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# TOPOLOGICAL CONJUGACY OF TOPOLOGICAL MARKOV SHIFTS AND RUELLE ALGEBRAS

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ABSTRACT. We will characterize topological conjugation for two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ in terms of the associated asymptotic Ruelle $C^*$-algebra $\mathcal{R}_A$ and its commutative $C^*$-subalgebra $C(\overline{X}_A)$ and the canonical circle action. We will also show that the extended Ruelle algebra $\widetilde{\mathcal{R}}_A$, which is a unital and purely infinite version of $\mathcal{R}_A$, together with its commutative $C^*$-subalgebra $C(\overline{X}_A)$ and the canonical torus action $\gamma^A$ is a complete invariant for topological conjugacy of $(\overline{X}_A, \overline{\sigma}_A)$. The diagonal action of $\gamma^A$ has a unique KMS-state on $\widetilde{\mathcal{R}}_A$, which is an extension of the Parry measure on $\overline{X}_A$.

KEYWORDS: *Topological Markov shift, topological conjugacy, étale groupoid, Ruelle algebra, Cuntz–Krieger algebra, K-group, KMS-state.*

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## 1. INTRODUCTION

A Smale space $(X, \phi)$ is a hyperbolic dynamical system having a local product structure (cf. [2], [35]). A two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ gives a typical example of Smale space. D. Ruelle in [33], [34] introduced $C^*$-algebras for a Smale space $(X, \phi)$. After Ruelle's work, I. Putnam in [22], [23] initiated the study of the structure of these $C^*$-algebras by using groupoid technique (for further studies, see [13], [24], [25], [26], [36], etc.). For a Smale space $(X, \phi)$, Putnam considered three kinds of $C^*$-algebras $S(X, \phi), U(X, \phi)$ and $A(X, \phi)$ and their crossed products $S(X, \phi) \rtimes \mathbb{Z}, U(X, \phi) \rtimes \mathbb{Z}$ and $A(X, \phi) \rtimes \mathbb{Z}$ induced by the original homeomorphisms $\phi$, respectively. The algebras $S(X, \phi), U(X, \phi)$ and $A(X, \phi)$ are the $C^*$-algebras of the groupoids of the stable equivalence relation on $X$, the unstable equivalence relation on $X$ and the asymptotic equivalence relation on $X$, respectively. I. Putnam has pointed out that if the Smale space $(X, \phi)$ is a two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ defined by an irreducible matrix $A$, the $C^*$-algebras $S(X, \phi), U(X, \phi)$ are isomorphic to AF-algebras $\mathcal{F}_A \otimes \mathcal{K}, \mathcal{F}_{A^{\mathrm{t}}} \otimes \mathcal{K}$, and $S(X,\phi) \rtimes \mathbb{Z}, U(X,\phi) \rtimes \mathbb{Z}$ are isomorphic to $\mathcal{O}_A \otimes \mathcal{K}, \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{K}$ where $\mathcal{O}_A, \mathcal{F}_A$ are the Cuntz–Krieger algebra, the canonical AF-subalgebra of $\mathcal{O}_A$, respectively for the matrix $A$, and $\mathcal{F}_{A^{\mathrm{t}}}, \mathcal{O}_{A^{\mathrm{t}}}$ are those ones for the transposed matrix $A^{\mathrm{t}}$ of $A$, and $\mathcal{K}$ is the $C^*$-algebra of compact operators on a separable infinite dimensional Hilbert space $\ell^2(\mathbb{N})$.

In [18], the author has introduced notions of asymptotic continuous orbit equivalence and asymptotic conjugacy of Smale spaces, and studied relationship with the crossed product $A(X,\phi) \rtimes \mathbb{Z}$ of the asymptotic Ruelle $C^*$-algebra. In this paper, we will restrict our interest to Smale spaces that arize from two-sided topological Markov shifts. Let $A$ be an $N \times N$ irreducible non-permutation matrix with entries in $\{0,1\}$. The shift space $\overline{X}_A$ of the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ is defined to be the compact metric space of bi-infinite sequences $(x_i)_{i\in\mathbb{Z}}$ satisfying $A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}$ with shift transformation $\overline{\sigma}_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$, where the metric $d$ on $\overline{X}_A$ is defined by

$$d((x_n)_{n\in\mathbb{Z}}, (y_n)_{n\in\mathbb{Z}}) = \begin{cases} 0 & \text{if } (x_n)_{n\in\mathbb{Z}} = (y_n)_{n\in\mathbb{Z}}, \\ 1 & \text{if } x_0 \neq y_0, \\ (\lambda_0)^{k+1} & \text{if } k = \max\{|n| : x_i = y_i \text{ for all } i \text{ with } |i| \leqslant n\}, \end{cases}$$

for some fixed real number $0 < \lambda_0 < 1$. Let $G_A^{\mathrm{a}}$ be the asymptotic étale groupoid for $(\overline{X}_A, \overline{\sigma}_A)$ defined by the asymptotic equivalence relation

$$G_A^{\mathrm{a}} = \Big\{(x,z) \in \overline{X}_A \times \overline{X}_A : \lim_{n\to\infty} d(\overline{\sigma}_A^n(x), \overline{\sigma}_A^n(z)) = \lim_{n\to\infty} d(\overline{\sigma}_A^{-n}(x), \overline{\sigma}_A^{-n}(z)) = 0\Big\}.$$

There are natural groupoid operations on $G_A^{\mathrm{a}}$ and a topology which makes the groupoid $G_A^{\mathrm{a}}$ étale (see [22], [23]). For the general theory of étale groupoids, see [1], [27], [28], [29], etc. As in [23], the $C^*$-algebra $A(\overline{X}_{\overline{\sigma}_A}, \overline{\sigma}_A) \rtimes \mathbb{Z}$ is realized as the $C^*$-algebra $C^*(G_A^{\mathrm{a}} \rtimes \mathbb{Z})$ of the étale groupoid

$$G_A^{\mathrm{a}} \rtimes \mathbb{Z} = \{(x,n,z) \in \overline{X}_A \times \mathbb{Z} \times \overline{X}_A : (\overline{\sigma}_A^k(x), \overline{\sigma}_A^l(z)) \in G_A^{\mathrm{a}}, n = k - l\}.$$

The $C^*$-algebra of $G_A^{\mathrm{a}} \rtimes \mathbb{Z}$ is denoted by $\mathcal{R}_A$ and called the asymptotic Ruelle algebra in this paper. Let $d_A : G_A^{\mathrm{a}} \rtimes \mathbb{Z} \to \mathbb{Z}$ be the groupoid homomorphism defined by $d_A(x,n,z) = n$. As the unit space $(G_A^{\mathrm{a}} \rtimes \mathbb{Z})^\circ$ of $G_A^{\mathrm{a}} \rtimes \mathbb{Z}$ is homeomorphic to $\overline{X}_A$, the commutative $C^*$-algebra $C(\overline{X}_A)$ can in a natural way be regarded as a subalgebra of $\mathcal{R}_A$. As the algebra $\mathcal{R}_A$ is a crossed product $A(\overline{X}_{\overline{\sigma}_A}, \overline{\sigma}_A) \rtimes \mathbb{Z}$, it has the dual action $\rho_t^A$ of $t \in \mathbb{T}$. In [18], an extended version $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ of the groupoid $G_A^{\mathrm{a}} \rtimes \mathbb{Z}$ was introduced by setting

$$G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 = \{(x,p,q,z) \in \overline{X}_A \times \mathbb{Z} \times \mathbb{Z} \times \overline{X}_A : (\overline{\sigma}_A^p(x), z) \in G_A^{\mathrm{s}}, (\overline{\sigma}_A^q(x), z) \in G_A^{\mathrm{u}}\}$$

where

$$G_A^{\mathrm{s}} = \Big\{(x,z) \in \overline{X}_A \times \overline{X}_A : \lim_{n\to\infty} d(\overline{\sigma}_A^n(x), \overline{\sigma}_A^n(z)) = 0\Big\},$$
$$G_A^{\mathrm{u}} = \Big\{(x,z) \in \overline{X}_A \times \overline{X}_A : \lim_{n\to\infty} d(\overline{\sigma}_A^{-n}(x), \overline{\sigma}_A^{-n}(z)) = 0\Big\}.$$

There are natural groupoid operations on $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ and a topology which makes $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ étale (see [18]). Let $c_A : G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to \mathbb{Z}^2$ be the groupoid homomorphism defined by $c_A(x,p,q,z) = (p,q)$. The groupoid $C^*$-algebra $C^*(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ is denoted by $\widetilde{\mathcal{R}}_A$ (it was denoted by $\mathcal{R}_A^{\mathrm{s,u}}$ in [18]). Since the unit space $(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)^\circ$ of $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ is $\{(x,0,0,x) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 : x \in \overline{X}_A\}$, which is homeomorphic to $\overline{X}_A$, the algebra $\widetilde{\mathcal{R}}_A$ includes $C(\overline{X}_A)$ as a subalgebra in natural way. There is a projection $E_A$ in the tensor product $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ such that $\widetilde{\mathcal{R}}_A$ is naturally isomorphic to $E_A(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)E_A$. Hence the algebra $\widetilde{\mathcal{R}}_A$ might be regarded as a bilateral Cuntz–Krieger algebra. The tensor product $\alpha_r^{A^{\mathrm{t}}} \otimes \alpha_s^A$ of the gauge actions $\alpha_r^{A^{\mathrm{t}}}$ on $\mathcal{O}_{A^{\mathrm{t}}}$ and $\alpha_s^A$ on $\mathcal{O}_A$ gives rise to an action $\gamma_{(r,s)}^A$ of $(r,s) \in \mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A$. It has been shown in [18] that the fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$ of $\widetilde{\mathcal{R}}_A$ under the diagonal action $\delta_t^A = \gamma_{(t,t)}^A$, $t \in \mathbb{T}$ is isomorphic to $\mathcal{R}_A$.

In [8] (cf. [6]), Cuntz–Krieger proved that the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes \mathcal{K}$ with its diagonal $C^*$-subalgebra $\mathcal{D}_A \otimes \mathcal{C}$ of $\mathcal{O}_A \otimes \mathcal{K}$, where $\mathcal{C}$ denotes the maximal abelian $C^*$-subalgebra of $\mathcal{K}$ consisting of diagonal operators on $\ell^2(\mathbb{N})$, and the stabilized gauge action $\alpha^A \otimes \mathrm{id}$, is invariant under topological conjugacy of the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ for irreducible non-permutation matrix $A$. T.M. Carlsen and J. Rout have recently proved in [5] that the converse also holds even for more general matrices without irreducibility and non-permutation. As a consequence, the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes \mathcal{K}$ with its diagonal $C^*$-subalgebra $\mathcal{D}_A \otimes \mathcal{C}$ and the stabilized gauge action $\alpha^A \otimes \mathrm{id}$ is a complete invariant of the topological conjugacy of the two-sided topological Markov shift. Inspired by this fact, we will in this paper show that the Ruelle algebra $\mathcal{R}_A$ with its subalgebra $C(\overline{X}_A)$ and the dual action $\rho^A$ is a complete invariant of the topological conjugacy class of the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$. We will also see that the $C^*$-algebra $\widetilde{\mathcal{R}}_A$ with its subalgebra $C(\overline{X}_A)$ and the action $\gamma^A$ of $\mathbb{T}^2$ is a complete invariant of the topological conjugacy class of $(\overline{X}_A, \overline{\sigma}_A)$. We will show the following theorem.

THEOREM 1.1. *Let $A, B$ be irreducible, non-permutation matrices with entries in $\{0,1\}$. The following six conditions are equivalent:*

(i) *the topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate;*

(ii) *the topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are asymptotically conjugate;*

(iii) *there exists an isomorphism $\varphi : G_A^{\mathrm{a}} \rtimes \mathbb{Z} \to G_B^{\mathrm{a}} \rtimes \mathbb{Z}$ of étale groupoids such that $d_B \circ \varphi = d_A$;*

(iv) *there exists an isomorphism $\widetilde{\varphi} : G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ of étale groupoids such that $c_B \circ \widetilde{\varphi} = c_A$;*

(v) *there exists an isomorphism $\Phi : \mathcal{R}_A \to \mathcal{R}_B$ of $C^*$-algebras such that $\Phi(C(\overline{X}_A)) = C(\overline{X}_B)$ and $\Phi \circ \rho_t^A = \rho_t^B \circ \Phi$ for $t \in \mathbb{T}$;*

(vi) *there exists an isomorphism* $\widetilde{\Phi} : \widetilde{\mathcal{R}}_A \to \widetilde{\mathcal{R}}_B$ *of* $C^*$*-algebras such that* $\widetilde{\Phi}(C(\overline{X}_A)) = C(\overline{X}_B)$ *and* $\widetilde{\Phi} \circ \gamma^A_{(r,s)} = \gamma^B_{(r,s)} \circ \widetilde{\Phi}$ *for* $(r,s) \in \mathbb{T}^2$.

The equivalences among (ii), (iii) and (v) come from [18]. The main assertion is the implication (ii) $\Rightarrow$ (i) which will be proved in Theorem 3.3. The other implications are not difficult and will be proved in Section 4.

Since the algebra $\widetilde{\mathcal{R}}_A$ is a unital, simple, purely infinite, nuclear $C^*$-algebra satisfying UCT, its isomorphism class is completely determined by its K-theory data by a general classification theorem ([15], [21], [31]). It follows from the Künneth formulas and the universal coefficient theorem that

$$K_0(\widetilde{\mathcal{R}}_A) \cong KK^1(\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A), \quad K_1(\widetilde{\mathcal{R}}_A) \cong KK(\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A).$$

It follows from Theorem 1.1 that the group $K_0(\widetilde{\mathcal{R}}_A)$ and the position of the class $[1_{\widetilde{\mathcal{R}}_A}]$ of the unit $1_{\widetilde{\mathcal{R}}_A}$ of $\widetilde{\mathcal{R}}_A$ in $K_0(\widetilde{\mathcal{R}}_A)$ is invariant under topological conjugacy of $(\overline{X}_A, \overline{\sigma}_A)$. We see that $(K_0(\widetilde{\mathcal{R}}_A), [1_{\widetilde{\mathcal{R}}_A}])$ is isomorphic to $(K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A), [E_A])$ and the class $[E_A]$ of the projection $E_A$ actually lives in the group $K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)$. We set the vector $e_i = \left[0, \dots, 0, \overset{i}{1}, 0, \dots, 0\right] \in \mathbb{Z}^N$ for $i = 1, \dots, N$. We have the following theorem.

THEOREM 1.2. *Suppose that* $A$ *is an* $N \times N$ *irreducible, non-permutation matrix with entries in* $\{0,1\}$*. The position* $[E_A]$ *of the projection* $E_A$ *in* $K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)$ *is* $\sum_{i=1}^{N} [e_i] \otimes [e_i]$ *in the group* $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$*. Hence it is invariant under topological conjugacy of the two-sided topological Markov shift* $(\overline{X}_A, \overline{\sigma}_A)$.

