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OPERATOR-VALUED LOCAL HARDY SPACES

RUNLIAN XIA and XIAO XIONG

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ABSTRACT. This paper gives a systematic study of operator-valued local Hardy spaces, which are localizations of the Hardy spaces defined by Mei. We prove the h_1-bmo duality and the h_p-h_q duality for any conjugate pair (p,q) when $1 < p < \infty$. We show that $\mathrm{h}_1(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{bmo}(\mathbb{R}^d, \mathcal{M})$ are also good endpoints of $L_p(L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M})$ for interpolation. We obtain the local version of Calderón–Zygmund theory, and then deduce that the Poisson kernel in our definition of the local Hardy norms can be replaced by any reasonable test function. Finally, we establish the atomic decomposition of the local Hardy space $\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

KEYWORDS: *Noncommutative L_p-spaces, operator-valued Hardy spaces, operator-valued* bmo *spaces, duality, interpolation, Calderón–Zygmund theory, characterization, atomic decomposition.*

MSC (2010): Primary 46L52, 42B30; Secondary 46L07, 47L65.

1. INTRODUCTION AND PRELIMINARIES

This paper is devoted to the study of operator-valued local Hardy spaces. It follows the current line of the investigation of noncommutative harmonic analysis. This field arose from the noncommutative integration theory developed by Murray and von Neumann, in order to provide a mathematical foundation for quantum mechanics. The objective was to construct and study a linear functional on an operator algebra which plays the role of the classical integral. In [30], Pisier and Xu developed a pioneering work on noncommutative martingale theory; since then, many classical results have been successfully transferred to the noncommutative setting, see for instance, [1], [3], [12], [14], [19], [20], [21], [22], [26], [28], [32], [33], [34].

Inspired by the above mentioned developments and the Littlewood–Paley-Stein theory of quantum Markov semigroups (cf. [15], [16], [18]), Mei [24] studied operator-valued Hardy spaces, which are defined by the Littlewood–Paley g-function and Lusin area integral function associated to the Poisson kernel. These

spaces are shown to be very useful for many aspects of noncommutative harmonic analysis. In [41], we obtain general characterizations of Mei's Hardy spaces, which state that the Poisson kernel can be replaced by any reasonable test function. This is done mainly by using the operator-valued Calderón–Zygmund theory.

In the classical setting, the theory of Hardy spaces is one of the most important topics in harmoic analysis. The local Hardy spaces $\mathrm{h}_p(\mathbb{R}^d)$ were first introduced by Goldberg [10]. These spaces are viewed as local or inhomogeneous counterparts of the classical real Hardy spaces $\mathcal{H}_p(\mathbb{R}^d)$ [6]. Goldberg's motivation of introducing these local spaces was the study of pseudo-differential operators. It is known that pseudo-differential operators are not necessarily bounded on the classical Hardy space $\mathcal{H}_1(\mathbb{R}^d)$, but bounded on $\mathrm{h}_1(\mathbb{R}^d)$ under some appropriate assumptions. Afterwards, many other inhomogeneous spaces have also been studied. Our references for the classical theory are [7], [10], [37]. However, they have not been investigated so far in the operator-valued case.

Motivated by [24], [41], [42], we provide a localization of Mei's operator-valued Hardy spaces on $\mathbb{R}^d$ in this paper. The norms of these spaces are partly given by the truncated versions of the Littlewood–Paley g-function and Lusin area integral function. Some techniques that we use to deal with our local Hardy spaces are modelled after those of [41]; however, some highly non-trivial modifications are needed. This happens since with the truncation, we only know the L_p-norms of the Poisson integrals of functions on the strip $\mathbb{R}^d \times (0,1)$, and lose information when the time is large. This brings some substantial difficulties that the non-local case does not have, for example, in the duality problem. Moreover, the noncommutative maximal function method is still unavailable in this setting, while in the classical case it is efficiently and frequently employed. However, based on tools developed recently, for instance, in [14], [18], [20], [24], [25], [30], [32], [33], we can overcome these difficulties.

Let us present here the four main results of this paper. The first family of results concerns of the operator-valued local Hardy spaces $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. The first major result of this part is the $\mathrm{h}_p^{\mathrm{c}}$-$\mathrm{bmo}_q^{\mathrm{c}}$ duality for $1 \leqslant p < 2$, where q denotes the conjugate index of p. In particular, when $p = 1$, we obtain the operator-valued local analogue of the classical Fefferman–Stein theorem. The pattern of the proof of this theorem is similar to that of Mei's non-local case. We also show that

$$\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \quad \text{for } 2 < q < \infty$$

like in the martingale and non-local settings. Thus the dual of $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ agrees with $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ when $1 < p \leqslant 2$.

The second major result shows that the local Hardy spaces behave well with respect to both complex and real interpolations. In particular, we have

$$(\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}), \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}))_{1/p} = \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$$

for $1 < p < \infty$. We reduce this interpolation problem to the corresponding one on the non-local Hardy spaces in order to use Mei's interpolation result in [24]. This proof is quite simple.

The third major result concerns the Calderón–Zygmund theory. We know that the usual $\mathcal{M}$-valued Calderón–Zygmund operators which satisfy the Hörmander condition are in general not bounded on inhomogeneous spaces. Thus, in order to guarantee the boundedness of a Calderón–Zygmund operator on $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, we need to impose an extra decay at infinity to the kernel.

The Calderón–Zygmund theory mentioned above will be applied to the general characterization of $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ with the Poisson kernel replaced by any reasonable test function. This characterization will play an important role in our recent study of (inhomogeneous) Triebel–Lizorkin spaces on $\mathbb{R}^d$, see [39].

1.1. NOTATION. In the following, we collect some notation which will be frequently used in this paper. Throughout, we will use the notation $A \lesssim B$, which is an inequality up to a constant: $A \leqslant cB$ for some constant $c > 0$. The relevant constants in all such inequalities may depend on the dimension d, the test function Φ or p, etc., but never on the function f in consideration. The equivalence $A \approx B$ will mean $A \lesssim B$ and $B \lesssim A$ simultaneously.

The Bessel and Riesz potentials are $J^\alpha = (1 - (2\pi)^{-2}\Delta)^{\alpha/2}$ and $I^\alpha = (-(2\pi)^{-2}\Delta)^{\alpha/2}$, respectively. If $\alpha = 1$, we will abbreviate J^1 as J and I^1 as I. We denote also $J_\alpha(\xi) = (1 + |\xi|^2)^{\alpha/2}$ on $\mathbb{R}^d$ and $I_\alpha(\xi) = |\xi|^\alpha$ on $\mathbb{R}^d \setminus \{0\}$. Then J_α and I_α are the symbols of the Fourier multipliers J^α and I^α, respectively.

We denote by $H_2^\sigma(\mathbb{R}^d)$ the potential Sobolev space, consisting of all tempered distributions f such that $J^\sigma(f) \in L_2(\mathbb{R}^d)$. If $\sigma > \frac{d}{2}$, the elements in $H_2^\sigma(\mathbb{R}^d)$ will serve as important convolution kernels in the sequel.

1.2. NONCOMMUTATIVE L_p-SPACES. Now we recall some preliminaries on noncommutative L_p-spaces and operator-valued Hardy spaces. We start with a brief introduction to noncommutative L_p-spaces. Let $\mathcal{M}$ be a von Neumann algebra equipped with a normal semifinite faithful trace τ and $S_{\mathcal{M}}^+$ be the set of all positive elements x in $\mathcal{M}$ with $\tau(s(x)) < \infty$, where $s(x)$ denotes the support of x, i.e., the smallest projection e such that $exe = x$. Let $S_{\mathcal{M}}$ be the linear span of $S_{\mathcal{M}}^+$. Then every $x \in S_{\mathcal{M}}$ has finite trace, and $S_{\mathcal{M}}$ is a w*-dense $*$-subalgebra of $\mathcal{M}$.

Let $1 \leqslant p < \infty$. For any $x \in S_{\mathcal{M}}$, the operator $|x|^p$ belongs to $S_{\mathcal{M}}^+$ (recalling $|x| = (x^*x)^{1/2}$). We define

$$\|x\|_p = (\tau(|x|^p))^{1/p}.$$

One can prove that $\|\cdot\|_p$ is a norm on $S_{\mathcal{M}}$. The completion of $(S_{\mathcal{M}}, \|\cdot\|_p)$ is denoted by $L_p(\mathcal{M})$, which is the usual noncommutative L_p-space associated to $(\mathcal{M}, \tau)$. In this paper, the norm of $L_p(\mathcal{M})$ will be often denoted simply by $\|\cdot\|_p$ if there is no confusion. But if different L_p-spaces appear in a same context, we

will precise their norms in order to avoid possible ambiguity. We refer the reader to [44] and [31] for further information on noncommutative L_p-spaces.

Let us now introduce noncommutative Hilbert space-valued L_p-spaces $L_p(\mathcal{M}; H^{\rm c})$ and $L_p(\mathcal{M}; H^{\rm r})$, which are studied at length in [18]. Let H be a Hilbert space and $v \in H$ with $\|v\| = 1$, and p_v be the orthogonal projection onto the one-dimensional subspace generated by v. Then define the following row and column noncommutative L_p-spaces:

$$L_p(\mathcal{M}; H^{\rm r}) = (p_v \otimes 1_{\mathcal{M}}) L_p(B(H)\overline{\otimes}\mathcal{M}) \quad \text{and}$$
$$L_p(\mathcal{M}; H^{\rm c}) = L_p(B(H)\overline{\otimes}\mathcal{M})(p_v \otimes 1_{\mathcal{M}}),$$

where the tensor product $B(H)\overline{\otimes}\mathcal{M}$ is equipped with the tensor trace while $B(H)$ is equipped with the usual trace, and where $1_{\mathcal{M}}$ denotes the unit of $\mathcal{M}$. For $f \in L_p(\mathcal{M}; H^{\rm c})$,

$$\|f\|_{L_p(\mathcal{M};H^{\rm c})} = \|(f^* f)^{1/2}\|_p.$$

A similar formula holds for the row space by passing to adjoint: $f \in L_p(\mathcal{M}; H^{\rm r})$ if and only if $f^* \in L_p(\mathcal{M}; H^{\rm c})$, and $\|f\|_{L_p(\mathcal{M};H^{\rm r})} = \|f^*\|_{L_p(\mathcal{M};H^{\rm c})}$. It is clear that $L_p(\mathcal{M}; H^{\rm c})$ and $L_p(\mathcal{M}; H^{\rm r})$ are 1-complemented subspaces of $L_p(B(H)\overline{\otimes}\mathcal{M})$ for any p.

1.3. OPERATOR-VALUED HARDY SPACES. Throughout the remainder of the paper, unless explicitly stated otherwise, $(\mathcal{M}, \tau)$ will be fixed as before and $\mathcal{N} = L_\infty(\mathbb{R}^d)\overline{\otimes}\mathcal{M}$, equipped with the tensor trace. In this subsection, we introduce Mei's operator-valued Hardy spaces. Contrary to the custom, we will use letters s, t to denote variables of $\mathbb{R}^d$ since letters x, y are reserved for operators in noncommutative L_p-spaces. Accordingly, a generic element of the upper half-space $\mathbb{R}^{d+1}_+$ will be denoted by (s, ε) with $\varepsilon > 0$, where $\mathbb{R}^{d+1}_+ = \{(s, \varepsilon) : s \in \mathbb{R}^d, \varepsilon > 0\}$.

Let P be the Poisson kernel on $\mathbb{R}^d$:

$$\mathrm{P}(s) = c_d \, \frac{1}{(|s|^2 + 1)^{(d+1)/2}}$$

with c_d the usual normalizing constant and $|s|$ the Euclidean norm of s. Let

$$\mathrm{P}_\varepsilon(s) = \frac{1}{\varepsilon^d} \mathrm{P}\Big(\frac{s}{\varepsilon}\Big) = c_d \, \frac{\varepsilon}{(|s|^2 + \varepsilon^2)^{(d+1)/2}}.$$

For any function f on $\mathbb{R}^d$ with values in $L_1(\mathcal{M}) + \mathcal{M}$, its Poisson integral, whenever it exists, will be denoted by $\mathrm{P}_\varepsilon(f)$:

$$\mathrm{P}_\varepsilon(f)(s) = \int_{\mathbb{R}^d} \mathrm{P}_\varepsilon(s - t) f(t) \mathrm{d}t, \quad (s, \varepsilon) \in \mathbb{R}^{d+1}_+.$$

Note that the Poisson integral of f exists if

$$f \in L_1\Big(\mathcal{M}; L_2^{\rm c}\Big(\mathbb{R}^d, \frac{\mathrm{d}t}{1 + |t|^{d+1}}\Big)\Big) + L_\infty\Big(\mathcal{M}; L_2^{\rm c}\Big(\mathbb{R}^d, \frac{\mathrm{d}t}{1 + |t|^{d+1}}\Big)\Big).$$

This space is the right space in which all functions considered in this paper live as far as only column spaces are involved. As it will appear frequently later, to simplify notation, we will denote the Hilbert space $L_2\Big(\mathbb{R}^d, \frac{\mathrm{d}t}{1+|t|^{d+1}}\Big)$ by R_d:

$$\mathrm{R}_d = L_2\Big(\mathbb{R}^d, \frac{\mathrm{d}t}{1+|t|^{d+1}}\Big). \tag{1.1}$$

The Lusin area square function of f is defined by

$$S^{\mathrm{c}}(f)(s) = \Big(\int\limits_{\Gamma} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\Big|^2 \frac{\mathrm{d}t\,\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\Big)^{1/2}, \quad s\in\mathbb{R}^d, \tag{1.2}$$

where Γ is the cone $\{(t,\varepsilon)\in\mathbb{R}^{d+1}_+ : |t|<\varepsilon\}$. For $1\leqslant p<\infty$ define the column Hardy space $\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$ to be

$$\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M}) = \{f : \|f\|_{\mathcal{H}^{\mathrm{c}}_p} = \|S^{\mathrm{c}}(f)\|_p<\infty\}.$$

Note that [24] uses the gradient of $\mathrm{P}_\varepsilon(f)$ instead of the sole radial derivative in the definition of S^{c} above, but this does not affect $\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$ (up to equivalent norms). At the same time, it is proved in [24] that $\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$ can be equally defined by the Littlewood–Paley g-function:

$$G^{\mathrm{c}}(f)(s) = \Big(\int\limits_0^\infty \varepsilon\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\Big|^2\mathrm{d}\varepsilon\Big)^{1/2}, \quad s\in\mathbb{R}^d. \tag{1.3}$$

Thus

$$\|f\|_{\mathcal{H}^{\mathrm{c}}_p}\approx\|G^{\mathrm{c}}(f)\|_p, \quad f\in\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M}).$$

The row Hardy space $\mathcal{H}^{\mathrm{r}}_p(\mathbb{R}^d,\mathcal{M})$ is the space of all f such that $f^*\in\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$, equipped with the norm $\|f\|_{\mathcal{H}^{\mathrm{r}}_p}=\|f^*\|_{\mathcal{H}^{\mathrm{c}}_p}$. Finally, we define the mixture space $\mathcal{H}_p(\mathbb{R}^d,\mathcal{M})$ as

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M}) + \mathcal{H}^{\mathrm{r}}_p(\mathbb{R}^d,\mathcal{M}) \quad \text{for } 1\leqslant p\leqslant 2$$

equipped with the sum norm

$$\|f\|_{\mathcal{H}_p} = \inf\{\|f_1\|_{\mathcal{H}^{\mathrm{c}}_p}+\|f_2\|_{\mathcal{H}^{\mathrm{r}}_p} : f=f_1+f_2\}, \quad \text{and}$$
$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})\cap\mathcal{H}^{\mathrm{r}}_p(\mathbb{R}^d,\mathcal{M}) \quad \text{for } 2<p<\infty$$

equipped with the intersection norm

$$\|f\|_{\mathcal{H}_p} = \max(\|f\|_{\mathcal{H}^{\mathrm{c}}_p}, \|f\|_{\mathcal{H}^{\mathrm{r}}_p}).$$

Observe that

$$\mathcal{H}^{\mathrm{c}}_2(\mathbb{R}^d,\mathcal{M}) = \mathcal{H}^{\mathrm{r}}_2(\mathbb{R}^d,\mathcal{M}) = L_2(\mathcal{N}) \quad \text{with equivalent norms.}$$

It is proved in [24] that for $1<p<\infty$

$$\mathcal{H}_p(\mathbb{R}^d,\mathcal{M}) = L_p(\mathcal{N}) \quad \text{with equivalent norms.}$$

The operator-valued BMO spaces are also studied in [24]. Let Q be a cube in $\mathbb{R}^d$ (with sides parallel to the axes) and $|Q|$ its volume. For a function f with values in $\mathcal{M}$, f_Q denotes its mean over Q:

$$f_Q = \frac{1}{|Q|}\int_Q f(t)\mathrm{d}t.$$

The column BMO norm of f is defined to be

$$\|f\|_{\mathrm{BMO^c}} = \sup_{Q\subset\mathbb{R}^d}\Big\|\frac{1}{|Q|}\int_Q |f(t)-f_Q|^2\mathrm{d}t\Big\|_{\mathcal{M}}^{1/2}. \tag{1.4}$$

Then

$$\mathrm{BMO^c}(\mathbb{R}^d,\mathcal{M}) = \{f\in L_\infty(\mathcal{M};\mathrm{R_d^c}) : \|f\|_{\mathrm{BMO^c}}<\infty\}.$$

Similarly, we define the row space $\mathrm{BMO^r}(\mathbb{R}^d,\mathcal{M})$ as the space of f with the property that f^* lies in $\mathrm{BMO^c}(\mathbb{R}^d,\mathcal{M})$, and $\mathrm{BMO}(\mathbb{R}^d,\mathcal{M}) = \mathrm{BMO^c}(\mathbb{R}^d,\mathcal{M})\cap \mathrm{BMO^r}(\mathbb{R}^d,\mathcal{M})$ with the intersection norm.

