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ON THE ABSOLUTE VALUE OF UNBOUNDED OPERATORS

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ABSTRACT. The primary purpose of the present paper is to investigate when relations of the types $|AB| = |A||B|$, $|A \pm B| \leqslant |A| + |B|$, $||A| - |B|| \leqslant |A \pm B|$ and $|\overline{\mathrm{Re} A}| \leqslant |A|$ (among others) hold in an unbounded operator setting. As consequences, we obtain a characterization of (unbounded) self-adjointness as well as a characterization of invertibility for the class of unbounded normal operators. Some examples accompany our results.

KEYWORDS: *Normal and self-adjoint operators, commutativity, Fuglede theorem.*

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1. INTRODUCTION

All operators considered here are linear but not necessarily bounded. If an operator is bounded and everywhere defined, then we say that it belongs to $B(H)$ (the algebra of all bounded linear operators on H).

Most unbounded operators that we encounter are defined on a subspace (called domain) of a Hilbert space. If the domain is dense, then we say that the operator is densely defined. In such case, the adjoint exists and is unique.

In order that the paper be as self-contained as possible, let us recall a few basic definitions about non-necessarily bounded operators. If S and T are two linear operators with domains $D(S)$ and $D(T)$ respectively, then T is said to be an extension of S, written as $S \subset T$, if $D(S) \subset D(T)$ and S and T coincide on $D(S)$.

An operator T is called closed if its graph is closed in $H \oplus H$. It is called closable if it has a closed extension. The smallest closed extension of it is called its closure and it is denoted by $\overline{T}$ (a standard result states that T is closable if and only if T^* has a dense domain and in which case $\overline{T} = T^{**}$).

If T is closable, then

$$S \subset T \Rightarrow \overline{S} \subset \overline{T}.$$

If T is densely defined, we say that T is self-adjoint when $T = T^*$; symmetric if $T \subset T^*$; normal if T is *closed* and $TT^* = T^*T$.

The product ST and the sum $S + T$ of two operators S and T are defined in the usual fashion on the natural domains:

$$D(ST) = \{x \in D(T) : Tx \in D(S)\} \quad \text{and} \quad D(S+T) = D(S) \cap D(T).$$

In the event that S, T and ST are densely defined, then

$$T^*S^* \subset (ST)^*,$$

with the equality occurring when $S \in B(H)$. Also, when S, T and $S + T$ are densely defined, then

$$S^* + T^* \subset (S+T)^*,$$

and the equality holding if $S \in B(H)$.

The product of two closed operators need not be closed and the sum of two closed operators is not closed either. However, it is known (among other results) that TS and $T + S$ are closed if T is closed and $S \in B(H)$.

The real and imaginary parts of a densely defined operator T are defined respectively, by

$$\mathrm{Re}T = \frac{T+T^*}{2} \quad \text{and} \quad \mathrm{Im}T = \frac{T-T^*}{2\mathrm{i}}.$$

Clearly, if T is closed, then ReT is symmetric but it is not always self-adjoint. In fact, it may even fail to be closed.

If $S \in B(H)$ and T is unbounded, then S commutes with T if $ST \subset TS$. In the previous case if also S and T are normal, then the previous amounts to the commutativity of their spectral measures (see e.g. [1] or [24]). Two unbounded normal operators are said to (strongly) commute if their corresponding spectral measures commute. If S and T are self-adjoint, then the strong commutativity of S and T may also be interpreted as

$$\mathrm{e}^{\mathrm{i}sT}\mathrm{e}^{\mathrm{i}tS} = \mathrm{e}^{\mathrm{i}tS}\mathrm{e}^{\mathrm{i}sT}$$

for all $s, t \in \mathbb{R}$.

If a self-adjoint T is such that $\langle Tx, x\rangle \geqslant 0$ for all $x \in D(T)$, then we say that T is positive. We can define the unique positive self-adjoint square root which we denote by $\sqrt{T}$ (see [26] for a new proof of the existence of the positive square root of unbounded positive self-adjoint operators). If T is positive and $ST \subset TS$ where $S \in B(H)$, then $S\sqrt{T} \subset \sqrt{T}S$ (see [2]). Also, if $ST \subset TS$ where $S \in B(H)$ is positive, then $\sqrt{S}T \subset T\sqrt{S}$ (see e.g. [17]).

If T is densely defined and closed, then T^*T (and TT^*) is self-adjoint and positive (a celebrated result due to von Neumann, see e.g. [24]). We digress a little bit to say that the self-adjointness of T^*T (say) alone does not imply that T is closed. A simple counterexample is prescribed in [25]. Sebestyén–Tarcsay went on in the same reference to show that the self-adjointness of *both* T^*T and TT^* implies the closedness of T. A re-proof may be found in [10].

As just observed, if T is closed then T^*T is self-adjoint and positive. Hence it is legitimate to define its square root. The unique positive self-adjoint square root of T^*T is denoted by $|T|$. It is customary to call it the absolute value or modulus of T.

If T is closed, then (see e.g. Lemma 7.1 in [24])

$$D(T) = D(|T|) \quad \text{and} \quad \|Tx\| = \||T|x\|, \quad \forall x \in D(T).$$

It is also plain that if T is normal, then $|T| = |T^*|$. Also, in the event of the normality of T, $\overline{\mathrm{Re}T}$ and $\overline{\mathrm{Im}T}$ are self-adjoint (see e.g. [29]).

The use of the Fuglede (–Putnam) theorem is unavoidable when working with normal operators. For convenience, we recall it next (see e.g. [5] for a proof).

THEOREM 1.1. *If T is a bounded operator and N and M are not necessarily bounded normal operators, then*

$$TN \subset MT \Rightarrow TN^* \subset M^*T.$$

We recall some more definitions. In the context of bounded operators, the following two definitions coincide.

DEFINITION 1.2 ([24], Definition 10.5, p. 230). Let A and B be two symmetric operators with domains $D(A)$ and $D(B)$, respectively. We write $A \succeq B$ if $D(A) \subset D(B)$ and

$$\langle Ax, x\rangle \geqslant \langle Bx, x\rangle, \quad \forall x \in D(A).$$

DEFINITION 1.3. Let T and S be unbounded positive self-adjoint operators. We say that $S \geqslant T$ if $D(S^{1/2}) \subseteq D(T^{1/2})$ and $\|S^{1/2}x\| \geqslant \|T^{1/2}x\|$ for all $x \in D(S^{1/2})$.

The Heinz inequality is valid for positive unbounded self-adjoint operators as well. In particular, if S and T are self-adjoint operators, then (see e.g. [24])

$$S \geqslant T \geqslant 0 \Rightarrow \sqrt{S} \geqslant \sqrt{T}.$$

Also, as in p. 200 in [29], if S and T are self-adjoint, T is boundedly invertible and $S \geqslant T \geqslant 0$, then S is boundedly invertible and $S^{-1} \leqslant T^{-1}$.