We put

$$e_A = \sum_{i=1}^{N} [e_i] \otimes [e_i] \quad \text{in } \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N.$$

We will actually see that the pair $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is a shift equivalence invariant (Proposition 5.4, Proposition 5.7). We will present an example of pairs of matrices $A = [A(i,j)]_{i,j=1}^N, B = [B(i,j)]_{i,j=1}^M$ such that $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B), \det(\mathrm{id} - A) = \det(\mathrm{id} - B)$, but the invariants $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ and $(\mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M \otimes \mathbb{Z}^M/(\mathrm{id} - B^{\mathrm{t}})\mathbb{Z}^M, e_B)$ are different (Proposition 5.5). Hence the invariant $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is strictly stronger than the Bowen–Franks group $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N$ and not invariant under flow equivalence of $(\overline{X}_A, \overline{\sigma}_A)$.

J. Cuntz in [7] studied the homotopy groups $\pi_n(\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K}))$ of the space $\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K})$ of endomorphisms of the $C^*$-algebra $\mathcal{O}_A \otimes \mathcal{K}$. He proved that natural maps $\epsilon_n : \pi_n(\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K})) \to KK^n(\mathcal{O}_A, \mathcal{O}_A)$ yield isomorphisms, and defined an element denoted by $\epsilon_1(\lambda^A)$ in $\mathrm{Ext}(\mathcal{O}_A) \otimes K_0(\mathcal{O}_A)$, where $\lambda^A$ denotes the gauge action $\alpha^A$ on $\mathcal{O}_A$. His observation shows that the element $\epsilon_1(\lambda^A)$ is nothing but the above element $e_A$ under the natural identification between

$\operatorname{Ext}(\mathcal{O}_A) \otimes K_0(\mathcal{O}_A)$ and $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$. He already states in [7] that the position $\epsilon_1(\lambda^A)$ in $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ is invariant under topological conjugacy of the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$.

We will finally study KMS states for the diagonal action $\delta_t^A = \gamma_{(t,t)}^A$ on $\widetilde{\mathcal{R}}_A$, and prove the following theorem.

THEOREM 1.3. *Assume that the matrix $A$ is aperiodic. A* KMS *state on $\widetilde{\mathcal{R}}_A$ for the action $\delta^A$ at the inverse temperature* $\log \gamma$ *exists if and only if $\gamma$ is the Perron–Frobenius eigenvalue $\beta$ of $A$. The admitted* KMS *state is unique. The restriction of the admitted* KMS *state to the subalgebra $C(\overline{X}_A)$ is the state defined by the Parry measure on $\overline{X}_A$.*

The Parry measure is the measure of maximal entropy (cf. [37]). Since $\log \beta$ is the topological entropy of the Markov shift $(\overline{X}_A, \overline{\sigma}_A)$, the inverse temperature expresses the entropy. This exactly corresponds to the result obtained by Enomoto–Fujii–Watatani in [10] on KMS states for the gauge action on the Cuntz–Krieger algebras $\mathcal{O}_A$.

Throughout the paper, we denote by $\mathbb{Z}_+$ the set of nonnegative integers and by $\mathbb{N}$ the set of positive integers.

This paper is a continuation of the paper [18].

## 2. PRELIMINARIES

We fix an irreducible, non-permutation matrix $A = [A(i,j)]_{i,j=1}^N$ with entries in $\{0,1\}$. Let $\mathcal{O}_A$ and $\mathcal{O}_{A^{\mathrm{t}}}$ be the Cuntz–Krieger algebras for the matrices $A$ and its transpose $A^{\mathrm{t}}$, respectively. We may take generating partial isometries $S_i, i = 1, \dots, N$ of $\mathcal{O}_A$ and $T_i, i = 1, \dots, N$ of $\mathcal{O}_{A^{\mathrm{t}}}$ such that

$$\sum_{i=1}^N S_i S_i^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \tag{2.1}$$

$$\sum_{i=1}^N T_i T_i^* = 1, \quad T_i^* T_i = \sum_{j=1}^N A^{\mathrm{t}}(i,j) T_j T_j^*. \tag{2.2}$$

In the $C^*$-algebra $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ of the tensor product, let us denote by $E_A$ the projection defined by

$$E_A = \sum_{i=1}^N T_i^* T_i \otimes S_i S_i^*. \tag{2.3}$$

By using the relations (2.1) and (2.2), it is easy to see that $E_A = \sum_{i=1}^N T_i T_i^* \otimes S_i^* S_i$. The $C^*$-algebra $\widetilde{R}_A$ is defined as the groupoid $C^*$-algebra $C^*(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ which is realized as the $C^*$-algebra ([18])

$$\widetilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)E_A.$$

The $C^*$-algebra $\widetilde{\mathcal{R}}_A$ was denoted by $\mathcal{R}_A^{\mathrm{s,u}}$ in [18]. Since both the algebras $\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A$ are simple and purely infinite, and $\widetilde{\mathcal{R}}_A \otimes \mathcal{K}$ is isomorphic to $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \otimes \mathcal{K}$, the $C^*$-algebra $\widetilde{\mathcal{R}}_A$ is simple and purely infinite under the condition that $A$ is irreducible and non-permutation (cf. Proposition 5.5 of [26]).

Let $B_n(\overline{X}_A)$ be the set of admissible words in $\overline{X}_A$ of length $n$. Let us denote by $B_*(\overline{X}_A)$ the set $\bigcup_{n=0}^{\infty} B_n(\overline{X}_A)$, where $B_0(\overline{X}_A)$ denotes the empty word. For a word $\xi = (\xi_1, \ldots, \xi_k), \mu = (\mu_1, \ldots, \mu_m) \in B_*(\overline{X}_A)$, we denote by $\overline{\xi} = (\xi_k, \ldots, \xi_1) \in B_k(\overline{X}_{A^{\mathrm{t}}})$ and set $T_{\overline{\xi}} = T_{\xi_k} \cdots T_{\xi_1}$ and $S_\mu = S_{\mu_1} \cdots S_{\mu_m}$. Let $\alpha^A, \alpha^{A^{\mathrm{t}}}$ be the gauge actions of $\mathcal{O}_A$ and $\mathcal{O}_{A^{\mathrm{t}}}$, respectively, which are defined by

$$\alpha_t^A(S_i) = \exp(\sqrt{-1}t)S_i, \quad \alpha_t^{A^{\mathrm{t}}}(T_i) = \exp(\sqrt{-1}t)T_i, \quad i = 1, \ldots, N$$

for $t \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}$. The fixed point algebras $(\mathcal{O}_A)^{\alpha^A}, (\mathcal{O}_{A^{\mathrm{t}}})^{\alpha^{A^{\mathrm{t}}}}$ of $\mathcal{O}_A, \mathcal{O}_{A^{\mathrm{t}}}$ under the gauge actions $\alpha^A, \alpha^{A^{\mathrm{t}}}$ are known to be AF-algebras, which are denoted by $\mathcal{F}_A, \mathcal{F}_{A^{\mathrm{t}}}$, respectively.

We first note the following facts which were seen in [18].

PROPOSITION 2.1. (i) *The groupoid $C^*$-algebra $C^*(G_A^{\mathrm{a}})$ of the groupoid $G_A^{\mathrm{a}}$ is isomorphic to the $C^*$-subalgebra of $\mathcal{F}_{A^{\mathrm{t}}} \otimes \mathcal{F}_A$ defined by*

$$\begin{aligned} C^*( \quad & T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \in \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A : \\ & \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\overline{X}_A), \\ & \overline{\xi} = (\xi_k, \ldots, \xi_1), \overline{\eta} = (\eta_l, \ldots, \eta_1) \in B_*(\overline{X}_{A^{\mathrm{t}}}), \\ & A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1, k = l, m = n \quad ). \end{aligned}$$

(ii) *The $C^*$-algebra $\mathcal{R}_A$ is isomorphic to the $C^*$-subalgebra of $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ defined by*

$$\begin{aligned} C^*( \quad & T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \in \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A : \\ & \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\overline{X}_A), \\ & \overline{\xi} = (\xi_k, \ldots, \xi_1), \overline{\eta} = (\eta_l, \ldots, \eta_1) \in B_*(\overline{X}_{A^{\mathrm{t}}}), \\ & A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1,\ k - l = n - m \quad ). \end{aligned}$$

(iii) *The $C^*$-algebra $\widetilde{\mathcal{R}}_A$ is isomorphic to the $C^*$-subalgebra of $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ defined by*

$$\begin{aligned} C^*( \quad & T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \in \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A : \\ & \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\overline{X}_A), \\ & \overline{\xi} = (\xi_k, \ldots, \xi_1), \overline{\eta} = (\eta_l, \ldots, \eta_1) \in B_*(\overline{X}_{A^{\mathrm{t}}}), \\ & A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1 \quad ). \end{aligned}$$

We note that for $i = 1, \ldots, N$ the identity

$$T_i \otimes S_i^* = \sum_{j,k=1}^{N} T_i T_j T_j^* \otimes S_k S_k^* S_i^* = \sum_{j,k=1}^{N} A(j,i) A(i,k) T_{ij} T_j^* \otimes S_k S_{ik}^*$$

holds. Since $A(j,i)A(i,k)T_{ij}T_j^* \otimes S_k S_{ik}^*$ belongs to $\mathcal{R}_A$, we see that $T_i \otimes S_i^*$ and hence $T_i^* \otimes S_i$ belong to $\mathcal{R}_A$.

Define the diagonal action $\delta^A$ on $\widetilde{R}_A$ by setting

$$\delta_t^A = \alpha_t^{A^{\mathrm{t}}} \otimes \alpha_t^A, \quad t \in \mathbb{R}/2\pi\mathbb{Z} = \mathbb{T}.$$

Since $\delta_t^A(E_A) = E_A$, the automorphisms $\delta_t^A, t \in \mathbb{T}$ define an action of $\mathbb{T}$ on $\widetilde{\mathcal{R}}_A$. For

$$\mu = (\mu_1, \dots, \mu_m), \quad \nu = (\nu_1, \dots, \nu_n) \in B_*(\overline{X}_A),$$
$$\overline{\xi} = (\xi_k, \dots, \xi_1), \quad \overline{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\overline{X}_{A^{\mathrm{t}}})$$

satisfying $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$, we see that

$$\delta_t^A(T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) = \exp(\sqrt{-1}(k - l + m - n)t) T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*$$

so that the following lemma holds.

LEMMA 2.2. *Keep the above notation. The element $T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*$ in $\widetilde{\mathcal{R}}_A$ belongs to $\mathcal{R}_A$ if and only if $k - l = n - m$.*

Hence we have the following proposition.

PROPOSITION 2.3 ([18], Theorem 9.6). *The fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$ of $\widetilde{\mathcal{R}}_A$ under $\delta^A$ is the asymptotic Ruelle algebra $\mathcal{R}_A$.*

As in Lemma 9.5 of [18], the $C^*$-subalgebra of $C^*(G_A^{\mathrm{a}})$ generated by elements $T_{\overline{\xi}} T_{\overline{\xi}}^* \otimes S_\mu S_\mu^*$, $\overline{\xi} = (\xi_k, \dots, \xi_1) \in B_*(\overline{X}_{A^{\mathrm{t}}}), \mu = (\mu_1, \dots, \mu_m) \in B_*(\overline{X}_A)$ with $A(\xi_k, \mu_1) = 1$ is canonically isomorphic to the commutative $C^*$-algebra $C(\overline{X}_A)$ of continuous functions on $\overline{X}_A$. In what follows, we identify the subalgebra with the algebra $C(\overline{X}_A)$ so that $C(\overline{X}_A)$ is a $C^*$-subalgebra of $\mathcal{R}_A$ and $\widetilde{\mathcal{R}}_A$.

## 3. ASYMPTOTIC CONJUGACY AND TOPOLOGICAL CONJUGACY

For $x = (x_n)_{n \in \mathbb{Z}} \in \overline{X}_A$, we set $x_+ = (x_n)_{n=0}^\infty$ and $x_- = (x_{-n})_{n=0}^\infty$. Let us denote by $X_A$ the compact Hausdorff space of right infinite sequences $(x_i)_{i \in \mathbb{Z}_+} \in \{1, \dots, N\}^{\mathbb{Z}_+}$ satisfying $A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}_+$. The right one-sided topological Markov shift $(X_A, \sigma_A)$ is defined by a topological dynamical system of shift transformation $\sigma_A((x_i)_{i \in \mathbb{Z}_+}) = (x_{i+1})_{i \in \mathbb{Z}_+}$ on $X_A$. For $x = (x_i)_{i \in \mathbb{Z}_+} \in X_A$ and $k \in \mathbb{Z}_+$, we set $x_{[k,\infty)} = \sigma_A^k(x) = (x_k, x_{k+1}, \dots) \in X_A$.

In [18], a notion of asymptotic conjugacy in Smale spaces was introduced. We apply the notion for topological Markov shifts and rephrase it in the following way.

DEFINITION 3.1 ([18]). Two topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are said to be *asymptotically conjugate* if there exists a homeomorphism $h : \overline{X}_A \to \overline{X}_B$ satisfying the following three conditions:

(i) There exists a nonnegative integer $K \in \mathbb{Z}_+$ such that:

$$\overline{\sigma}_B^{K+1}(h(x))_+ = \overline{\sigma}_B^{K}(h(\overline{\sigma}_A(x)))_+ \quad \text{for } x \in \overline{X}_A, \tag{3.1}$$

$$\overline{\sigma}_B^{-K+1}(h(x))_- = \overline{\sigma}_B^{-K}(h(\overline{\sigma}_A(x)))_- \quad \text{for } x \in \overline{X}_A, \tag{3.2}$$

$$\overline{\sigma}_A^{K+1}(h^{-1}(y))_+ = \overline{\sigma}_A^{K}(h^{-1}(\overline{\sigma}_B(y)))_+ \quad \text{for } y \in \overline{X}_B, \tag{3.3}$$

$$\overline{\sigma}_A^{-K+1}(h^{-1}(y))_- = \overline{\sigma}_A^{-K}(h^{-1}(\overline{\sigma}_B(y)))_- \quad \text{for } y \in \overline{X}_B. \tag{3.4}$$

(ii) There exists a continuous function $m_1 : G_A^{\mathrm{a}} \to \mathbb{Z}_+$ such that

$$\overline{\sigma}_B^{m_1(x,z)}(h(x))_+ = \overline{\sigma}_B^{m_1(x,z)}(h(z))_+ \quad \text{for } (x,z) \in G_A^{\mathrm{a}},$$
$$\overline{\sigma}_B^{-m_1(x,z)}(h(x))_- = \overline{\sigma}_B^{-m_1(x,z)}(h(z))_- \quad \text{for } (x,z) \in G_A^{\mathrm{a}}.$$

(iii) There exists a continuous function $m_2 : G_B^{\mathrm{a}} \to \mathbb{Z}_+$ such that

$$\overline{\sigma}_A^{m_2(y,w)}(h^{-1}(y))_+ = \overline{\sigma}_A^{m_2(y,w)}(h^{-1}(w))_+ \quad \text{for } (y,w) \in G_B^{\mathrm{a}},$$
$$\overline{\sigma}_A^{-m_2(y,w)}(h^{-1}(y))_- = \overline{\sigma}_A^{-m_2(y,w)}(h^{-1}(w))_- \quad \text{for } (y,w) \in G_B^{\mathrm{a}}.$$

Let $A = [A(i,j)]_{i,j=1}^N, B = [B(i,j)]_{i,j=1}^M$ be irreducible matrices with entries in $\{0,1\}$. The following proposition is key in this section.

