishes for $\ell > 0$. So $a \in A \cap V(J) \cap L(0)$ becomes 0 in $\mathcal{O}(\mathcal{E}, J)$. Conversely, assume that $J$ is contained in $K^{\perp}$. We must show that the map $A \to \mathcal{O}(\mathcal{E}, J)$ is faithful. We prove a slightly more general claim, which will be needed below. Namely, we treat $x \in \mathcal{T}_{0,N}$ instead of $a \in A$. We may write $x = \sum_{i=0}^{N} x_i$ with $x_i \in \mathbb{K}(\mathcal{E}_i)$. In the Fock representation, this acts by the diagonal operator with entries

$$y_{\ell} := \sum_{i=0}^{\min\{\ell,N\}} \vartheta_i^{\ell}(x_i)$$

for $i \in \mathbb{N}$. We assume $x \in V(J)$, that is, $y_{\ell} \in \mathbb{B}(\mathcal{E}_{\ell}, \mathcal{E}_{\ell} J)$ for all $\ell \in \mathbb{N}$.

CLAIM 5.9. *An element $x \in \mathcal{T}_{0,N} \cap V(J)$ satisfies $x \in L(0)$ if and only if $y_{\ell} = 0$ for all $\ell \geqslant N$.*

If $N = 0$, then $\mathcal{T}_{0,0} = A$ and the claim says that $x = 0$ if $x \in A \cap V(J) \cap L(0)$, which is what we have to prove. So the following proof of the claim will also finish the proof of the proposition. The claim does not follow from Lemma 5.5 because we do not yet know the vanishing and covariance ideals of the homomorphism $A \to \mathcal{O}(\mathcal{E}, J)$.

The proof of Theorem 5.3 shows that elements for which there is $M \in \mathbb{N}$ with $y_{\ell} = 0$ for all $\ell > M$ are dense in $\mathcal{T}_{0,N} \cap V(J) \cap L(0)$. To prove the claim, we must show that if there is $M \in \mathbb{N}$ with $y_{\ell} = 0$ for all $\ell > M$, then already $y_{\ell} = 0$ for all $\ell \geqslant N$. Let $M$ be maximal with $y_M \neq 0$. We assume $M > N$ in order to get to a contradiction. Let $\xi_1, \xi_2 \in \mathcal{E}_M$ and $\eta \in \mathcal{E}_i$ for some $i > 0$. Then $\xi_2 \cdot \eta \in \mathcal{E}_{M+i}$ and so $x \cdot \xi_2 \cdot \eta = y_{M+i}(\xi_2 \cdot \eta) = 0$ because $y_{\ell} = 0$ for $\ell > M$. (Here the product $x \cdot \xi_2 \cdot \eta$ takes place in $\mathcal{T}$, whereas $y_{M+i}(\xi_2 \cdot \eta) = 0$ is an element of $\mathcal{E}_{M+i} \subseteq \mathcal{T}$.) We may also write $x \cdot \xi_2 \cdot \eta = y_M(\xi_2) \cdot \eta$, and

$$0 = \xi_1^* \cdot x \cdot \xi_2 \cdot \eta = \langle \xi_1 \,|\, y_M(\xi_2) \rangle \cdot \eta.$$

The right hand side is the product of $\langle\xi_1 \mid y_M(\xi_2)\rangle \in A$ with $\eta \in \mathcal{E}_i$. Since this vanishes for all $\eta \in \mathcal{E}_i$ for all $i > 0$, we get $\langle\xi_1 \mid y_M(\xi_2)\rangle \in K$. Hence $y_M(\xi_2) \in \mathcal{E}_M \cdot K$. We also assumed $y_M(\xi_2) \in \mathcal{E}_M \cdot J$. So $y_M(\xi_2) \in \mathcal{E}_M \cdot (J \cap K)$. Hence $y_M(\xi_2) = 0$ because we assumed $J \perp K$. Since $\xi_2 \in \mathcal{E}_M$ is arbitrary, this implies $y_M = 0$. This is the desired contradiction, which proves the claim. ∎

Theorem 5.8 shows that the covariance ideal of the representation of $\mathcal{E}$ in its Katsura algebra is maximal among the covariance ideals of faithful representations. In other words, the Katsura algebra is the smallest quotient of $\mathcal{T}$ for which the canonical map $A \to \mathcal{T} \to \mathcal{O}_{\mathrm{Katsura}}(\mathcal{E})$ is injective. This is the design principle of Katsura's construction of a $C^*$-algebra for a $C^*$-correspondence in [9].

Product systems are easier than partial product systems because for them the maps $\vartheta_m^n : \mathbb{K}(\mathcal{E}_m) \to \mathbb{K}(\mathcal{E}_n)$ for $m \leqslant n$ satisfy $\overline{\vartheta}_m^n \circ \vartheta_\ell^m = \vartheta_\ell^n$ for all $\ell \leqslant m \leqslant n$, where $\overline{\vartheta}_m^n$ is the canonical extension of $\vartheta_m^n$ to $\mathbb{B}(\mathcal{E}_m)$. Hence $\bigcap_{\ell=1}^{\infty} \ker \vartheta_0^\ell = \ker \vartheta_0^1$ for product systems.