Finally, we recall the definition of an unbounded hyponormal operator.

DEFINITION 1.4. A densely defined operator T with domain $D(T)$ is called *hyponormal* if

$$D(T) \subset D(T^*) \quad \text{and} \quad \|T^*x\| \leqslant \|Tx\|, \quad \forall x \in D(T).$$

As in the bounded case, we have the following proposition (the proof may be found in [7]).

PROPOSITION 1.5. *Let T be a closed hyponormal operator. Then*

$$T^*T \geqslant TT^*.$$

The following recently proved result will be needed in some of our proofs.

THEOREM 1.6 ([7]). *Let T be a closed hyponormal operator. Then*

$$|\langle Tx, x\rangle| \leqslant \langle |T|x, x\rangle \quad \text{for all } x \in D(T).$$

Despite the fact that we have recalled most results which will be used in our proofs, we refer readers to [24] and [27] for further reading. We also refer readers to [3] or [24] for the spectral theorem.

The absolute value plays a prominent role in operator theory. For example, it intervenes in any version of the polar decompositions of an operator. We also utilize it when defining singular values in matrix theory.

As alluded, we are mainly (but not only) interested in investigating when relations of the types

$$|AB| = |A||B|, \quad |A \pm B| \leqslant |A| + |B|, \quad ||A| - |B|| \leqslant |A \pm B| \quad \text{and} \quad |\overline{\mathrm{Re}A}| \leqslant |A|$$

are valid in an unbounded setting. The all-bounded case has already been treated in [20]. So, there is no need to reiterate that the previous inequalities, without any a priori conditions, are false in general (see e.g. [21]). Going back to unbounded operators, and to the best of our knowledge, we are only aware of simple inequalities of the type

$$|A + \alpha I| \leqslant |A| + |\alpha| I,$$

valid for an unbounded self-adjoint operator A and for some scalar α. This appeared on p. 335 of the classical monograph by T. Kato ([15]). This inequality becomes a simple consequence of Proposition 3.7 below. The paper contains, besides some nice examples, some interesting consequences. For instance, we give a nonspectral condition which forces an unbounded normal operator to be self-adjoint. Also, we characterize the invertibility of normal operators in terms of their real and imaginary parts and hence so we do with the spectrum. We also obtain a very simple and short proof of the realness of the spectrum of an unbounded self-adjoint operator.

2. THE ABSOLUTE VALUE AND PRODUCTS

We start with the following known result which is recalled for readers convenience.

PROPOSITION 2.1. *Let A be a self-adjoint positive operator and let $B \in B(H)$ be positive (hence also self-adjoint). If $BA \subset AB$, then AB (and $\overline{BA}$) is self-adjoint, positive and $\overline{BA} = AB$. Moreover,*

$$\sqrt{AB} = \sqrt{A}\sqrt{B}.$$

Proof. The self-adjointness and the positiveness of AB follow by invoking the spectral theorem as carried out in Lemma 3.1 of [14]. The equality $\sqrt{AB} =$

$\sqrt{A}\sqrt{B}$ may also be obtained via a similar argument. Alternatively, we may proceed as follows: we have

$$BA \subset AB \Rightarrow \sqrt{B}A \subset A\sqrt{B} \Rightarrow \sqrt{B}\sqrt{A} \subset \sqrt{A}\sqrt{B}.$$

Now, by the first part of the proof, $\sqrt{A}\sqrt{B}$ is self-adjoint (and positive) and hence so is $(\sqrt{A}\sqrt{B})^2$. But

$$(\sqrt{A}\sqrt{B})^2 = \sqrt{A}\sqrt{B}\sqrt{A}\sqrt{B} \subset \sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B} = AB.$$

Next, as both $(\sqrt{A}\sqrt{B})^2$ and AB are self-adjoint, then by the maximality of self-adjoint operators (see e.g. [24], cf. [16]), we obtain

$$(\sqrt{A}\sqrt{B})^2 = AB.$$

Accordingly, by the uniqueness of the positive square root, we infer that

$$\sqrt{AB} = \sqrt{A}\sqrt{B},$$

as needed. ∎

The following perhaps known result will be used without further notice. Its power lies in the fact that B is not assumed to be normal (and so it cannot be established using the spectral theorem).

PROPOSITION 2.2. *Let A be normal and let $B \in B(H)$. Then*

$$BA \subset AB \Rightarrow |B||A| \subset |A||B|,$$

i.e. $|B||A|x = |A||B|x$ *whenever* $x \in D(B|A|) = D(|A|) = D(A)$.

Proof. Since $BA \subset AB$ and A is normal, we get $BA^* \subset A^*B$. Hence

$$BA^*A \subset A^*BA \subset A^*AB.$$

Therefore, $B|A| \subset |A|B$. As $|A|$ is self-adjoint, it follows by taking adjoints that $B^*|A| \subset |A|B^*$. Thus,

$$B|A| \subset |A|B \Rightarrow B^*B|A| \subset B^*|A|B \subset |A|B^*B.$$

Accordingly,

$$|B||A| \subset |A||B|,$$

as required. ∎

COROLLARY 2.3. *Let A be normal and let $B \in B(H)$. If $BA \subset AB$, then A^*AB^*B is self-adjoint.*

Proof. As above, we obtain $B^*BA^*A \subset A^*AB^*B$. Since B^*B and A^*A are self-adjoint, Proposition 2.1 implies the self-adjointness of A^*AB^*B. ∎

THEOREM 2.4. *Let A be normal and let $B \in B(H)$. If $BA \subset AB$, then*

$$|\overline{BA}| = |AB| = |A||B| = \overline{|B||A|}.$$

Proof. The equality $|A||B| = \overline{|B||A|}$ is established as follows: by Proposition 2.2, we have $|B||A| \subset |A||B|$. Since $|B|$ and $|A|$ are self-adjoint, it follows by Proposition 2.1 that

$$\overline{|B||A|} = |A||B|.$$

Now, by the general theory, $BA \subset AB$ gives $B^*A^* \subset A^*B^*$. Also and as above, the normality of A yields

$$BA \subset AB \Rightarrow BA^* \subset A^*B.$$

Hence by passing to adjoints, $B^*A \subset AB^*$. Therefore,

$$(AB)^*AB \supset B^*A^*AB = B^*AA^*B \supset B^*ABA^* \supset B^*BAA^*.$$

The closedness of AB gives the self-adjointness of $(AB)^*AB$. The previous "inclusion" becomes after taking adjoints

$$(AB)^*AB \subset AA^*B^*B = A^*AB^*B.$$

Since $(AB)^*AB$ and A^*AB^*B are self-adjoint, by the maximality of self-adjoint operators, we get

$$(AB)^*AB = A^*AB^*B.$$

We infer by a glance at Proposition 2.1 that

$$|AB| = \sqrt{(AB)^*AB} = \sqrt{A^*AB^*B} = \sqrt{A^*A}\sqrt{B^*B} = |A||B|.$$

To prove the last equality, we proceed as before. We have

$$(BA)^*BA = A^*B^*BA \supset B^*A^*BA \supset B^*BA^*A.$$

Since BA is closable and $B^*B \in B(H)$, it follows by passing to adjoints that

$$(BA)^*\overline{BA} \subset [(BA)^*BA]^* \subset A^*AB^*B.$$

As above, we obtain

$$(BA)^*\overline{BA} = A^*AB^*B.$$

Accordingly,

$$|\overline{BA}| = \sqrt{(\overline{BA})^*\overline{BA}} = \sqrt{(BA)^*\overline{BA}} = \sqrt{A^*AB^*B} = |A||B|. \quad ∎$$