PROPOSITION 3.2. *If the topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are asymptotically conjugate, then they are topologically conjugate.*

*Proof.* Let $h : \overline{X}_A \to \overline{X}_B$ be a homeomorphism and $K \in \mathbb{Z}_+$ a nonnegative integer satisfying (3.1), (3.2), (3.3), (3.4). We define two continuous maps $h_+ : \overline{X}_A \to X_B$ and $h_+^{-1} : \overline{X}_B \to X_A$ by setting

$$h_+(x) = \overline{\sigma}_B^K(h(x))_+, \quad x \in \overline{X}_A,$$
$$h_+^{-1}(y) = \overline{\sigma}_A^K(h^{-1}(y))_+, \quad y \in \overline{X}_B.$$

It then follows that by (3.1),

$$h_+(\overline{\sigma}_A(x)) = \overline{\sigma}_B^K(h(\overline{\sigma}_A(x)))_+ = \overline{\sigma}_B^{K+1}(h(x))_+ = [\overline{\sigma}_B^{K+1}(h(x))]_{[0,\infty)} = [\overline{\sigma}_B(h(x))]_{[K,\infty)}.$$

On the other hand,

$$\sigma_B(h_+(x)) = \sigma_B([\overline{\sigma}_B^K(h(x))]_{[0,\infty)}) = [\overline{\sigma}_B^K(h(x))]_{[1,\infty)} = [\overline{\sigma}_B(h(x))]_{[K,\infty)}.$$

Therefore we have

$$h_+(\overline{\sigma}_A(x)) = \sigma_B(h_+(x)) \quad \text{for } x \in \overline{X}_A.$$

Hence the continuous map $h_+ : \overline{X}_A \to X_B$ is a sliding block code (cf. [16]). Therefore there exists a block map $\Phi : B_{m+n+1}(\overline{X}_A) \to \{1, 2, \ldots, M\}$ for some $m, n \in \mathbb{Z}_+$ such that

$$h_+((x_i)_{i\in\mathbb{Z}}) = \Phi([x_{i-m}, \ldots, x_{i+n}])_{i\in\mathbb{Z}_+} \quad \text{for } x = (x_i)_{i\in\mathbb{Z}} \in \overline{X}_A.$$

Similarly the continuous map $h_+^{-1} : \overline{X}_B \to X_A$ satisfies $h_+^{-1}(\overline{\sigma}_B(y)) = \sigma_A(h_+^{-1}(y))$ for $y \in \overline{X}_B$ and there exists a block map $\Psi : B_{m'+n'+1}(\overline{X}_B) \to \{1, 2, \dots, N\}$ for some $m', n' \in \mathbb{Z}_+$ such that

$$h_+^{-1}((y_i)_{i\in\mathbb{Z}}) = \Psi([y_{i-m'}, \dots, y_{i+n'}])_{i\in\mathbb{Z}_+} \quad \text{for } y = (y_i)_{i\in\mathbb{Z}} \in \overline{X}_B.$$

By using these block maps

$$\Phi : B_{m+n+1}(\overline{X}_A) \to \{1, 2, \dots, M\}, \quad \Psi : B_{m'+n'+1}(\overline{X}_B) \to \{1, 2, \dots, N\},$$

we define two sliding block codes $\Phi_\infty : \overline{X}_A \to \overline{X}_B$ and $\Psi_\infty : \overline{X}_B \to \overline{X}_A$ by setting

$$\Phi_\infty((x_i)_{i\in\mathbb{Z}}) = \Phi([x_{i-m}, \dots, x_{i+n}])_{i\in\mathbb{Z}} \in \overline{X}_B \quad \text{for } x = (x_i)_{i\in\mathbb{Z}} \in \overline{X}_A,$$
$$\Psi_\infty((y_i)_{i\in\mathbb{Z}}) = \Psi([y_{i-m'}, \dots, y_{i+n'}])_{i\in\mathbb{Z}} \in \overline{X}_A \quad \text{for } y = (y_i)_{i\in\mathbb{Z}} \in \overline{X}_B.$$

We note that

$$\Phi_\infty((x_i)_{i\in\mathbb{Z}})_+ = h_+((x_i)_{i\in\mathbb{Z}}) \in X_B \quad \text{for } x = (x_i)_{i\in\mathbb{Z}} \in \overline{X}_A,$$
$$\Psi_\infty((y_i)_{i\in\mathbb{Z}})_+ = h_+^{-1}((y_i)_{i\in\mathbb{Z}}) \in X_A \quad \text{for } y = (y_i)_{i\in\mathbb{Z}} \in \overline{X}_B.$$

For $y = (y_i)_{i\in\mathbb{Z}} \in \overline{X}_B$, we have

$$\begin{aligned}[\Psi_\infty(y)]_{[K,\infty)} &= [\overline{\sigma}_A^K(\Psi_\infty(y))]_{[0,\infty)} = [\Psi_\infty(\overline{\sigma}_B^K(y))]_{[0,\infty)} = h_+^{-1}(\overline{\sigma}_B^K(y)) \\ &= [\overline{\sigma}_A^K(h^{-1}(\overline{\sigma}_B^K(y)))]_{[0,\infty)} = [h^{-1}(\overline{\sigma}_B^K(y))]_{[K,\infty)}.\end{aligned}$$

As $\Phi_\infty$ is a sliding block code with memory $m$, the condition $[\Psi_\infty(y)]_{[K,\infty)} = [h^{-1}(\overline{\sigma}_B^K(y))]_{[K,\infty)}$ implies

$$[\Phi_\infty(\Psi_\infty(y))]_{[K+m,\infty)} = [\Phi_\infty(h^{-1}(\overline{\sigma}_B^K(y)))]_{[K+m,\infty)}.$$

It then follows that

$$\begin{aligned}[\Phi_\infty(\Psi_\infty(y))]_{[K+m,\infty)} &= [\Phi_\infty(h^{-1}(\overline{\sigma}_B^K(y)))]_{[K+m,\infty)} = [h_+(h^{-1}(\overline{\sigma}_B^K(y)))]_{[K+m,\infty)} \\ &= [(\overline{\sigma}_B^K \circ h)(h^{-1}(\overline{\sigma}_B^K(y)))]_{[K+m,\infty)} \\ &= [\overline{\sigma}_B^K(\overline{\sigma}_B^K(y))]_{[K+m,\infty)} = [\overline{\sigma}_B^{2K}(y)]_{[K+m,\infty)}\end{aligned}$$

so that

$$[\Phi_\infty(\Psi_\infty(y))]_{[K+m,\infty)} = [\overline{\sigma}_B^{2K}(y)]_{[K+m,\infty)} \quad \text{for } y \in \overline{X}_B.$$

Since $\Phi_\infty \circ \Psi_\infty$ is a sliding block code, we obtain that

$$\Phi_\infty \circ \Psi_\infty = \overline{\sigma}_B^{2K}.$$

Hence $\Phi_\infty$ is surjective. Similarly we have that $\Psi_\infty \circ \Phi_\infty = \overline{\sigma}_A^{2K}$ so that $\Phi_\infty$ is injective. Therefore we have a topological conjugacy $\Phi_\infty : \overline{X}_A \to \overline{X}_B$. ∎

We remark that the above proof needs only the equalities (3.1) and (3.3). We thus conclude the following theorem.

THEOREM 3.3. *Two topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are asymptotically conjugate if and only if they are topologically conjugate.*

For the proof it is direct to see that topological conjugacy implies asymptotic conjugacy. Hence the assertion follows from the preceding proposition.

## 4. CONJUGACY, GROUPOID ISOMORPHISM AND $C^*$-ALGEBRAS

We consider the groupoid $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ and its $C^*$-algebra written $\widetilde{\mathcal{R}}_A$. Recall that an action $\gamma^A$ of $\mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)E_A$ is defined by setting

$$\gamma^A_{(r,s)} = \alpha_r^{A^{\mathrm{t}}} \otimes \alpha_s^A \quad \text{on } \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \text{ for } (r,s) \in \mathbb{T}^2.$$

Since $\gamma^A_{(r,s)}(E_A) = E_A$, we have an action $\gamma^A$ of $\mathbb{T}^2$ on $\widetilde{\mathcal{R}}_A$, which defines two kinds of actions of $\mathbb{T}$ on $\widetilde{\mathcal{R}}_A$ such that

$$\delta_t^A = \gamma^A_{(t,t)} \quad \text{and} \quad \rho_t^A = \gamma^A_{(-t/2,t/2)} \quad \text{for } t \in \mathbb{T}.$$

We regard the groupoid $C^*$-algebra $C^*(G_A^{\mathrm{a}} \rtimes \mathbb{Z})$ as the crossed product $C^*$-algebra $C^*(G_A^{\mathrm{a}}) \rtimes \mathbb{Z}$ in a natural way. Let us denote by $\widehat{\sigma}^A$ the dual action on $C^*(G_A^{\mathrm{a}}) \rtimes \mathbb{Z}$. In the following lemma, the $C^*$-algebra $\widetilde{\mathcal{R}}_A$ is regarded as a $C^*$-subalgebra of $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ as in Proposition 2.1(ii).

LEMMA 4.1. *There exists an isomorphism $\Psi : C^*(G_A^{\mathrm{a}}) \rtimes \mathbb{Z} \to \mathcal{R}_A$ such that*

$$\Psi(C(\overline{X}_A)) = C(\overline{X}_A) \quad \textit{and} \quad \Psi \circ \widehat{\sigma}_t^A = \rho_t^A \circ \Psi, \quad t \in \mathbb{T}.$$

*Proof.* Let $U_A$ be the unitary in $\mathcal{R}_A$ defined by $U_A = \sum_{i=1}^N T_i^* \otimes S_i$. As in Proposition 9.9 of [18], $\mathrm{Ad}(U_A)$ corresponds to the shift operation on $C(\overline{X}_A)$. Since

$$\rho_t^A(U_A) = \sum_{i=1}^N \alpha_{-t/2}(T_i^*) \otimes \alpha_{t/2}(S_i) = \exp(\sqrt{-1}t) \sum_{i=1}^N T_i^* \otimes S_i = \exp(\sqrt{-1}t) U_A,$$

we have the assertion. ∎

Now we will give a proof of Theorem 1.1 stated in Introduction.

*Proof of Theorem* 1.1. The equivalence between (i) and (ii) is proved in Theorem 3.3. The equivalences among (ii), (iii) and (v) are shown in [18]. We will prove the three implications (i) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (v).

(i) $\Rightarrow$ (iv) Suppose that there exists a topological conjugacy $h : \overline{X}_A \to \overline{X}_B$ so that $h \circ \overline{\sigma}_A = \overline{\sigma}_B \circ h$. For $(x,p,q,z) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$, the conditions $(\overline{\sigma}_A^p(x), z) \in G_A^{\mathrm{s}}$ and $(\overline{\sigma}_A^q(x), z) \in G_A^{\mathrm{u}}$ imply $(\overline{\sigma}_B^p(h(x)), h(z)) \in G_B^{\mathrm{s}}$ and $(\overline{\sigma}_A^q(h(x)), h(z)) \in G_B^{\mathrm{u}}$, so that we have $(h(x), p, q, h(z)) \in G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$. It is routine to show that the correspondence

$$\widetilde{\varphi} : (x,p,q,z) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to (h(x), p, q, h(z)) \in G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$$

yields an isomorphism of étale groupoids. It is then clear that $c_B \circ \widetilde{\varphi} = c_A$. This shows the condition (iv).

(iv) $\Rightarrow$ (vi) Suppose that there exists an isomorphism $\widetilde{\varphi} : G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ of étale groupoids such that $c_B \circ \widetilde{\varphi} = c_A$. Since both groupoids $G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ and $G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ are amenable and étale by Proposition 7.2 and Lemma 7.3 of [18], the $C^*$-algebras $\widetilde{\mathcal{R}}_A$ and $\widetilde{\mathcal{R}}_B$ are represented on the Hilbert $C^*$-modules $\ell^2(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ and $\ell^2(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$, respectively as in [18]. As $\widetilde{\varphi} : G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$ is an isomorphism of étale groupoids, there exist a homeomorphism $h : \overline{X}_A \to \overline{X}_B$ and a continuous groupoid homomorphism $c : G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$\varphi(x,p,q,z) = (h(x), c(x,p,q,z), h(z)), \quad (x,p,q,z) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2.$$

The condition $c_B \circ \widetilde{\varphi} = c_A$ implies that $c(x,p,q,z) = (p,q)$ so that we have

$$\varphi(x,p,q,z) = (h(x), p, q, h(z)), \quad (x,p,q,z) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2.$$

Let us consider the unitaries $V_h : \ell^2(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2) \to \ell^2(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ and $V_{h^{-1}} : \ell^2(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2) \to \ell^2(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ defined by

$$[V_h\zeta](x,p,q,z) = \zeta(h(x),p,q,h(z)), \quad [V_{h^{-1}}\xi](y,m,n,w) = \xi(h^{-1}(y),m,n,h^{-1}(w))$$

for $\zeta \in \ell^2(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2), (x,p,q,z) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$, $\xi \in \ell^2(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2), (y,m,n,w) \in G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2$. Put $\widetilde{\Phi} = \mathrm{Ad}(V_h)$ which satisfies $\widetilde{\Phi}(C_c(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)) = C_c(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)$ so that $\widetilde{\Phi}(\widetilde{\mathcal{R}}_A) = \widetilde{\mathcal{R}}_B$. Since $\overline{X}_A, \overline{X}_B$ are identified with the unit spaces

$$(G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)^\circ = \{(x,0,0,x) \in G_A^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 : x \in \overline{X}_A\},$$
$$(G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2)^\circ = \{(y,0,0,y) \in G_B^{\mathrm{s,u}} \rtimes \mathbb{Z}^2 : y \in \overline{X}_B\},$$

respectively, it follows that $\widetilde{\Phi}(C(\overline{X}_A)) = C(\overline{X}_B)$. It is also direct to see that the identity $\widetilde{\Phi} \circ \gamma^A_{(r,s)} = \gamma^B_{(r,s)} \circ \widetilde{\Phi}$ for $(r,s) \in \mathbb{T}^2$ holds, because of the equality $c_B \circ \widetilde{\varphi} = c_A$.