It is showed in [24] that the dual of $\mathcal{H}_1^c(\mathbb{R}^d,\mathcal{M})$ can be identified with $\mathrm{BMO^c}(\mathbb{R}^d,\mathcal{M})$. This is the operator-valued analogue of the celebrated Fefferman–Stein H_1-BMO duality theorem.

On the other hand, one of the main results of [41] asserts that the Poisson kernel in the definition of Hardy spaces can be replaced by more general test functions.

Take any Schwartz function Φ with vanishing mean. We will assume that Φ is nondegenerate in the following sense:

$$\forall\,\xi\in\mathbb{R}^d\setminus\{0\}\ \exists\,\varepsilon>0,\ \text{such that}\ \widehat{\Phi}(\varepsilon\xi)\neq 0. \tag{1.5}$$

Set $\Phi_\varepsilon(s)=\varepsilon^{-d}\Phi(\frac{s}{\varepsilon})$ for $\varepsilon>0$. The radial and conic square functions of f associated to Φ are defined by replacing the partial derivative of the Poisson kernel P in $S^c(f)$ and $G^c(f)$ by Φ :

$$S_\Phi^c(f)(s) = \Big(\int_\Gamma |\Phi_\varepsilon * f(s+t)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\Big)^{1/2},\quad s\in\mathbb{R}^d \quad\text{and} \tag{1.6}$$

$$G_\Phi^c(f)(s) = \Big(\int_0^\infty |\Phi_\varepsilon * f(s)|^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}. \tag{1.7}$$

The following two lemmas are taken from [41]. The first one says that the two square functions above define equivalent norms in $\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$.

LEMMA 1.1. *Let $1\leqslant p<\infty$ and $f\in L_1(\mathcal{M};\mathrm{R_d^c})+L_\infty(\mathcal{M};\mathrm{R_d^c})$. Then $f\in\mathcal{H}_p^c(\mathbb{R}^d,\mathcal{M})$ if and only if $G_\Phi^c(f)\in L_p(\mathcal{N})$ if and only if $S_\Phi^c(f)\in L_p(\mathcal{N})$. If this is the case, then*

$$\|G_\Phi^c(f)\|_p \approx \|S_\Phi^c(f)\|_p \approx \|f\|_{\mathcal{H}_p^c}$$

with the relevant constants depending only on p,d and Φ.

The above square functions G_Φ^{c} and S_Φ^{c} can be discretized as follows:

$$
\begin{aligned}
G_\Phi^{\mathrm{c},D}(f)(s) &= \Big(\sum_{j=-\infty}^{\infty} |\Phi_{2^{-j}} * f(s)|^2 \Big)^{1/2} \\
S_\Phi^{\mathrm{c},D}(f)(s) &= \Big(\sum_{j=-\infty}^{\infty} 2^{dj} \int\limits_{B(s,2^{-j})} |\Phi_{2^{-j}} * f(t)|^2 \mathrm{d}t \Big)^{1/2}.
\end{aligned} \tag{1.8}
$$

Here $B(s,r)$ denotes the ball of $\mathbb{R}^d$ with center s and radius r. To prove that these discrete square functions also describe our Hardy spaces, we need to impose the following condition on the previous Schwartz function Φ, which is stronger than (1.5):

$$
\forall\, \xi \in \mathbb{R}^d \setminus \{0\}\ \exists\, 0 < 2a \leqslant b < \infty \ \text{ such that } \ \widehat{\Phi}(\varepsilon\xi) \neq 0,\ \forall\, \varepsilon \in (a,\, b]. \tag{1.9}
$$

The following is the discrete version of Lemma 1.1.

LEMMA 1.2. *Let $1 \leqslant p < \infty$ and $f \in L_1(\mathcal{M};\mathrm{R}_d^{\mathrm{c}}) + L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$. Then $f \in \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ if and only if $G_\Phi^{\mathrm{c},D}(f) \in L_p(\mathcal{N})$ if and only if $S_\Phi^{\mathrm{c},D}(f) \in L_p(\mathcal{N})$. Moreover,*

$$
\|G_\Phi^{\mathrm{c},D}(f)\|_p \approx \|S_\Phi^{\mathrm{c},D}(f)\|_p \approx \|f\|_{\mathcal{H}_p^{\mathrm{c}}}
$$

with the relevant constants depending only on p, d and Φ.

Finally, let us give some easy facts on operator-valued functions. The first one is the following Cauchy–Schwarz type inequality for the operator-valued square function,

$$
\Big| \int_{\mathbb{R}^d} \phi(s) f(s) \mathrm{d}s \Big|^2 \leqslant \int_{\mathbb{R}^d} |\phi(s)|^2 \mathrm{d}s \int_{\mathbb{R}^d} |f(s)|^2 \mathrm{d}s, \tag{1.10}
$$

where $\phi : \mathbb{R}^d \to \mathbb{C}$ and $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$ are functions such that all integrations of the above inequality make sense. We also require the operator-valued version of the Plancherel formula. For sufficiently nice functions $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$, for example, for $f \in L_2(\mathbb{R}^d) \otimes L_2(\mathcal{M})$, we have

$$
\int_{\mathbb{R}^d} |f(s)|^2 \mathrm{d}s = \int_{\mathbb{R}^d} |\widehat{f}(\xi)|^2 \mathrm{d}\xi. \tag{1.11}
$$

Given two nice functions f and g, the polarized version of the above equality is

$$
\int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s = \int_{\mathbb{R}^d} \widehat{f}(\xi) \widehat{g}(\xi)^* \mathrm{d}\xi. \tag{1.12}
$$

The paper is organized as follows. In the next section, we give the definitions of operator-valued local Hardy and bmo spaces. Section 3 is devoted to the proofs of duality results, including the h_1-bmo duality and the h_p-h_q duality for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Section 4 gives the results on interpolation. In Section 5, we develop Calderón–Zymund theory that is suitable for our local version

of Hardy spaces. In Section 6, we prove general characterizations of $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, and then connect the local Hardy spaces $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ with Mei's non-local Hardy spaces $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. In the last section of this paper, we give the atomic decomposition of $\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$.

2. OPERATOR-VALUED LOCAL HARDY SPACES

2.1. OPERATOR-VALUED LOCAL HARDY SPACES. In this subsection, we give the definition of operator-valued local Hardy spaces as well as some basic facts on them. Let $f\in L_1(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})+L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$ (recalling that the Hilbert space R_d is defined by (1.1)). Then the Poisson integral of f is well-defined and takes values in $L_1(\mathcal{M})+\mathcal{M}$. Now we define the local analogue of the Lusin area square function of f by

$$s^{\mathrm{c}}(f)(s)=\Big(\int\limits_{\widetilde{\Gamma}}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\Big|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\Big)^{1/2},\quad s\in\mathbb{R}^d,$$

where $\widetilde{\Gamma}$ is the truncated cone $\{(t,\varepsilon)\in\mathbb{R}_+^{d+1}:|t|<\varepsilon<1\}$. It is the intersection of the cone $\{(t,\varepsilon)\in\mathbb{R}_+^{d+1}:|t|<\varepsilon\}$ and the strip $S\subset\mathbb{R}_+^{d+1}$ defined by:

$$S=\{(s,\varepsilon):s\in\mathbb{R}^d,0<\varepsilon<1\}.$$

For $1\leqslant p<\infty$ define the column local Hardy space $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ to be

$$\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})=\{f\in L_1(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})+L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}}):\|f\|_{\mathrm{h}_p^{\mathrm{c}}}<\infty\},$$

where the $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$-norm of f is defined by

$$\|f\|_{\mathrm{h}_p^{\mathrm{c}}}=\|s^{\mathrm{c}}(f)\|_{L_p(\mathcal{N})}+\|\mathrm{P}*f\|_{L_p(\mathcal{N})}.$$

The row local Hardy space $\mathrm{h}_p^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})$ is the space of all f such that $f^*\in\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, equipped with the norm $\|f\|_{\mathrm{h}_p^{\mathrm{r}}}=\|f^*\|_{\mathrm{h}_p^{\mathrm{c}}}$. Moreover, define the mixture space $\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})$ as follows:

$$\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})+\mathrm{h}_p^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})\quad\text{for } 1\leqslant p\leqslant 2$$

equipped with the sum norm

$$\|f\|_{\mathrm{h}_p}=\inf\{\|g\|_{\mathrm{h}_p^{\mathrm{c}}}+\|h\|_{\mathrm{h}_p^{\mathrm{r}}}:f=g+h,g\in\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),h\in\mathrm{h}_p^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})\},\quad\text{and}$$
$$\mathrm{h}_p(\mathbb{R}^d,\mathcal{M})=\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})\cap\mathrm{h}_p^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})\quad\text{for } 2<p<\infty$$

equipped with the intersection norm

$$\|f\|_{\mathrm{h}_p}=\max\{\|f\|_{\mathrm{h}_p^{\mathrm{c}}},\|f\|_{\mathrm{h}_p^{\mathrm{r}}}\}.$$

The local analogue of the Littlewood–Paley g-function of f is defined by

$$g^{\mathrm{c}}(f)(s) = \Big(\int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s) \Big|^2 \varepsilon \mathrm{d}\varepsilon \Big)^{1/2}, \quad s \in \mathbb{R}^d.$$

We will see in Section 6 that, for all $1 \leqslant p < \infty$,

$$\|s^{\mathrm{c}}(f)\|_p + \|\mathrm{P} * f\|_p \approx \|g^{\mathrm{c}}(f)\|_p + \|\mathrm{P} * f\|_p.$$

In the following, we give some easy facts that will be frequently used later. Firstly, we have

$$\|s^{\mathrm{c}}(f)\|_2^2 + \|\mathrm{P} * f\|_2^2 \approx \|f\|_2^2. \tag{2.1}$$

Indeed, by (1.11), we have

$$\int_{\mathbb{R}^d} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s) \Big|^2 \mathrm{d}s = \int_{\mathbb{R}^d} \Big| \widehat{\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon}(\xi) \Big|^2 |\widehat{f}(\xi)|^2 \mathrm{d}\xi = \int_{\mathbb{R}^d} 4\pi^2 |\xi|^2 |\widehat{f}(\xi)|^2 \mathrm{e}^{-4\pi\varepsilon|\xi|} \mathrm{d}\xi.$$

Then

$$\int_{\mathbb{R}^d} \int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s) \Big|^2 \varepsilon \mathrm{d}\varepsilon \mathrm{d}s = \frac{1}{4} \int_{\mathbb{R}^d} (1 - \mathrm{e}^{-4\pi|\xi|} - 4\pi|\xi| \mathrm{e}^{-4\pi|\xi|}) |\widehat{f}(\xi)|^2 \mathrm{d}\xi.$$

Therefore

$$\begin{aligned} \|s^{\mathrm{c}}(f)\|_2^2 &= \tau \int_{\mathbb{R}^d} \int_{\widetilde{\Gamma}} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s+t) \Big|^2 \frac{\mathrm{d}\varepsilon \mathrm{d}t}{\varepsilon^{d-1}} \mathrm{d}s = \tau \int_{\mathbb{R}^d} \int_0^1 \int_{B(s,\varepsilon)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \frac{\mathrm{d}\varepsilon \mathrm{d}t}{\varepsilon^{d-1}} \mathrm{d}s \\ &= c_d \tau \int_{\mathbb{R}^d} \int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s) \Big|^2 \varepsilon \mathrm{d}\varepsilon \mathrm{d}s = \frac{c_d}{4} \tau \int_{\mathbb{R}^d} (1 - \mathrm{e}^{-4\pi|\xi|} - 4\pi|\xi| \mathrm{e}^{-4\pi|\xi|}) |\widehat{f}(\xi)|^2 \mathrm{d}\xi, \end{aligned}$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Meanwhile,

$$\|\mathrm{P} * f\|_2^2 = \tau \int_{\mathbb{R}^d} \mathrm{e}^{-4\pi|\xi|} |\widehat{f}(\xi)|^2 \mathrm{d}\xi.$$

Then we deduce (2.1) from the equality

$$\frac{4}{c_d} \|s^{\mathrm{c}}(f)\|_2^2 + \|\mathrm{P} * f\|_2^2 = \tau \int_{\mathbb{R}^d} (1 - 4\pi|\xi| \mathrm{e}^{-4\pi|\xi|}) |\widehat{f}(\xi)|^2 \mathrm{d}\xi$$

and the fact that $0 \leqslant 4\pi|\xi| \mathrm{e}^{-4\pi|\xi|} \leqslant \frac{1}{\mathrm{e}}$ for every $\xi \in \mathbb{R}^d$. Passing to adjoint, (2.1) also tells us that $\|f\|_{\mathrm{h}_2^{\mathrm{r}}(\mathbb{R}^d, \mathcal{M})} \approx \|f^*\|_2 = \|f\|_2$. Then we have, with equivalent norms,

$$\mathrm{h}_2^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_2^{\mathrm{r}}(\mathbb{R}^d, \mathcal{M}) = L_2(\mathcal{N}). \tag{2.2}$$

Next, if we apply (1.12) instead of (1.11) in the above proof, we get the following polarized version of (2.1),

$$\begin{aligned}\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s &= 4\int_{\mathbb{R}^d}\int_0^1 \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\,\mathrm{d}\varepsilon\mathrm{d}s\\ &\quad+\int_{\mathbb{R}^d}\mathrm{P}{*}f(s)(\mathrm{P}*g(s))^*\mathrm{d}s+4\pi\int_{\mathbb{R}^d}\mathrm{P}{*}f(s)(I(\mathrm{P})*g(s))^*\mathrm{d}s\\ &= \frac{4}{c_d}\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s+t)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\mathrm{d}s\\ &\quad+\int_{\mathbb{R}^d}\mathrm{P}{*}f(s)(\mathrm{P}{*}g(s))^*\mathrm{d}s+4\pi\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*\mathrm{d}s\end{aligned}\tag{2.3}$$

for nice $f, g \in L_1(\mathcal{M};\mathrm{R}_d^{\mathrm{c}}) + L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$ (recalling that I is the Riesz potential of order 1).