THEOREM 5.10. *Let $\mathcal{E}$ be a product system and let $H \triangleleft \mathcal{T}$ be a gauge-invariant ideal with $A \cap H = 0$. Then its covariance ideal is contained in*

$$J_{\max} := (\vartheta_0^1)^{-1}(\mathbb{K}(\mathcal{E}_1)) \cap (\ker \vartheta_0^1)^\perp.$$

*Any ideal $J \triangleleft J_{\max}$ is the covariance ideal of a unique gauge-invariant ideal $H \triangleleft \mathcal{T}$ with $A \cap H = 0$, namely, $H = V(J) \cap L(0)$. The quotient $\mathcal{T}/H$ is the $J$-covariance algebra $\mathcal{O}(\mathcal{E}, J)$. The Katsura algebra and the Pimsner algebra of $\mathcal{E}$ are the $C^*$-algebras defined already by Katsura and Pimsner in this case.*

*Proof.* Let $J$ be the covariance ideal of $H$. Then $H = V(J) \cap L(0)$ by Theorem 5.3. Theorem 5.8 implies $J \subseteq (\ker \vartheta_0^1)^\perp$. We must prove $\vartheta_0^1(J) \subseteq \mathbb{K}(\mathcal{E}_1)$.

Let $N \geqslant 1$ and let $x_i \in \mathbb{K}(\mathcal{E}_i)$ for $i = 0, \dots, N$ be such that $\sum_{i=0}^{N} x_i \in \mathcal{T}_{0,N}$ belongs to $H = V(J) \cap L(0)$. For $0 \leqslant \ell \leqslant N$, define

$$y_\ell := \sum_{i=0}^{\ell} \vartheta_i^\ell(x_i) \in \mathbb{B}(\mathcal{E}_\ell).$$

CLAIM 5.11. *$y_\ell \in \mathbb{K}(\mathcal{E}_\ell)$ and $\sum_{i=0}^{\ell-1} x_i + (x_\ell - y_\ell) \in H$ for $1 \leqslant \ell \leqslant N$.*

*Proof.* We prove this recursively for $\ell = N, N-1, N-2, \dots$. Claim 5.9 implies $y_N = 0$. So the claim for $\ell = N$ is our assumption $\sum_{i=0}^{N} x_i \in H$. Assume the claim has been shown for some $\ell > 1$. We prove the claim for $\ell - 1$. Since $\mathcal{E}$ is a global product system, the unique strictly continuous extension $\overline{\vartheta}_{\ell-1}^\ell$ of $\vartheta_{\ell-1}^\ell$ to

$\mathbb{B}(\mathcal{E}_m)$ satisfies $\overline{\vartheta}^{\ell}_{\ell-1} \circ \vartheta^{\ell-1}_i = \vartheta^{\ell}_i$ for $0 \leqslant i \leqslant \ell - 1$. So

$$x_\ell - y_\ell = -\sum_{i=0}^{\ell-1} \overline{\vartheta}^{\ell}_{\ell-1}(\vartheta^{\ell-1}_i(x_i)) = -\overline{\vartheta}^{\ell}_{\ell-1}(y_{\ell-1}).$$

We first prove $y_{\ell-1} \in \mathbb{K}(\mathcal{E}_{\ell-1})$. This is clear if $\ell = 1$ because $y_0 = x_0 \in A = \mathbb{K}(\mathcal{E}_0)$. So let $\ell > 1$. Then $\mu_{\ell-1,1} : \mathcal{E}_{\ell-1} \otimes_A \mathcal{E}_1 \to \mathcal{E}_\ell$ is unitary. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit for $\mathbb{K}(\mathcal{E}_{\ell-1})$. Then $\lim u_\lambda = 1$ in the strong topology on $\mathcal{E}_{\ell-1}$. Since the net $(u_\lambda)$ is self-adjoint and bounded, this implies strong convergence $\lim u_\lambda \otimes_A \mathrm{id}_{\mathcal{E}_1} = 1$ on $\mathcal{E}_{\ell-1} \otimes_A \mathcal{E}_1$. This is equivalent to the strong convergence $\lim \overline{\vartheta}^{\ell}_{\ell-1}(u_\lambda) = 1$ in $\mathbb{B}(\mathcal{E}_\ell)$. Since the net $\overline{\vartheta}^{\ell}_{\ell-1}(u_\lambda)$ is self-adjoint and bounded, this is equivalent to strict convergence. The operator $\overline{\vartheta}^{\ell}_{\ell-1}(y_{\ell-1})$ is compact by the claim for $\ell$. Hence the strict convergence of $\overline{\vartheta}^{\ell}_{\ell-1}(u_\lambda)$ implies norm convergence $\lim_\lambda \overline{\vartheta}^{\ell}_{\ell-1}(u_\lambda \cdot y_{\ell-1}) = \overline{\vartheta}^{\ell}_{\ell-1}(y_{\ell-1})$. So $\overline{\vartheta}^{\ell}_{\ell-1}(y_{\ell-1}) \in \overline{\vartheta}^{\ell}_{\ell-1}(\mathbb{K}(\mathcal{E}_{\ell-1}))$. This is equivalent to $y_{\ell-1} \in \mathbb{K}(\mathcal{E}_{\ell-1}) + \ker \overline{\vartheta}^{\ell}_{\ell-1}$ because

$$\overline{\vartheta}^{\ell}_{\ell-1}(\mathbb{K}(\mathcal{E}_{\ell-1})) \cong (\mathbb{K}(\mathcal{E}_{\ell-1}) + \ker \overline{\vartheta}^{\ell}_{\ell-1}) / \ker \overline{\vartheta}^{\ell}_{\ell-1}.$$