(vi) $\Rightarrow$ (v) Suppose that there exists an isomorphism $\widetilde{\Phi} : \widetilde{\mathcal{R}}_A \to \widetilde{\mathcal{R}}_B$ of $C^*$-algebras such that $\widetilde{\Phi}(C(\overline{X}_A)) = C(\overline{X}_B)$ and $\widetilde{\Phi} \circ \gamma^A_{(r,s)} = \gamma^B_{(r,s)} \circ \widetilde{\Phi}$ for $(r,s) \in \mathbb{T}^2$. As the action $\delta^A_t = \gamma^A_{(t,t)}$ of $t \in \mathbb{T}$ act on $\widetilde{\mathcal{R}}_A$ and its fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$ is $\mathcal{R}_A$. Let us denote by $\Phi$ the restriction of $\widetilde{\Phi}$ to the fixed point algebra $\mathcal{R}_A$. It induces an isomorphism $\Phi : \mathcal{R}_A \to \mathcal{R}_B$. Then it is clear that the action $\rho^A_t = \gamma^A_{(-t/2,t/2)}$ on $\mathcal{R}_A$ satisfies $\Phi \circ \rho^A_t = \rho^B_t \circ \Phi$. This shows the condition (v). ∎

## 5. K-THEORETIC INVARIANTS

It follows from Theorem 1.1 that the isomorphism classes of the $C^*$-algebras $\mathcal{R}_A$ and $\widetilde{\mathcal{R}}_A$ are invariant under topological conjugacy of two-sided topological Markov shifts. Concerning the asymptotic Ruelle algebra $\mathcal{R}_A$, its K-group formula has been obtained by Putnam ([22], p. 129; cf. [11], [14]). We focus on studying the K-group $K_0(\widetilde{\mathcal{R}}_A)$ of the latter algebra $\widetilde{\mathcal{R}}_A$. Under the assumption that the matrix $A$ is irreducible and non-permutation, the algebra $\widetilde{\mathcal{R}}_A$ is a unital, simple, purely infinite, nuclear $C^*$-algebra satisfying UCT, so that its isomorphism class is completely determined by its K-theory date by a general classification theory of Kirchberg ([15]) and Phillips ([21]). Hence the following is a corollary of Theorem 1.1.

PROPOSITION 5.1. *The pair* $(K_0(\widetilde{\mathcal{R}}_A), [1_{\widetilde{\mathcal{R}}_A}])$ *of the* $K_0$*-group of* $\widetilde{\mathcal{R}}_A$ *and the position of the unit* $1_{\widetilde{\mathcal{R}}_A}$ *of* $\widetilde{\mathcal{R}}_A$ *in* $K_0(\widetilde{\mathcal{R}}_A)$ *is invariant under topological conjugacy of two-sided topological Markov shift* $(\overline{X}_A, \overline{\sigma}_A)$.

Recall that the projection $E_A$ is defined in (2.3). We have the following proposition.

PROPOSITION 5.2. *There exists an isomorphism* $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \to \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \otimes \mathcal{K}$ *such that the induced isomorphism* $\Phi_* : K_0(\widetilde{\mathcal{R}}_A) \to K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$ *satisfies* $\Phi_*([1_{\widetilde{\mathcal{R}}_A}]) = [E_A]$.

*Proof.* Since the $C^*$-algebra $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ is unital and simple, the projection $E_A$ in (2.3) is a full projection in $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$. Brown's theorem [4] tells us that there exists an isometry $v_A$ in the multiplier algebra $M(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \otimes \mathcal{K})$ of $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \otimes \mathcal{K}$ such that $v_A^* v_A = 1$ and $v_A v_A^* = E_A \otimes 1$. Define an isomorphism $\Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \to \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A \otimes \mathcal{K}$ by $\Phi = \mathrm{Ad}(v_A^*)$. Let $p_0$ be a rank one projection in $\mathcal{K}$. We then have

$$\Phi_*([1_{\widetilde{\mathcal{R}}_A}]) = \Phi_*([E_A \otimes p_0]) = [v_A^*(E_A \otimes p_0)v_A] = [E_A \otimes p_0] = [E_A]$$

in $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$. ∎

Hence the position $[E_A]$ in $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$ as well as the group $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$ are invariant under topological conjugacy of $(\overline{X}_A, \overline{\sigma}_A)$. By the Künneth formulas [32] of the K-groups of the tensor product $C^*$-algebras, we know that

$$\begin{aligned}
K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A) &\cong (K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)) \oplus (K_1(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_1(\mathcal{O}_A)),\\
K_1(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A) &\cong (K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_1(\mathcal{O}_A)) \oplus (K_1(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A))\\
&\qquad \oplus \mathrm{Tor}_1^{\mathbb{Z}}(K_0(\mathcal{O}_{A^{\mathrm{t}}}), K_0(\mathcal{O}_A)).
\end{aligned}$$

By the universal coefficient theorem for KK-groups, the K-group $K_i(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$ is isomorphic to the KK-group $KK^{i+1}(\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A)$ for $i = 0, 1$, so that

$$K_0(\widetilde{\mathcal{R}}_A) \cong KK^1(\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A), \quad K_1(\widetilde{\mathcal{R}}_A) \cong KK(\mathcal{O}_{A^{\mathrm{t}}}, \mathcal{O}_A).$$

Since $K_0(\mathcal{O}_{A^{\mathrm{t}}})$ is isomorphic to $K_0(\mathcal{O}_A)$ and $K_1(\mathcal{O}_A)$ is the torsion free part of $K_0(\mathcal{O}_A)$, the groups $K_i(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A), i = 0, 1$ do not have any further information than the group $K_0(\mathcal{O}_A)$ by the above Künneth formulas. As $K_0(\mathcal{O}_A) = \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$, it is a direct sum $\mathbb{Z}^n \oplus T_A$ of its torsion free part $\mathbb{Z}^n$ and its torsion part $T_A = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_k\mathbb{Z}$, where $m_i | m_{i+1}$ with $m_i \geqslant 2$, $i = 1, \dots, k-1$. It is easy to see that

$$
\begin{aligned}
&\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N \\
&\quad \cong \mathbb{Z}^{n^2} \oplus (T_A)^n \oplus (T_A)^n \oplus (T_A \otimes T_A) \\
&\quad \cong \mathbb{Z}^{n^2} \oplus (\mathbb{Z}/m_1\mathbb{Z})^{2n+2k-1} \oplus (\mathbb{Z}/m_2\mathbb{Z})^{2n+2k-3} \oplus \cdots \oplus (\mathbb{Z}/m_k\mathbb{Z})^{2n+1}.
\end{aligned}
$$

Hence the groups $K_i(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A), i = 0,1$ give us the same information as the group $K_0(\mathcal{O}_A)$.

The position $[E_A]$ in $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$ however gives us more information than the group $K_0(\mathcal{O}_A)$. In the above Künneth formula for $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$, the element $[E_A]$ lives in $K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)$ as the element $\sum_{i=1}^N [T_i^* T_i] \otimes [S_i S_i^*]$ by definition of $E_A$. Therefore the position $[E_A]$ of the projection $E_A$ in $K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)$ is invariant under topological conjugacy of $(\overline{X}_A, \overline{\sigma}_A)$. We set the vector $e_i = \left[0,\dots,0,\overset{i}{1},0,\dots,0\right]$ for $i = 1,\dots,N$. We summarize the above discussion as Theorem 1.2 in Introduction and give its proof.

*Proof of Theorem* 1.2. Let $A = [A(i,j)]_{i,j=1}^N, B = [B(i,j)]_{i,j=1}^M$ be irreducible non-permutation matrices such that $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are topologically conjugate. By William's theorem [38], the matrices $A, B$ are strong shift equivalent. Therefore there exist two rectangular nonnegative integer matrices $C, D$ such that $A = CD, B = DC$. By Theorem 4.6 of [17], there exists an isomorphism $\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ of $C^*$-algebras such that the diagram

$$
\begin{array}{ccc}
K_0(\mathcal{O}_A) & \xrightarrow{\Phi_*} & K_0(\mathcal{O}_B) \\
\epsilon_A \Big\downarrow & & \Big\downarrow \epsilon_B \\
\mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N & \xrightarrow{m_{C^{\mathrm{t}}}} & \mathbb{Z}^M/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^M
\end{array}
$$

commutes, where $m_{C^{\mathrm{t}}}$ is the isomorphism induced by multiplying the matrix $C^{\mathrm{t}}$ from the left and $\epsilon_A : K_0(\mathcal{O}_A) \to \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N$ is an isomorphism defined by $\epsilon_A([S_i S_i^*]) = [e_i]$ the class of the vector $e_i$ in $\mathbb{Z}^N$. Since the identities $A^{\mathrm{t}} = D^{\mathrm{t}} C^{\mathrm{t}}, B^{\mathrm{t}} = C^{\mathrm{t}} D^{\mathrm{t}}$ also hold, we similarly have an isomorphism $\Phi^{\mathrm{t}} : \mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{K} \to \mathcal{O}_{B^{\mathrm{t}}} \otimes \mathcal{K}$ of $C^*$-algebras such that the diagram

$$
\begin{array}{ccc}
K_0(\mathcal{O}_{A^{\mathrm{t}}}) & \xrightarrow{\Phi^{\mathrm{t}}_*} & K_0(\mathcal{O}_{B^{\mathrm{t}}}) \\
\epsilon_{A^{\mathrm{t}}} \Big\downarrow & & \Big\downarrow \epsilon_{B^{\mathrm{t}}} \\
\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N & \xrightarrow{m_D} & \mathbb{Z}^M/(\mathrm{id}-B)\mathbb{Z}^M
\end{array}
$$

commutes. We then have a commutative diagram:

$$
\begin{CD}
K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A) @>{\Phi_*^{\mathrm{t}} \otimes \Phi_*}>> K_0(\mathcal{O}_{B^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_B) \\
@V{\epsilon_{A^{\mathrm{t}}} \otimes \epsilon_A}VV @VV{\epsilon_{B^{\mathrm{t}}} \otimes \epsilon_B}V \\
\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N @>{m_D \otimes m_{C^{\mathrm{t}}}}>> \mathbb{Z}^M/(\mathrm{id}-B)\mathbb{Z}^M \otimes \mathbb{Z}^M/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^M.
\end{CD}
$$

We note that

$$
\begin{aligned}
\sum_{i=1}^N \epsilon_{A^{\mathrm{t}}}([T_i^* T_i]) \otimes \epsilon_A([S_i S_i^*]) &= \sum_{i=1}^N \epsilon_{A^{\mathrm{t}}}([T_i T_i^*]) \otimes \epsilon_A([S_i S_i^*]) \\
&= \sum_{i=1}^N [e_i] \otimes [e_i] \quad \text{in } \mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N,
\end{aligned}
$$

and set the specific element as

$$
e_A = \sum_{i=1}^N [e_i] \otimes [e_i] \quad \text{in } \mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N. \tag{5.1}
$$

We will show that $(m_D \otimes m_{C^{\mathrm{t}}})(e_A) = e_B$. In the computation below, the vectors $e_i$, and $f_j$ denote the $N \times 1$ matrix in $\mathbb{Z}^N$ whose $i$th component is one and zero elsewhere, and the $M \times 1$ matrix in $\mathbb{Z}^M$ whose $j$th component is one and zero elsewhere, respectively. We have

$$
\begin{aligned}
&\sum_{i=1}^N De_i \otimes C^{\mathrm{t}} e_i \\
&= \sum_{i=1}^N \begin{bmatrix} D(1,i) \\ D(2,i) \\ \vdots \\ D(M,i) \end{bmatrix} \otimes \begin{bmatrix} C(i,1) \\ C(i,2) \\ \vdots \\ C(i,M) \end{bmatrix} = \sum_{i=1}^N \begin{bmatrix} D(1,i) \\ D(2,i) \\ \vdots \\ D(M,i) \end{bmatrix} \otimes \sum_{j=1}^M C(i,j) f_j \\
&= \sum_{i=1}^N \sum_{j=1}^M \begin{bmatrix} D(1,i)C(i,j) \\ D(2,i)C(i,j) \\ \vdots \\ D(M,i)C(i,j) \end{bmatrix} \otimes f_j = \sum_{j=1}^M \begin{bmatrix} \sum_{i=1}^N D(1,i)C(i,j) \\ \sum_{i=1}^N D(2,i)C(i,j) \\ \vdots \\ \sum_{i=1}^N D(N,i)C(i,j) \end{bmatrix} \otimes f_j \\
&= \sum_{j=1}^M \begin{bmatrix} B(1,j) \\ B(2,j) \\ \vdots \\ B(N,j) \end{bmatrix} \otimes f_j = \sum_{j=1}^M Bf_j \otimes f_j.
\end{aligned}
$$

Hence we have

$$
\sum_{i=1}^N De_i \otimes C^{\mathrm{t}} e_i - \sum_{j=1}^M f_j \otimes f_j = \sum_{j=1}^M (B - \mathrm{id}) f_j \otimes f_j \tag{5.2}
$$

so that

$$(m_D \otimes m_{C^{\mathrm{t}}})(e_A) = \sum_{i=1}^{N} [De_i] \otimes [C^{\mathrm{t}} e_i] = \sum_{j=1}^{M} [f_j] \otimes [f_j] = e_B$$

thus proving Theorem 1.2. ∎

REMARK 5.3. (i) The pair $(\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is a complete invariant for the isomorphism class of the $C^*$-algebra $\widetilde{\mathcal{R}}_A$, because the group structure of $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N$ determines the groups $K_i(\mathcal{O}_A)$, $K_i(\mathcal{O}_{A^{\mathrm{t}}}), i = 0, 1$ and also the pair determines the position $[E_A]$ in $K_0(\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A)$. Hence by Proposition 5.2, the pair $(K_0(\widetilde{\mathcal{R}}_A), [E_A])$ and the group $K_1(\widetilde{\mathcal{R}}_A)$ are determined by the pair $(\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N, e_A)$.

(ii) Since the projection $E_A$ is regarded as an element of the $C^*$-algebra $\mathcal{F}_{A^{\mathrm{t}}} \otimes \mathcal{F}_A$ such that $C^*(G_A^{\mathrm{a}}) = E_A(\mathcal{F}_{A^{\mathrm{t}}} \otimes \mathcal{F}_A)E_A$, we have another topological conjugacy invariant $(K_0(\mathcal{F}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{F}_A), [E_A])$, the position $[E_A]$ in the group $K_0(\mathcal{F}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{F}_A)$. We discuss this kind of invariants in [19].

(iii) J. Cuntz in [7] studied the homotopy groups $\pi_n(\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K}))$ of the space of endomorphisms $\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K})$ of the $C^*$-algebra $\mathcal{O}_A \otimes \mathcal{K}$. He proved that natural maps $\epsilon_n : \pi_n(\mathrm{End}(\mathcal{O}_A \otimes \mathcal{K})) \to KK^n(\mathcal{O}_A, \mathcal{O}_A)$ yield isomorphisms, and defined an element denoted by $\epsilon_1(\lambda^A)$ in the group $\mathrm{Ext}(\mathcal{O}_A) \otimes K_0(\mathcal{O}_A)$, where $\lambda^A$ denotes the gauge action $\alpha^A$ on $\mathcal{O}_A$. By the Kaminker–Putnam's K-theoretic duality between $\mathrm{Ext}(\mathcal{O}_A)$ and $K_0(\mathcal{O}_{A^{\mathrm{t}}})$ ([12]), the element $\epsilon_1(\lambda^A)$ can be regarded as an element in $K_0(\mathcal{O}_{A^{\mathrm{t}}}) \otimes K_0(\mathcal{O}_A)$. Cuntz's observation in [7] shows that the element $\epsilon_1(\lambda^A)$ is nothing but the above element $e_A$ under the identification between $\mathrm{Ext}(\mathcal{O}_A) \otimes K_0(\mathcal{O}_A)$ and $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N$. He already states in [7] that the position $\epsilon_1(\lambda^A)$ in $\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N$ is invariant under topological conjugacy of the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$.