REMARK 2.5. The condition $BA \subset AB$ may be obtained if AB is normal, A and B are self-adjoint (one of them is positive) and $B \in B(H)$ (see [11], [17], cf. [14]).

COROLLARY 2.6. *Let A be a normal operator and let $B \in B(H)$. If $BA \subset AB$, then*

$$|AB| = |A^*B| = |\overline{BA}| = |\overline{BA^*}|.$$

COROLLARY 2.7. *Let A be a boundedly invertible normal operator. Then*

$$|A^{-1}| = |A|^{-1}.$$

Proof. We may use the spectral theorem. Otherwise, by hypothesis,

$$A^{-1}A \subset AA^{-1} = I$$

where $A^{-1} \in B(H)$. Hence

$$|A||A^{-1}| = |AA^{-1}| = |I| = I.$$

Since $|A|$ is self-adjoint and right invertible, it is invertible (see [6]) and clearly

$$|A^{-1}| = |A|^{-1}. \quad \blacksquare$$

By using the spectral theorem for (unbounded) normal operators, it is easy to see that $|A^n| = |A|^n$ for any $n \in \mathbb{N}$ and A an unbounded normal operator. As a consequence of a recent result due to Jabłoński–Jung–Stochel in [12], we may extend the above to quasinormal operators (a class for which there is no spectral theorem). For the readers convenience, we recall that a closed densely defined operator A is said to be quasinormal if $U|A| \subset |A|U$ where $A = U|A|$ is the (non-unitary) polar decomposition of A.

PROPOSITION 2.8 ([12], Lemma 3.5). *If A is unbounded and quasinormal, then for all $n \in \mathbb{N}$*

$$(A^*A)^n = A^{*n}A^n = (A^n)^*A^n.$$

Hence, we have the following corollary.

COROLLARY 2.9. *If A is an unbounded quasinormal operator, then*

$$|A^n| = |A|^n$$

for all $n \in \mathbb{N}$.

The previous corollary is not true for the weaker class of hyponormal operators, even bounded ones. Indeed, we have the following example.

EXAMPLE 2.10. Let $S \in B(\ell^2)$ be the usual (unilateral) shift. Remember that $SS^* \neq I$ and $S^*S = I$ where I is the identity operator on ℓ^2. Now, take $A = S + I$ so that A is hyponormal. Then

$$|A|^2 = |S+I|^2 = (S^*+I)(S+I) = 2I + S + S^*.$$

On the other hand,

$$|A^2| = \sqrt{(S^*+I)^2(S+I)^2} = \sqrt{6I + 4S^* + 4S + S^{*2} + S^2}.$$

If $|A^2| = |A|^2$ held, then we would obtain $|A^2|^2 = |A|^4$. Working out details would then yield $SS^* = I$ which is impossible.

REMARK 2.11. We have just seen that Corollary 2.9 does not hold for the class of bounded hyponormal operators. Obviously, it does not hold for the class of (unbounded) symmetric and closed operators either. Let A be a closed, symmetric and semi-bounded operator as in [4]. Then $D(A^2) = \{0\}$ and so "$|A^2|$" does not make sense whereas $|A|^2$ is self-adjoint!

Before passing to inequalities, we generalize Theorem 2.4 to a finite family of normal operators. First, we have the following proposition.

PROPOSITION 2.12. *Let A and B be two strongly commuting normal operators. Then $\overline{AB} = \overline{BA}$ are normal and*

$$|\overline{BA}| = |\overline{AB}| = \overline{|A||B|} = \overline{|B||A|}.$$

Proof. The proof makes use of the spectral theorem. Since A and B are normal, by the spectral theorem we may write

$$A = \int_{\mathbb{C}} z\mathrm{d}E_A \quad \text{and} \quad B = \int_{\mathbb{C}} z'\mathrm{d}F_B,$$

where E_A and F_B designate the associated spectral measures. By the strong commutativity, we have

$$E_A(I)F_B(J) = F_B(J)E_A(I)$$

for all Borel sets I and J in $\mathbb{C}$. Hence

$$E_{A,B}(z,z') = E_A(z)F_B(z')$$

defines a two parameter spectral measure. Thus

$$C = \int_{\mathbb{C}}\int_{\mathbb{C}} zz'\mathrm{d}E_{A,B}$$

defines a normal operator, such that $C = \overline{AB} = \overline{BA}$. Therefore, as $|zz'| = |z||z'|$ for all z, z', then (see e.g. p. 78 of [24])

$$|\overline{AB}| = \int_{\mathbb{R}}\int_{\mathbb{R}} |zz'|\mathrm{d}E_{A,B} = \overline{|A||B|},$$

as wished. ∎

COROLLARY 2.13. *Let A and B be two strongly commuting normal operators. Then*

$$|\overline{AB}| = |\overline{A^*B}| = |\overline{AB^*}| = |\overline{A^*B^*}| = |\overline{BA}| = |\overline{BA^*}| = |\overline{B^*A}| = |\overline{B^*A^*}|.$$

Proof. The proof follows easily from Proposition 2.12 and from $|A^*| = |A|$ and $|B^*| = |B|$. ∎

Using the polar decomposition of (unbounded) normal operators, the following result then becomes obvious.

COROLLARY 2.14. *Let A and B be two strongly commuting normal operators. Then*

$$\overline{BA} = \overline{AB} = U\overline{|A||B|} = U\overline{|B||A|} = \overline{|B||A|}U = \overline{|A||B|}U$$

for some unitary operator $U \in B(H)$.

Before carrying on, we give the following probably known result on the important notion of the adjoint of products.