The homomorphism $\overline{\vartheta}^{\ell}_{\ell-1}$ is injective on the ideal $\mathbb{B}(\mathcal{E}_{\ell-1}, \mathcal{E}_{\ell-1}J)$ by Lemma 2.4 and because $J \cap \ker \vartheta^1_0 = 0$. Lemma 5.5 implies $y_{\ell-1} \in \mathbb{B}(\mathcal{E}_{\ell-1}, \mathcal{E}_{\ell-1}J)$. This is orthogonal to $\ker \overline{\vartheta}^{\ell}_{\ell-1}$. Hence $y_{\ell-1} \in \mathbb{K}(\mathcal{E}_{\ell-1}) + \ker \overline{\vartheta}^{\ell}_{\ell-1}$ implies $y_{\ell-1} \in \mathbb{K}(\mathcal{E}_{\ell-1})$.

Now $\sum_{i=0}^{\ell-2} x_i + (x_{\ell-1} - y_{\ell-1}) \in \mathcal{T}_{0,\ell-1}$ makes sense. It belongs to $V(J)$ because $\sum_{i=0}^{\ell-1} x_i \in V(J)$ by the claim for $\ell$ and $y_{\ell-1} \in \mathbb{B}(\mathcal{E}_{\ell-1}, \mathcal{E}_{\ell-1}J)$ by Lemma 5.5. And $\sum_{i=0}^{\ell-2} x_i + (x_{\ell-1} - y_{\ell-1})$ belongs to $L(0)$ because $\sum_{i=0}^{\ell-2} \vartheta^{\ell-1}_i(x_i) + (x_{\ell-1} - y_{\ell-1}) = 0$ implies $\sum_{i=0}^{\ell-2} \vartheta^k_i(x_i) + \vartheta^k_{\ell-1}(x_{\ell-1} - y_{\ell-1}) = 0$ for all $k \geqslant \ell - 1$ because $\overline{\vartheta}^{k}_{\ell-1} \circ \vartheta^{\ell-1}_i = \vartheta^k_i$. This finishes the proof of the claim. ∎

The proof of Theorem 5.3 shows that the set of elements $x_0 \in A$ for which there are $N \in \mathbb{N}$ and $x_i \in \mathbb{K}(\mathcal{E}_i)$ for $i = 1, \dots, N$ with $\sum_{i=0}^{N} x_i \in H$ is dense in $J$. The claim for $\ell = 1$ says that $\vartheta^1_0(x_0) \in \mathbb{K}(\mathcal{E}_1)$ and $x_0 - \vartheta^1_0(x_0) \in H$ for all such $x_0 \in A$. Hence $\vartheta^1_0(J) \subseteq \mathbb{K}(\mathcal{E}_1)$. This completes the proof that the covariance ideal of $H$ is contained in $J_{\max}$. In the Fock representation, $x - \vartheta^1_0(x)$ for $x \in A$ with $\vartheta^1_0(x) \in \mathbb{K}(\mathcal{E}_1)$ acts by $x \cdot P_0$, where $P_0$ is the orthogonal projection onto $\mathcal{E}_0$. So the claim also shows that $xP_0 \in H$ for any $x \in J$. Conversely, if $xP_0 \in H$ for some $x \in A$, then $\vartheta^1_0(x) \in \mathbb{K}(\mathcal{E}_1)$ follows, and so $x \in J$. Thus the covariance ideal $J$ of $H$ is the set of all $x \in A$ with $xP_0 \in H$. The standard definition of a relative Cuntz–Pimsner in Definition 2.18 of [14] is to take the quotient of $\mathcal{T}$

by the ideal generated by $J \cdot P_0$ for some ideal $J \subseteq J_{\max}$. The argument above shows that this relative Cuntz–Pimsner has the covariance ideal $J$. Hence any $J \triangleleft J_{\max}$ is a covariance ideal for some gauge-invariant ideal $H$ with $H \cap A = 0$, and the resulting covariance algebra is the usual relative Cuntz–Pimsner algebra. In particular, we get the $C^*$-algebra defined by Katsura [9] for $J = J_{\max}$. We have already observed after Definition 5.6 that $\mathcal{O}_{\mathrm{Pimsner}}(\mathcal{E})$ is the $C^*$-algebra associated to the $C^*$-correspondence $\mathcal{E}_1$ by Pimsner [15]. ∎

REMARK 5.12. Theorem 5.10 says that the covariance ideal of any representation of a product system $\mathcal{E}$ with $K = 0$ is contained in $(\vartheta_0^1)^{-1}(\mathbb{K}(\mathcal{E}_1))$. This may fail if $K \neq 0$. Namely, it can happen that $\vartheta_0^2(a) = 0$ for some $a \in A$ for which $\vartheta_0^1(a)$ is not compact. Then $a \in L(0)$, although $\vartheta_0^1(a)$ is not compact. The Pimsner algebra in such a case is not a relative Cuntz–Pimsner algebra.