In [38], Williams introduced an equivalence relation of nonnegative integral square matrices called shift equivalence. It is weaker than strong shift equivalence. The shift equivalence relation has played a crucial rôle in the classification theory of symbolic dynamical systems (cf. [16]). Two matrices $A, B$ are said to be shift equivalent if there exist a positive integer $\ell$ and rectangular nonnegative integer matrices $R, S$ such that

$$AR = RB, \quad SA = BS, \quad A^{\ell} = RS, \quad B^{\ell} = SR. \tag{5.3}$$

We strengthen Theorem 1.2 in the following way.

PROPOSITION 5.4. *The pair $(\mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is invariant under shift equivalence.*

*Proof.* Suppose that matrices $A = [A(i,j)]_{i,j=1}^N, B = [B(i,j)]_{i,j=1}^M$ are shift equivalent. Let $\ell$ be a positive integer and $R, S$ rectangular nonnegative integer matrices satisfying (5.3). Then the map $m_S : \mathbb{Z}^N/(\mathrm{id}-A)\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id}-B)\mathbb{Z}^M$ defined by the left multiplication of the matrix $S$ yields an isomorphism of the abelian groups. We similarly see that $m_{R^{\mathrm{t}}} : \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id} - B^{\mathrm{t}})\mathbb{Z}^B$ defined by the left multiplication of the matrix $R^{\mathrm{t}}$ yields an isomorphism of the abelian groups. A similar computation proving the equality (5.2) in the proof of Theorem 1.2 shows that the equality

$$\sum_{i=1}^{N} Se_i \otimes R^{\mathrm{t}} e_i - \sum_{j=1}^{M} f_j \otimes f_j = \sum_{j=1}^{M} (SR - \mathrm{id}) f_j \otimes f_j = \sum_{j=1}^{M} (B^{\ell} - \mathrm{id}) f_j \otimes f_j$$

holds. As $B^{\ell} - \mathrm{id} = (B - \mathrm{id})(B^{\ell-1} + \cdots + B + \mathrm{id})$, we know $(m_S \otimes m_{R^{\mathrm{t}}})(e_A) = e_B$ so that the map

$$m_S \otimes m_{R^{\mathrm{t}}} : \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N \to \mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M \otimes \mathbb{Z}^M/(\mathrm{id} - B^{\mathrm{t}})\mathbb{Z}^M$$

gives rise to an isomorphism between $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ and $(\mathbb{Z}^M/(\mathrm{id} - B)\mathbb{Z}^M \otimes \mathbb{Z}^M/(\mathrm{id} - B^{\mathrm{t}})\mathbb{Z}^M, e_B)$. ∎

We will present an example showing that the invariant $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is strictly finer than the K-group $K_0(\mathcal{O}_A)$. We note that Enomoto–Fujii–Watatani in [9] listed a complete classification table of Cuntz–Krieger algebras $\mathcal{O}_A$ in terms of their K-groups for which the sizes of matrices are three.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Since $(\mathrm{id} - A) \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -m-n \\ -l-n \\ -l-m \end{bmatrix}$, the map

$$\varphi : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{Z}^3 \to [a+b+c] \in \mathbb{Z}/2\mathbb{Z}$$

induces an isomorphism $\overline{\varphi} : \mathbb{Z}^3/(\mathrm{id} - A)\mathbb{Z}^3 \to \mathbb{Z}/2\mathbb{Z}$. Hence we have an isomorphism

$$\widetilde{\varphi} := \overline{\varphi} \otimes \overline{\varphi} : \mathbb{Z}^3/(\mathrm{id} - A)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^3 \to \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

Since $\widetilde{\varphi}(e_i \otimes e_i) = \overline{\varphi}(e_i) \otimes \overline{\varphi}(e_i) = 1 \otimes 1$, we then have

$$\widetilde{\varphi}(e_A) = [1 \otimes 1] + [1 \otimes 1] + [1 \otimes 1] = [1] \quad \text{in } \mathbb{Z}/2\mathbb{Z}$$

so that

$$(\mathbb{Z}^3/(\mathrm{id} - A)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^3, e_A) \cong (\mathbb{Z}/2\mathbb{Z}, [1]).$$

On the other hand, let $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and hence $B^{\mathrm{t}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Since

$$(\mathrm{id} - B) \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -m-n \\ -l \\ -l-m+n \end{bmatrix}, \quad (\mathrm{id} - B^{\mathrm{t}}) \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} -m-n \\ -l-n \\ -l+n \end{bmatrix},$$

the maps

$$\psi : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{Z}^3 \to [a+b+c] \in \mathbb{Z}/2\mathbb{Z}, \quad \psi^{\mathrm{t}} : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{Z}^3 \to [b+c] \in \mathbb{Z}/2\mathbb{Z}$$

satisfy

$$\psi\Big((\mathrm{id}-B)\begin{bmatrix} l \\ m \\ n \end{bmatrix}\Big) = 2(-l-m), \quad \psi^{\mathrm{t}}\Big((\mathrm{id}-B^{\mathrm{t}})\begin{bmatrix} l \\ m \\ n \end{bmatrix}\Big) = -2l$$

so that they induce isomorphisms

$$\overline{\psi} : \mathbb{Z}^3/(\mathrm{id}-B)\mathbb{Z}^3 \to \mathbb{Z}/2\mathbb{Z}, \quad \overline{\psi}^{\mathrm{t}} : \mathbb{Z}^3/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^3 \to \mathbb{Z}/2\mathbb{Z}$$

and

$$\widetilde{\psi} := \overline{\psi} \otimes \overline{\psi}^{\mathrm{t}} : \mathbb{Z}^3/(\mathrm{id}-B)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^3 \to \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

Since

$$\widetilde{\psi}(e_i \otimes e_i) = \overline{\psi}(e_i) \otimes \overline{\psi}^{\mathrm{t}}(e_i) = \begin{cases} [1 \otimes 0] = [0] & \text{if } i = 1, \\ [1 \otimes 1] = [1] & \text{if } i = 2, 3, \end{cases}$$

we then have

$$\widetilde{\psi}(e_A) = [1 \otimes 0] + [1 \otimes 1] + [1 \otimes 1] = [0] \quad \text{in } \mathbb{Z}/2\mathbb{Z}$$

so that

$$(\mathbb{Z}^3/(\mathrm{id}-B)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^3, e_B) \cong (\mathbb{Z}/2\mathbb{Z}, [0]).$$

We thus have the following proposition.

PROPOSITION 5.5. *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. *Then* $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)(\cong \mathbb{Z}/2\mathbb{Z})$ *and* $\det(\mathrm{id}-A) = \det(\mathrm{id}-B)(= -2)$. *However*

$$(\mathbb{Z}^3/(\mathrm{id}-A)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id}-A^{\mathrm{t}})\mathbb{Z}^3, e_A) \cong (\mathbb{Z}/2\mathbb{Z}, [1]),$$
$$(\mathbb{Z}^3/(\mathrm{id}-B)\mathbb{Z}^3 \otimes \mathbb{Z}^3/(\mathrm{id}-B^{\mathrm{t}})\mathbb{Z}^3, e_B) \cong (\mathbb{Z}/2\mathbb{Z}, [0]).$$

In the rest of this section, we will deal with square matrices with nonnegative integers. Such matrices are called nonnegative integral matrices. A nonnegative integral matrix is said to be essential if none of its rows or columns is zero vector. Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ essential nonnegative integral matrix. The matrix defines a finite directed graph $G_A = (V_A, E_A)$ with $N$ vertices $V_A = \{v_1, \dots, v_N\}$ and $A(i,j)$ directed edges from the vertex $v_i$ to the vertex $v_j$ for $i,j = 1, \dots, N$. The directed edges are denoted by $\{a_1, \dots, a_{N_A}\} = E_A$. For an edge $a_k \in E_A$, denote by $s(a_k), t(a_k)$ its source vertex, terminal vertex, respectively. The directed graph $G_A$ has the $N_A \times N_A$ transition matrix $A^G = [A^G(i,j)]_{i,j=1}^{N_A}$ of edges defined by

$$A^G(i,j) = \begin{cases} 1 & \text{if } t(a_i) = s(a_j), \\ 0 & \text{otherwise,} \end{cases} \qquad i,j = 1,\dots,N_A.$$

As in Remark 2.16 of [8] and Section 4 of [30], the Cuntz–Krieger algebra $\mathcal{O}_A$ for the nonnegative integral matrix $A$ is defined to be the Cuntz–Krieger algebra $\mathcal{O}_{A^G}$ for the matrix $A^G$ with entries in $\{0,1\}$. It is well-known that there exist rectangular nonnegative integral matrices $R, S$ such that $A = RS, A^G = SR$ (cf. [16]). As in Lemma 4.5 of [17], the left multiplication of the matrix $S^{\mathrm{t}}$ induces an isomorphism $m_{S^{\mathrm{t}}} : \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^{\mathrm{t}})\mathbb{Z}^{N_A} \to \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ such that $m_{S^{\mathrm{t}}}([1_{N_A}]) = [1_N]$, where $1_{N_A} = [1,\dots,1] \in \mathbb{Z}^{N_A}, 1_N = [1,\dots,1] \in \mathbb{Z}^N$. Let $1_{\mathcal{O}_A}$ be the unit of the Cuntz–Krieger algebra $\mathcal{O}_A$. By Proposition 3.1 of [6], there exists an isomorphism from $K_0(\mathcal{O}_{A^G})$ to $\mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^{\mathrm{t}})\mathbb{Z}^{N_A}$ that sends the class $[1_{\mathcal{O}_A^G}]$ of $1_{\mathcal{O}_A^G}$ to the class $[1_{N_A}]$ of $1_{N_A}$. Hence for a nonnegative integral matrix $A$, there exists an isomorphism from $K_0(\mathcal{O}_A)$ to $\mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ that sends the class of the unit $[1_{\mathcal{O}_A}]$ of $\mathcal{O}_A$ to the class $[1_N]$ of $1_N$. We define the element $[e_A]$ in the group $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ by the same formula (5.1) as that for matrices with entries in $\{0,1\}$. We then have the following lemma.

LEMMA 5.6. *There exists an isomorphism $\Phi$ of groups from $\mathbb{Z}^{N_A}/(\mathrm{id} - A^G)\mathbb{Z}^{N_A} \otimes \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^{\mathrm{t}})\mathbb{Z}^{N_A}$ onto $\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ such that $\Phi(e_{A^G}) = e_A$.*

*Proof.* Let $R, S$ be rectangular nonnegative integral matrices $R, S$ satisfying $A = RS, A^G = SR$. As in the proof of Theorem 1.2, the isomorphism $m_R \otimes m_{S^{\mathrm{t}}} : \mathbb{Z}^{N_A}/(\mathrm{id} - A^G)\mathbb{Z}^{N_A} \otimes \mathbb{Z}^{N_A}/(\mathrm{id} - (A^G)^{\mathrm{t}})\mathbb{Z}^{N_A} \to \mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N$ satisfies $m_R \otimes m_{S^{\mathrm{t}}}(e_{A^G}) = e_A$. ∎

The following proposition can be proved in a similar way to how Proposition 5.4 was proved.

PROPOSITION 5.7. *Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ essential nonnegative integral matrix. The pair $(\mathbb{Z}^N/(\mathrm{id} - A)\mathbb{Z}^N \otimes \mathbb{Z}^N/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^N, e_A)$ is invariant under shift equivalence.*

We will present an example of nonnegative integral matrix $A$ such that the two $C^*$-algebras $\widetilde{\mathcal{R}}_A$ and $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ are not isomorphic.

Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$. Since $(\mathrm{id} - A)\begin{bmatrix} l \\ m \end{bmatrix} = \begin{bmatrix} -3l - m \\ -l + m \end{bmatrix}$, the map $\varphi : \begin{bmatrix} l \\ m \end{bmatrix} \in \mathbb{Z}^2 \to [l + m] \in \mathbb{Z}/4\mathbb{Z}$ induces isomorphisms $\overline{\varphi} : \mathbb{Z}^2/(\mathrm{id} - A)\mathbb{Z}^2 \to \mathbb{Z}/4\mathbb{Z}$ and

$$\widetilde{\varphi} := \overline{\varphi} \otimes \overline{\varphi} : \mathbb{Z}^2/(\mathrm{id} - A)\mathbb{Z}^2 \otimes \mathbb{Z}^2/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^2 \to \mathbb{Z}/4\mathbb{Z} \otimes \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z}.$$

Since $\widetilde{\varphi}(e_i \otimes e_i) = \overline{\varphi}(e_i) \otimes \overline{\varphi}(e_i) = 1 \otimes 1$, we have

$$\widetilde{\varphi}(e_A) = [1 \otimes 1] + [1 \otimes 1] = [2] \quad \text{in } \mathbb{Z}/4\mathbb{Z}.$$

On the other hand, we have $\widetilde{\varphi}([1_2] \otimes [1_2]) = \varphi\Big( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big) \otimes \varphi^{\mathrm{t}}\Big( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big) = [2 \otimes 2] = [0]$ in $\mathbb{Z}/4\mathbb{Z}$. We thus have

$$(\mathbb{Z}^2/(\mathrm{id} - A)\mathbb{Z}^2 \otimes \mathbb{Z}^2/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^2, e_A) \cong (\mathbb{Z}/4\mathbb{Z}, [2]),$$
$$(\mathbb{Z}^2/(\mathrm{id} - A)\mathbb{Z}^2 \otimes \mathbb{Z}^2/(\mathrm{id} - A^{\mathrm{t}})\mathbb{Z}^2, [1_2] \otimes [1_2]) \cong (\mathbb{Z}/4\mathbb{Z}, [0]),$$

so that the algebras $\widetilde{\mathcal{R}}_A$ and $\mathcal{O}_{A^{\mathrm{t}}} \otimes \mathcal{O}_A$ are not isomorphic by the classification theorem of unital, purely infinite, simple nuclear $C^*$-algebras ([15], [21]).

## 6. KMS STATES ON $\widetilde{\mathcal{R}}_A$

In this section, we will study KMS states on the $C^*$-algebra $\widetilde{\mathcal{R}}_A$ for the diagonal action $\delta^A$. Following after [3], we will define KMS states in the following way. For a one-parameter automorphism group $\alpha_t, t \in \mathbb{R}$ on a $C^*$-algebra $\mathcal{A}$ and a real number $\gamma \in \mathbb{R}$, a state $\psi$ on $\mathcal{A}$ is called a KMS state for the action $\alpha$ if $\psi$ satisfies

$$\psi(X\alpha_{i\gamma}(Y)) = \psi(YX) \tag{6.1}$$

for all $X, Y$ in a norm dense $\alpha$-invariant $*$-subalgebra of the set of entire analytic elements for $\alpha$ in $\mathcal{A}$. The value $\gamma$ is called the inverse temperature and the condition (6.1) is called the KMS condition.