2.2. OPERATOR-VALUED BMO SPACES. Now we introduce the noncommutative analogue of bmo spaces defined in [10]. For any cube $Q \subset \mathbb{R}^d$, we denote its center by c_Q, its side length by $l(Q)$, and its volume by $|Q|$. Let $f \in L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$. The mean value of f over Q is denoted by $f_Q := \frac{1}{|Q|}\int_Q f(s)\mathrm{d}s$. We set

$$\|f\|_{\mathrm{bmo}^{\mathrm{c}}} = \max\Big\{\sup_{|Q|<1}\Big\|\Big(\frac{1}{|Q|}\int_Q |f-f_Q|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}, \sup_{|Q|=1}\Big\|\Big(\int_Q |f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}\Big\}.\tag{2.4}$$

Then we define

$$\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) = \{f \in L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}}) : \|f\|_{\mathrm{bmo}^{\mathrm{c}}} < \infty\}.$$

Respectively, define $\mathrm{bmo}^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})$ to be the space of all $f \in L^\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{r}})$ such that

$$\|f^*\|_{\mathrm{bmo}^{\mathrm{c}}} < \infty$$

with the norm $\|f\|_{\mathrm{bmo}^{\mathrm{r}}} = \|f^*\|_{\mathrm{bmo}^{\mathrm{c}}}$. And $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ is defined as the intersection of these two spaces

$$\mathrm{bmo}(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \cap \mathrm{bmo}^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})$$

equipped with the norm

$$\|f\|_{\mathrm{bmo}} = \max\{\|f\|_{\mathrm{bmo}^{\mathrm{c}}}, \|f\|_{\mathrm{bmo}^{\mathrm{r}}}\}.$$

REMARK 2.1. Let Q be a cube with volume $k^d \leqslant |Q| < (k+1)^d$ for some positive integer k. Then Q can be covered by at most $(k+1)^d$ cubes with volume 1, say Q_j's. Evidently,

$$\frac{1}{|Q|}\int_Q |f|^2\mathrm{d}t \leqslant k^{-d}\int_Q |f|^2\mathrm{d}t \leqslant k^{-d}\sum_{j=1}^{(k+1)^d}\int_{Q_j}|f|^2\mathrm{d}t.$$

Hence,

$$\sup_{|Q|\geqslant 1}\Big\|\Big(\frac{1}{|Q|}\int_Q |f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}} \leqslant 2^{d/2}\sup_{|Q|=1}\Big\|\Big(\int_Q |f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}.$$

Thus, if we replace the second supremum in (2.4) over all cubes of volume one by that over all cubes of volume not less than one, we get an equivalent norm of $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$.

PROPOSITION 2.2. *Let $f\in \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. Then*

$$\|f\|_{L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})}\lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}}.$$

Moreover, $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$, $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ *and* $\mathrm{bmo}^{\mathrm{r}}(\mathbb{R}^d,\mathcal{M})$ *are Banach spaces.*

Proof. Let Q_0 be the cube centered at the origin with side length 1 and $Q_m = Q_0+m$ for each $m\in\mathbb{Z}^d$. For $f\in L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$,

$$\begin{aligned}\|f\|^2_{L_\infty(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})} &= \Big\|\int_{\mathbb{R}^d}\frac{|f(t)|^2}{1+|t|^{d+1}}\mathrm{d}t\Big\|_{\mathcal{M}} \leqslant \sum_{m\in\mathbb{Z}^d}\Big\|\int_{Q_m}\frac{|f(t)|^2}{1+|t|^{d+1}}\mathrm{d}t\Big\|_{\mathcal{M}}\\ &\lesssim \sum_{m\in\mathbb{Z}^d}\Big\|\frac{1}{1+|m|^{d+1}}\int_{Q_m}|f(t)|^2\mathrm{d}t\Big\|_{\mathcal{M}} \lesssim \|f\|^2_{\mathrm{bmo}^{\mathrm{c}}}\sum_{m\in\mathbb{Z}^d}\frac{1}{1+|m|^{d+1}}\lesssim \|f\|^2_{\mathrm{bmo}^{\mathrm{c}}}.\end{aligned}$$

It is then easy to check that $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ is a Banach space. ∎

PROPOSITION 2.3. *We have the inclusion* $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})\subset \mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. *More precisely, there exists a constant C depending only on the dimension d, such that for any $f\in\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$,*

$$\|f\|_{\mathrm{BMO}^{\mathrm{c}}}\leqslant C\|f\|_{\mathrm{bmo}^{\mathrm{c}}}. \tag{2.5}$$

Proof. By virtue of Remark 2.1, it suffices to compare $\Big\|\Big(\frac{1}{|Q|}\int_Q|f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}$ and $\Big\|\Big(\frac{1}{|Q|}\int_Q|f-f_Q|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}$ for $|Q|\geqslant 1$. By the triangle inequality and (1.10), we have the following which leads immediately to (2.5):

$$\begin{aligned}\Big\|\Big(\frac{1}{|Q|}\int_Q|f-f_Q|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}} &\leqslant \Big\|\Big(\frac{1}{|Q|}\int_Q|f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}+\|f_Q\|_{\mathcal{M}}\\ &\leqslant 2\Big\|\Big(\frac{1}{|Q|}\int_Q|f|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}.\end{aligned}$$ ∎

Classically, BMO functions are related to Carleson measures (see [9]). A similar relation still holds in the present noncommutative local setting. We say

that an $\mathcal{M}$-valued measure $d\lambda$ on the strip $S = \mathbb{R}^d \times (0,1)$ is a Carleson measure if

$$N(\lambda) = \sup_{|Q|<1} \Big\{ \frac{1}{|Q|} \Big\| \int_{T(Q)} d\lambda \Big\|_{\mathcal{M}} : Q \subset \mathbb{R}^d \text{ cube} \Big\} < \infty,$$

where $T(Q) = Q \times (0, l(Q)]$.

LEMMA 2.4. *Let $g \in \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Then $d\lambda_g = |\frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g)(s)|^2 \varepsilon ds d\varepsilon$ is an $\mathcal{M}$-valued Carleson measure on the strip S and*

$$\max\{N(\lambda_g)^{1/2}, \ \|\mathrm{P} * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}.$$

Proof. Given a cube Q with $|Q| < 1$, denote by $2Q$ the cube with the same center and twice the side length of Q. We decompose $g = g_1 + g_2 + g_3$, where $g_1 = (g - g_{2Q})\mathbb{1}_{2Q}$ and $g_2 = (g - g_{2Q})\mathbb{1}_{\mathbb{R}^d \setminus 2Q}$. Since $\int \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(s) ds = 0$ for any $\varepsilon > 0$, we have $\frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g) = \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g_1) + \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g_2)$. By (1.10),

$$N(\lambda_g) \leqslant 2(N(\lambda_{g_1}) + N(\lambda_{g_2})).$$

We first deal with $N(\lambda_{g_1})$. By (1.11) and (2.5), we have

$$\begin{aligned}
\int_{T(Q)} \Big| \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g_1)(s) \Big|^2 \varepsilon ds d\varepsilon &\leqslant \int_{\mathbb{R}^d} \int_0^\infty \Big| \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g_1)(s) \Big|^2 \varepsilon ds d\varepsilon \\
&= \int_{\mathbb{R}^d} \int_0^\infty \Big| \widehat{\frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon}(\xi) \Big|^2 |\widehat{g}_1(\xi)|^2 \varepsilon d\varepsilon ds \\
&\lesssim \int_{\mathbb{R}^d} |g_1(s)|^2 ds = \int_{2Q} |g - g_{2Q}|^2 ds \lesssim |Q| \, \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}}.
\end{aligned}$$

Thus, $N(\lambda_{g_1}) \lesssim \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}}$. Since $\Big| \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(s) \Big| \lesssim \frac{1}{(\varepsilon + |s|)^{d+1}}$, applying (1.10), we obtain

$$\Big| \frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g_2)(s) \Big|^2 \lesssim \frac{1}{\varepsilon} \int_{\mathbb{R}^d \setminus 2Q} \frac{|g(t) - g_{2Q}|^2}{(\varepsilon + |s-t|)^{d+1}} dt.$$

The integral on the right hand side of the above inequality can be treated by a standard argument as follows: for any $(s, \varepsilon) \in T(Q)$,

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus 2Q} \frac{|g(t) - g_{2Q}|^2}{(\varepsilon + |s-t|)^{d+1}} dt &\lesssim \int_{\mathbb{R}^d \setminus 2Q} \frac{|g(t) - g_{2Q}|^2}{|t - c_Q|^{d+1}} dt \lesssim \sum_{k \geqslant 1} \int_{2^{k+1}Q \setminus 2^k Q} \frac{|g(t) - g_{2Q}|^2}{|t - c_Q|^{d+1}} dt \\
&\lesssim \frac{1}{l(Q)} \sum_{k \geqslant 1} 2^{-k} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |g(t) - g_{2Q}|^2 dt \lesssim \frac{1}{l(Q)} \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}},
\end{aligned}$$

where c_Q is the center of Q. Then, it follows that $N(\lambda_{g_2}) \lesssim \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}}$.

Now we deal with the term $\|\mathrm{P}*g(s)\|_{\mathcal{M}}$. Let $Q_m = Q_0 + m$ be the translate of the cube with volume one centered at the origin, so $\mathbb{R}^d = \bigcup_{m\in\mathbb{Z}^d} Q_m$. By (1.10), for any $s \in \mathbb{R}^d$, we have

$$\begin{aligned}\|\mathrm{P}*g(s)\|_{\mathcal{M}} &= \Big\| \sum_m \int_{Q_m} \mathrm{P}(t)g(s-t)\mathrm{d}t\Big\|_{\mathcal{M}}\\ &\leqslant \sum_m \Big(\int_{Q_m} |\mathrm{P}(t)|^2\mathrm{d}t\Big)^{1/2} \cdot \sup_{m\in\mathbb{Z}^d} \Big\| \Big(\int_{Q_m} |g(s-t)|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}}\\ &\lesssim \sup_{|Q|=1} \Big\| \Big(\frac{1}{|Q|}\int_Q |g(t)|^2\mathrm{d}t\Big)^{1/2}\Big\|_{\mathcal{M}} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}.\end{aligned}$$

Thus, $\|\mathrm{P}*g\|_{L_\infty(\mathcal{N})} = \sup_{s\in\mathbb{R}^d} \|\mathrm{P}*g(s)\|_{\mathcal{M}} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}$, which completes the proof. ∎

Reexamining the above proof, we find that the facts used to prove $N(\lambda_g)^{1/2} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}$ are:

(•) $\int_{\mathbb{R}^d} \varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s)\mathrm{d}s = 0$ for $\forall\,\varepsilon < 0$;

(•) $\sup_{\xi\in\mathbb{R}^d} \int_0^\infty \Big|\widehat{\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon}(\xi)\Big|^2 \frac{\mathrm{d}\varepsilon}{\varepsilon} < \infty$;

(•) $\Big|\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s)\Big| \lesssim \frac{\varepsilon}{(\varepsilon+|s|)^{d+1}}$.

It is easy to see that if we replace $\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon$ above by $\Psi_\varepsilon = \frac{1}{\varepsilon^d}\Psi(\frac{\cdot}{\varepsilon})$, where Ψ is a Schwartz function such that $\widehat{\Psi}(0) = 0$, the corresponding three conditions still hold. On the other hand, the only fact used for proving the inequality $\|\mathrm{P}*g\|_{L_\infty(\mathcal{N})} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}$ is that

$$\sum_m \Big(\int_{Q_m} |\mathrm{P}(t)|^2\mathrm{d}t\Big)^{1/2} < \infty.$$

Recall that $H_2^\sigma(\mathbb{R}^d)$ denotes the potential Sobolev space, consisting of distributions f such that $J^\sigma(f) \in L_2(\mathbb{R}^d)$. It is equipped with the norm $\|f\|_{H_2^\sigma(\mathbb{R}^d)} = \|J^\sigma f\|_{L_2(\mathbb{R}^d)}$. If ψ is a function on $\mathbb{R}^d$ such that $\widehat{\psi} \in H_2^\sigma(\mathbb{R}^d)$ for some $\sigma > \frac{d}{2}$, we have

$$\sum_m \Big(\int_{Q_m} |\psi(s)|^2\mathrm{d}t\Big)^{1/2} \lesssim \Big(\sum_m \frac{1}{(1+|m|^2)^\sigma}\Big)^{1/2} \Big(\int_{\mathbb{R}^d} (1+|s|^2)^\sigma |\psi(s)|^2\mathrm{d}s\Big)^{1/2} \lesssim \|\widehat{\psi}\|_{H_2^\sigma}.$$

Based on the above observation, we have the following generalization of Lemma 2.4.

LEMMA 2.5. *Let ψ be the (inverse) Fourier transform of a function in $H_2^\sigma(\mathbb{R}^d)$, and Ψ be a Schwartz function such that $\widehat{\Psi}(0) = 0$. If $g \in \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, then $\mathrm{d}\mu_g =$*

$|\Psi_\varepsilon * g(s)|^2 \frac{\mathrm{d}\varepsilon\mathrm{d}s}{\varepsilon}$ is an $\mathcal{M}$-valued Carleson measure on the strip S and

$$\max\{N(\mu_g)^{1/2},\ \|\psi * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}. \tag{2.6}$$

In particular,

$$\max\{N(\mu_g)^{1/2},\ \|J(\mathrm{P}) * g\|_{L_\infty(\mathcal{N})}\} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}}. \tag{2.7}$$

Proof. (2.6) follows from the above discussion; (2.7) is ensured by (2.6) and the fact that $(1+|\xi|^2)^{1/2}\mathrm{e}^{-2\pi|\xi|} \in H_2^\sigma(\mathbb{R}^d)$, which can be checked by a direct computation. ∎

REMARK 2.6. We will see in the next section that the converse inequality of (2.7) also holds.

3. THE DUAL SPACE OF $\mathrm{h}_p^{\mathrm{c}}$ FOR $1 \leqslant p < 2$

In this section, we describe the dual of $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ for $1 \leqslant p < 2$ as bmo type spaces. We call these spaces $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ (with q the conjugate index of p). The argument used here is modelled on the one used in [10] when studying the duality between $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. However, due to the truncation of the square functions, some highly non-trivial modifications are needed.

3.1. DEFINITION OF $\mathrm{bmo}_q^{\mathrm{c}}$. Let $2 < q \leqslant \infty$. We define $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ to be the space of all $f \in L_q(\mathcal{M}; \mathrm{R}_d^{\mathrm{c}})$ such that

$$\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}} = \Big(\Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|}\int_Q |f(t)-f_Q|^2\mathrm{d}t \Big\|_{q/2}^{q/2} + \Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|=1}}{}^{+} \frac{1}{|Q|}\int_Q |f(t)|^2\mathrm{d}t \Big\|_{q/2}^{q/2} \Big)^{1/q} < \infty.$$

If $q=\infty$, $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ coincides with the space $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ introduced in the previous section.

Note that the norm $\Big\| \sup_i{}^{+} a_i \Big\|_{q/2}$ is just an intuitive notation since the pointwise supremum does not make any sense in the noncommutative setting. This is the norm of the Banach space $L_{q/2}(\mathcal{N}; \ell_\infty)$; we refer to [14], [22], [29] for more information.

If $1 \leqslant p < \infty$ and $(a_i)_{i\in\mathbb{Z}}$ is a sequence of positive elements in $L_p(\mathcal{N})$, it has been proved by Junge (see Remark 3.7 of [14]) that

$$\Big\| \sup_i{}^{+} a_i \Big\|_p = \sup\Big\{ \sum_{i\in\mathbb{Z}} \tau(a_i b_i) : b_i \in L_q(\mathcal{N}),\ b_i \geqslant 0,\ \Big\| \sum_{i\in\mathbb{Z}} b_i \Big\|_q \leqslant 1 \Big\}. \tag{3.1}$$

It is also known that a positive sequence $(x_i)_i$ belongs to $L_p(\mathcal{N}; \ell_\infty)$ if and only if there is an $a \in L_p(\mathcal{N})$ such that $x_i \leqslant a$ for all i, and moreover,

$$\|(x_i)\|_{L_p(\mathcal{N};\ell_\infty)} = \inf\{\|a\|_p : a \in L_p(\mathcal{N}), x_i \leqslant a, \forall\, i\}.$$

Then we get the following fact (which can be taken as an equivalent definition): $f \in \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ if and only if

$$\exists\, a \in L_{q/2}(\mathcal{N}) \text{ such that } \frac{1}{|Q|}\int_Q |f(t) - f_Q|^2 \mathrm{d}t \leqslant a(s),$$

$$\forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| < 1 \quad \text{and} \tag{3.2}$$

$$\exists\, b \in L_{q/2}(\mathcal{N}) \text{ such that } \frac{1}{|Q|}\int_Q |f(t)|^2 \mathrm{d}t \leqslant b(s),$$

$$\forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| = 1. \tag{3.3}$$

If this is the case, then

$$\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}} = \inf\{(\|a\|_{q/2}^{q/2} + \|b\|_{q/2}^{q/2})^{1/q} : a, b \text{ as in (3.2) and (3.3), respectively}\}.$$

In fact, the cubes considered in the definition of $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ can be reduced to cubes with dyadic lengths. Let Q_s^k denote the cube centered at s and with side length 2^{-k}, $k \in \mathbb{Z}$. Set

$$f_k^{\#}(s) = \frac{1}{|Q_s^k|}\int_{Q_s^k} |f(t) - f_{Q_s^k}|^2 \mathrm{d}t \quad \text{and} \quad f^{\#}(s) = \frac{1}{|Q_s^0|}\int_{Q_s^0} |f(t)|^2 \mathrm{d}t.$$

LEMMA 3.1. *If $q > 2$, then*

$$\Big(\Big\| \sup_{k\geqslant 1}{}^{+} f_k^{\#}\Big\|_{q/2}^{q/2} + \|f^{\#}\|_{q/2}^{q/2}\Big)^{1/q}$$

gives an equivalent norm in $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

Proof. It is obvious from the definition that

$$\Big\| \sup_{k\geqslant 1}{}^{+} f_k^{\#}\Big\|_{q/2}^{1/2} \leqslant \|f\|_{\mathrm{bmo}_q^{\mathrm{c}}} \quad \text{and} \quad \|f^{\#}\|_{q/2}^{1/2} \leqslant \|f\|_{\mathrm{bmo}_q^{\mathrm{c}}}.$$

We notice that for any cube Q with $|Q| < 1$ and $s \in Q$, there exists $k \geqslant -1$ such that $Q \subset Q_s^k$ and $|Q_s^k| \leqslant 4^d|Q|$. Thus

$$\frac{1}{4^d}\Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|<1}}{}^{+} \frac{1}{|Q|}\int_Q |f(t) - f_Q|^2 \mathrm{d}t\Big\|_{q/2}^{1/2} \lesssim \Big\| \sup_{k\geqslant -1}{}^{+} f_k^{\#}\Big\|_{q/2}^{1/2} \lesssim 2^d \Big\| \sup_{k\geqslant 1}{}^{+} f_k^{\#}\Big\|_{q/2}^{1/2}.$$

Similarly,

$$\frac{1}{4^d}\Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|=1}}{}^{+} \frac{1}{|Q|}\int_Q |f(t)|^2 \mathrm{d}t\Big\|_{q/2}^{1/2} \leqslant 2^d \|f^{\#}\|_{q/2}^{1/2}.$$

Thus the lemma is proved. ∎

From the proofs of Proposition 2.2 and Lemma 2.4, it is easy to see that their q-analogues still hold in the present setting. We leave the proofs to the reader.