PROPOSITION 2.15. *Let A and B be two strongly commuting normal operators. Then*

$$(AB)^* = (\overline{AB})^* = \overline{B^*A^*} = \overline{A^*B^*} = (BA)^*.$$

Proof. The proof is contained in the proof of Proposition 2.12 by remembering that

$$A^* = \int_{\mathbb{C}} \bar{z} \mathrm{d}E_A \quad \text{and} \quad B^* = \int_{\mathbb{C}} \overline{z'} \mathrm{d}F_B. \quad \blacksquare$$

PROPOSITION 2.16. *Let A and C be strongly commuting normal operators and let $B \in B(H)$. If $BA \subset AB$, $BC \subset CB$ and AC is densely defined, then*

$$|\overline{ABC}| = |\overline{AC}||B| = \overline{|A||C|}|B|.$$

The proof of the preceding proposition relies upon a Fuglede-like result.

LEMMA 2.17. *Let $B \in B(H)$ and let T be a densely defined operator such that $\overline{T}$ is normal. If $BT \subset TB$, then*

$$BT^* \subset T^*B, \quad B\overline{T} \subset \overline{T}B \quad \textit{and} \quad B^*\overline{T} \subset \overline{T}B^*.$$

Proof. Since $\overline{T}$ is normal, $\overline{T}^* = T^*$ stays normal. Now,

$$\begin{aligned} BT \subset TB \Rightarrow & B^*T^* \subset T^*B^* \quad \text{(by taking adjoints)} \\ \Rightarrow & B^*\overline{T} \subset \overline{T}B^* \quad \text{(use the Fuglede theorem)} \\ \Rightarrow & BT^* \subset T^*B \quad \text{(by taking adjoints)} \\ \Rightarrow & B\overline{T} \subset \overline{T}B \quad \text{(apply the Fuglede theorem again).} \end{aligned}$$

This marks the end of the proof. $\blacksquare$

REMARK 2.18. It is worth noting that (see Lemma 2.8 of [28]): if $B \in B(H)$ and T is densely defined such that $\overline{T}$ exists, then

$$BT \subset TB \Longleftrightarrow B\overline{T} \subset \overline{T}B.$$

Now, we give a proof of Proposition 2.16.

Proof. By assumptions,

$$B(AC) \subset ABC \subset (AC)B.$$

Since AC is densely defined and $D(AC) = D(BAC) \subset D(ABC)$, clearly ABC is densely defined. Hence

$$(ABC)^* \subset [B(AC)]^* = (AC)^*B^*.$$

As $\overline{AC}$ is normal, then Lemma 2.17 gives

$$B(AC)^* \subset (AC)^*B \text{ or } B^*\overline{AC} \subset \overline{AC}B^*.$$

Also,

$$ABC \subset ACB \subset \overline{AC}B$$

and so

$$\overline{ABC} \subset \overline{AC}B$$

for $\overline{AC}B$ is closed. Hence, we may write

$$(ABC)^*\overline{ABC} \subset (AC)^*B^*\overline{ABC} \subset (AC)^*B^*\overline{AC}B \subset (AC)^*\overline{AC}B^*B.$$

Corollary 2.3 gives the self-adjointness of $(AC)^*\overline{AC}B^*B$ and because $\overline{ABC}$ is closed, we then get

$$(ABC)^*\overline{ABC} = (AC)^*\overline{AC}B^*B.$$

Ergo, Proposition 2.1 (and Corollary 2.3 again) yields

$$|\overline{ABC}| = |\overline{AC}||B| = \overline{|A||C|}|B|,$$

establishing the result. ∎

COROLLARY 2.19. *Let A be a normal operator and $B \in B(H)$. If $BA \subset AB$, then*

$$|\overline{ABA}| = |A|^2|B| = |A|\overline{|B||A|}.$$

COROLLARY 2.20. *Let $(A_i)_{i=1,\dots,n}$ be pairwise strongly commuting normal operators. Then*

$$|\overline{A_1A_2\cdots A_n}| = \overline{|A_1||A_2|\cdots|A_n|}.$$

The proof follows by induction using n parameter spectral measures.

We finish with an ultimate generalization.

PROPOSITION 2.21. *Let $(A_i)_{i=1,\dots,n}$ be pairwise strongly commuting normal operators such that $A_1A_2\cdots A_n$ is densely defined. Let $B \in B(H)$ be such that $BA_i \subset A_iB$ for $i = 1,2,\dots,n$. Then*

$$|\overline{BA_1A_2\cdots A_n}| = |\overline{A_1A_2\cdots A_nB}| = |\overline{A_1A_2\cdots A_iBA_{i+1}\cdots A_n}| = \overline{|A_1||A_2|\cdots|A_n|}|B|.$$

Proof. These equalities are simple consequences of the results obtained above. For example, to prove the last equality, observe that $A_1A_2\cdots A_i$ and $A_{i+1}\cdots A_n$ are normal. Then

$$|\overline{A_1A_2\cdots A_iBA_{i+1}\cdots A_n}| = |\overline{(A_1A_2\cdots A_i)(A_{i+1}\cdots A_n)}||B| = \overline{|A_1||A_2|\cdot s|A_n|}|B|,$$

as required. ∎

3. THE ABSOLUTE VALUE: SUMS AND INEQUALITIES

THEOREM 3.1. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then*

$$|A + B| \leqslant |A| + |B|.$$

The proof requires the following result which is also interesting in its own.

THEOREM 3.2. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then A^*B is hyponormal, that is,*

$$\|(A^*B)^*x\| \leqslant \|A^*Bx\|$$

for all $x \in D(A^*B) \subset D[(A^*B)^*]$.

Proof. Since $BA \subset AB$, by the Fuglede theorem $BA^* \subset A^*B$ because A is normal. By Corollary 2.6 we have

$$|A^*B| = |\overline{BA^*}|.$$

Since A^*B is closed (by the general theory), we obtain

$$D(A^*B) = D(|A^*B|) = D(|\overline{BA^*}|) = D(\overline{BA^*}).$$

On the other hand, as A^*B is densely defined, we always have

$$B^*A^{**} = B^*A \subset (A^*B)^*.$$

Hence

$$\overline{B^*A} \subset (A^*B)^*$$

and so $D(A^*B) = D(\overline{B^*A}) \subset D[(A^*B)^*]$.

Now, for all $x \in D(A) = D(A^*) = D(BA^*) \subset D(A^*B) \subset D[(A^*B)^*]$, we have (using the hyponormality of B)

$$\|(A^*B)^*x\| = \|B^*Ax\| \leqslant \|BAx\| = \|ABx\| = \|A^*Bx\|,$$

proving the hyponormality of A^*B, as wished. ∎

COROLLARY 3.3. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then AB is hyponormal.*

Proof. Clearly, $BA \subset AB$ gives $BA^* \subset A^*B$. Now the foregoing theorem yields the hyponormality of $A^{**}B = AB$. ∎

We are ready to prove Theorem 3.1.