Let $\beta$ be the Perron–Frobenius eigenvalue for an irreducible matrix $A$ with entries in $\{0, 1\}$. It has been shown in [10] that KMS states for gauge action on Cuntz–Krieger algebra $\mathcal{O}_A$ exist if and only if its inverse temperature is $\log \beta$, and the admitted KMS state is unique. Let us denote by $\varphi$ the unique KMS state for gauge action on $\mathcal{O}_A$. Similarly we denote by $\varphi^{\mathrm{t}}$ the unique KMS state for gauge action on $\mathcal{O}_{A^{\mathrm{t}}}$. As in [10], the vector $\begin{bmatrix} \varphi(S_1 S_1^*) \\ \vdots \\ \varphi(S_N S_N^*) \end{bmatrix}$ gives rise to the unique normalized positive eigenvector of $A$ for the eigenvalue $\beta$. Hence we have

$$\beta \begin{bmatrix} \varphi(S_1 S_1^*) \\ \vdots \\ \varphi(S_N S_N^*) \end{bmatrix} = \begin{bmatrix} A(1,1) & \cdots & A(1,N) \\ \vdots & & \vdots \\ A(N,1) & \cdots & A(N,N) \end{bmatrix} \begin{bmatrix} \varphi(S_1 S_1^*) \\ \vdots \\ \varphi(S_N S_N^*) \end{bmatrix} = \begin{bmatrix} \varphi(S_1^* S_1) \\ \vdots \\ \varphi(S_N^* S_N) \end{bmatrix}$$

so that $\beta\varphi(S_i S_i^*) = \varphi(S_i^* S_i), i = 1, \ldots, N$ and more generally

$$\beta^m \varphi(S_{\mu_1 \cdots \mu_m} S_{\mu_1 \cdots \mu_m}^*) = \varphi(S_{\mu_1 \cdots \mu_m}^* S_{\mu_1 \cdots \mu_m}), \quad (\mu_1, \ldots, \mu_m) \in B_m(\overline{X}_A).$$

Therefore we have
(6.2)
$$\varphi(S_{\mu_m}S_{\mu_m}^*) = \frac{1}{\beta}\varphi(S_{\mu_m}^*S_{\mu_m}) = \frac{1}{\beta}\varphi(S_{\mu_1\cdots\mu_m}^*S_{\mu_1\cdots\mu_m}) = \beta^{m-1}\varphi(S_{\mu_1\cdots\mu_m}S_{\mu_1\cdots\mu_m}^*)$$

and similarly

$$\varphi^{\mathrm{t}}(T_{\xi_1}T_{\xi_1}^*) = \beta^{k-1}\varphi^{\mathrm{t}}(T_{\xi_k\cdots\xi_1}T_{\xi_k\cdots\xi_1}^*), \quad (\xi_k,\dots,\xi_1) \in B_k(\overline{X}_{A^{\mathrm{t}}}). \tag{6.3}$$

Let $[a_i]_{i=1}^N$ and $[b_i]_{i=1}^N$ be the positive eigenvectors of $A$ and $A^{\mathrm{t}}$ for the eigenvalue $\beta$, respectively satisfying

$$\sum_{i=1}^N a_i b_i = 1.$$

For admissible words $\xi = (\xi_1,\dots,\xi_k) \in B_k(\overline{X}_A)$ and $\nu = (\nu_1,\dots,\nu_n) \in B_n(\overline{X}_A)$, put $\xi\nu = (\xi_1,\dots,\xi_k,\nu_1,\dots,\nu_n) \in B_{k+n}(\overline{X}_A)$. For $i \in \mathbb{Z}$, denote by $U_{[\xi\nu]_i^{i+k+n-1}}$ the cylinder set of $\overline{X}_A$ such that

$$U_{[\xi\nu]_i^{i+k+n-1}} = \{(x_j)_{j\in\mathbb{Z}} \in \overline{X}_A : x_i = \xi_1,\dots,x_{i+k-1} = \xi_k, x_{i+k} = \nu_1,\dots,x_{i+k+n-1} = \nu_n\}.$$

In [20], W. Parry proved that there exists a unique invariant measure $\mu$ on $\overline{X}_A$ of maximal entropy. It is called the Parry measure, which satisfies the following equality:

$$\mu(U_{[\xi\nu]_i^{i+k+n-1}}) = b_{\xi_1}a_{\nu_n}\beta^{-(k+n-1)}, \quad i \in \mathbb{Z}. \tag{6.4}$$

Let $C^*(G_A^{\mathrm{a}})$ be the groupoid $C^*$-algebra for the groupoid $G_A^{\mathrm{a}}$. As in Putnam's paper [22] and his lecture notes [23], the algebra is an AF-algebra with a tracial state Tr defined by

$$\mathrm{Tr}(f) = \int_{\overline{X}_A} f(x,x)\,\mathrm{d}\mu(x) \quad \text{for } f \in C_{\mathrm{c}}(G_A).$$

Let us define a state $\widetilde{\varphi}$ on $\widetilde{\mathcal{R}}_A$ by setting:

$$\widetilde{\varphi} = \frac{1}{\sum_{j=1}^N \varphi^{\mathrm{t}}(T_jT_j^*)\varphi(S_j^*S_j)}\varphi^{\mathrm{t}}\otimes\varphi \quad \text{on } \widetilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A^{\mathrm{t}}}\otimes\mathcal{O}_A)E_A.$$

Since $(\varphi^{\mathrm{t}}\otimes\varphi)(E_A) = \sum_{j=1}^N \varphi^{\mathrm{t}}(T_jT_j^*)\varphi(S_j^*S_j)$, we know that $\widetilde{\varphi}$ gives rise to a state on $\widetilde{\mathcal{R}}_A$. We know more about $\widetilde{\varphi}$ in the following way.

PROPOSITION 6.1. (i) *The state $\widetilde{\varphi}$ is a* KMS *state on $\widetilde{\mathcal{R}}_A$ for the diagonal action $\delta^A$ at the inverse temperature* $\log\beta$.

(ii) *The restriction of $\widetilde{\varphi}$ to the subalgebra $C(\overline{X}_A)$ coincides with the Parry measure $\mu$ on $\overline{X}_A$.*

(iii) *The formula*

$$\widetilde{\varphi}(Y) = \mathrm{Tr}\Big( \iint_{\mathbb{T}^2} \gamma^A_{r,s}(Y) \mathrm{d}r\mathrm{d}s \Big) \quad \textit{for } Y \in \widetilde{\mathcal{R}}_A \tag{6.5}$$

*holds.*

*Proof.* (i) For

$\mu = (\mu_1, \dots, \mu_m), \nu = (\nu_1, \dots, \nu_n),\ \mu' = (\mu'_1, \dots, \mu'_{m'}), \nu' = (\nu'_1, \dots, \nu'_{n'}) \in B_*(\overline{X}_A)$,
$\overline{\xi} = (\xi_k, \dots, \xi_1), \overline{\eta} = (\eta_l, \dots, \eta_1),\ \overline{\xi}' = (\xi'_{k'}, \dots, \xi'_1), \overline{\eta}' = (\eta'_{l'}, \dots, \eta'_1) \in B_*(\overline{X}_{A^{\mathrm{t}}})$
with $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = A(\xi'_{k'}, \mu'_1) = A(\eta'_{l'}, \nu'_1) = 1$, put

$$x = T_{\overline{\xi}} T^*_{\overline{\eta}} \otimes S_\mu S^*_\nu, \quad x' = T_{\overline{\xi}'} T^*_{\overline{\eta}'} \otimes S_{\mu'} S^*_{\nu'} \in \widetilde{\mathcal{R}}_A.$$

It then follows that

$$\begin{aligned}
&(\varphi^{\mathrm{t}} \otimes \varphi)(E_A) \cdot \widetilde{\varphi}(x' \delta^A_{i\log\beta}(x)) \\
&\quad = (\varphi^{\mathrm{t}} \otimes \varphi)((T_{\overline{\xi}'} T^*_{\overline{\eta}'} \otimes S_{\mu'} S^*_{\nu'})(\alpha^{A^{\mathrm{t}}}_{i\log\beta}(T_{\overline{\xi}} T^*_{\overline{\eta}}) \otimes \alpha^A_{i\log\beta}(S_\mu S^*_\nu))) \\
&\quad = \varphi^{\mathrm{t}}(T_{\overline{\xi}'} T^*_{\overline{\eta}'} \alpha^{A^{\mathrm{t}}}_{i\log\beta}(T_{\overline{\xi}} T^*_{\overline{\eta}})) \varphi(S_{\mu'} S^*_{\nu'} \alpha^A_{i\log\beta}(S_\mu S^*_\nu)) \\
&\quad = \varphi^{\mathrm{t}}(T_{\overline{\xi}} T^*_{\overline{\eta}} T_{\overline{\xi}'} T^*_{\overline{\eta}'}) \varphi(S_\mu S^*_\nu S_{\mu'} S^*_{\nu'}) \\
&\quad = (\varphi^{\mathrm{t}} \otimes \varphi)((T_{\overline{\xi}} T^*_{\overline{\eta}} \otimes S_\mu S^*_\nu)(T_{\overline{\xi}'} T^*_{\overline{\eta}'} \otimes S_{\mu'} S^*_{\nu'})) \\
&\quad = (\varphi^{\mathrm{t}} \otimes \varphi)(x x') = (\varphi^{\mathrm{t}} \otimes \varphi)(E_A) \cdot \widetilde{\varphi}(x x'),
\end{aligned}$$

thus proving that $\widetilde{\varphi}$ is a KMS state on $\widetilde{\mathcal{R}}_A$ for the diagonal action $\delta^A$ at the inverse temperature $\log\beta$.

(ii) Put

$$\overline{a}_i = \frac{a_i}{\sum_{i=1}^N a_i} = \varphi(S_i S_i^*), \quad \overline{b}_i = \frac{b_i}{\sum_{i=1}^N b_i} = \varphi^{\mathrm{t}}(T_i T_i^*)$$

so that

$$\sum_{i=1}^N \varphi(S_i S_i^*) \varphi^{\mathrm{t}}(T_i T_i^*) = \frac{1}{\Big(\sum_{i=1}^N a_i\Big) \cdot \Big(\sum_{i=1}^N b_i\Big)}.$$

It then follows that

$$\begin{aligned}
\mu(U_{[\xi\nu]_i^{i+k+n-1}}) &= \overline{b}_{\xi_1} \cdot \Big(\sum_{i=1}^N b_i\Big) \cdot \overline{a}_{\nu_n} \Big(\sum_{i=1}^N a_i\Big) \cdot \beta^{-(k+n-1)} \\
&= \varphi^{\mathrm{t}}(T_{\xi_1} T^*_{\xi_1}) \Big(\sum_{i=1}^N b_i\Big) \cdot \varphi(S_{\nu_n} S^*_{\nu_n}) \Big(\sum_{i=1}^N a_i\Big) \cdot \beta^{-(k+n-1)} \\
&= \frac{1}{\sum_{i=1}^N \varphi^{\mathrm{t}}(T_i T_i^*) \varphi(S_i S_i^*)} \cdot \varphi^{\mathrm{t}}(T_{\xi_1} T^*_{\xi_1}) \varphi(S_{\nu_n} S^*_{\nu_n}) \cdot \beta^{-(k+n-1)}.
\end{aligned}$$

By using (6.2) and (6.3) we thus have

$$\begin{aligned}
\mu(U_{[\xi\nu]_i^{i+k+n-1}}) &= \frac{1}{\sum_{i=1}^N \varphi^{\mathrm{t}}(T_iT_i^*)\varphi(S_iS_i^*)}\cdot\frac{1}{\beta}\cdot\varphi^{\mathrm{t}}(T_{\xi_k\cdots\xi_1}T^*_{\xi_k\cdots\xi_1})\varphi(S_{\nu_1\cdots\nu_n}S^*_{\nu_1\cdots\nu_n})\\
&= \frac{1}{\sum_{i=1}^N \varphi^{\mathrm{t}}(T_iT_i^*)\varphi(S_i^*S_i)}\cdot\varphi^{\mathrm{t}}(T_{\overline{\xi}}T^*_{\overline{\xi}})\varphi(S_\nu S_\nu^*)\\
&= \frac{1}{(\varphi^{\mathrm{t}}\otimes\varphi)(E_A)}\cdot\varphi^{\mathrm{t}}(T_{\overline{\xi}}T^*_{\overline{\xi}})\varphi(S_\nu S_\nu^*) = \widetilde{\varphi}(T_{\overline{\xi}}T^*_{\overline{\xi}}\otimes S_\nu S_\nu^*).
\end{aligned}$$

(iii) For

$$\begin{aligned}
&\mu=(\mu_1,\dots,\mu_m),\ \nu=(\nu_1,\dots,\nu_n)\in B_*(\overline{X}_A),\\
&\overline{\xi}=(\xi_k,\dots,\xi_1),\ \overline{\eta}=(\eta_l,\dots,\eta_1)\in B_*(\overline{X}_{A^{\mathrm{t}}}),
\end{aligned}$$

satisfying $A(\xi_k,\mu_1)=A(\eta_l,\nu_1)=1$, it is direct to see the following equalities:

$$\begin{aligned}
\widetilde{\varphi}(T_{\overline{\xi}}T^*_{\overline{\eta}}\otimes S_\mu S_\nu^*) &= \varphi^{\mathrm{t}}(T_{\overline{\xi}}T^*_{\overline{\eta}})\varphi(S_\mu S_\nu^*)\\
&= \begin{cases}\varphi^{\mathrm{t}}(T_{\overline{\xi}}T^*_{\overline{\xi}})\varphi(S_\nu S_\nu^*) & \text{if } \overline{\xi}=\overline{\eta}, \mu=\nu,\\ 0 & \text{otherwise,}\end{cases}\\
&= \begin{cases}\mu(U_{[\xi\nu]_i^{i+k+n-1}}) & \text{if } \overline{\xi}=\overline{\eta}, \mu=\nu,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Since the above value coincides with

$$\mathrm{Tr}\Big(\iint_{\mathbb{T}^2}\gamma^A_{r,s}(T_{\overline{\xi}}T^*_{\overline{\eta}}\otimes S_\mu S_\nu^*)\mathrm{d}r\mathrm{d}s\Big),$$

the formula (6.5) holds. ∎

We finally prove that a KMS state on $\widetilde{\mathcal{R}}_A$ for the diagonal action $\delta^A$ exists only at the inverse temperature $\log\beta$. We will further show that the admitted KMS state is unique. In order to avoid non essential difficulty, we assume that the irreducible matrix $A$ with entries in $\{0,1\}$ is aperiodic so that there exists $n_0\in\mathbb{N}$ such that $A^{n_0}(i,j)\geqslant 1$ for all $i,j=1,\dots,N$. In case when $A$ is not aperiodic, but irreducible with period $p$ and non-permutation, it is not difficult to see that the equation (6.12) holds for $n=pm, m\geqslant m_0$ for some $m_0\in\mathbb{N}$ and $i,j$ in the same period class (cf. [16]), so that Theorem 1.3 holds. If $A$ is not irreducible, the unicity of a KMS state is broken, so that Theorem 1.3 does not hold any longer.

Let $\psi$ be a KMS state on $\widetilde{\mathcal{R}}_A$ for the diagonal action $\delta^A$ at the inverse temperature $\log\gamma$ for $1<\gamma\in\mathbb{R}$. We will prove that $\gamma=\beta$, the Perron–Frobenius eigenvalue of $A$ and $\psi=\widetilde{\varphi}$.