PROPOSITION 3.2. *Let $q > 2$ and $f \in \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$. Then*

$$\|f\|_{L_q(\mathcal{M};\mathrm{R}^{\mathrm{c}}_d)} \lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

LEMMA 3.3. *Let $f \in \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$ and assume that the operators a and b satisfy* (3.2) *and* (3.3)*, respectively. Then $\mathrm{d}\lambda_f$ is a q-Carleson measure in the following sense:*

$$\frac{1}{|Q|}\int\limits_{T(Q)} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\Big|^2 \varepsilon \mathrm{d}t\mathrm{d}\varepsilon \lesssim a(s), \quad \forall\, s \in Q \text{ and } \forall\, Q \subset \mathbb{R}^d \text{ with } |Q| < 1.$$

Moreover, $|\psi * f(s)|^2 \lesssim b(s)$ for any $s \in \mathbb{R}^d$, if ψ is the (inverse) Fourier transform of a function in $H^\sigma_2(\mathbb{R}^d)$.

3.2. A BOUNDED MAP. In the sequel, we equip the truncated cone $\widetilde{\Gamma} = \{(s,\varepsilon) \in \mathbb{R}^{d+1}_+ : |s| < \varepsilon < 1\}$ with the measure $\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}$. For any $1 \leqslant p < \infty$, we will embed $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ into a larger space $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$. Here $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$ is the ℓ_p-direct sum of the Banach spaces $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))$ and $L_p(\mathcal{N})$, equipped with the norm

$$\|(f,g)\| = (\|f\|^p_{L_p(\mathcal{N};L^{\mathrm{c}}_2(\widetilde{\Gamma}))} + \|g\|^p_{L_p(\mathcal{N})})^{1/p}$$

for $f \in L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))$ and $g \in L_p(\mathcal{N})$, with the usual modification for $p = \infty$.

DEFINITION 3.4. We define a map $\mathbf{E}$ from $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ to $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$ by

$$\mathbf{E}(f)(s,t,\varepsilon) = \Big(\varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t), \mathrm{P} * f(s)\Big),$$

and a map $\mathbf{F}$ for sufficiently nice $h = (h', h'') \in L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$ by

$$\mathbf{F}(h)(u) = \int\limits_{\mathbb{R}^d}\Big[\frac{4}{c_d}\iint\limits_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d} + h''(s)(\mathrm{P}+4\pi I(\mathrm{P}))(s-u)\Big]\mathrm{d}s\,.$$

By definition, the map $\mathbf{E}$ embeds $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ isometrically into $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$. The following results, Theorems 3.8 and 3.18 show that by identifying $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ as a subspace of $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$ via $\mathbf{E}$, $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ is complemented in $L_p(\mathcal{N}; L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus\limits_p L_p(\mathcal{N})$ for every $1 < p < \infty$ by virtue of the map $\mathbf{F}$.

PROPOSITION 3.5. *Let $1 \leqslant p < \infty$. Then for any nice $f \in L_1(\mathcal{M};\mathrm{R}^{\mathrm{c}}_d) + L_\infty(\mathcal{M};\mathrm{R}^{\mathrm{c}}_d)$, we have*

$$\mathbf{F}(\mathbf{E}(f)) = f.$$

Proof. Applying (2.3), we get the following which completes the proof, for any nice function g,

$$\begin{aligned}\int_{\mathbb{R}^d}\mathbf{F}(\mathbf{E}(f))(u)g(u)\mathrm{d}u &= \int_{\mathbb{R}^d}\Big[\frac{4}{c_d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}g(u)\mathrm{d}u\\ &\qquad + \mathrm{P}*f(s)\int(\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u))g(u)\mathrm{d}u\Big]\mathrm{d}s\\ &= \int_{\mathbb{R}^d}\Big[\frac{4}{c_d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s+t)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\\ &\qquad +\mathrm{P}*f(s)(\mathrm{P}*g+4\pi I(\mathrm{P})*g)(s)\Big]\mathrm{d}s = \int_{\mathbb{R}^d}f(u)g(u)\mathrm{d}u\,. \quad \blacksquare\end{aligned}$$

The following dyadic covering lemma is known. Tao Mei [24] proved this lemma for the d-torus and also for the real line. For the case $\mathbb{R}^d$ with $d>1$, we refer the interested readers to [5], [13], [23] for more details. In the following, we give a sketch of the way how we choose the dyadic covering.

LEMMA 3.6. *There exist a constant $C>0$, depending only on d, and $d+1$ dyadic increasing filtrations $\mathcal{D}^i=\{\mathcal{D}^i_j\}_{j\in\mathbb{Z}}$ of σ-algebras on $\mathbb{R}^d$ for $0\leqslant i\leqslant d$, such that for any cube $Q\subset\mathbb{R}^d$, there is a cube $D^i_{m,j}\in\mathcal{D}^i_j$ satisfying $Q\subset D^i_{m,j}$ and $|D^i_{m,j}|\leqslant C|Q|$.*

Proof. Let $\{\alpha^i\}_{i=0}^d$ be a sequence in the interval $(0,1)$ such that $\min_{i\neq i'}|\alpha^i-\alpha^{i'}|>0$. Then we define

$$\alpha^i_j=\begin{cases}\alpha^i & j\geqslant 0,\\ \alpha^i+\frac{1}{3}(2^{-j}-1) & j<0 \text{ and } -j \text{ even},\\ \alpha^i-\frac{1}{3}(2^{-j}+1) & j<0 \text{ and } -j \text{ odd}.\end{cases}\tag{3.4}$$

The σ-algebra $\mathcal{D}^i_j$ is generated by the cubes

$$D^i_{m,j}=(\alpha^i_j+m_1 2^{-j},\alpha^i_j+(m_1+1)2^{-j}]\times\cdots\times(\alpha^i_j+m_d2^{-j},\alpha^i_j+(m_d+1)2^{-j}],$$

for all $m=(m_1,\dots,m_d)\in\mathbb{Z}^d$.

For any cube $Q\subset\mathbb{R}^d$, there exist a constant C, depending only on $\{\alpha^i\}_{i=0}^d$ and d, and a dyadic cube $D^i_{m,j}$ such that $Q\subset D^i_{m,j}$ and $|D^i_{m,j}|\leqslant C|Q|$. $\blacksquare$

To show the boundedness of the map $\mathbf{F}$, we need the following assertion by Mei, see Proposition 3.2 of [24]; we include a proof for this lemma, since the one in [24] is the one dimensional case. Let $1\leqslant p<\infty$, and $f\in L_p(\mathcal{N})$ be a positive function. Let Q be a cube centered at the origin, and denote $Q_t=t+Q$. Then we define

$$f^Q(t)=\frac{1}{|Q|}\int_{Q_t}f(s)\mathrm{d}s.$$

LEMMA 3.7. *Let $1 \leqslant p < \infty$ and let $(f_k)_{k\in\mathbb{Z}}$ be a positive sequence in $L_p(\mathcal{N})$ and $(Q^k)_{k\in\mathbb{Z}}$ be a sequence of cubes centered at the origin. Then*

$$\Big\| \sum_{k\in\mathbb{Z}} (f_k)^{Q^k} \Big\|_p \lesssim \Big\| \sum_{k\in\mathbb{Z}} f_k \Big\|_p.$$

Proof. Similarly to the proof of Proposition 3.2 in [24], we are going to apply Theorem 0.1 of [14] for noncommutative martingales. By Lemma 3.6, we can cover every Q^k by some D^i_{m',j_k}, and thus by some D^i_{m,j_k-1}, which has twice the side length of D^i_{m',j_k}. Moreover, $|D^i_{m,j_k-1}| \leqslant C|Q^k|$. Obviously, $t + Q^k$ is still covered by $t + D^i_{m,j_k-1}$, but the later is not necessarily a dyadic cube in $\mathcal{D}^i_{j_k-1}$. Let us adjust the translation vector $t = (t_1, \dots, t_d)$ as follows. Write $Q^k = (-a, a] \times \cdots \times (-a, a]$ and $D^i_{m,j_k-1} = (b_1, b_2] \times \cdots \times (b_1, b_2]$, then either $b_2 - a \geqslant 2^{-j_k}$ or $-a - b_1 \geqslant 2^{-j_k}$. Without loss of generality, we can assume $b_2 - a \geqslant 2^{-j_k}$. Now set $\widetilde{t} = (\widetilde{t}_1, \dots, \widetilde{t}_d)$ with $\widetilde{t}_j$ the largest real number in the set $2^{-j_k}\mathbb{Z}$ less than t_j. Then it is easy to see that $t + Q^k$ is covered by $\widetilde{t} + D^i_{m,j_k-1}$ and that the later is a dyadic cube. Thus,

$$(f_k)^{Q^k} \leqslant C \sum_{0\leqslant i\leqslant d} \mathbb{E}(f_k|\mathcal{D}^i_{j_k}),$$

where $\mathbb{E}(\cdot|\mathcal{D}^i_j)$ denotes the conditional expectation with respect to $\mathcal{D}^i_j$. Then the lemma follows from Theorem 0.1 in [14]. ∎

THEOREM 3.8. *For $2 < p \leqslant \infty$, the map $\mathbf{F}$ extends to a bounded map from $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$ to $\mathrm{bmo}_p^c(\mathbb{R}^d, \mathcal{M})$.*

Proof. We have to show that for any $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$,

$$\|\mathbf{F}(h)\|_{\mathrm{bmo}_p^c} \lesssim \|h\|_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\,\oplus_p\, L_p(\mathcal{N})}.$$

Fix $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$ and set $\varphi = \mathbf{F}(h)$. We will apply Lemma 3.1 to estimate the bmo_p^c-norm of $\mathbf{F}(h)$. For $v \in \mathbb{R}^d$ and $k \in \mathbb{N}$, denote by Q^k_v the cube centered at v and with side length 2^{-k}, then we have $Q^k_v = v + Q^k_0$. We set

$$h'_1(s,t,\varepsilon) = h'(s,t,\varepsilon)\mathbb{1}_{Q^{k-1}_v}(s), \quad h'_2(s,t,\varepsilon) = h'(s,t,\varepsilon)\mathbb{1}_{(Q^{k-1}_v)^c}(s) \quad \text{and}$$

$$\varphi^{\#}_k(v) = \frac{1}{|Q^k_v|}\int_{Q^k_v} |\varphi(u) - \varphi_{Q^k_v}|^2 \mathrm{d}u.$$

Let

$$B^{Q^k_0}(v) = \int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}} \Big(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon\Big)^{Q^k_0}(s,t,v) h'_2(s,t,\varepsilon)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s$$

with $(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon)^{Q_0^k}(s,t,v)=\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\mathrm{d}u$. Then, we have

$$\begin{aligned}\varphi_k^{\#}(v)&\lesssim\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}|\varphi(u)-B^{Q_0^k}(v)|^2\mathrm{d}u\\&\lesssim\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\Big|\int\limits_{(Q_v^{k-1})^{\mathrm{c}}}\iint\limits_{\widetilde{\Gamma}}h_2'(s,t,\varepsilon)\Big[\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)-\Big(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon\Big)^{Q_0^k}(s,t,v)\Big]\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s\Big|^2\mathrm{d}u\\&\quad+\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\Big|\int\limits_{Q_v^{k-1}}\iint\limits_{\widetilde{\Gamma}}h_1'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s\Big|^2\mathrm{d}u\\&\quad+\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\Big|\int\limits_{\mathbb{R}^d}h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]\mathrm{d}s\Big|^2\mathrm{d}u.\end{aligned}$$

When $s\in(Q_v^{k-1})^{\mathrm{c}}$, $u\in Q_v^k$ and $(t,\varepsilon)\in\widetilde{\Gamma}$, we have $|s+t-u|+\varepsilon\approx|s-v|+\varepsilon$ with uniform constants. Then,

$$\begin{aligned}&\iint\limits_{\widetilde{\Gamma}}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)-\Big(\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon\Big)^{Q_0^k}(s,t,v)\Big|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\\&\quad\lesssim\iint\limits_{\widetilde{\Gamma}}\Big(\frac{2^{-k}}{(|s+t-u|+\varepsilon)^{d+2}}\Big)^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\lesssim\int\limits_0^1\int\limits_{B(0,\varepsilon)}\frac{2^{-2k}}{(|s-v|^2+\varepsilon^2)^{d+2}}\mathrm{d}t\,\frac{\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\\&\quad=c_d\int\limits_0^1\frac{2^{-2k}\varepsilon}{(|s-v|^2+\varepsilon^2)^{d+2}}\mathrm{d}\varepsilon\lesssim\frac{2^{-2k}}{|s-v|^{2d+2}}.\end{aligned}$$

Let $(a_k)_{k\in\mathbb{N}}$ be a positive sequence such that $\Big\|\sum\limits_{k\geqslant1}a_k\Big\|_{(p/2)'}\leqslant1$, where r' denotes the conjugate index of r. Let

$$\begin{aligned}\mathrm{A}&=\sum_{k\geqslant1}\tau\int\limits_{\mathbb{R}^d}\int\limits_{(Q_v^{k-1})^{\mathrm{c}}}\frac{2^{-2k}}{|s-v|^{d+1}}\mathrm{d}s\cdot\int\limits_{(Q_v^{k-1})^{\mathrm{c}}}\frac{1}{|s-v|^{d+1}}\iint\limits_{\widetilde{\Gamma}}|h_2'(s,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s\cdot a_k(v)\mathrm{d}v\\\mathrm{B}&=\sum_{k\geqslant1}\tau\int\limits_{\mathbb{R}^d}\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\Big|\int\limits_{Q_v^{k-1}}\iint\limits_{\widetilde{\Gamma}}h_1'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s\Big|^2\mathrm{d}u\cdot a_k(v)\mathrm{d}v\\\mathrm{C}&=\sum_{k\geqslant1}\tau\int\limits_{\mathbb{R}^d}\frac{1}{|Q_v^k|}\int\limits_{Q_v^k}\Big|\int\limits_{\mathbb{R}^d}h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]\mathrm{d}s\Big|^2\mathrm{d}u\cdot a_k(v)\mathrm{d}v.\end{aligned}$$

Then,

$$\sum_{k\geqslant1}\tau\int\varphi_k^{\#}(v)a_k(v)\mathrm{d}v\lesssim\mathrm{A}+\mathrm{B}+\mathrm{C}.$$