Proof. First, as usual we have $BA \subset AB$ and $BA^* \subset A^*B$. Whence, if $x \in D(A) = D(A^*)$, then $Bx \in D(A) = D(A^*)$. Moreover, the self-adjointness of $|A|$ and $|B|$ gives the self-adjointness (and positiveness) of $|A| + |B|$ given that $|B| \in B(H)$. Since A is closed and $B \in B(H)$, $A + B$ is closed. Hence $|A + B|$ makes sense and besides

$$D(|A| + |B|) = D(|A|) = D(A) = D(A + B) = D(|A + B|).$$

Now, since $|A + B|$ is self-adjoint and positive, to show the required triangle inequality, by using the Heinz inequality (unbounded version), it suffices to show that

$$|A + B|^2 \leqslant (|A| + |B|)^2.$$

Let $x \in D(A)$. Then

$$\begin{aligned}
\||A+B|x\|^2 &= \|(A+B)x\|^2 = \langle (A+B)x, (A+B)x\rangle \\
&= \|Ax\|^2 + \|Bx\|^2 + \langle Bx, Ax\rangle + \langle Ax, Bx\rangle \\
&= \|Ax\|^2 + \|Bx\|^2 + \langle Bx, Ax\rangle + \overline{\langle Bx, Ax\rangle} \\
&= \|Ax\|^2 + \|Bx\|^2 + 2\mathrm{Re}\langle Bx, Ax\rangle \\
&= \|Ax\|^2 + \|Bx\|^2 + 2\mathrm{Re}\langle A^*Bx, x\rangle \\
&\leqslant \|Ax\|^2 + \|Bx\|^2 + 2|\langle A^*Bx, x\rangle|.
\end{aligned}$$

But A^*B is closed. It is also hyponormal by Theorem 3.2 and so Theorem 1.6 yields

$$|\langle A^*Bx, x\rangle| \leqslant \langle |A^*B|x, x\rangle$$

for each x. By Proposition 2.2 and Theorem 2.4, we have for each $x \in D(A)$:

$$|A^*B|x = |A^*||B|x = |A||B|x = |B||A|x.$$

Hence

$$2|\langle A^*Bx, x\rangle| \leqslant 2\langle |A^*B|x, x\rangle = \langle |A||B|x, x\rangle + \langle |B||A|x, x\rangle.$$

Accordingly,

$$\||A+B|x\|^2 \leqslant \|Ax\|^2 + \|Bx\|^2 + \langle |A||B|x, x\rangle + \langle |B||A|x, x\rangle = \|(|A|+|B|)x\|^2$$

or merely

$$\||A+B|x\| \leqslant \|(|A|+|B|)x\|$$

for all $x \in D(A)$. By Definition 1.3, this just means that

$$|A+B|^2 \leqslant (|A|+|B|)^2,$$

as needed above. ∎

Before moving forward, we give several examples.

EXAMPLES 3.4. (i) Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$. Then $\mathcal{F}$ is unitary and so

$$\mathcal{F} = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\lambda} \mathrm{d}E_\lambda$$

in terms of a spectral measure E_λ. Now, set

$$A = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$$

which is unbounded and self-adjoint and, by construction, it strongly commutes with $\mathcal{F}$. Therefore, $A + \mathcal{F}$ satisfies

$$|A + \mathcal{F}| \leqslant |A| + I.$$

(ii) Let $a \in \mathbb{R}$. Let U be the unitary operator defined on $L^2(\mathbb{R})$ by

$$(Uf)(x) = f(x+a), \quad x \in \mathbb{R}.$$

Let $Af(x) = xf(x)$ defined on $D(A) = \{f \in L^2(\mathbb{R}) : xf \in L^2(\mathbb{R})\}$. Then $|A|f(x) = |x|f(x)$ and $|U| = I$. Also, U commutes with A. By Theorem 3.1,

$$|A+U| \leqslant M$$

where $Mf(x) = (|x|+1)f(x)$ defined on $D(A)$.

(iii) Let A be a boundedly invertible normal operator and let A^{-1} be its everywhere bounded inverse. Then

$$|A+A^{-1}| \leqslant |A| + |A^{-1}| = |A| + |A|^{-1}.$$

COROLLARY 3.5. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then*

$$|A-B| \leqslant |A| + |B|.$$

REMARK 3.6. The proof of the previous theorem in the bounded case (as it appeared in [20]) used the known fact that

$$|\mathrm{Re}T| \leqslant |T| \tag{3.1}$$

whenever $T \in B(H)$ is hyponormal. So, the proof above is a new proof of Theorem 3.4 in the above reference. The natural question is whether we can expect Inequality (1) to hold for unbounded closed hyponormal operators? The answer is no in general, however, we have the following related and interesting result.

PROPOSITION 3.7. *Let A be a closed hyponormal operator. Then*

$$|\overline{\mathrm{Re}A}| \leqslant |A|$$

where $\overline{\mathrm{Re}A}$ denotes the closure of $\mathrm{Re}A$.

Proof. Since A is hyponormal, we have $D(A) \subset D(A^*)$ and so $A + A^*$ is densely defined. Since A is closed, $A + A^*$ is symmetric. Therefore, $\mathrm{Re}A \subset \overline{\mathrm{Re}A}$. Let $x \in D(A)$. Then

$$(\mathrm{Re}A)x = (\overline{\mathrm{Re}A})x.$$

As above, it suffices to show that

$$|\overline{\mathrm{Re}A}|^2 \leqslant |A|^2.$$

By the closedness of A,

$$D(|A|) = D(A) = D(\mathrm{Re}A) \subset D(\overline{\mathrm{Re}A}) = D(|\overline{\mathrm{Re}A}|).$$

Let $x \in D(|A|)$. The use of the hyponormality of A implies that

$$\||\overline{\mathrm{Re}A}|x\| = \|(\overline{\mathrm{Re}A})x\| = \|(\mathrm{Re}A)x\| \leqslant \frac{1}{2}(\|Ax\| + \|A^*x\|) \leqslant \|Ax\| = \||A|x\|,$$

i.e. $|\overline{\mathrm{Re}A}|^2 \leqslant |A|^2$. Using the Heinz inequality, we infer that

$$|\overline{\mathrm{Re}A}| \leqslant |A|,$$

as coveted. ∎

Mutatis mutandis, we may prove the following proposition.

PROPOSITION 3.8. *Let A be a closed hyponormal operator. Then*

$$|\overline{\mathrm{Im}A}| \leqslant |A|$$

where $\overline{\mathrm{Im}A}$ denotes the closure of $\mathrm{Im}A$*.*

The following result gives an important application to self-adjointness. Notice that its first proof was a little longer. Then, we came across the paper [13] and the proof became shorter. The next theorem is also a partial answer in an unbounded setting to a famous conjecture posed by Fong–Tsui in [9] (asked in the $B(H)$ context). The analogous result on $B(H)$ was already obtained in [19].