Next we study the restriction of a Fell bundle $\mathcal{B} = (B_n)_{n\in\mathbb{Z}}$ over $\mathbb{Z}$. The multiplication and involution of the Fell bundle give a $*$-algebra structure on the direct sum $\bigoplus\limits_{n\in\mathbb{Z}} B_n$. This $*$-algebra has a maximal $C^*$-norm. Its completion for this norm is the *section $C^*$-algebra* $C^*(\mathcal{B})$ of the Fell bundle. It carries a canonical $\mathbb{T}$-action where the subspaces $B_n \subseteq C^*(\mathcal{B})$ for $n \in \mathbb{Z}$ are the homogeneous subspaces. Since the maximal $C^*$-seminorm on $\bigoplus\limits_{n\in\mathbb{Z}} B_n$ is a $C^*$-norm, the canonical maps $B_n \to C^*(\mathcal{B})$ for $n \in \mathbb{Z}$ are all injective, even isometric. In particular, $B_0 \hookrightarrow C^*(\mathcal{B})$.

THEOREM 5.13. *Let $\mathcal{B}$ be a Fell bundle over $\mathbb{Z}$. The section $C^*$-algebra $C^*(\mathcal{B})$ is naturally isomorphic to the Katsura algebra of the restriction $\mathcal{B}|_{\mathbb{N}}$ of $\mathcal{B}$ to a partial product system over $\mathbb{N}$. The covariance ideal of the canonical representation of $\mathcal{B}|_{\mathbb{N}}$ in $C^*(\mathcal{B})$ is the closed linear span of $\sum\limits_{n=1}^{\infty} \langle\langle B_n \mid B_n\rangle\rangle$.*

*Proof.* The restriction $\mathcal{B}|_{\mathbb{N}}$ is a partial product system by Proposition 3.3. Let $\mathcal{T}$ be its Toeplitz $C^*$-algebra. The canonical maps $B_n \to C^*(\mathcal{B})$ for $n \in \mathbb{N}$ form a representation of $\mathcal{B}|_{\mathbb{N}}$. Hence they induce a $*$-homomorphism $\mathcal{T} \to C^*(\mathcal{B})$. It is $\mathbb{T}$-equivariant, and it is also surjective because its range contains the dense $*$-subalgebra $\bigoplus\limits_{n\in\mathbb{N}} B_n$. Thus $C^*(\mathcal{B})$ is the quotient of $\mathcal{T}$ by a gauge-invariant ideal $H$ in $\mathcal{T}$. We have $H \cap A = 0$ because $A = B_0 \hookrightarrow C^*(\mathcal{B})$. Let $J$ denote the covariance ideal of the representation of $\mathcal{B}|_{\mathbb{N}}$ in $C^*(\mathcal{B})$. It follows from Theorem 5.3 that $C^*(\mathcal{B})$ is the covariance algebra $\mathcal{O}(\mathcal{B}|_{\mathbb{N}}, J)$. Theorem 5.8 shows that $H_{\max} := L(0) \cap V(K^\perp)$ with $K = \bigcap\limits_{n=1}^{\infty} \ker(\vartheta_0^n)$ is the maximal gauge-invariant ideal $H_{\max} \triangleleft \mathcal{T}$ with $H_{\max} \cap A = 0$. We claim that $H = H_{\max}$. Since $H \cap A = 0$, the maximality of $H_{\max}$ gives $H \subseteq H_{\max}$. For the converse inclusion, we show that $L \cap A \neq 0$ for any gauge-invariant ideal $L \triangleleft \mathcal{T}$ with $H \subsetneq L$. Since $H \subseteq L$, the quotient $\mathcal{T}/L$ is a quotient of $C^*(\mathcal{B})$ by a gauge-invariant ideal. The gauge-invariant ideals in

$C^*(\mathcal{B})$ are naturally in bijection with the $\mathcal{B}$-invariant ideals in $B_0$. Since $H \neq L$, the map $B_0 \to C^*(\mathcal{B})/(L/H)$ is not injective. Thus $L \cap A \neq 0$ as claimed if $H \subsetneq L$. So $H = H_{\max}$ and $C^*(\mathcal{B})$ is the Katsura algebra of $\mathcal{B}|_{\mathbb{N}}$.