First, we estimate the term A. Applying the Fubini theorem and the Hölder inequality, we arrive at

$$\begin{aligned}
\mathrm{A} &\lesssim \sum_{k\geqslant 1}\tau\int_{\mathbb{R}^d} 2^{-k}\int_{(Q_s^{k-1})^{\mathrm{c}}}|v-s|^{-d-1}\iint_{\widetilde{\Gamma}}|h_2'(s,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s\,a_k(v)\mathrm{d}v\\
&\leqslant \Big\|\iint_{\widetilde{\Gamma}}|h_2'(\cdot,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\Big\|_{p/2}\cdot\Big\|\sum_{k\geqslant 1}2^{-k}\int_{(Q_s^{k-1})^{\mathrm{c}}}|v-s|^{-d-1}a_k(v)\mathrm{d}v\Big\|_{(p/2)'}\\
&\lesssim \|h'\|^2_{L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))}\cdot\Big\|\sum_{k\geqslant 1}2^{-k}\sum_{j\leqslant k}\int_{Q_s^{j-2}\setminus Q_s^{j-1}}2^{(j-1)(d+1)}a_k(v)\mathrm{d}v\Big\|_{(p/2)'}.
\end{aligned}$$

Here and in the context below, $\|\cdot\|_{(p/2)'}$ is the norm of $L_{(p/2)'}(\mathcal{N})$ with respect to the variable $s\in\mathbb{R}^d$. Now we apply Lemma 3.7 to estimate the second factor of the last term:

$$\begin{aligned}
\Big\|\sum_{k\geqslant 1}\sum_{j\leqslant k}2^{(j-1)d}\int_{Q_s^{j-2}\setminus Q_s^{j-1}}2^{j-k-1}a_k(v)\mathrm{d}v\Big\|_{(p/2)'} &\lesssim \Big\|\sum_{j\in\mathbb{Z}}\sum_{\substack{k\geqslant j\\ k\geqslant 1}}2^{j-k-1}a_k\Big\|_{(p/2)'}\\
&\lesssim \Big\|\sum_{k\geqslant 1}a_k\Big\|_{(p/2)'}\leqslant 1.
\end{aligned}$$

Then we move to the estimate of B:

$$\begin{aligned}
\mathrm{B} &\leqslant \sum_{k\geqslant 1}\int_{\mathbb{R}^d}2^{kd}\tau\int_{\mathbb{R}^d}\Big|\int_{Q_v^{k-1}}\iint_{\widetilde{\Gamma}}h_1'(s,t,\varepsilon)a_k^{1/2}(v)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s\Big|^2\mathrm{d}u\mathrm{d}v\\
&\leqslant \sum_{k\geqslant 1}\int_{\mathbb{R}^d}2^{kd}\sup_{\|f\|_2=1}\Big|\tau\int_{Q_v^{k-1}}\iint_{\widetilde{\Gamma}}h_1'(s,t,\varepsilon)a_k^{1/2}(v)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s+t)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s\Big|^2\mathrm{d}v.
\end{aligned}$$

Since $\mathrm{h}_2^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})=L_2(\mathcal{N})$ with equivalent norms, by the Cauchy–Schwarz inequality and Lemma 3.7, we get

$$\begin{aligned}
\mathrm{B} &\leqslant \sum_{k\geqslant 1}\int_{\mathbb{R}^d}2^{kd}\tau\int_{Q_v^{k-1}}\iint_{\widetilde{\Gamma}}|h_1'(s,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s\,a_k(v)\mathrm{d}v\cdot\|f\|_{\mathrm{h}_2^{\mathrm{c}}}\\
&\lesssim \sum_{k\geqslant 1}\tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}|h_1'(s,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}2^{kd}\int_{Q_s^{k-1}}a_k(v)\mathrm{d}v\mathrm{d}s\\
&\leqslant \|h'\|^2_{L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))}\Big\|\sum_{k\geqslant 1}2^{kd}\int_{Q_s^{k-1}}a_k(v)\mathrm{d}v\Big\|_{(p/2)'}\\
&\leqslant 2^d\|h'\|^2_{L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))}\Big\|\sum_{k\geqslant 1}a_k\Big\|_{(p/2)'}\leqslant 2^d\|h'\|^2_{L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))}.
\end{aligned}$$

The techniques used to estimate the term C are similar to that of B:

$$\begin{aligned}\mathrm{C} &= \sum_{k\geqslant 1}\tau\int_{\mathbb{R}^d} 2^{kd}\int_{Q_v^{k-1}}\Big|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]\mathrm{d}s\Big|^2 a_k(v)\mathrm{d}v\mathrm{d}u\\ &\leqslant \Big\|\sum_{k\geqslant 1} 2^{kd}\int_{Q_s^{k-1}} a_k(v)\mathrm{d}v\Big\|_{(p/2)'}\Big\|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-\cdot)+4\pi I(\mathrm{P})(s-\cdot)]\mathrm{d}s\Big|^2\Big\|_{p/2}\\ &\lesssim \Big\|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-\cdot)+4\pi I(\mathrm{P})(s-\cdot)]\mathrm{d}s\Big|^2\Big\|_{p/2}.\end{aligned}$$

Take $f\in L_{p'}(\mathcal{N})$ with norm one such that

$$\Big\|\Big|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]\mathrm{d}s\Big|^2\Big\|_{p/2}=\Big|\tau\int_{\mathbb{R}^d} h''(s)[\mathrm{P}*f(s)+4\pi I(\mathrm{P})*f(s)]\mathrm{d}s\Big|^2.$$

Then

$$\begin{aligned}\Big|\tau\int_{\mathbb{R}^d} h''(s)[\mathrm{P}*f(s)+4\pi I(\mathrm{P})*f(s)]\mathrm{d}s\Big|^2 &\leqslant \|h''\|_p^2\|\mathrm{P}*f+4\pi I(\mathrm{P})*f\|_{p'}^2\\ &\lesssim \|h''\|_p^2\|f\|_{p'}^2=\|h''\|_p^2.\end{aligned}$$

Combining the estimates of A, B and C with (3.1), we obtain

$$\Big\|\sup_{k\geqslant 1}{}^{+}\varphi_k^{\#}\Big\|_{p/2}\lesssim \|h\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}.$$

It remains to establish the $L_{p/2}$-norm of $\varphi^{\#}(s)=\frac{1}{|Q_s^0|}\int_{Q_s^0}|\varphi(t)|^2\mathrm{d}t$, which is relatively easy. For any positive operator a such that $\|a\|_{L_{(p/2)'}(\mathcal{N})}\leqslant 1$, we have

$$\begin{aligned}\tau\int\varphi^{\#}(v)a(v)\mathrm{d}v &\lesssim \tau\int_{\mathbb{R}^d}\int_{Q_v^0}\Big|\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s\Big|^2\mathrm{d}u\cdot a(v)\mathrm{d}v\\ &\quad+\tau\int_{\mathbb{R}^d}\int_{Q_v^0}\Big|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-u)+4\pi I(\mathrm{P})(s-u)]\mathrm{d}s\Big|^2\mathrm{d}u\cdot a(v)\mathrm{d}v\\ &\overset{\mathrm{def}}{=}\mathrm{B}'+\mathrm{C}'.\end{aligned}$$

The terms B′ and C′ are treated in the same way as B and C, respectively. The results are

$$\begin{aligned}\mathrm{B}' &\leqslant \tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}|h'(s,t,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\int_{Q_s^0}a(v)\mathrm{d}v\mathrm{d}s\leqslant\|h'\|^2_{L_p(\mathcal{N};L_2^c(\widetilde{\Gamma}))}\|a\|_{(p/2)'},\\ \mathrm{C}' &\leqslant \Big\|\int_{Q_s^0}a(v)\mathrm{d}v\Big\|_{(p/2)'}\Big\|\int_{\mathbb{R}^d} h''(s)[\mathrm{P}(s-\cdot)+4\pi I(\mathrm{P})(s-\cdot)]\mathrm{d}s\Big|^2\Big\|_{p/2}\lesssim\|h''\|_p^2.\end{aligned}$$

So we obtain

$$\|\varphi^{\#}\|_{p/2} \lesssim \|h\|^2_{L_p(\mathcal{N};L_2^{\rm c}(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}.$$

Thus, Lemma 3.1 ensures the following which proves the theorem:

$$\|\mathbf{F}(h)\|_{{\rm bmo}_p^{\rm c}} \lesssim \|h\|_{L_p(\mathcal{N};L_2^{\rm c}(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}. \quad \blacksquare$$

COROLLARY 3.9. *Let $1 \leqslant p < 2$. For any $f \in L_p(\mathcal{M}; L_2^{\rm c}(\mathbb{R}^d, (1+|t|^{d+1})\mathrm{d}t))$, we have*

$$\|f\|_{{\rm h}_p^{\rm c}} \lesssim \|f\|_{L_p(\mathcal{M};L_2^{\rm c}(\mathbb{R}^d,(1+|t|^{d+1})\mathrm{d}t))}.$$

Proof. To simplify notation, we denote $L_2(\mathbb{R}^d, (1+|t|^{d+1})\mathrm{d}t)$ by W_d. Let q be the conjugate index of p. By duality, we can choose $h = (h', h'') \in L_q(\mathcal{N}; L_2^{\rm c}) \bigoplus_q L_q(\mathcal{N})$ with norm one such that

$$\begin{aligned}
&\|s^{\rm c}(f)\|_p + \|\mathrm{P} * f\|_p \\
&\qquad = \Big|\tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)h'^*(s,t,\varepsilon)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s + \tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)h''^*(s)\mathrm{d}s\Big| \\
&\qquad = \Big|\tau\int f(u)\widetilde{\mathbf{F}}(h)^*(u)\mathrm{d}u\Big|,
\end{aligned}$$

where

$$\widetilde{\mathbf{F}}(h)(u) = \int_{\mathbb{R}^d}\Big[\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(s+t-u)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d} + h''(s)\mathrm{P}(s-u)\Big]\mathrm{d}s\,. \tag{3.5}$$

Following the proof of Theorem 3.8, it is easy to see that $\widetilde{\mathbf{F}}$ is also bounded from $L_q(\mathcal{N}; L_2^{\rm c}(\widetilde{\Gamma}))\bigoplus_q L_q(\mathcal{N})$ to ${\rm bmo}_q^{\rm c}(\mathbb{R}^d, \mathcal{M})$. They applying Proposition 3.2 and Theorem 3.8, we have the following and thus we obtain the desired assertion:

$$\begin{aligned}
\Big|\tau\int f(s)\widetilde{\mathbf{F}}(h)^*(s)\mathrm{d}s\Big| &\lesssim \sup_{\|\varphi\|_{{\rm bmo}_q^{\rm c}(\mathbb{R}^d,\mathcal{M})}\leqslant 1}\Big|\tau\int f(s)\varphi^*(s)\mathrm{d}s\Big| \\
&\lesssim \sup_{\|\varphi\|_{L_q(\mathcal{M};\mathrm{R}_d^{\rm c})}\leqslant 1}\Big|\tau\int(1+|s|^{d+1})f(s)\varphi^*(s)\frac{\mathrm{d}s}{1+|s|^{d+1}}\Big| \\
&= \|(1+|s|^{d+1})f\|_{L_p(\mathcal{M};\mathrm{R}_d^{\rm c})} = \|f\|_{L_p(\mathcal{M};\mathrm{W}_d^{\rm c})}. \quad \blacksquare
\end{aligned}$$

3.3. DUALITY. Now we are going to present the ${\rm h}_p^{\rm c}$-${\rm bmo}_q^{\rm c}$ duality for $1 \leqslant p < 2$. We begin this subsection by a lemma which will be very useful in the sequel.

LEMMA 3.10. *Let $1 \leqslant p \leqslant 2$ and q be its conjugate index. For $f \in {\rm h}_p^{\rm c}(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in {\rm bmo}_q^{\rm c}(\mathbb{R}^d, \mathcal{M})$,*

$$\Big|\tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s\Big| \lesssim \|f\|_{{\rm h}_p^{\rm c}}\|g\|_{{\rm bmo}_q^{\rm c}}.$$

Proof. It suffices to prove the lemma for compactly supported (relative to the variable of $\mathbb{R}^d$) $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. We assume that f is sufficiently nice that all calculations below are legitimate. We need two auxiliary square functions. For $s \in \mathbb{R}^d$ and $\varepsilon \in [0,1]$, we define

$$s^{\mathrm{c}}(f)(s,\varepsilon) = \Big(\int_{\varepsilon}^{1} \int_{B(s,r-\varepsilon/2)} \Big| \frac{\partial}{\partial r} \mathrm{P}_r(f)(t) \Big|^2 \frac{\mathrm{d}t\mathrm{d}r}{r^{d-1}} \Big)^{1/2}, \tag{3.6}$$

$$\overline{s}^{\mathrm{c}}(f)(s,\varepsilon) = \Big(\int_{\varepsilon}^{1} \int_{B(s,r/2)} \Big| \frac{\partial}{\partial r} \mathrm{P}_r(f)(t) \Big|^2 \frac{\mathrm{d}t\mathrm{d}r}{r^{d-1}} \Big)^{1/2}. \tag{3.7}$$

Both $\overline{s}^{\mathrm{c}}(f)(s,\varepsilon)$ and $s^{\mathrm{c}}(f)(s,\varepsilon)$ are decreasing in ε and $s^{\mathrm{c}}(f)(s,0) = s^{\mathrm{c}}(f)(s)$. In addition, it is clear that $\overline{s}^{\mathrm{c}}(f)(s,\varepsilon) \leqslant s^{\mathrm{c}}(f)(s,\varepsilon)$. Let $(e_i)_{i\in I}$ be an increasing family of τ-finite projections of $\mathcal{M}$ such that e_i converges to $1_{\mathcal{M}}$ in the strong operator topology. Then we can approximate $s^{\mathrm{c}}(f)(s,\varepsilon)$ by $s^{\mathrm{c}}(e_i f e_i)(s,\varepsilon)$. Thus we can assume that τ is finite; under this finiteness assumption, for any small $\delta > 0$ (which will tend to zero in the end of the proof), consider $s^{\mathrm{c}}(f)(s,\varepsilon) + \delta 1_{\mathcal{M}}$ instead of $s^{\mathrm{c}}(f)(s,\varepsilon)$, we can assume that $s^{\mathrm{c}}(f)(s,\varepsilon)$ is invertible in $\mathcal{M}$ for every $(s,\varepsilon) \in S$. By (2.3) and the Fubini theorem, we have

$$\begin{aligned}
\Big|\tau\int f(s)g^*(s)\mathrm{d}s\Big| &\lesssim \Big|\tau \int_{\mathbb{R}^d} \int_0^1 \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\,\mathrm{d}\varepsilon\mathrm{d}s\Big| \\
&\quad + \Big|\tau\int_{\mathbb{R}^d} \mathrm{P}*f(s)(\mathrm{P}*g(s))^*\mathrm{d}s + \tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*\mathrm{d}s\Big| \\
&= \Big|\frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\varepsilon/2)} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(t)\frac{\mathrm{d}\varepsilon\mathrm{d}t}{\varepsilon^{d-1}}\mathrm{d}s\Big| \\
&\quad + \Big|\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*\mathrm{d}s + \tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*\mathrm{d}s\Big|.
\end{aligned}$$

Then,

$$\begin{aligned}
&\Big|\tau\int f(s)g^*(s)\mathrm{d}s\Big| \\
&\lesssim \Big|\frac{2^d}{c_d}\tau\int_{\mathbb{R}^d}\int_0^1\int_{B(s,\varepsilon/2)} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)s^{\mathrm{c}}(f)(s,\varepsilon)^{(p-2)/2}s^{\mathrm{c}}(f)(s,\varepsilon)^{(2-p)/2}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(t)\frac{\mathrm{d}\varepsilon\mathrm{d}t}{\varepsilon^{d-1}}\mathrm{d}s\Big| \\
&\quad + \Big(\Big|\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(\mathrm{P}*g(s))^*\mathrm{d}s\Big| + \Big|\tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)(I(\mathrm{P})*g(s))^*\mathrm{d}s\Big|\Big) \overset{\mathrm{def}}{=} \mathrm{I}+\mathrm{II}.
\end{aligned}$$

The term II is easy to deal with. By the Hölder inequality and (2.7), we get

$$\mathrm{II} \leqslant \|\mathrm{P} * f\|_p \|\mathrm{P} * g\|_q + \|\mathrm{P} * f\|_p \|I(\mathrm{P}) * g\|_q.$$

Then by Proposition V.3 and Lemma V.3.2 of [35] we have

$$\|\mathrm{P} * g\|_q \lesssim \|J(\mathrm{P}) * g\|_q, \quad \text{and} \quad \|I(\mathrm{P}) * g\|_q \lesssim \|J(\mathrm{P}) * g\|_q.$$

Hence, by Lemma 3.3,

$$\mathrm{II} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q} \|f\|_{\mathrm{h}^{\mathrm{c}}_p}.$$