THEOREM 3.9. *Let A be a normal operator such that*

$$|A| \leqslant |\overline{\mathrm{Re}A}|.$$

Then A is self-adjoint.

Proof. Since A is normal, it is closed and hyponormal and so by Proposition 3.7, $|\overline{\mathrm{Re}A}| \leqslant |A|$. A glance at the hypothesis of the theorem yields

$$|A| = |\overline{\mathrm{Re}A}|.$$

By the normality of A, we have $D(A) = D(A^*)$. Since $\overline{\mathrm{Re}A}$ is self-adjoint, we may write

$$D[(\mathrm{Re}A)^*] = D[(\overline{\mathrm{Re}A})^*] = D(\overline{\mathrm{Re}A}) = D(|A|) = D(A) \quad \text{and}$$
$$\|A^*x\| = \|Ax\| = \||A|x\| = \||\overline{\mathrm{Re}A}|x\| = \|\overline{\mathrm{Re}A}x\| = \|(\mathrm{Re}A)^*x\|.$$

Therefore, the conditions of Theorem 3 in [13] are fulfilled and so $A = A^*$, as needed. ∎

As alluded to above, the power of Theorem 3.1 say, lies in the fact that one operator is hyponormal and so there is no room for the (two parameter) spectral theorem. If we work with strongly commuting normal operators (even both unbounded), then the proof becomes a slightly simpler. Besides, this will be used to generalize some of the results above.

THEOREM 3.10. *Let A and B be two strongly commuting unbounded normal operators. Then,*

$$|\overline{A+B}| \leqslant |A| + |B|.$$

Proof. Since A and B are normal, by the spectral theorem we may write

$$A = \int_{\mathbb{C}} z \mathrm{d}E_A \quad \text{and} \quad B = \int_{\mathbb{C}} z' \mathrm{d}F_B,$$

where E_A and F_B denote the associated spectral measures. As A and B strongly commute, then so do $|A|$ and $|B|$. Since $|A|$ and $|B|$ are also self-adjoint and positive, by Lemma 4.15.1 in [22], it follows that $|A| + |B|$ is self-adjoint (hence closed) and positive. As for domains, we have

$$D(|A|+|B|)=D(|A|)\cap D(|B|)=D(A)\cap D(B)=D(A+B)\subset D(\overline{A+B})\subset D(|\overline{A+B}|).$$

By the strong commutativity, we have

$$E_A(I)F_B(J) = F_B(J)E_A(I)$$

for all Borel sets I and J in $\mathbb{C}$. Hence

$$E_{A,B}(z,z') = E_A(z)F_B(z')$$

defines a two parameter spectral measure. Thus

$$C = \iint_{\mathbb{C}^2} (z+z')\mathrm{d}E_{A,B}$$

defines a normal operator such that $C = \overline{A+B}$. Since $|z+z'| \leqslant |z| + |z'|$ for all z, z', it follows that

$$|\overline{A+B}| = \iint_{\mathbb{R}^2} |z+z'|\mathrm{d}E_{A,B}(z,z') \leqslant |A| + |B|$$

(we used the positivity of the measure $\langle E_{A,B}(\Delta)x, x\rangle$ with Δ being a Borel subset of $\mathbb{R}^2$). In fact, what we have proved so far is "only"

$$|\overline{A+B}| \preceq |A| + |B|.$$

In other language, we have shown that

$$\langle |\overline{A+B}|x, x\rangle \leqslant \langle (|A| + |B|)x, x\rangle$$

for all $x \in D(|A| + |B|) \subset D(|\overline{A+B}|)$.

Since $|\overline{A+B}|$ and $|A| + |B|$ are self-adjoint and positive, by Lemma 10.10 in [24], "$\preceq$" becomes "$\leqslant$", that is, we have established the desired inequality:

$$|\overline{A+B}| \leqslant |A| + |B|. \quad \blacksquare$$

REMARK 3.11. The strong commutativity of the normal A and B is not sufficient to make the sum $A+B$ closed even when A and B are self-adjoint as shown by the usual trick: consider an unbounded self-adjoint operator A with a *non-closed* domain $D(A)$. Then $B = -A$ is also self-adjoint and evidently strongly commutes with A but the bounded operator $A+B = 0$ on $D(A)$ is not closed.

EXAMPLES 3.12. (i) Let $A = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ and $B = \frac{\mathrm{d}}{\mathrm{d}x}$ be acting on the respective Sobolev spaces $H^2(\mathbb{R})$ and $H^1(\mathbb{R})$. Both A and B are closed. Moreover,

$$ABf(x) = BAf(x) = f'''(x)$$

for any f in the common domain $\{f \in L^2(\mathbb{R}) : f''' \in L^2(\mathbb{R})\}$. It is known that A is self-adjoint and that both B and AB are normal. Therefore, A and B strongly commute (by [8], say).

One way of finding explicitly $|A|$ and $|B|$ is via the L^2-Fourier transform which we denote by $\mathcal{F}$. Clearly

$$(\mathcal{F}^*|A|\mathcal{F})f(t) = t^2 f(t) \quad \text{and} \quad (\mathcal{F}^*|B|\mathcal{F})f(t) = |t|f(t).$$

Observe also that using the Fourier transform, we may show that $A+B$ is normal (hence closed). Therefore, we obtain

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right| \leqslant \left|\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right| + \left|\frac{\mathrm{d}}{\mathrm{d}x}\right|.$$

(ii) Consider the so-called Cauchy–Riemann operator

$$\bar{\partial} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big)$$

and define $T = 2\bar{\partial}$, $A = \frac{\partial}{\partial x}$ and $B = \frac{\partial}{\partial y}$ on $L^2(\mathbb{R}^2)$. The domains are

$$D(A) = \{f \in L^2(\mathbb{R}^2) : Af \in L^2(\mathbb{R}^2)\}, \quad D(B) = \{f \in L^2(\mathbb{R}^2) : Bf \in L^2(\mathbb{R}^2)\}$$

and $D(T) = D(A) \cap D(B)$.

We can show that A and B commute strongly as before (i.e. using the self-adjoint "wave operator" AB. cf. [18]). Alternatively, we can show via the Stone theorem that $-\mathrm{i}A$ and $-\mathrm{i}B$ are the infinitesimal generators of the following unitary groups (defined on $L^2(\mathbb{R}^2)$)

$$U_1(s) = \mathrm{e}^{\mathrm{i}sA} \quad \text{and} \quad U_2(t) = \mathrm{e}^{\mathrm{i}tB}$$

where

$$U_1(s)f(x,y) = f(x+s,y) \quad \text{and} \quad U_2(t)f(x,y) = f(x,y+t)$$

(readers may wish to consult [23]).