Finally, we show that the covariance ideal $J$ of $C^*(\mathcal{B})$ is the closed linear span of $\sum_{n=1}^{\infty} \langle\langle B_n \mid B_n\rangle\rangle$. The relation $\langle\langle x \mid y\rangle\rangle = x \cdot y^*$ for $x, y \in B_n$, $n \in \mathbb{N}$ holds in $C^*(\mathcal{B})$. Hence $\langle\langle x \mid y\rangle\rangle \in B_{\geqslant 1} \cdot C^*(\mathcal{B})$ if $n \geqslant 1$. Thus the closed linear span of the ideals $\langle\langle B_n \mid B_n\rangle\rangle$ for $n \in \mathbb{N}_{\geqslant 1}$ is contained in the covariance ideal. The right ideal $\mathcal{E}_{\geqslant 1} C^*(\mathcal{B})$ in $C^*(\mathcal{B})$ is the closed linear span of $B_\ell \cdot B_n$ for $\ell \in \mathbb{N}_{\geqslant 1}$, $n \in \mathbb{Z}$. Since $B_\ell \cdot B_\ell^* \cdot B_\ell = B_\ell$, it is equal to the closed linear span of $B_\ell \cdot B_\ell^* \cdot B_n = \langle\langle B_\ell \mid B_\ell\rangle\rangle \cdot B_n$. In particular, its gauge-invariant part is the closed linear span of $\langle\langle B_\ell \mid B_\ell\rangle\rangle \cdot B_0 = \langle\langle B_\ell \mid B_\ell\rangle\rangle$. The covariance ideal must be contained in this ideal. This inclusion and the reverse inclusion proved above give the assertion. ∎

Now we may answer the question when a partial product system of Hilbert bimodules over $\mathbb{N}$ is the restriction of a Fell bundle over $\mathbb{Z}$. We mean here that the Fell bundle is such that it induces both the multiplication maps and the inner products in the partial product system.

THEOREM 5.14. *A weak partial product system $\mathcal{E} = (A, \mathcal{E}_n, \mu_{n,m})_{n,m\in\mathbb{N}}$ is the restriction to $\mathbb{N}$ of a Fell bundle over $\mathbb{Z}$ if and only if each $\mathcal{E}_n$ is a Hilbert $A$-bimodule and*

$$\langle\langle \mathcal{E}_n \mid \mathcal{E}_n\rangle\rangle \cdot \mathcal{E}_m \subseteq \mu_{n,m-n}(\mathcal{E}_n \otimes_A \mathcal{E}_{m-n}), \tag{5.3}$$

$$\mathcal{E}_m \cdot \langle \mathcal{E}_n \mid \mathcal{E}_n\rangle \subseteq \mu_{m-n,n}(\mathcal{E}_{m-n} \otimes_A \mathcal{E}_n) \tag{5.4}$$

*as submodules of $\mathcal{E}_m$ for all $m, n \in \mathbb{N}$ with $m \geqslant n$; here $\langle\langle \cdot \mid \cdot\rangle\rangle$ denotes the left inner product. The extension to a Fell bundle over $\mathbb{Z}$ is unique up to a canonical isomorphism. The reverse inclusions to those in* (5.3) *and* (5.4) *always hold, so that these inclusions are equivalent to equalities.*

*Proof.* The left action of the ideal $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n\rangle\rangle \triangleleft A$ is nondegenerate on $\mathcal{E}_n$ and hence also on $\mathcal{E}_n \otimes_A \mathcal{E}_m$. Therefore, $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n\rangle\rangle \cdot \mathcal{E}_m \supseteq \mu_{n,m-n}(\mathcal{E}_n \otimes_A \mathcal{E}_{m-n})$ holds for any weak partial product system of Hilbert bimodules. A dual proof shows the inclusion $\mathcal{E}_m \cdot \langle \mathcal{E}_n \mid \mathcal{E}_n\rangle \supseteq \mu_{m-n,n}(\mathcal{E}_{m-n} \otimes_A \mathcal{E}_n)$. Thus the reverse inclusions to those in (5.3) and (5.4) always hold.

Now assume that $\mathcal{E}$ is the restriction of a Fell bundle over $\mathbb{Z}$, which we also denote by $(\mathcal{E}_n)_{n\in\mathbb{Z}}$. We may rewrite $\langle\langle x \mid y\rangle\rangle = x \cdot y^*$ and $\langle x \mid y\rangle = x^* \cdot y$ for all $x, y \in \mathcal{E}_n$, $n \in \mathbb{N}$. Hence

$$\langle\langle \mathcal{E}_n \mid \mathcal{E}_n\rangle\rangle \cdot \mathcal{E}_m = \mathcal{E}_n \cdot \mathcal{E}_n^* \cdot \mathcal{E}_m \subseteq \mathcal{E}_n \cdot \mathcal{E}_{m-n},$$

which is equivalent to (5.3). A dual argument gives (5.4). Hence these two conditions are necessary for a weak partial product system to be the restriction of a Fell bundle over $\mathbb{Z}$. Now we assume these two conditions. We are going to prove that our weak partial product system extends to a Fell bundle over $\mathbb{Z}$.

First, we show that it is a partial product system. Then we prove that $J := \sum_{n=1}^{\infty} \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle$ is the covariance ideal for a gauge-invariant ideal $H \triangleleft \mathcal{T}$ with $H \cap A = 0$. So we get a faithful representation of the partial product system $(\mathcal{E}_n)_{n\in\mathbb{N}}$ in $\mathcal{T}/H$. Finally, we show that the images $B_n$ of $\mathcal{E}_n$ in $\mathcal{T}/H$ for $n \in \mathbb{N}$ and their adjoints $B_{-n} := B_n^*$ form a concrete Fell bundle over $\mathbb{Z}$, that is, $B_n^* = B_{-n}$ for all $n \in \mathbb{Z}$ and $B_n \cdot B_m \subseteq B_{n+m}$ for all $n, m \in \mathbb{Z}$. Since the representation of $\mathcal{E}$ in $\mathcal{T}/H$ is faithful, this is the desired extension of $(B_n)_{n\in\mathbb{Z}}$ to a Fell bundle over $\mathbb{Z}$.