Now we estimate the term I. By the Cauchy–Schwarz inequality

$$\frac{c_d^2}{4^d}\,\mathrm{I}^2 \leqslant \tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) s^{\mathrm{c}}(f)(s,\varepsilon)^{p-2} \mathrm{d}\varepsilon \mathrm{d}s$$

$$\cdot\, \tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) s^{\mathrm{c}}(f)(s,\varepsilon)^{2-p} \mathrm{d}\varepsilon \mathrm{d}s \stackrel{\mathrm{def}}{=} A \cdot B.$$

Note here that $s^{\mathrm{c}}(f)(s,\varepsilon)$ is the function of two variables defined by (3.6), which is differentiable in the w*-sense. We first deal with A. Using $\overline{s}^{\mathrm{c}}(f)(s,\varepsilon) \leqslant s^{\mathrm{c}}(f)(s,\varepsilon)$, we have

$$\begin{aligned} A &\leqslant \tau \int_{\mathbb{R}^d} \int_0^1 \int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \overline{s}^{\mathrm{c}}(f)(s,\varepsilon)^{p-2} \frac{\mathrm{d}\varepsilon \mathrm{d}t}{\varepsilon^{d-1}} \mathrm{d}s \\ &= -\tau \int_{\mathbb{R}^d} \int_0^1 \Big(\frac{\partial}{\partial \varepsilon} \overline{s}^{\mathrm{c}}(f)(s,\varepsilon)^2 \Big) \overline{s}^{\mathrm{c}}(f)(s,\varepsilon)^{p-2} \mathrm{d}\varepsilon \mathrm{d}s \\ &= -2\tau \int_{\mathbb{R}^d} \int_0^1 \overline{s}^{\mathrm{c}}(f)(s,\varepsilon)^{p-1} \frac{\partial}{\partial \varepsilon} \overline{s}^{\mathrm{c}}(f)(s,\varepsilon) \mathrm{d}\varepsilon \mathrm{d}s. \end{aligned}$$

Since $1 \leqslant p < 2$ and $\overline{s}^{\mathrm{c}}(f)(s,\varepsilon)$ is decreasing in ε, $\overline{s}^{\mathrm{c}}(f)(s,\varepsilon)^{p-1} \leqslant \overline{s}^{\mathrm{c}}(f)(s,0)^{p-1}$. At the same time, $\frac{\partial}{\partial \varepsilon} \overline{s}^{\mathrm{c}}(f)(s,\varepsilon) \leqslant 0$. Therefore,

$$A \lesssim -\tau \int_{\mathbb{R}^d} \overline{s}^{\mathrm{c}}(f)(s,0)^{p-1} \int_0^1 \frac{\partial}{\partial \varepsilon} \overline{s}(f)^{\mathrm{c}}(s,\varepsilon) \mathrm{d}\varepsilon \mathrm{d}s \lesssim \tau \int_{\mathbb{R}^d} s^{\mathrm{c}}(f)(s,0)^p \mathrm{d}s = \|f\|^p_{\mathrm{h}^{\mathrm{c}}_p}.$$

The estimate of B is harder. For any $j \in \mathbb{N}$, we need to create a square net partition in $\mathbb{R}^d$ as follows:

$$Q_{m,j} = \Big(\frac{1}{\sqrt{d}}(m_1 - 1)2^{-j}, \frac{1}{\sqrt{d}} m_1 2^{-j} \Big] \times \cdots \times \Big(\frac{1}{\sqrt{d}}(m_d - 1)2^{-j}, \frac{1}{\sqrt{d}} m_d 2^{-j} \Big]$$

with $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$. Let $c_{m,j}$ denote the center of $Q_{m,j}$. Define

$$\mathbb{S}^{\mathrm{c}}(f)(s,j) = \Big(\int_{2^{-j}}^{1} \int_{B(c_{m,j},r)} \Big| \frac{\partial}{\partial r} \mathrm{P}_r(f)(t) \Big|^2 \frac{\mathrm{d}t\mathrm{d}r}{r^{d-1}} \Big)^{1/2} \quad \text{if } s \in Q_{m,j}. \tag{3.8}$$

For any $s \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$ ($\mathbb{N}_0$ being the set of nonnegative integers), we define

$$d(s,k) = \mathbb{S}^{\mathrm{c}}(f)(s,k)^{2-p} - \mathbb{S}^{\mathrm{c}}(f)(s,k-1)^{2-p}.$$

Since $B(s, r - \frac{\varepsilon}{2}) \subset B(c_{m,j}, r)$ whenever $s \in Q_{m,j}$ and $\varepsilon \geqslant 2^{-j}$, we have

$$s^{\mathrm{c}}(f)(s,\varepsilon) \leqslant \mathbb{S}^{\mathrm{c}}(f)(s,j), \quad \forall\, s \in Q_{m,j}, \varepsilon \geqslant 2^{-j}.$$

It is clear that $\mathbb{S}^{\mathrm{c}}(f)(s,j)$ is increasing in j, so $d(s,k) \geqslant 0$. At the same time, $d(s,k)$ is constant on $Q_{m,k}$ and $\sum_{k \leqslant j} d(s,k) = \mathbb{S}^{\mathrm{c}}(f)(s,j)^{2-p}$. Therefore,

$$\begin{aligned}
B &\lesssim \tau \sum_{m \in \mathbb{Z}^d} \sum_{j \geqslant 1} \int_{Q_{m,j}} \int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) \mathbb{S}^{\mathrm{c}}(f)(s,j)^{2-p} \mathrm{d}\varepsilon \mathrm{d}s \\
&= \tau \int_{\mathbb{R}^d} \sum_{j \geqslant 1} \mathbb{S}^{\mathrm{c}}(f)(s,j)^{2-p} \int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) \mathrm{d}\varepsilon \mathrm{d}s \\
&= \tau \int_{\mathbb{R}^d} \sum_{j \geqslant 1} \sum_{1 \leqslant k \leqslant j} d(s,k) \int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) \mathrm{d}\varepsilon \mathrm{d}s \\
&= \tau \int_{\mathbb{R}^d} \sum_{k \geqslant 1} d(s,k) \sum_{j \geqslant k} \int_{2^{-j}}^{2^{-j+1}} \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) \mathrm{d}\varepsilon \mathrm{d}s \\
&= \tau \sum_m \sum_{k \geqslant 1} d(s,k) \int_{Q_{m,k}} \int_{0}^{2^{-k+1}} \Big(\int_{B(s,\varepsilon/2)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \frac{\mathrm{d}t}{\varepsilon^{d-1}} \Big) \mathrm{d}\varepsilon \mathrm{d}s \,.
\end{aligned}$$

Since $g \in \mathrm{bmo}^{\mathrm{c}}_q$, Lemma 3.3 ensures the existence of a positive operator $a \in L_{q/2}(\mathcal{N})$ such that $\|a\|_{q/2} \lesssim \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}_q}$ and

$$\frac{1}{|Q|} \int_{T(Q)} \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(t) \Big|^2 \varepsilon \mathrm{d}t \mathrm{d}\varepsilon \leqslant a(s) \text{ and for } s \in Q \text{ and for all cubes } Q \text{ with } |Q| < 1.$$

Let $\widetilde{Q}_{m,k}$ be the cube concentric with $Q_{m,k}$ and having side length 2^{-k+1}. By the Fubini theorem and Lemma 2.4, we have

$$\begin{aligned}\int_{Q_{m,k}}\int_0^{2^{-k+1}}\Big(\int_{B(s,\varepsilon/2)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)\Big|^2\frac{\mathrm{d}t}{\varepsilon^{d-1}}\Big)\mathrm{d}\varepsilon\mathrm{d}s &\leqslant 2^d\int_{\widetilde{Q}_{m,k}}\int_0^{2^{-k+1}}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)\Big|^2\varepsilon\mathrm{d}\varepsilon\mathrm{d}s\\ &= 2^d\int_{T(\widetilde{Q}_{m,k})}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)\Big|^2\varepsilon\mathrm{d}\varepsilon\mathrm{d}s\\ &\lesssim\int_{Q_{m,k}}a(s)\mathrm{d}s.\end{aligned}$$

Then we deduce

$$\begin{aligned}B&\lesssim\tau\sum_m\sum_{k\geqslant1}\int_{Q_{m,k}}d(s,k)a(s)\mathrm{d}s=\tau\int_{\mathbb{R}^d}\sum_{k\geqslant1}d(s,k)a(s)\mathrm{d}s\\ &=\tau\int_{\mathbb{R}^d}\mathbb{S}^{\mathrm{c}}(f)(s,+\infty)^{2-p}a(s)\mathrm{d}s=\tau\int_{\mathbb{R}^d}S^{\mathrm{c}}(f)(s)^{2-p}a(s)\mathrm{d}s\leqslant\|S^{\mathrm{c}}(f)\|_p^{2-p}\|a\|_{q/2}\\ &\leqslant\|f\|_{\mathrm{h}_p^{\mathrm{c}}}^{2-p}\|a\|_{q/2}\lesssim\|f\|_{\mathrm{h}_p^{\mathrm{c}}}^{2-p}\|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}^2.\end{aligned}$$

Combining the estimates of A, B and II, we complete the proof. ∎

The following is the main theorem of this section.

THEOREM 3.11. *Let $1\leqslant p<2$ and q be its conjugate index. We have $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})^*=\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ with equivalent norms. More precisely, every $g\in\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ defines a continuous linear functional on $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ by*

$$\ell_g(f)=\tau\int f(s)g^*(s)\mathrm{d}s,\quad\forall\,f\in L_p(\mathcal{M};\mathrm{W}_d^{\mathrm{c}}).$$

Conversely, every $\ell\in\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})^*$ can be written as above and is associated to some $g\in\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ with

$$\|\ell\|_{(\mathrm{h}_p^{\mathrm{c}})^*}\approx\|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}.$$

Proof. First, by Lemma 3.10, we get

$$|\ell_g(f)|\lesssim\|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}\|f\|_{\mathrm{h}_p^{\mathrm{c}}}. \tag{3.9}$$

Now we prove the converse. Suppose that $\ell\in\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})^*$. By the Hahn–Banach theorem, ℓ extends to a continuous functional on $L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_p L_p(\mathcal{N})$ with the same norm. Thus, there exists $h=(h',h'')\in L_q(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_q L_q(\mathcal{N})$

such that

$$\|h\|_{L_q(\mathcal{N};L_2^c(\widetilde{\Gamma}))\oplus_q L_q(\mathcal{N})} = \|\ell\|_{(\mathrm{h}_p^c)^*} \quad \text{and}$$

$$\ell(f) = \tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)h'^*(s,t,\varepsilon)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^d}\mathrm{d}s + \tau\int_{\mathbb{R}^d}\mathrm{P}*f(s)h''^*(s)\mathrm{d}s,$$

$$= \tau\int_{\mathbb{R}^d} f(u)\widetilde{\mathbf{F}}(h)^*(u)\mathrm{d}u,$$

where $\widetilde{\mathbf{F}}$ is the map defined in (3.5).

Let $g = \widetilde{\mathbf{F}}(h)$. Following the proof of Theorem 3.8, we have

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \|\ell\|_{(\mathrm{h}_p^c)^*} \quad \text{and}$$

$$\ell(f) = \tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s, \quad \forall\, f\in L_p(\mathcal{M};\mathrm{W}_d^c).$$

Thus, we have accomplished the proof of the theorem. ∎

The following corollary gives an equivalent norm of the space bmo_q^c. Note that it is a strengthening of the one-sided estimates in Lemmas 2.4 and 2.5.

COROLLARY 3.12. *Let $2 < q \leqslant \infty$. Then $g \in \mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ if and only if $\mathrm{d}\lambda_g = |\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)|^2\varepsilon\mathrm{d}s\mathrm{d}\varepsilon$ is an $\mathcal{M}$-valued Carleson q-measure on S and $\|J(\mathrm{P})*g\|_q < \infty$. Furthermore,*

$$\|g\|_{\mathrm{bmo}_q^c} \approx \Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^{+}\frac{1}{|Q|}\int_{T(Q)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)\Big|^2\varepsilon\mathrm{d}t\mathrm{d}\varepsilon\Big\|_{q/2}^{1/2} + \|J(\mathrm{P})*g\|_q.$$

Proof. From the proof of Lemma 3.10, it is easy to see that if

$$\mathrm{d}\lambda_g = \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)\Big|^2\varepsilon\mathrm{d}s\mathrm{d}\varepsilon$$

is an $\mathcal{M}$-valued Carleson q-measure on S and $J(\mathrm{P})*g\in L_q(\mathcal{N})$, then g defines a continuous functional on $\mathrm{h}_p^c(\mathbb{R}^d,\mathcal{M})$:

$$\ell(f) = \tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s, \quad \text{and}$$

$$\|\ell\|_{(\mathrm{h}_p^c)^*} \lesssim \Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^{+}\frac{1}{|Q|}\int_{T(Q)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)\Big|^2\varepsilon\mathrm{d}t\mathrm{d}\varepsilon\Big\|_{q/2}^{1/2} + \|J(\mathrm{P})*g\|_q.$$

According to Theorem 3.11, there exists a function $g'\in\mathrm{bmo}_q^c(\mathbb{R}^d,\mathcal{M})$ such that

$$\|g'\|_{\mathrm{bmo}_q^c} \lesssim \Big\|\sup_{\substack{s\in Q\subset\mathbb{R}^d\\|Q|<1}}{}^{+}\frac{1}{|Q|}\int_{T(Q)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)\Big|^2\varepsilon\mathrm{d}t\mathrm{d}\varepsilon\Big\|_{q/2}^{1/2} + \|J(\mathrm{P})*g\|_q$$

and that

$$\tau \int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s = \tau \int_{\mathbb{R}^d} f(s)g'^*(s)\mathrm{d}s,$$

for any $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Thus, $g = g'$ with

$$\|g\|_{\mathrm{bmo}_q^{\mathrm{c}}} \lesssim \Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(t)\Big|^2 \varepsilon \mathrm{d}t\mathrm{d}\varepsilon \Big\|_{q/2}^{1/2} + \|J(\mathrm{P}) * g\|_q.$$

The inverse inequality is already contained in Lemmas 2.4 and 2.5. We obtain the desired assertion. ∎

3.4. THE EQUIVALENCE $\mathrm{h}_q = \mathrm{bmo}_q$. We now show that $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ for $2 < q < \infty$. Thus according to the duality obtained in the last subsection, the dual of $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ agrees with $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ when $1 < p < 2$. Let us begin with two lemmas concerning the comparison of $s^{\mathrm{c}}(f)$ and $g^{\mathrm{c}}(f)$. We require an auxiliary truncated square function. For $s \in \mathbb{R}^d$ and $\varepsilon \in [0, \frac{2}{3}]$, we define:

$$\widetilde{g}^{\mathrm{c}}(f)(s,\varepsilon) = \Big(\int_{\varepsilon}^{2/3} \Big|\frac{\partial}{\partial r}\mathrm{P}_r(f)(s)\Big|^2 r\mathrm{d}r\Big)^{1/2}. \tag{3.10}$$

LEMMA 3.13. *We have*

$$\widetilde{g}^{\mathrm{c}}(f)(s,\varepsilon) \lesssim s^{\mathrm{c}}(f)(s, \tfrac{\varepsilon}{2}),$$

where the relevant constant depends only on the dimension d.