Also, T is normal as it is unitarily equivalent to the multiplication operator by the complex-valued function $(\lambda,\mu) \mapsto \mathrm{i}\lambda - \mu$ on $L^2(\mathbb{R}^2)$ (readers should not be mislead to think that e.g. A is the real part of T for A is not self-adjoint).

Therefore,

$$\left|\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right| \leqslant \left|\frac{\partial}{\partial x}\right| + \left|\frac{\partial}{\partial y}\right|.$$

REMARK 3.13. Both examples above can be treated fairly easily using the L^2-Fourier transform. For example, for the first case:

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right| \leqslant \mathcal{F}M\mathcal{F}^*$$

with $Mf(t) = (t^2 + |t|)f(t)$.

In the second case, we use the Fourier transform $\mathcal{F}$ (on $L^2(\mathbb{R}^2)$ this time) to obtain:

$$\left|\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right| \leqslant \mathcal{F}M\mathcal{F}^*$$

with $Mf(\lambda,\mu) = (|\lambda| + |\mu|)f(\lambda,\mu)$.

COROLLARY 3.14. *Let A be a normal operator. Then*

$$|A| \leqslant |\overline{\mathrm{Re}A}| + |\overline{\mathrm{Im}A}|.$$

Proof. Since A is normal, $\overline{\mathrm{Re}A}$ and $\overline{\mathrm{Im}A}$ are strongly commuting self-adjoint operators. Hence $\overline{\mathrm{Re}A}$ and $\mathrm{i}\overline{\mathrm{Im}A}$ are strongly commuting normal operators. Now, apply Theorem 3.10. ∎

COROLLARY 3.15. *Let A and B be normal operators where $B \in B(H)$. If $BA \subset AB$, then*

$$|A+B| \leqslant |A| + |B|.$$

Proof. Since $B \in B(H)$, the condition $BA \subset AB$ amounts to the strong commutativity of A and B. It is also plain that $A+B$ is closed and this finishes the proof. ∎

As an interesting consequence, we obtain a characterization of invertibility for the class of unbounded normal operators.

PROPOSITION 3.16. *Let A be a normal operator. Then*

$$A \text{ is invertible} \Leftrightarrow |\overline{\mathrm{Re}A}| + |\overline{\mathrm{Im}A}| \text{ is invertible.}$$

Particularly, if $\lambda = \alpha + \mathrm{i}\beta$, then

$$\lambda \in \sigma(A) \Leftrightarrow |\overline{\mathrm{Re}A} - \alpha I| + |\overline{\mathrm{Im}A} - \beta I| \text{ is not invertible.}$$

Proof. Since A is invertible, so is $|A|$. Hence by Corollary 3.14, $|\overline{\mathrm{Re}A}| + |\overline{\mathrm{Im}A}|$ too is invertible. Conversely, by Propositions 3.7 and 3.8

$$|\overline{\mathrm{Re}A}| \leqslant |A| \quad \text{and} \quad |\overline{\mathrm{Im}A}| \leqslant |A|.$$

Hence, by the same arguments as in the previous proofs, we may easily establish

$$|\overline{\mathrm{Re}A}| + |\overline{\mathrm{Im}A}| \leqslant 2|A|.$$

Thus, the invertibility of $|\overline{\mathrm{Re}A}| + |\overline{\mathrm{Im}A}|$ implies the invertibility of $|A|$. Since the latter means that the normal operator A is left invertible, by [6] we get that A is invertible. ∎

Another consequence is the following.

COROLLARY 3.17. *Let A be a normal operator. Then*

$$\sigma(A) \subset \sigma(\overline{\mathrm{Re}A}) + \mathrm{i}\sigma(\overline{\mathrm{Im}A})$$

where the sum of sets is defined in the usual fashion.

Proof. Let $\lambda = \alpha + \mathrm{i}\beta \in \sigma(A)$. Proposition 3.16 tells us that $|\overline{\mathrm{Re}A} - \alpha I| + |\overline{\mathrm{Im}A} - \beta I|$ is not invertible. If either $|\overline{\mathrm{Re}A} - \alpha I|$ or $|\overline{\mathrm{Im}A} - \beta I|$ is invertible, then $|\overline{\mathrm{Re}A} - \alpha I| + |\overline{\mathrm{Im}A} - \beta I|$ would be invertible! Therefore, both $|\overline{\mathrm{Re}A} - \alpha I|$ and $|\overline{\mathrm{Im}A} - \beta I|$ are not invertible, i.e. $\overline{\mathrm{Re}A} - \alpha I$ and $\overline{\mathrm{Im}A} - \beta I$ are not invertible. In other words, $\alpha \in \sigma(\overline{\mathrm{Re}A})$ and $\beta \in \sigma(\overline{\mathrm{Im}A})$. Consequently, $\lambda \in \sigma(\overline{\mathrm{Re}A}) + \mathrm{i}\sigma(\overline{\mathrm{Im}A})$, as needed. ∎

We also obtain a very simple proof of the “realness” of the spectrum of unbounded self-adjoint operators.

COROLLARY 3.18. *Let A be a self-adjoint operator. Then $\sigma(A) \subset \mathbb{R}$.*

Proof. Let $\lambda \notin \mathbb{R}$, i.e. $\lambda = \alpha + i\beta$ ($\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^*$). Since $A - \alpha I$ is self-adjoint, it follows that $A - \alpha I - i\beta I$ is normal. By the invertibility of $|\beta|I$, it follows that of $|A - \alpha I| + |\beta I|$ ($\geqslant |\beta|I$). By Proposition 3.16, this just means that $A - \lambda I$ is invertible, that is, $\lambda \notin \sigma(A)$. ∎

Using a proof by induction, we can extend Theorem 3.10 to a finite family of normal operators.

THEOREM 3.19. *Let $(A_i)_{i=1,\dots,n}$ be pairwise strongly commuting normal operators. Then*

$$|\overline{A_1 + A_2 + \cdots + A_n}| \leqslant |A_1| + |A_2| + \cdots + |A_n|.$$

COROLLARY 3.20. *Let $(A_i)_{i=1,\dots,n}$ be pairwise commuting normal operators. If $B \in B(H)$ is hyponormal and $BA_i \subset A_iB$ for $i = 1, 2, \dots, n$, then*

$$|\overline{A_1 + A_2 + \cdots + A_n} + B| \leqslant |A_1| + |A_2| + \cdots + |A_n| + |B|.$$

Next, we pass to another triangle inequality.