Let $n, m \in \mathbb{N}_{>0}$ satisfy $m \geqslant n$. The operator $S_n(x) : \mathcal{E}_{m-n} \to \mathcal{E}_m$, $y \mapsto x \otimes y$, is an adjointable map onto the Hilbert submodule $\mu_{n,m-n}(\mathcal{E}_n \otimes_A \mathcal{E}_{m-n}) \subseteq \mathcal{E}_m$. Equation (5.3) identifies this with $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \cdot \mathcal{E}_m$. Hence $\langle\langle y \mid z \rangle\rangle S_n(x) = S_n(\langle\langle y \mid z \rangle\rangle x)$ for $x, y, z \in \mathcal{E}_n$ is an adjointable operator $\mathcal{E}_{m-n} \to \mathcal{E}_m$. Since any element of $\mathcal{E}_n$ may be written as $\langle\langle x \mid y \rangle\rangle z$ for suitable $x, y, z \in \mathcal{E}_n$, this shows that the operators $S_n(x) : \mathcal{E}_{m-n} \to \mathcal{E}_m$ are adjointable. And

$$S_n(\mathcal{E}_n)^* \mathcal{E}_m = S_n(\mathcal{E}_n)^* \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_m = S_n(\mathcal{E}_n)^* \mu_{n,m-n}(\mathcal{E}_n \otimes_A \mathcal{E}_{m-n}) \subseteq \mathcal{E}_{m-n}.$$

Thus the Fock representation of our weak partial product system exists and is a representation. Equivalently, $\mathcal{E}$ is a partial product system.

Now let $J \triangleleft A$ be the closure of $\sum_{n=1}^{\infty} \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle$. We claim that the gauge-invariant ideal $V(J) \cap L(0) \triangleleft \mathcal{T}$ has zero intersection with $A$. If $b \in \bigcap_{\ell=1}^{\infty} \ker \vartheta_0^\ell$, then $b \cdot \mathcal{E}_\ell = 0$ and hence $b \cdot \langle\langle \mathcal{E}_\ell \mid \mathcal{E}_\ell \rangle\rangle = 0$ for all $\ell > 0$. So $b \cdot J = 0$. Hence $A \cap V(J) \cap L(0) = 0$ by Theorem 5.8. The covariance ideal of $V(J) \cap L(0)$ is always contained in $J$. We prove the reverse inclusion. Let $a \in \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle$ for $n \geqslant 1$. Then $\vartheta_0^n(a)$ is a compact operator on $\mathcal{E}_n$ because the left action on a Hilbert bimodule maps $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle$ isomorphically onto the compact operators. We claim that $a - \vartheta_0^n(a) \in \mathcal{T}$ belongs to $V(J) \cap L(0)$. Since $\vartheta_0^n(a) \in \mathcal{E}_{\geqslant n}\mathcal{T} \subseteq \mathcal{E}_{\geqslant 1}\mathcal{T}$, this implies that $a$ belongs to the covariance ideal.

We prove the claim. Let $m \geqslant n$. Then the operators $\vartheta_0^m(a)^*$ and $\vartheta_n^m(\vartheta_0^n(a))^*$ on $\mathcal{E}_n$ agree on the Hilbert submodule $\mathcal{E}_n \cdot \mathcal{E}_{m-n}$. This is equal to $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_m$ by assumption and hence contains $(\vartheta_0^m(a) - \vartheta_n^m(\vartheta_0^n(a)))b$ for all $b \in \mathcal{E}_m$. So

$$(\vartheta_0^m(a) - \vartheta_n^m(\vartheta_0^n(a)))^* \cdot (\vartheta_0^m(a) - \vartheta_n^m(\vartheta_0^n(a))) \cdot b = 0.$$

This shows $\vartheta_0^m(a)^* = \vartheta_n^m(\vartheta_0^n(a))^*$ for $m \geqslant n$ as needed. Now let $\ell < n$. Then $a - \vartheta_0^n(a) \in \mathcal{T}_0$ acts on the summand $\mathcal{E}_\ell$ in the Fock representation by $\vartheta_0^\ell(a)$. It maps $\mathcal{E}_\ell$ into $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \cdot \mathcal{E}_\ell$. We claim that this is contained in $\mathcal{E}_\ell \langle\langle \mathcal{E}_{n-\ell} \mid \mathcal{E}_{n-\ell} \rangle\rangle \subseteq \mathcal{E}_\ell J$. The proof of this claim will finish the proof that $a - \vartheta_0^n(a) \in V(J) \cap L(0)$, which is all that remains to prove that the covariance ideal of $V(J) \cap L(0)$ is $J$.