Proof. By translation, it suffices to prove this inequality for $s = 0$. Given $\varepsilon \in [0, \frac{2}{3}]$, for any r such that $\varepsilon \leqslant r \leqslant \frac{2}{3}$, let us denote the ball centered at $(0, r)$ and tangent to the boundary of the cone $\Big\{(t,u) \in \mathbb{R}_+^{d+1} : |t| < \frac{r-(\varepsilon/2)}{r}u\Big\}$ by $\widetilde{B}_r$. We notice that the radius of $\widetilde{B}_r$ is greater than or equal to $\frac{r}{\sqrt{5}}$. By the harmonicity of $\frac{\partial}{\partial r}\mathrm{P}_r(f)$, we have

$$\frac{\partial}{\partial r}\mathrm{P}_r(f)(0) = \frac{1}{|\widetilde{B}_r|} \int_{\widetilde{B}_r} \frac{\partial}{\partial u}\mathrm{P}_u(f)(t)\mathrm{d}t.$$

Then by (1.10), we arrive at

$$\Big|\frac{\partial}{\partial r}\mathrm{P}_r(f)(0)\Big|^2 \leqslant \frac{\sqrt{5}^{d+1}}{c_{d+1}r^{d+1}} \int_{\widetilde{B}_r} \Big|\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)\Big|^2 \mathrm{d}t,$$

where c_{d+1} is the volume of the unit ball of $\mathbb{R}^{d+1}$. Integrating the above inequality, we get

$$\int_{\varepsilon}^{\frac{2}{3}} \Big|\frac{\partial}{\partial r}\mathrm{P}_r(f)(0)\Big|^2 r\mathrm{d}r \leqslant \int_{\varepsilon}^{\frac{2}{3}} \frac{\sqrt{5}^{d+1}}{c_{d+1}r^d} \int_{\widetilde{B}_r} \Big|\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)\Big|^2 \mathrm{d}t\mathrm{d}u\mathrm{d}r. \tag{3.11}$$

Since $(t,u) \in \widetilde{B}_r$ implies $\frac{\sqrt{5}}{\sqrt{5}+1}u \leqslant r \leqslant \frac{\sqrt{5}}{\sqrt{5}-1}u$ and $\frac{\varepsilon}{2} \leqslant u \leqslant 1$, the right hand side of (3.11) can be majorized by

$$\frac{\sqrt{5}^{d+1}}{c_{d+1}} \int_{\varepsilon/2}^{1} \int_{\widetilde{B}_r} \Big|\frac{\partial}{\partial u}\mathrm{P}_u(f)(t)\Big|^2 \int_{u/2}^{2u} \frac{1}{r^d}\mathrm{d}r\mathrm{d}t\mathrm{d}u \leqslant C|s^{\mathrm{c}}(f)(0,\tfrac{\varepsilon}{2})|^2,$$

where C is a constant depending only on d. Therefore, $\widetilde{g}^{\mathrm{c}}(f)(0,\varepsilon) \lesssim s^{\mathrm{c}}(f)(0,\frac{\varepsilon}{2})$. ∎

LEMMA 3.14. *Let $1 \leqslant p < \infty$. Then for any $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$, we have*

$$\|s^{\mathrm{c}}(f)\|_p + \|\mathrm{P} * f\|_p \lesssim \|g^{\mathrm{c}}(f)\|_p + \|\mathrm{P} * f\|_p.$$

Proof. We first deal with the case when $1 \leqslant p < 2$. Let g be a function in $\mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$ (q is the conjugate index of p). Following a similar calculation as (2.3), it is easy to see that

$$\begin{aligned}
&\tau \int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s \\
&\quad = 4\tau \int_{\mathbb{R}^d} \int_0^{2/3} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)^*(s)\varepsilon\mathrm{d}\varepsilon\mathrm{d}s \\
&\qquad + \Big(\tau \int_{\mathbb{R}^d} \mathrm{P} * f(s)(\mathrm{P}_{1/3} * g(s))^*\mathrm{d}s + \frac{8\pi}{3}\tau \int_{\mathbb{R}^d} \mathrm{P} * f(s)(I(\mathrm{P}_{1/3}) * g(s))^*\mathrm{d}s\Big) \\
&\quad \stackrel{\mathrm{def}}{=} \mathrm{I} + \mathrm{II}.
\end{aligned}$$

The term II can be treated in the same way as in the proof of Lemma 3.10:

$$\mathrm{II} \lesssim \|\mathrm{P} * f\|_p \cdot \|J(\mathrm{P}_{1/3}) * f\|_p.$$

Applying Lemma 3.3, we have

$$\mathrm{II} \lesssim \|\mathrm{P} * f\|_p \cdot \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

Concerning the term I, we have

$$
\begin{aligned}
&|\mathrm{I}|^2\\
&\lesssim \tau \int_{\mathbb{R}^d}\int_0^{2/3} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)\Big|^2 \widetilde{g}^{\mathrm{c}}(f)(s,\varepsilon)^{p-2}\varepsilon \mathrm{d}\varepsilon \mathrm{d}s \cdot \tau \int_{\mathbb{R}^d}\int_0^{2/3} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)\Big|^2 \widetilde{g}^{\mathrm{c}}(f)(s,\varepsilon)^{2-p}\varepsilon \mathrm{d}\varepsilon \mathrm{d}s\\
&\overset{\mathrm{def}}{=} A' \cdot B'.
\end{aligned}
$$

Following the argument for the estimate of A in the proof of Lemma 3.10, we deduce similarly that $A' \lesssim \|\widetilde{g}^{\mathrm{c}}(f)\|_p^p$. Now we deal with term B'. By Lemma 3.13, we have

$$
B' \leqslant \tau \int_{\mathbb{R}^d}\int_0^{2/3} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s)\Big|^2 s^{\mathrm{c}}(f)(s,\tfrac{\varepsilon}{2})\varepsilon \mathrm{d}\varepsilon \mathrm{d}s.
$$

Then we apply almost the same argument as in the estimate of B. There is only one minor difference: when $\varepsilon \geqslant 2^{-j}$ and $s \in Q_{m,j}$, we have $s^{\mathrm{c}}(f)(s,\frac{\varepsilon}{2}) \leqslant \mathbb{S}^{\mathrm{c}}(f)(s,j+1)$. We conclude that

$$
B' \lesssim \|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}^2 \|s^{\mathrm{c}}(f)\|_p^{2-p}.
$$

Combining the estimates of I, A' and B' with Theorem 3.11, we get

$$
\|s^{\mathrm{c}}(f)\|_p + \|\mathrm{P}*f\|_p \lesssim \|\widetilde{g}^{\mathrm{c}}(f)\|_p + \|\mathrm{P}*f\|_p \lesssim \|g^{\mathrm{c}}(f)\|_p + \|\mathrm{P}*f\|_p.
$$

The case $p=2$ is obvious. For $p>2$, choose a positive $g \in L_{(p/2)'}(\mathcal{N})$ with norm one such that,

$$
\begin{aligned}
\|s^{\mathrm{c}}(f)\|_p^2 &= \Big\| \iint_{\widetilde{\Gamma}} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(\cdot+t)\Big|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\Big\|_{p/2}\\
&= \tau \int_{\mathbb{R}^d} \iint_{\widetilde{\Gamma}} \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s+t)\Big|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}} g(s)\mathrm{d}s\\
&= \tau \int_{\mathbb{R}^d}\int_0^1 \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\Big|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}} \int_{B(t,\varepsilon)} g(s)\mathrm{d}s.
\end{aligned}
$$

By the noncommutative Hardy–Littlewood maximal inequality (the one dimension $\mathbb{R}$ case is given by Theorem 3.3 of [24], the case $\mathbb{R}^d$ is a simple corollary of (3.1) and Lemma 3.7), there exists a positive $a \in L_{(p/2)'}(\mathcal{N})$ such that $\|a\|_{(p/2)'} \leqslant 1$ and

$$
\frac{1}{|B(t,2^{-k})|}\int_{B(t,2^{-k})} g(s)\mathrm{d}s \leqslant a(t), \quad \forall\, t \in \mathbb{R}^d,\ \forall\, \varepsilon > 0.
$$

Therefore,

$$\|s^{\mathrm{c}}(f)\|_p^2 = \tau \int_{\mathbb{R}^d} \int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}} \int_{B(t,\varepsilon)} g(s)\mathrm{d}s \leqslant c_d \tau \int_{\mathbb{R}^d} \int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \varepsilon a(t) \mathrm{d}t\mathrm{d}\varepsilon$$

$$\leqslant c_d \Big\| \int_0^1 \Big| \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(t) \Big|^2 \varepsilon \mathrm{d}t\mathrm{d}\varepsilon \Big\|_{p/2} \|a\|_{(p/2)'} \leqslant c_d \|g^{\mathrm{c}}(f)\|_p.$$

Then the assertion for the case $p > 2$ is also proved. ∎

To proceed further, we introduce the definition of tent spaces. These spaces are closely related to Hardy spaces and were initially defined by Coifman, Meyer and Stein in [4]. In the noncommutative setting, the operator-valued tent spaces were first defined and studied by Mei [25].

DEFINITION 3.15. For any function defined on $\mathbb{R}^d \times (0,1) = S$ with values in $L_1(\mathcal{M}) + \mathcal{M}$, whenever it exists, we define

$$A^{\mathrm{c}}(f)(s) = \Big(\int_{\widetilde{\Gamma}} |f(t+s,\varepsilon)|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}} \Big)^{1/2}, \quad s \in \mathbb{R}^d.$$

For $1 \leqslant p < \infty$, we define

$$T_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \{f : A^{\mathrm{c}}(f) \in L_p(\mathcal{N})\}$$

equipped with the norm $\|f\|_{T_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})} = \|A^{\mathrm{c}}(f)\|_p$. For $p = \infty$, define the operator-valued column T_∞^{c} norm of f as

$$\|f\|_{T_\infty^{\mathrm{c}}} = \sup_{|Q| \leqslant 1} \Big\| \Big(\frac{1}{|Q|} \int_{T(Q)} |f(s,\varepsilon)|^2 \frac{\mathrm{d}s\mathrm{d}\varepsilon}{\varepsilon} \Big)^{1/2} \Big\|_{\mathcal{M}},$$

and the corresponding space is

$$T_\infty^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \{f : \|f\|_{T_\infty^{\mathrm{c}}} < \infty\}.$$

REMARK 3.16. By the same arguments used in the proof of Theorem 3.11, it is straightforward to prove the duality that $T_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})^* = T_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ for $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For the case $p = 1$, it suffices to replace $\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)(s)$ and $\frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)(s)$ in the proof of Lemma 3.10 by $f(s,\varepsilon)$ and $g(s,\varepsilon)$, respectively. A similar argument will give us the inclusion that $T_\infty^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \subset T_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})^*$. On the other hand, since $L_\infty(\mathcal{N}; L_2^{\mathrm{c}}(\widetilde{\Gamma})) \subset T_\infty^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, we get the reverse inclusion. For $1 < p < \infty$, the tent space $T_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ we define above is a complemented subspace of the column tent space defined in [24]. So by Remark 4.6 in [41], we obtain the duality that $T_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})^* = T_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

THEOREM 3.17. *For* $2 < q < \infty$, $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms.*

Proof. First, we show the inclusion $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. By Theorem 3.11, it suffices to show that $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})^*$. Applying (2.3), for any $f \in \mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ and $g \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, we have

$$\tau\int_{\mathbb{R}^d} g(s)f^*(s)\mathrm{d}s = \frac{4}{c_d}\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}} \frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)(s+t)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s+t)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d-1}}\mathrm{d}s + \int_{\mathbb{R}^d}\mathrm{P}*g(s)(\mathrm{P}*f(s))^*\mathrm{d}s + 4\pi\int_{\mathbb{R}^d} I(\mathrm{P})*g(s)(\mathrm{P}*f(s))^*\mathrm{d}s.$$

Then, by the Hölder inequality,

$$\begin{aligned}\left|\tau\int_{\mathbb{R}^d} g(s)f^*(s)\mathrm{d}s\right| &\leqslant \left\|\varepsilon\cdot\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(g)\right\|_{L_p(\mathcal{N};L_2(\widetilde{\Gamma}))}\left\|\varepsilon\cdot\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)\right\|_{L_q(\mathcal{N};L_2(\widetilde{\Gamma}))}\\ &\quad + \|(\mathrm{P}+I(\mathrm{P}))*g\|_p\cdot\|\mathrm{P}*f\|_q\\ &\leqslant (\|s^{\mathrm{c}}(g)\|_p + \|(\mathrm{P}+I(\mathrm{P}))*g\|_p)\|f\|_{\mathrm{h}_q^{\mathrm{c}}}.\end{aligned}$$

Now, we show that for any $1 \leqslant p < 2$ and $g \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, we have $\|(\mathrm{P}+I(\mathrm{P}))*g\|_p \lesssim \|g\|_{\mathrm{h}_p^{\mathrm{c}}}$. Since $2 < q < \infty$, we have $1 < \frac{q}{2} < \infty$. Applying the noncommutative Hardy–Littlewood maximal inequality, we get

$$\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}} \lesssim \left\|\sup_{s\in Q\subset\mathbb{R}^d}{}^+\frac{1}{|Q|}\int_Q |f(t)|^2\mathrm{d}t\right\|_{q/2}^{1/2} \lesssim \||f|^2\|_{q/2}^{1/2} = \|f\|_q.$$

This implies that $L_q(\mathcal{N}) \subset \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ for any $2 < q \leqslant \infty$. Then by Theorem 3.11, we get $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset L_p(\mathcal{N})$. Therefore we deduce that

$$\|(\mathrm{P}+I(\mathrm{P}))*g\|_p \lesssim \|g\|_p \lesssim \|g\|_{\mathrm{h}_p^{\mathrm{c}}}. \tag{3.12}$$

Thus,

$$\left|\tau\int_{\mathbb{R}^d} g(s)f^*(s)\mathrm{d}s\right| \lesssim \|f\|_{\mathrm{h}_q^{\mathrm{c}}}\|g\|_{\mathrm{h}_p^{\mathrm{c}}}.$$

We have proved $\mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$.

Let us turn to the reverse inclusion $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. We need to make use of the tent spaces in Definition 3.15. We claim that for $q > 2$, every $f \in \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ induces a linear functional on $T_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})\bigoplus_p L_p(\mathcal{N})$. Indeed, for any $h = (h',h'') \in T_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})\bigoplus_p L_p(\mathcal{N})$, we define

$$\begin{aligned}\ell_f(h) &= \tau\int_{\mathbb{R}^d}\int_0^1 h'(s,\varepsilon)\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)^*(s)\mathrm{d}\varepsilon\mathrm{d}s\\ &\quad + \tau\int_{\mathbb{R}^d} h''(s)[(\mathrm{P}*f)^*(s) + 4\pi(I(\mathrm{P})*f)^*(s)]\mathrm{d}s.\end{aligned} \tag{3.13}$$

Set

$$A^{\mathrm{c}}(h')(s,\varepsilon)=\int_{\varepsilon}^{1}\int_{B(s,r-\varepsilon/2)}|h'(s,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}r}{r^{d+1}},\quad \overline{A}^{\mathrm{c}}(h')(s,\varepsilon)=\int_{\varepsilon}^{1}\int_{B(s,r/2)}|h'(s,\varepsilon)|^2\frac{\mathrm{d}t\mathrm{d}r}{r^{d+1}}.$$

Then by the Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned}|\ell_f(h)|\lesssim{}&\Big(\tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\varepsilon/2)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\Big|^2\frac{\mathrm{d}t}{\varepsilon^{d-1}}\Big)\overline{A}^{\mathrm{c}}(h')(s,\varepsilon)^{p-2}\mathrm{d}\varepsilon\mathrm{d}s\Big)^{1/2}\\&\cdot\Big(\tau\int_{\mathbb{R}^d}\int_0^1\Big(\int_{B(s,\varepsilon/2)}\Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t)\Big|^2\frac{\mathrm{d}t}{\varepsilon^{d-1}}\Big)\overline{A}^{\mathrm{c}}(h')(s,\varepsilon)^{2-p}\mathrm{d}\varepsilon\mathrm{d}s\Big)^{1/2}\\&+\Big|\tau\int_{\mathbb{R}^d}h''(s)(\mathrm{P}*f(s))^*\mathrm{d}s\Big|+\Big|\tau\int_{\mathbb{R}^d}h''(s)(I(\mathrm{P})*f(s))^*\mathrm{d}s\Big|.\end{aligned}$$

Following a similar argument as in the proof of Lemma 3.10, we obtain that

$$|\ell_f(h)|\lesssim(\|h'\|_{T_p^{\mathrm{c}}}+\|h''\|_{L_p})\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}}\lesssim\|h\|_{T_p^{\mathrm{c}}\oplus_p L_p}\cdot\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}},$$

which implies that $\|\ell_f\|\leqslant c_q\|f\|_{\mathrm{bmo}_q^{\mathrm{c}}}$. So the claim is proved.