THEOREM 3.21. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then*

$$||A| - |B|| \leqslant |A + B|.$$

Proof. The proof is in essence fairly similar to that of Theorem 3.1 and we omit some details. Clearly, $|A| - |B|$ and $A + B$ are closed and

$$D(|A+B|) = D(A+B) = D(A) = D(|A|) = D(|A|-|B|) = D(||A|-|B||).$$

Let $x \in D(A)$. Then

$$\begin{aligned}
\|||A| - |B||x\|^2 &= \|(|A| - |B|)x\|^2 = \langle(|A| - |B|)x, (|A| - |B|)x\rangle \\
&= \||A|x\|^2 + \||B|x\|^2 - \langle|A|x, |B|x\rangle - \langle|B|x, |A|x\rangle \\
&= \||A|x\|^2 + \||B|x\|^2 - \langle|B||A|x, x\rangle - \langle|A||B|x, x\rangle \\
&= \||A|x\|^2 + \||B|x\|^2 - 2\langle|A||B|x, x\rangle \\
&= \||A|x\|^2 + \||B|x\|^2 - 2\langle|A^*||B|x, x\rangle \\
&= \||A|x\|^2 + \||B|x\|^2 - 2\langle|A^*B|x, x\rangle.
\end{aligned}$$

Since A^*B is closed and hyponormal, we have for all x:

$$|\langle A^*Bx, x\rangle| \leqslant \langle|A^*B|x, x\rangle$$

and so

$$-2\langle|A^*B|x, x\rangle \leqslant -2|\langle A^*Bx, x\rangle| \leqslant 2\mathrm{Re}(\langle A^*Bx, x\rangle).$$

Hence

$$\begin{aligned}\||A|-|B||x\|^2 &\leqslant \||A|x\|^2+\||B|x\|^2+2\mathrm{Re}(\langle A^*Bx,x\rangle)\\ &= \||A|x\|^2+\||B|x\|^2+\langle A^*Bx,x\rangle+\overline{\langle A^*Bx,x\rangle}\\ &= \||A|x\|^2+\||B|x\|^2+\langle A^*Bx,x\rangle+\langle x,A^*Bx\rangle\\ &= \||A|x\|^2+\||B|x\|^2+\langle Bx,Ax\rangle+\langle Ax,Bx\rangle\\ &= \|Ax\|^2+\|Bx\|^2+\langle Bx,Ax\rangle+\langle Ax,Bx\rangle\\ &= \|(A+B)x\|^2=\||A+B|x\|^2.\end{aligned}$$

Therefore,

$$\||A|-|B||x\| \leqslant \||A+B|x\|,$$

i.e. we have shown that

$$||A|-|B||^2 \leqslant |A+B|^2$$

or merely

$$||A|-|B|| \leqslant |A+B|. \quad ∎$$

COROLLARY 3.22. *Let A be a normal operator and let $B \in B(H)$ be hyponormal. If $BA \subset AB$, then*

$$||A|-|B|| \leqslant |A-B|.$$

One may wonder whether we may adapt the proof of Theorem 3.10 to generalize the previous theorem to a couple of unbounded and strongly commuting normal operators. The first observation leads to an issue with domains which seems to be an insurmontable difficulty unless we impose an extra condition. For instance, we have the following theorem.

THEOREM 3.23. *Let A and B be two strongly commuting unbounded normal operators. If $D(\overline{A+B}) \subset D(\overline{|A|-|B|})$, then*

$$|\overline{|A|-|B|}| \leqslant |\overline{A+B}|.$$

Proof. Most details will be omitted. Write

$$A=\int_{\mathbb{C}} z\mathrm{d}E_A \quad \text{and} \quad B=\int_{\mathbb{C}} z'\mathrm{d}F_B.$$

Then

$$|\overline{|A|-|B|}| = \iint_{\mathbb{R}^2} ||z|-|z'||\mathrm{d}E_{A,B} \quad \text{and} \quad |\overline{A+B}| = \iint_{\mathbb{R}^2} |z+z'|\mathrm{d}E_{A,B}.$$

Argue as in Theorem 3.10 to obtain the desired inequality:

$$|\overline{|A|-|B|}| \leqslant |\overline{A+B}|. \quad ∎$$

REMARK 3.24. The condition $D(\overline{A+B}) \subset D(\overline{|A|-|B|})$ may be dropped at the cost of assuming that B (say) is A-bounded with a relative bounded strictly less than one. In such case $A+B$ is closed. Since $|B|$ is also $|A|$-bounded, $|A|-|B|$ too is closed. Everything else remains unchanged.

We finish with a related inequality.

PROPOSITION 3.25. *If A is a normal operator, then*

$$|\overline{|\overline{\mathrm{Re}A}|-|\overline{\mathrm{Im}A}|}| \leqslant |A|.$$

Proof. We know that $||\mathrm{Re}z|-|\mathrm{Im}z|| \leqslant |z|$ for any complex z. Also,

$$|\overline{\mathrm{Re}A}| = \int_{\mathbb{R}} |\mathrm{Re}z| \mathrm{d}E_{\overline{\mathrm{Re}A}} \quad \text{and} \quad |\overline{\mathrm{Im}A}| = \int_{\mathbb{R}} |\mathrm{Im}z| \mathrm{d}F_{\overline{\mathrm{Im}A}}.$$

Since $|\overline{\mathrm{Re}A}|$ and $|\overline{\mathrm{Im}A}|$ are commuting self-adjoint operators thanks to the normality of A, we may form the double integral defining $|\overline{|\overline{\mathrm{Re}A}|-|\overline{\mathrm{Im}A}|}|$. Therefore, we need only check the wanted inclusion of domains (and then proceed as in the proof of Theorem 3.10). Since A is normal, $D(A) = D(A^*)$. Hence

$$D(A) = D(\mathrm{Re}A) \subset D(\overline{\mathrm{Re}A}) = D(|\overline{\mathrm{Re}A}|) \quad \text{and} \quad D(A) = D(|\overline{\mathrm{Im}A}|)$$

and so

$$D(|A|) = D(A) \subset D(|\overline{\mathrm{Re}A}|) \cap D(|\overline{\mathrm{Im}A}|) = D(|\overline{|\overline{\mathrm{Re}A}|-|\overline{\mathrm{Im}A}|}|),$$

finishing the proof. ∎

4. QUESTIONS

Although we have more or less succeeded in dealing with the absolute value of unbounded operators, some questions remain unfortunately unanswered. For instance:

(i) If $A, B \in B(H)$ are hyponormal and commuting, then do we have

$$|A+B| \leqslant |A| + |B|?$$

This question is interesting in the sense that there are no counterexamples if $\dim H < \infty$ for it is shown that the previous inequality holds for commuting normal operators (and as is known the classes of normal and hyponormal operators coincide when $\dim H < \infty$).

(ii) Can Theorem 3.9 be weakened to the class of closed hyponormal operators without making further assumptions?

(iii) As observed above, can we prove

$$|\overline{|A|-|B|}| \leqslant |\overline{A+B}|$$

by solely assuming that A and B are strongly commuting normal operators?

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