Assumption (5.3) implies that the range ideal of $\mathcal{E}_\ell \otimes_A \mathcal{E}_{n-\ell}$ is $\langle\langle \mathcal{E}_\ell \mid \mathcal{E}_\ell \rangle\rangle \cap \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle$. So the submodule $(\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_\ell) \otimes_A \mathcal{E}_{n-\ell}$ has the same range ideal and hence is equal to it. Since its range ideal is equal to the range ideal of $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_\ell$, the source ideal of $\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_\ell$ contains the range ideal $\langle\langle \mathcal{E}_{n-\ell} \mid \mathcal{E}_{n-\ell} \rangle\rangle$ of $\mathcal{E}_{n-\ell}$.

Hence

$$\langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_\ell = \langle\langle \mathcal{E}_n \mid \mathcal{E}_n \rangle\rangle \mathcal{E}_\ell \cdot \langle\langle \mathcal{E}_{n-\ell} \mid \mathcal{E}_{n-\ell} \rangle\rangle \subseteq \mathcal{E}_\ell \cdot \langle\langle \mathcal{E}_{n-\ell} \mid \mathcal{E}_{n-\ell} \rangle\rangle.$$

This proves the claim and shows that the covariance ideal of $V(J) \cap L(0)$ is $J$.

We have already seen that $A \cap V(J) \cap L(0) = 0$. Hence the canonical representation $\omega_n : \mathcal{E}_n \to \mathcal{T}/(V(J) \cap L(0))$ is faithful. We define $B_n := \omega_n(\mathcal{E}_n)$ for $n \in \mathbb{N}$ and $B_{-n} := B_n^* \subseteq \mathcal{T}/(V(J) \cap L(0))$. This satisfies $B_n^* = B_{-n}$ for all $n \in \mathbb{Z}$ by construction. We claim that

$$B_n \cdot B_m \subseteq B_{n+m} \tag{5.5}$$

holds for all $n, m \in \mathbb{Z}$. Thus $(B_n)_{n \in \mathbb{Z}}$ is a concrete Fell bundle. It extends the partial product system of Hilbert bimodules $\mathcal{E}$.

If (5.5) holds for $(n, m) \in \mathbb{Z}^2$, then also $B_{-m} \cdot B_{-n} = B_m^* \cdot B_n^* = (B_n \cdot B_m)^* \subseteq B_{n+m}^* = B_{-n-m}$, that is, (5.5) holds for $(-m, -n) \in \mathbb{Z}^2$. Thus it suffices to prove (5.5) for those cases with $n + m \geqslant 0$.

Let $n, m \in \mathbb{Z}$ satisfy $n + m \geqslant 0$. Conditions (i), (ii) and (iv) in Definition 1.1 imply (5.5) if $m \geqslant 0$ and either $n \geqslant 0$ or $n = -m$ or $-m < n < 0$. This covers all $(n, m) \in \mathbb{Z}^2$ with $n + m \geqslant 0$ and $n \geqslant 0$. It remains to treat $(n, m) \in \mathbb{Z}^2$ with $m < 0$ and $n + m \geqslant 0$. So $\ell := -m$ satisfies $0 < \ell \leqslant n$. We compute

$$B_n B_{-\ell} = B_n B_\ell^* B_\ell B_\ell^* = B_n \langle B_\ell \mid B_\ell \rangle B_\ell^* \subseteq B_{n-\ell} B_\ell B_\ell^* = B_{n-\ell} \langle\langle B_\ell \mid B_\ell \rangle\rangle \subseteq B_{n-\ell};$$

here the first step uses $B_{-\ell} = B_\ell^* = B_\ell^* B_\ell B_\ell^*$; the second step uses the relation $x^* y = \langle x \mid y \rangle$ for all $x, y \in B_\ell$ in the Toeplitz algebra; the third step uses (5.4); the fourth step uses $xy^* = \langle\langle x \mid y \rangle\rangle$ for all $x, y \in B_\ell$. Hence (5.5) holds for all $n, m \in \mathbb{Z}$. ∎

PROPOSITION 5.15. *As in Proposition* 3.8*, let* $E = \bigsqcup_{n \in \mathbb{N}} E_n$ *be the path category of a directed graph* $\Gamma$*, where generators in* $\Gamma$ *may have degrees different from* 1*. The Katsura algebra of the partial product system defined by* $E$ *is the graph* $C^*$*-algebra of* $\Gamma$*, with the gauge action where the partial isometry associated to* $e \in E_n$ *is* $n$*-homogeneous for all* $n \in \mathbb{N}$*.*