For $i,j\in\{1,\dots,N\}$ and words $\mu=(\mu_1,\dots,\mu_m)$, $\nu=(\nu_1,\dots,\nu_n)$ such that $(i,\mu_1,\dots,\mu_m,j)\in B_{m+2}(\overline{X}_A)$, $(i,\nu_1,\dots,\nu_n,j)\in B_{n+2}(\overline{X}_A)$, we define a partial isometry

$$V_{\nu,\mu}(i,j)=T_i^*T_{\nu_1}^*\cdots T_{\nu_n}^*T_j^*\otimes S_iS_{\mu_1}\cdots S_{\mu_m}S_j. \tag{6.6}$$

Since $T_i^* \otimes S_i, T_j^* \otimes S_j \in \widetilde{\mathcal{R}}_A$, we know that $V_{\nu,\mu}(i,j)$ belongs to $\widetilde{\mathcal{R}}_A$. We then have the identities

$$\begin{aligned} & V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^* \\ & \qquad = T_i^* T_{\nu_1}^* \cdots T_{\nu_n}^* T_j^* T_j T_{\nu_n} \cdots T_{\nu_1} T_i \otimes S_i S_{\mu_1} \cdots S_{\mu_m} S_j S_j^* S_{\mu_m}^* \cdots S_{\mu_1}^* S_i^* \\ & \qquad = T_i^* T_i \otimes S_i S_{\mu_1} \cdots S_{\mu_m} S_j S_j^* S_{\mu_m}^* \cdots S_{\mu_1}^* S_i^* \quad \text{and} \\ & V_{\nu,\mu}(i,j)^* V_{\nu,\mu}(i,j) \\ & \qquad = T_j T_{\nu_n} \cdots T_{\nu_1} T_i T_i^* T_{\nu_1}^* \cdots T_{\nu_n}^* T_j^* \otimes S_j^* S_{\mu_m}^* \cdots S_{\mu_1}^* S_i^* S_i S_{\mu_1} \cdots S_{\mu_m} S_j \\ & \qquad = T_j T_{\nu_n} \cdots T_{\nu_1} T_i T_i^* T_{\nu_1}^* \cdots T_{\nu_n}^* T_j^* \otimes S_j^* S_j. \end{aligned}$$

For $p \in \mathbb{Z}$, denote by $\widetilde{\mathcal{R}}_A^{\delta^A}(p)$ the $p$th spectral subspace of $\widetilde{\mathcal{R}}_A$ for the action $\delta^A$.

LEMMA 6.2. *Suppose that $X \in \widetilde{\mathcal{R}}_A$ belongs to $\widetilde{\mathcal{R}}_A^{\delta^A}(p)$ for some $p \neq 0$. Then we have $\psi(X) = 0$*

*Proof.* We may assume $p > 0$. For $i,j = 1,\dots,N$, let $\mu = (\mu_1,\dots,\mu_{n_0+p})$ be an admissible word such that $(i,\mu_1,\dots,\mu_{n_0+p},j) \in B_{n_0+p+2}(\overline{X}_A)$. Take $\nu = (\nu_1,\dots,\nu_{n_0})$ with $(i,\nu_1,\dots,\nu_{n_0},j) \in B_{n_0}(\overline{X}_A)$ and consider the partial isometry

$$V_{\nu,\mu}(i,j) = T_i^* T_{\nu_1}^* \cdots T_{\nu_{n_0}}^* T_j^* \otimes S_i S_{\mu_1} \cdots S_{\mu_{n_0+p}} S_j.$$

The partial isometry $V_{\nu,\mu}(i,j)$ belongs to $\widetilde{\mathcal{R}}_A^{\delta^A}(p)$ and satisfies

$$V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^* = T_i^* T_i \otimes S_i S_{\mu_1} \cdots S_{\mu_{n_0+p}} S_j S_j^* S_{\mu_{n_0+p}}^* \cdots S_{\mu_1}^* S_i^*.$$

We then have

$$E_A = \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^*.$$

It then follows that

$$\begin{aligned} \psi(X) = \psi(E_A X) &= \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} \psi(V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^* X) \\ &= \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} \psi(V_{\nu,\mu}(i,j)^* X \delta^A_{i\log\gamma}(V_{\nu,\mu}(i,j))) \\ &= \frac{1}{\gamma^p} \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} \psi(V_{\nu,\mu}(i,j)^* X V_{\nu,\mu}(i,j)) \\ &= \frac{1}{\gamma^p} \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} \psi(V_{\nu,\mu}(i,j) \delta^A_{i\log\gamma}(V_{\nu,\mu}(i,j)^* X)) \\ &= \frac{1}{\gamma^p} \sum_{i,j=1}^N \sum_{\mu \in B_{n_0+p}(\overline{X}_A)} \psi(V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^* X) = \frac{1}{\gamma^p}\psi(E_A X). \end{aligned}$$

Since $\gamma > 1$, we have $\psi(X) = 0$. ∎

Since $\mathcal{R}_A$ is the fixed point algebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$ of $\widetilde{\mathcal{R}}_A$ under $\delta^A$, we may define a conditional expectation $\mathcal{E}_A : \widetilde{\mathcal{R}}_A \to \mathcal{R}_A$ by

$$\mathcal{E}_A(X) = \int_{\mathbb{T}} \delta_t^A(X)\mathrm{d}t, \quad X \in \widetilde{\mathcal{R}}_A. \tag{6.7}$$

The preceding lemma implies the following lemma.

LEMMA 6.3. *Let $\psi_0$ be the restriction of $\psi$ to the subalgebra $(\widetilde{\mathcal{R}}_A)^{\delta^A}$. Then $\psi_0$ is a tracial state on $\mathcal{R}_A$ such that $\psi = \psi_0 \circ \mathcal{E}_A$.*

Hence the value of KMS state is determined on the subalgebra $\mathcal{R}_A$. Recall that $U_A$ denotes the unitary $U_A = \sum_{i=1}^{N} T_i^* \otimes S_i$ which belongs to $\mathcal{R}_A$.

LEMMA 6.4. $\psi(U_A X U_A^*) = \psi(X)$ *for all* $X \in \widetilde{\mathcal{R}}_A$.

*Proof.* Since $U_A$ is fixed under the action $\delta^A$, we have

$$\psi(X) = \psi(U_A^* U_A X) = \psi(U_A X \delta^A_{i\log\gamma}(U_A^*)) = \psi(U_A X U_A^*). \quad ∎$$

As in Proposition 9.9 of [18], the automorphism $\mathrm{Ad}(U_A)$ behaves like the shift on $\widetilde{\mathcal{R}}_A$. Lemma 6.4 tells us that the KMS state is invariant nuder the shift. The following lemma is crucial in our discussions.

LEMMA 6.5. *Let $X = T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \in \mathcal{R}_A$ where*

$$\mu = (\mu_1, \dots, \mu_m), \nu = (\nu_1, \dots, \nu_n) \in B_*(\overline{X}_A),$$
$$\overline{\xi} = (\xi_k, \dots, \xi_1), \overline{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\overline{X}_{A^t}).$$

*Suppose that $\psi(X) \neq 0$. Then we have $k = l, m = n$ and $\mu = \nu, \overline{\xi} = \overline{\eta}$.*

*Proof.* Since $X$ belongs to $\mathcal{R}_A$, we have $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$ and $k - l = n - m$. We may assume that $k \geqslant l$ and hence $n \geqslant m$. It then follows that

$$\begin{aligned}
\psi(X) &= \psi((T_{\xi_k} \cdots T_{\xi_1} T_{\eta_1}^* \cdots T_{\eta_l}^* \otimes S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^*) \cdot (T_{\eta_l} T_{\eta_l}^* \otimes S_{\nu_1} S_{\nu_1}^*)) \\
&= \psi((T_{\eta_l} T_{\eta_l}^* \otimes S_{\nu_1} S_{\nu_1}^*) \cdot \delta^A_{i\log\gamma}(T_{\xi_k} \cdots T_{\xi_1} T_{\eta_1}^* \cdots T_{\eta_l}^* \otimes S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^*)) \\
&= \frac{1}{\gamma^{k+m-l-n}} \psi((T_{\eta_l} T_{\eta_l}^* \otimes S_{\nu_1} S_{\nu_1}^*) \cdot (T_{\xi_k} \cdots T_{\xi_1} T_{\eta_1}^* \cdots T_{\eta_l}^* \otimes S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^*)) \\
&= \psi(T_{\eta_l} T_{\eta_l}^* \cdot T_{\xi_k} \cdots T_{\xi_1} T_{\eta_1}^* \cdots T_{\eta_l}^* \otimes S_{\nu_1} S_{\nu_1}^* \cdot S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^*).
\end{aligned}$$

By the assumption $\psi(X) \neq 0$, we get $\eta_l = \xi_k$ and $\nu_1 = \mu_1$, and we have

$$\psi(X) = \psi(T_{\eta_l} T_{\xi_{k-1}} \cdots T_{\xi_1} T_{\eta_1}^* \cdots T_{\eta_l}^* \otimes S_{\mu_1} \cdots S_{\mu_m} S_{\nu_n}^* \cdots S_{\nu_1}^*).$$

As

$$\begin{aligned}
&U_A \cdot T_{\eta_l} T_{\xi_{k-1}} \cdots T_{\xi_1} T^*_{\eta_1} \cdots T^*_{\eta_l} \otimes S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} \cdot U^*_A \\
&\qquad = T_{\xi_{k-1}} \cdots T_{\xi_1} T^*_{\eta_1} \cdots T^*_{\eta_{l-1}} \otimes S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l},
\end{aligned}$$

Lemma 6.4 shows us the equality

$$\psi(X) = \psi(T_{\xi_{k-1}} \cdots T_{\xi_1} T^*_{\eta_1} \cdots T^*_{\eta_{l-1}} \otimes S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l}). \tag{6.8}$$

We apply the same argument above to the right hand side of (6.8), and continue these procedures so that we finally get

$$\eta_{l-1} = \xi_{k-1}, \eta_{l-2} = \xi_{k-2}, \ldots, \eta_1 = \xi_{k-l+1}$$

and the identity

$$\psi(X) = \psi(T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} T_{\eta_1} \otimes S_{\eta_1} S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2} S^*_{\eta_1}).$$

As $\xi_{k-l+1} = \eta_1$, we see that $A(\xi_{k-l}, \eta_1) = 1$ and hence $T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_1}$ belongs to the algebra $\widetilde{\mathcal{R}}_A$ such that

$$\delta^A_{i \log \gamma}(T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_1}) = \frac{1}{\gamma^{k-l}} T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_1}.$$

Hence we have

$$\begin{aligned}
\psi(X) &= \psi((T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_1}) \cdot (T_{\eta_1} \otimes S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2} S^*_{\eta_1})) \\
&= \psi((T_{\eta_1} \otimes S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2} S^*_{\eta_1}) \\
&\qquad \cdot \delta^A_{i \log \gamma}(T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_1})) \\
&= \frac{1}{\gamma^{k-l}} \psi(T_{\eta_1} T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2} S^*_{\eta_1} S_{\eta_1}).
\end{aligned}$$

Since $S^*_{\eta_2} S^*_{\eta_1} S_{\eta_1} = S^*_{\eta_2}$, we have

$$\begin{aligned}
\psi(X) &= \frac{1}{\gamma^{k-l}} \psi(T_{\eta_1} T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \otimes S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2}) \\
&= \frac{1}{\gamma^{k-l}} \psi(U^{*\,l-1}_A (T_{\eta_1} T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} \\
&\qquad \otimes S_{\eta_2} \cdots S_{\eta_l} S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1} S^*_{\eta_l} \cdots S^*_{\eta_2} U^{l-1}_A)) \\
&= \frac{1}{\gamma^{k-l}} \psi(T_{\eta_l} \cdots T_{\eta_2} T_{\eta_1} T_{\xi_{k-l}} \cdots T_{\xi_1} T^*_{\eta_1} T^*_{\eta_2} \cdots T^*_{\eta_l} \otimes S_{\mu_1} \cdots S_{\mu_m} S^*_{\nu_n} \cdots S^*_{\nu_1}).
\end{aligned}$$

As $(\eta_l, \ldots, \eta_1) = (\xi_k, \ldots, \xi_{k-l+1})$, we finally obtain that

$$\psi(X) = \frac{1}{\gamma^{k-l}} \psi(X)$$

so that $k = l$ and hence $\eta = \xi$. We similarly see that $\mu = \nu$. ∎

Since any element $X$ of $\mathcal{R}_A$ is approximated by finite linear combinations of elements of the form $T_{\overline{\xi}}T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \in \mathcal{R}_A$, we have the following proposition by using Lemma 6.3.

PROPOSITION 6.6. *If an element $X \in \widetilde{\mathcal{R}}_A$ satisfies $\psi(X) \neq 0$, then $X$ belongs to $C(\overline{X}_A)$.*

We will next show that the restriction of the KMS state $\psi$ to the commutative subalgebra $C(\overline{X}_A)$ coincides with the state defined by the Parry measure on $\overline{X}_A$.

Recall the partial isometry $V_{\nu,\mu}(i,j)$ defined by (6.6) for $i,j \in \{1,\dots,N\}$ and $\mu = (\mu_1,\dots,\mu_m)$, $\nu = (\nu_1,\dots,\nu_n)$ such that $(i,\mu_1,\dots,\mu_m,j) \in B_{m+2}(\overline{X}_A)$, $(i,\nu_1,\dots,\nu_n,j) \in B_{n+2}(\overline{X}_A)$. We set

$$p_{m,\mu}(i,j) = \psi(T_iT_i^* \otimes S_{\mu_1}\cdots S_{\mu_m}S_jS_j^*S_{\mu_m}^*\cdots S_{\mu_1}^*).$$

The following lemma holds.

LEMMA 6.7. (i) $\psi(V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^*) = p_{m,\mu}(i,j)$.
(ii) $\psi(V_{\nu,\mu}(i,j)^*V_{\nu,\mu}(i,j)) = p_{n,\nu}(i,j)$.
(iii) $\psi(V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^*) = \gamma^{n-m}\psi(V_{\nu,\mu}(i,j)^*V_{\nu,\mu}(i,j))$.