*Proof.* Let $\mathcal{E}$ be the partial product system associated to $E$ by Proposition 3.8 and let $\mathcal{T}$ be its Toeplitz $C^*$-algebra. The graph $C^*$-algebra $C^*(\Gamma)$ receives a representation of the partial product system $\mathcal{E}$ associated to $E$. This induces a homomorphism $\mathcal{T} \to C^*(\Gamma)$. It is surjective because all the partial isometries and projections generating $C^*(\Gamma)$ belong to its range. It is $\mathbb{T}$-equivariant for the $\mathbb{T}$-action specified in the proposition. Hence $C^*(\Gamma)$ is a quotient of $\mathcal{T}$ by a gauge-invariant ideal. The canonical map $C_0(E_0) \to C^*(\Gamma)$ is injective. Hence $C^*(\Gamma)$ is the covariance algebra for some ideal $J$ in $C_0(E_0)$. The gauge-invariant ideals in $C^*(\Gamma)$ are described in Theorem 3.6 of [4] through a hereditary and saturated subset of $E_0$ and a set of breaking vertices. As a consequence, the map $C_0(E_0) \to C^*(\Gamma)/H$ for a non-zero gauge-invariant ideal $H \triangleleft C^*(\Gamma)$ is not injective. Now it follows as in the proof of Theorem 5.13 that $C^*(\Gamma)$ is the Katsura algebra of $\mathcal{E}$. ∎

REMARK 5.16. All examples of covariance algebras treated above are quotients of the Toeplitz algebra by relations of the form $a \sim \vartheta_0^\ell(a)$ for some $a \in A$, $\ell \in \mathbb{N}_{\geqslant 1}$ with $\vartheta_0^\ell(a) \in \mathbb{K}(\mathcal{E}_\ell)$. These relations are direct analogues of the usual Cuntz–Pimsner covariance condition. We should allow all $\ell \geqslant 1$ because $\mathcal{E}_1 = 0$ may happen. And even for a global product system, it is possible to have covariance ideals that are larger than $(\vartheta_0^1)^{-1}(\mathbb{K}(\mathcal{E}_1))$. Our formalism also allows relations of the form $\sum_{i=0}^{N} x_i \sim 0$ for $x_i \in \mathbb{K}(\mathcal{E}_i)$. We do not know an example of a covariance algebra that cannot be obtained from relations of the simpler Cuntz–Pimsner form $a \sim \vartheta_0^\ell(a)$.

## REFERENCES


[1] B. ABADIE, S. EILERS, R. EXEL, Morita equivalence for crossed products by Hilbert $C^*$-bimodules, *Trans. Amer. Math. Soc.* **350**(1998), 3043–3054.

[2] S. ALBANDIK, R. MEYER, Product systems over Ore monoids, *Doc. Math.* **20**(2015), 1331–1402.

[3] S. ALBANDIK, R. MEYER, Colimits in the correspondence bicategory, *Münster J. Math.* **9**(2016), 51–76.

[4] T. BATES, J.H. HONG, I. RAEBURN, W. SZYMAŃSKI, The ideal structure of the $C^*$-algebras of infinite graphs, *Illinois J. Math.* **46**(2002), 1159–1176.

[5] P. BIKRAM, CAR flows on type III factors and their extendability, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **17**(2014).

[6] R. EXEL, Circle actions on $C^*$-algebras, partial automorphisms, and a generalized Pimsner–Voiculescu exact sequence, *J. Funct. Anal.* **122**(1994), 361–401.

[7] R. EXEL, *Partial Dynamical Systems, Fell Bundles and Applications*, Math. Surveys Monogr., vol. 224, Amer. Math. Soc., Providence, RI 2017.

[8] N.J. FOWLER, Discrete product systems of Hilbert bimodules, *Pacific J. Math.* **204**(2002), 335–375.

[9] T. KATSURA, On $C^*$-algebras associated with $C^*$-correspondences, *J. Funct. Anal.* **217**(2004), 366–401.

[10] T. KATSURA, Ideal structure of $C^*$-algebras associated with $C^*$-correspondences, *Pacific J. Math.* **230**(2007), 107–145.

[11] E.C. LANCE, *Hilbert $C^*$-Modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[12] O.T. MARGETTS, R. SRINIVASAN, Invariants for $E_0$-semigroups on $II_1$ factors, *Comm. Math. Phys.* **323**(2013), 1155–1184.

[13] R. MEYER, C.F. SEHNEM, A bicategorical interpretation for relative Cuntz–Pimsner algebras, *Math. Scand.*, to appear.

[14] P.S. MUHLY, B. SOLEL, Tensor algebras over $C^*$-correspondences: representations, dilations, and $C^*$-envelopes, *J. Funct. Anal.* **158**(1998), 389–457.

[15] M.V. PIMSNER, A class of $C^*$-algebras generalizing both Cuntz–Krieger algebras and crossed products by **Z**, in *Free Probability Theory, (Waterloo, ON) 1995*, Fields Inst. Commun., vol. 12, Amer. Math. Soc., Providence, RI 1997, pp. 189–212.

[16] I. RAEBURN, *Graph Algebras*, CBMS Reg. Conf. Ser. Math, vol. 103, Amer. Math. Soc., Providence, RI 2005.

RALF MEYER, MATHEMATISCHES INSTITUT, UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3–5, 37073 GÖTTINGEN, GERMANY
*E-mail address*: rmeyer2@uni-goettingen.de

DEVARSHI MUKHERJEE, MATHEMATISCHES INSTITUT, UNIVERSITÄT GÖTTINGEN, BUNSENSTRASSE 3–5, 37073 GÖTTINGEN, GERMANY
*E-mail address*: devarshi.mukherjee@mathematik.uni-goettingen.de

Received February 20, 2018; revised November 23, 2018.