*Proof.* (i) Since $T_i^* \otimes S_i$ belongs to $\widetilde{\mathcal{R}}_A$ such that $\delta^A_{i\log\gamma}(T_i^*\otimes S_i) = T_i^*\otimes S_i$, we have

$$\begin{aligned}\psi(V_{\nu,\mu}(i,j)V_{\nu,\mu}(i,j)^*) &= \psi(T_i^*T_i \otimes S_iS_{\mu_1}\cdots S_{\mu_m}S_jS_j^*S_{\mu_m}^*\cdots S_{\mu_1}^*S_i^*)\\ &= \psi((T_i^*\otimes S_i)\cdot(T_i\otimes S_{\mu_1}\cdots S_{\mu_m}S_jS_j^*S_{\mu_m}^*\cdots S_{\mu_1}^*S_i^*))\\ &= \psi((T_i\otimes S_{\mu_1}\cdots S_{\mu_m}S_jS_j^*S_{\mu_m}^*\cdots S_{\mu_1}^*S_i^*)\cdot(T_i^*\otimes S_i))\\ &= \psi(T_iT_i^*\otimes S_{\mu_1}\cdots S_{\mu_m}S_jS_j^*S_{\mu_m}^*\cdots S_{\mu_1}^*S_i^*S_i) = p_{m,\mu}(i,j).\end{aligned}$$

(ii) We have $V_{\nu,\mu}(i,j)^*V_{\nu,\mu}(i,j) = T_jT_{\nu_n}\cdots T_{\nu_1}T_iT_i^*T_{\nu_1}^*\cdots T_{\nu_n}^*T_j^*\otimes S_j^*S_j$ and hence

$$U_A^{n+1}V_{\nu,\mu}(i,j)^*V_{\nu,\mu}(i,j)U_A^{*n+1} = T_iT_i^*\otimes S_{\nu_1}\cdots S_{\nu_n}S_jS_j^*S_{\nu_n}^*\cdots S_{\nu_1}^*.$$

By Lemma 6.4, we have the desired identity.

(iii) As $\delta^A_{i\log\gamma}(V_{\nu,\mu}(i,j)^*) = \gamma^{n-m}V_{\nu,\mu}(i,j)^*$, the KMS condition for $\psi$ ensures us the desired identity. ∎

The preceding lemma tells us that the values $p_{m,\mu}(i,j)$ and $p_{n,\nu}(i,j)$ are equal for $m = n$ as long as $(i,\mu_1,\dots,\mu_m,j) \in B_{m+2}(\overline{X}_A)$, $(i,\nu_1,\dots,\nu_n,j) \in B_{n+2}(\overline{X}_A)$. Hence the value $p_{n,\nu}(i,j)$ does not depend on the choice of the word $\nu$ as long as the length of $\nu$ is $n$ and $(i,\nu_1,\dots,\nu_n,j) \in B_{n+2}(\overline{X}_A)$. We may thus define $p_n(i,j)$ by $p_{n,\nu}(i,j)$ for some $\nu$ with $(i,\nu_1,\dots,\nu_n,j) \in B_{n+2}(\overline{X}_A)$. If there is no word $\nu$ such as $(i,\nu_1,\dots,\nu_n,j) \in B_{n+2}(\overline{X}_A)$, then we define $p_n(i,j)$ to be zero.

LEMMA 6.8. *Let $i,j = 1,\dots,n$ and $n \in \mathbb{Z}_+$.*
(i) *Assume $A^{n+1}(i,j) > 0$ and $A^{n+2}(i,j) > 0$. Then $p_n(i,j) = \gamma p_{n+1}(i,j)$.*

(ii) *Assume* $A^{n+1}(i,j) > 0$. *Then*

$$p_n(i,j) = \sum_{k=1}^{N} A(j,k) p_{n+1}(i,k) = \sum_{h=1}^{N} A(h,i) p_{n+1}(h,j).$$

(iii) *Assume* $A^n(i,j) > 0$ *and* $A^{n+1}(i,j) > 0$. *Then we have*

$$\gamma p_n(i,j) = \sum_{k=1}^{N} A(j,k) p_n(i,k) = \sum_{h=1}^{N} A(h,i) p_n(h,j).$$

*Proof.* (i) Since $A^{n+1}(i,j)$, $A^{n+2}(i,j) > 0$, we may find $\nu = (\nu_1, \dots, \nu_n)$ and $\mu = (\mu_1, \dots, \mu_{n+1})$ such that $(i, \nu_1, \dots, \nu_n, j) \in B_{n+2}(\overline{X}_A)$, $(i, \mu_1, \dots, \mu_{n+1}, j) \in B_{n+3}(\overline{X}_A)$. Consider $V_{\nu,\mu}(i,j) = T_i^* T_{\nu_1}^* \cdots T_{\nu_n}^* T_j^* \otimes S_i S_{\mu_1} \cdots S_{\mu_{n+1}} S_j$. It then follows that

$$\begin{aligned} p_n(i,j) &= \psi(V_{\nu,\mu}(i,j)^* V_{\nu,\mu}(i,j)) = \psi(V_{\nu,\mu}(i,j) \delta^A_{i \log \gamma}(V_{\nu,\mu}(i,j)^*)) \\ &= \gamma \psi(V_{\nu,\mu}(i,j) V_{\nu,\mu}(i,j)^*) = \gamma p_{n+1,\mu}(i,j) = \gamma p_{n+1}(i,j). \end{aligned}$$

(ii) Since $A^{n+1}(i,j) > 0$, we may find $\nu = (\nu_1, \dots, \nu_n)$ such that $(i, \nu_1, \dots, \nu_n, j) \in B_{n+2}(\overline{X}_A)$. It then follows that

$$\begin{aligned} p_n(i,j) &= p_{n,\nu}(i,j) = \psi(T_i T_i^* \otimes S_{\nu_1} \cdots S_{\nu_n} S_j S_j^* S_{\nu_n}^* \cdots S_{\nu_1}^*) \\ &= \sum_{k=1}^{N} A(j,k) \psi(T_i T_i^* \otimes S_{\nu_1} \cdots S_{\nu_n} S_j S_k S_k^* S_j^* S_{\nu_n}^* \cdots S_{\nu_1}^*) \\ &= \sum_{k=1}^{N} A(j,k) p_{n+1,\nu j}(i,k) = \sum_{k=1}^{N} A(j,k) p_{n+1}(i,k). \end{aligned}$$

We also see that

$$\begin{aligned} p_n(i,j) &= \sum_{h=1}^{N} A^{\mathrm{t}}(i,h) \psi(T_i T_h T_i^* T_h^* \otimes S_{\nu_1} \cdots S_{\nu_n} S_j S_j^* S_{\nu_n}^* \cdots S_{\nu_1}^*) \\ &= \sum_{h=1}^{N} A(h,i) p_{n+1,i\nu j}(h,j) = \sum_{h=1}^{N} A(h,i) p_{n+1}(h,j). \end{aligned}$$

The assertion (iii) follows from (i) and (ii). ∎

LEMMA 6.9. *For* $i = 1, \dots, N$ *and* $n \in \mathbb{Z}_+$, *we have:*

(i) $\sum_{j=1}^{N} A^{n+1}(i,j) p_n(i,j) = \psi(T_i^* T_i \otimes S_i S_i^*)$ *and hence* $\sum_{i,j=1}^{N} A^{n+1}(i,j) p_n(i,j) = 1$.

(ii) $\sum_{j=1}^{N} A^{n+1}(j,i) p_n(i,j) = \psi(T_i T_i^* \otimes S_i^* S_i)$ *and hence* $\sum_{i,j=1}^{N} A^{n+1}(j,i) p_n(i,j) = 1$.

*Proof.* (i) We have the following identities:

$$
\begin{aligned}
\psi(T_i^* T_i \otimes S_i S_i^*) &= \sum_{\mu_1=1}^{N} A(i,\mu_1)\psi(T_i^* T_i \otimes S_i S_{\mu_1} S_{\mu_1}^* S_i^*) \\
&= \sum_{j=1}^{N} \sum_{\mu_1,\dots,\mu_n=1}^{N} A(i,\mu_1)A(\mu_1,\mu_2)\cdots A(\mu_n,j)\psi(T_i^* T_i \otimes S_{i\mu_1\cdots\mu_n j} S_{i\mu_1\cdots\mu_n j}^*) \\
&= \sum_{j=1}^{N} A^{n+1}(i,j) p_n(i,j).
\end{aligned}
$$

We also have $\sum_{i=1}^{N} \psi(T_i^* T_i \otimes S_i S_i^*) = \psi(E_A) = 1$.

(ii) is similarly shown to (i) . ∎

We notice that $\psi(T_i^* T_i \otimes S_i S_i^*) = \psi(T_i T_i^* \otimes S_i^* S_i)$ because of the equality $\delta^A_{i \log\gamma}(T_i^* \otimes S_i) = T_i^* \otimes S_i$ and the KMS condition for $\psi$. Recall that we are assuming the matrix $A$ is aperiodic so that there exists $n_0 \in \mathbb{N}$ such that $A^n(i,j) > 0$ for all $i, j = 1,\dots,N$ and $n \geqslant n_0$.

LEMMA 6.10. *We have $\gamma = \beta$.*

*Proof.* Lemma 6.8 and Lemma 6.9 imply that the vector $[p_n(i,k)]_{k=1}^N$ is a nonnegative eigenvector of the matrix $A$ with eigenvalue $\gamma$ for each $n \in \mathbb{N}$ and $i = 1,\dots,N$. Since $A$ is aperiodic, $[p_n(i,k)]_{k=1}^N$ is actually a positive eigenvector with eigenvalue $\gamma$. By the Perron–Frobenius theorem, $\gamma$ coincides with the Perron–Frobenius eigenvalue $\beta$. ∎

We have seen that $\gamma$ must be the Perron–Frobenius eigenvalue of the matrix $A$ by Lemma 6.10. Its proof does not need the assumption $\gamma > 1$ that we had first assumed. Now the matrix $A$ is aperiodic and not any permutation so that its Perron–Frobenius eigenvalue is always greater than one. Hence $\gamma (= \beta)$ is automatically greater than one.

Recall that $[a_j]_{j=1}^N, [b_i]_{i=1}^N$ be the positive eigenvectors of $A$ and $A^{\mathrm{t}}$ for the eigenvalue $\beta$ respectively such that $\sum_{i=1}^{N} a_i b_i = 1$. We have the following lemma.

LEMMA 6.11. *For $n \geqslant n_0$ and $i, j = 1,\dots,N$, we have*

$$
p_n(i,j) = \frac{b_i a_j}{(\sum_{h=1}^N b_h)\cdot(\sum_{k=1}^N a_k)} \sum_{h,k=1}^{N} p_n(h,k). \tag{6.9}
$$

*Proof.* We fix $n \geqslant n_0$. For a fixed $i = 1,\dots,N$, the vector $[p_n(i,k)]_{k=1}^N$ is a positive eigenvector of the matrix $A$ for the eigenvalue $\beta$. By the uniqueness of the positive eigenvector of $A$, we may find a positive real number $c_{n,i}$ such that

$$
p_n(i,j) = c_{n,i} a_j \quad \text{for } j = 1,\dots,N. \tag{6.10}
$$

By Lemma 6.8, we know that the vector $\Big[\sum_{j=1}^{N} p_n(i,j)\Big]_{i=1}^{N}$ is a positive eigenvector of the matrix $A^{\mathrm{t}}$ for the eigenvalue $\beta$. Hence the normalized positive eigenvectors $\Big[\frac{\sum_{j=1}^{N} p_n(i,j)}{\sum_{h,k=1}^{N} p_n(h,k)}\Big]_{i=1}^{N}$ and $\Big[\frac{b_i}{\sum_{k=1}^{N} b_k}\Big]_{i=1}^{N}$ coincide, so that we have

$$\sum_{j=1}^{N} p_n(i,j) = b_i \frac{\sum_{h,k=1}^{N} p_n(h,k)}{\sum_{k=1}^{N} b_k} \quad \text{for } i = 1,\ldots,N. \tag{6.11}$$

By (6.10) and (6.11), we have

$$c_{n,i} = \frac{\sum_{j=1}^{N} p_n(i,j)}{\sum_{j=1}^{N} a_j} = \frac{b_i \sum_{h,k=1}^{N} p_n(h,k)}{(\sum_{k=1}^{N} b_k)(\sum_{j=1}^{N} a_j)}$$

so that we know (6.9) by using (6.10) again. ∎

We thus obtain the following lemma.

LEMMA 6.12. *For $n \geqslant n_0$ and $i,j = 1,\ldots,N$, we have*

$$p_n(i,j) = \frac{1}{\beta^{n+1}} b_i a_j. \tag{6.12}$$

*Proof.* We fix $n \geqslant n_0$. By Lemma 6.11 together with Lemma 6.9, we have

$$1 = \sum_{i,j=1}^{N} A^{n+1}(i,j) p_n(i,j) = \sum_{i,j=1}^{N} \frac{A^{n+1}(i,j) b_i a_j}{(\sum_{h=1}^{N} b_h)\cdot(\sum_{k=1}^{N} a_k)} \sum_{h,k=1}^{N} p_n(h,k).$$

As $[a_j]_{j=1}^{N}$ is a positive eigenvector of $A$ for the eigenvalue $\beta$, we have

$$\sum_{i,j=1}^{N} A^{n+1}(i,j) b_i a_j = \sum_{i=1}^{N} \beta^{n+1} b_i a_i = \beta^{n+1}$$

so that the equalities

$$1 = \frac{\beta^{n+1}}{(\sum_{h=1}^{N} b_h)\cdot(\sum_{k=1}^{N} a_k)} \sum_{h,k=1}^{N} p_n(h,k)$$

and

$$\sum_{h,k=1}^{N} p_n(h,k) = \frac{1}{\beta^{n+1}} \Big(\sum_{i=1}^{N} b_i\Big)\cdot\Big(\sum_{j=1}^{N} a_j\Big) \tag{6.13}$$

hold. By (6.9) and (6.13), we get the desired equality. ∎

Consequently, we know the following proposition.

PROPOSITION 6.13. *The restriction of a* KMS *state $\psi$ on $\widetilde{\mathcal{R}}_A$ to the commutative $C^*$-subalgebra $C(\overline{X}_A)$ coincides with the state defined by the Parry measure on $\overline{X}_A$.*

*Proof.* For $n \geqslant n_0$ and $\xi = (i, \nu_1, \dots, \nu_n, j) \in B_{n+2}(\overline{X}_A)$, Lemma 6.12 shows that

$$\psi(T_i T_i^* \otimes S_{\nu_1 \cdots \nu_n j} S^*_{\nu_1 \cdots \nu_n j}) = \frac{1}{\beta^{n+1}} b_i a_j.$$

Let $\mu$ be the Parry measure on $\overline{X}_A$. Since the Parry measure of the cylinder set $U_{[\xi]_m^{m+n+1}} \subset \overline{X}_A, m \in \mathbb{Z}$ for the word $\xi$ is given by

$$\mu(U_{[\xi]_m^{m+n+1}}) = \frac{1}{\beta^{n+1}} b_i a_j$$

by the formula (6.4). Let $\chi_{U_{[\xi]_m^{m+n+1}}}$ be the characteristic function of the cylinder set $U_{[\xi]_m^{m+n+1}}$. Since

$$\psi(T_i T_i^* \otimes S_{\nu_1 \cdots \nu_n j} S^*_{\nu_1 \cdots \nu_n j}) = \psi(\chi_{U_{[\xi]_m^{m+n+1}}}),$$

we obtain

$$\mu(U_{[\xi]_m^{m+n+1}}) = \psi(\chi_{U_{[\xi]_m^{m+n+1}}}).$$

Any cylinder set on $\overline{X}_A$ is a finite union of cylinder sets of words having length greater than $n_0 + 1$. Hence we may conclude that the restriction of $\psi$ to the commutative $C^*$-subalgebra $C(\overline{X}_A)$ of $\widetilde{\mathcal{R}}_A$ coincides with the state defined by the Parry measure on $\overline{X}_A$. ∎

Therefore we may summarize the above discussion in Theorem 1.3.

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