Next we show that $\|f\|_{\mathrm{h}_q^{\mathrm{c}}}\leqslant C_q\|\ell_f\|$. By definition, we can regard T_p^{c} as a closed subspace of $L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))$ in the natural way. Then, ℓ_f extends to a linear functional on $L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_p L_p(\mathcal{N})$. Thus, there exists $g=(g',g'')\in L_q(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_q L_q(\mathcal{N})$ such that

$$\|g\|_{L_q(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\oplus_q L_q(\mathcal{N})}\leqslant\|\ell_f\|$$

and for any $h=(h',h'')\in L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_p L_p(\mathcal{N})$,

$$\begin{aligned}\ell_f(h)&=\tau\int_{\mathbb{R}^d}\iint_{\widetilde{\Gamma}}h'(s,\varepsilon)g'^*(s,t,\varepsilon)\frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}}\mathrm{d}s+\tau\int_{\mathbb{R}^d}h''(s)g''^*(s)\mathrm{d}s\\&=\tau\int_{\mathbb{R}^d}\int_0^1h'(s,\varepsilon)\int_{B(s,\varepsilon)}g'^*(s,t,\varepsilon)\mathrm{d}t\frac{\mathrm{d}s\mathrm{d}\varepsilon}{\varepsilon^{d+1}}+\tau\int_{\mathbb{R}^d}h''(s)g''^*(s)\mathrm{d}s.\end{aligned}$$

Comparing the equalities above with (3.13), we get

$$\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(s)=\frac{1}{\varepsilon^{d+1}}\int_{B(0,\varepsilon)}g'(s,t,\varepsilon)\mathrm{d}t\quad\text{and}\quad\mathrm{P}*f+4\pi I(\mathrm{P})*f=g''.$$

By Lemma 3.14, we have

$$\begin{aligned}\|f\|_{\mathrm{h}^{\mathrm{c}}_q} &\lesssim \Big\|\Big(\int_0^1 \Big|\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)\Big|^2\varepsilon\mathrm{d}\varepsilon\Big)^{1/2}\Big\|_q + \|\mathrm{P}*f\|_q\\ &\leqslant c_d\Big\|\Big(\int_0^1\frac{1}{\varepsilon^{d+1}}\int_{B(0,\varepsilon)}|g'(s,t,\varepsilon)|^2\mathrm{d}t\mathrm{d}\varepsilon\Big)^{1/2}\Big\|_q + \|\mathrm{P}*f\|_q\\ &\lesssim \|g'\|_{L_q(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))} + \|\mathrm{P}*f\|_q.\end{aligned}$$

Now let us majorize the second term $\|\mathrm{P}*f\|_q$ by $\|g''\|_q$. Indeed, consider the function

$$G(s) = 2\pi\int_0^\infty \mathrm{e}^{-2\pi\varepsilon}\mathrm{P}_\varepsilon(s)\mathrm{d}\varepsilon.$$

It is easy to see that $G\in L_1(\mathbb{R}^d)$, $\|G\|_1\leqslant 1$ and $\widehat{G}(\xi) = (1+|\xi|)^{-1}$. This means that the operator $(1+I)^{-1}$ is a contractive Fourier multiplier on $L_q(\mathcal{N})$. Therefore,

$$\|\mathrm{P}*f\|_q \leqslant \|(\mathrm{P}+I(\mathrm{P}))*f\|_q \leqslant 4\pi\|g''\|_q.$$

Finally, we conclude that $\|f\|_{\mathrm{h}^{\mathrm{c}}_q}\lesssim\|\ell_f\|\lesssim\|f\|_{\mathrm{bmo}^{\mathrm{c}}_q}$ and then we have $\mathrm{h}^{\mathrm{c}}_q(\mathbb{R}^d,\mathcal{M}) = \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d,\mathcal{M})$ with equivalent norms. ∎

Armed with the theorem above, we are able to extend the content of Theorem 3.8.

THEOREM 3.18. (i) *The map* $\mathbf{F}$ *extends to a bounded map from* $L_\infty(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_\infty L_\infty(\mathcal{N})$ *into* $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ *and*

$$\|\mathbf{F}(h)\|_{\mathrm{bmo}^{\mathrm{c}}}\lesssim\|h\|_{L_\infty(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\oplus_\infty L_\infty(\mathcal{N})}.$$

(ii) *For* $1<p<\infty$, $\mathbf{F}$ *extends to a bounded map from* $L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\bigoplus_p L_p(\mathcal{N})$ *into* $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$ *and*

$$\|\mathbf{F}(h)\|_{\mathrm{h}^{\mathrm{c}}_p}\lesssim\|h\|_{L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}.$$

Proof. (i) is already contained in Theorem 3.8. When $p>2$, (ii) follows from Theorem 3.8 and Theorem 3.17. The case $p=2$ is trivial. For the case $1<p<2$, according to Theorem 3.11, we have

$$\|\mathbf{F}(h)\|_{\mathrm{h}^{\mathrm{c}}_p}\lesssim\sup_{\|f\|_{\mathrm{bmo}^{\mathrm{c}}_q}\leqslant 1}\Big|\tau\int_{\mathbb{R}^d}\mathbf{F}(h)(s)f^*(s)\mathrm{d}s\Big|.$$

Then, by Theorem 3.17 and (3.12), for $h = (h', h'') \in L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$, we have the following and the desired inequality is proved:

$$\begin{aligned}
&\sup_{\|f\|_{\mathrm{bmo}_q^c}\leqslant 1} \Big|\tau \int_{\mathbb{R}^d} \mathbf{F}(h)(s) f^*(s) \mathrm{d}s\Big| \\
&\lesssim \sup_{\|f\|_{\mathrm{h}_q^c}\leqslant 1} \Big|\tau \int_{\mathbb{R}^d} \Big[\iint_{\widetilde{\Gamma}} h'(s,t,\varepsilon) \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)^*(s+t) \mathrm{d}t\mathrm{d}\varepsilon + h''(s)([\mathrm{P}+4\pi I(\mathrm{P})] * f^*(s))\Big] \mathrm{d}s\Big| \\
&\lesssim \|h\|_{L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma}))\oplus_p L_p(\mathcal{N})}. \quad \blacksquare
\end{aligned}$$

The above theorem shows that, for any $1 < p < \infty$, $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N}; L_2^c(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$. Thus, we deduce the following duality theorem.

THEOREM 3.19. *We have* $\mathrm{h}_p^c(\mathbb{R}^d, \mathcal{M})^* = \mathrm{h}_q^c(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms for any* $1 < p < \infty$.

4. INTERPOLATION

In this section we study the interpolation of local Hardy and bmo spaces by transferring the problem to that of the operator-valued Hardy and BMO spaces defined in [24]. We begin with an easy observation on the difference between bmo_q^c and BMO_q^c norms.

LEMMA 4.1. *For* $2 < q \leqslant \infty$, *we have*

$$\|g\|_{\mathrm{bmo}_q^c} \approx (\|g\|_{\mathrm{bmo}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q)^{1/q}.$$

Proof. Repeating the proof of Proposition 2.3 with $\|\cdot\|_{\mathcal{M}}$ replaced by $\|\cdot\|_{L_{q/2}(\mathcal{N};\ell_\infty)}$, we have $\|g\|_{\mathrm{BMO}_q^c} \lesssim \|g\|_{\mathrm{bmo}_q^c}$. By Lemma 3.3, it is also evident that $\|J(\mathrm{P}) * g\|_q \lesssim \|g\|_{\mathrm{bmo}_q^c}$. Then we obtain

$$(\|g\|_{\mathrm{bmo}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q)^{1/q} \lesssim \|g\|_{\mathrm{bmo}_q^c}.$$

On the other hand, by Corollary 3.12, we have

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} \Big|\frac{\partial}{\partial\varepsilon} \mathrm{P}_\varepsilon(g)(t)\Big|^2 \varepsilon \mathrm{d}t\mathrm{d}\varepsilon\Big\|_{q/2}^{1/2} + \|J(\mathrm{P}) * g\|_q.$$

Clearly, the first term on the right side can be estimated from above by $\|g\|_{\mathrm{BMO}_q^c}$ (see Theorem 3.4 of [41]). Therefore, we have the following and thus, the lemma is proved:

$$\|g\|_{\mathrm{bmo}_q^c} \lesssim \|g\|_{\mathrm{BMO}_q^c} + \|J(\mathrm{P}) * g\|_q \approx (\|g\|_{\mathrm{bmo}_q^c}^q + \|J(\mathrm{P}) * g\|_q^q)^{1/q}. \quad \blacksquare$$

Define $F_q(\mathcal{N})$ to be the space of all $f \in L_q(\mathcal{M};\mathrm{R}_d^{\mathrm{c}})$ such that $\|J(\mathrm{P}) * f\|_q < \infty$. From the above lemma, we see that $\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \bigoplus_q F_q(\mathcal{N})$ have equivalent norms. By the interpolation between $\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ (see [24] for more details), we deduce the following lemma.

LEMMA 4.2. *Let $2 < q < \infty$ and $0 < \theta < 1$. Then*

$$(\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta \subset \mathrm{bmo}_\varrho^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \quad \textit{with } \varrho = \frac{q}{1-\theta}.$$

Proof. By Lemma 4.1, we see that

$$\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) = \mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \bigoplus_q F_q(\mathcal{N})$$

with equivalent norms. Define a map

$$\begin{aligned} Y_q : F_q(\mathcal{N}) &\longrightarrow L_q(\mathcal{N}) \\ f &\longmapsto J(\mathrm{P}) * f. \end{aligned}$$

Thus, Y_q defines an isometric embedding of $F_q(\mathcal{N})$ into $L_q(\mathcal{N})$. Then by the interpolation between $\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, we get the following which completes the proof:

$$\begin{aligned} &(\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta \\ &\qquad = \Big(\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \bigoplus_q F_q(\mathcal{N}), \mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \bigoplus_\infty F_\infty(\mathcal{N})\Big)_\theta \\ &\qquad = (\mathrm{BMO}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta \bigoplus_\varrho (F_q(\mathcal{N}),F_\infty(\mathcal{N}))_\theta \\ &\qquad \subset \mathrm{BMO}_\varrho^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \bigoplus_\varrho F_\varrho(\mathcal{N}) = \mathrm{bmo}_\varrho^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}). \quad \blacksquare \end{aligned}$$

THEOREM 4.3. *Let $1 < p < \infty$. We have*

$$(\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_{1/p} = \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}).$$

Proof. Let $1 < p < 2$ and $\frac{1}{p'} = \frac{1-\theta}{p} + \theta$. Since the map $\mathbf{E}$ in Definition 3.4 is an isometry from $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ to $L_p(\mathcal{N};L_2^{\mathrm{c}}(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$, we have

$$(\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta \subset \mathrm{h}_{p'}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}). \tag{4.1}$$

By Theorem 3.19, $\mathrm{h}_p^{\mathrm{c}}$ is a reflexive Banach space. Then applying Corollary 4.5.2 of [2], we know that the dual of $(\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta$ is $(\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}), \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta$. Therefore, if the inclusion (4.1) is proper, we will get the proper inclusion

$$\mathrm{bmo}_\varrho^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subsetneq (\mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta,$$

which is in contradiction with Lemma 4.2. Thus, we have

$$(\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d,\mathcal{M}))_\theta = \mathrm{h}^{\mathrm{c}}_{p'}(\mathbb{R}^d,\mathcal{M}). \tag{4.2}$$

By duality and Corollary 4.5.2 of [2] again, the above equality implies that for $q' = \frac{q}{1-\theta}$,

$$(\mathrm{h}^{\mathrm{c}}_q(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}))_\theta = \mathrm{h}^{\mathrm{c}}_{q'}(\mathbb{R}^d,\mathcal{M}). \tag{4.3}$$

For the case where $1 < p_1, p_2 < \infty$, the interpolation of $\mathrm{h}^{\mathrm{c}}_{p_1}(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{h}^{\mathrm{c}}_{p_2}(\mathbb{R}^d,\mathcal{M})$ is much easier to handle. Indeed, by Theorem 3.18, we have, for any $1 < p < \infty$, $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N};L^{\mathrm{c}}_2(\widetilde{\Gamma}))\bigoplus_p L_p(\mathcal{N})$ via the maps $\mathbf{E}$ and $\mathbf{F}$ in Definition 3.4. This implies that, for any $1 < p_1, p_2 < \infty$,

$$(\mathrm{h}^{\mathrm{c}}_{p_1}(\mathbb{R}^d,\mathcal{M}),\mathrm{h}^{\mathrm{c}}_{p_2}(\mathbb{R}^d,\mathcal{M}))_\theta = \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M}),$$

with $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. Combining this equivalence with (4.2), (4.3), and applying Wolff's interpolation theorem (see [38]), we get the desired assertion. ∎

The following theorem is the mixed version of Theorem 4.3, which states that $\mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ and $\mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ are also good endpoints of $L_p(\mathcal{N})$.

THEOREM 4.4. *Let $1 < p < \infty$. We have $(X,Y)_{1/p} = L_p(\mathcal{N})$ with equivalent norms, where $X = \mathrm{bmo}(\mathbb{R}^d,\mathcal{M})$ or $L_\infty(\mathcal{N})$, and $Y = \mathrm{h}_1(\mathbb{R}^d,\mathcal{M})$ or $L_1(\mathcal{N})$.*

Proof. By the same argument as in the proof of Theorem 4.3, we have the inclusion

$$(\mathrm{bmo}_q(\mathbb{R}^d,\mathcal{M}),\mathrm{bmo}(\mathbb{R}^d,\mathcal{M}))_\theta \subset \mathrm{bmo}_{q'}(\mathbb{R}^d,\mathcal{M}) \quad q' = \frac{q}{\theta},$$

which ensures by duality that

$$(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}))_\theta \supset \mathrm{h}_{p'}(\mathbb{R}^d,\mathcal{M}) = L_{p'}(\mathcal{N})$$

for $\frac{1}{p'} = \frac{1-\theta}{p} + \theta$. Then by Proposition 6.18,

$$L_{p'}(\mathcal{N}) \subset (\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}))_\theta = (L_p(\mathcal{N}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}))_\theta.$$

Since $\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}) \subset L_1(\mathcal{N})$, then

$$(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}))_\theta \subset (L_p(\mathcal{N}),L_1(\mathcal{N}))_\theta = L_{p'}(\mathcal{N}).$$

Combining the estimates above, we have

$$(\mathrm{h}_p(\mathbb{R}^d,\mathcal{M}),\mathrm{h}_1(\mathbb{R}^d,\mathcal{M}))_\theta = L_{p'}(\mathcal{N}).$$

Again, using duality and Wolff's interpolation theorem, we conclude the proof by the same trick as in the proof of Theorem 4.3. ∎

We end this section by some real interpolation results.

COROLLARY 4.5. *Let $1 < p < \infty$. Then we have:*

(i) $(\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}), \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}))_{1/p,p} = \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms;*

(ii) $(X, Y)_{1/p,p} = L_p(\mathcal{N})$ *with equivalent norms, where* $X = \mathrm{bmo}(\mathbb{R}^d, \mathcal{M})$ *and* $Y = \mathrm{h}_1(\mathbb{R}^d, \mathcal{M})$, *or* $X = L_\infty(\mathcal{N})$ *and* $Y = L_1(\mathcal{N})$.

Proof. Both (i) and (ii) follow from Theorem 4.7.2 of [2]; we only prove (i). Let $1 < p_1 < p < p_2 < \infty$ with $\frac{1}{p} = \frac{1-\eta}{p_1} + \frac{\eta}{p_2}$. By Theorem 4.7.2 of [2], we write

$$\begin{aligned}(4.4)\quad &(\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}), \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}))_{1/p,p}\\ &= ((\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}), \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}))_{1/p_1}, (\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}), \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}))_{1/p_2})_{\eta,p}.\end{aligned}$$

Then (i) follows from Theorem 4.4 and the facts that $(L_{p_1}(\mathcal{N}), L_{p_2}(\mathcal{N}))_{\eta,p} = L_p(\mathcal{N})$ and that $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ is a complemented subspace of $L_p(\mathcal{N}; L_2^{\mathrm{c}}(\widetilde{\Gamma})) \bigoplus_p L_p(\mathcal{N})$. ∎

5. CALDERÓN–ZYGMUND THEORY

We introduce the Calderón–Zygmund theory on operator-valued local Hardy spaces in this section. It is closely related to the similar results of [11], [17], [27] and [43]. The results in the following will be used in the next section to investigate various square functions that characterize local Hardy spaces.

Let K be an $L_1(\mathcal{M}) + \mathcal{M}$)-valued tempered distribution which coincides on $\mathbb{R}^d \setminus \{0\}$ with a locally integrable $L_1(\mathcal{M}) + \mathcal{M}$-valued function. We define the left singular integral operator K^{c} associated to K by

$$K^{\mathrm{c}}(f)(s) = \int_{\mathbb{R}^d} K(s-t) f(t) \mathrm{d}t,$$

and the right singular integral operator K^{r} associated to K by

$$K^{\mathrm{r}}(f)(s) = \int_{\mathbb{R}^d} f(t) K(s-t) \mathrm{d}t.$$

Both $K^{\mathrm{c}}(f)$ and $K^{\mathrm{r}}(f)$ are well-defined for sufficiently nice functions f with values in $L_1(\mathcal{M}) \cap \mathcal{M}$, for instance, for $f \in \mathcal{S} \otimes (L_1(\mathcal{M}) \cap \mathcal{M})$.

Let $\mathrm{bmo}_0^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ denote the subspace of $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ consisting of compactly supported functions. The following lemma is an analogue of Lemma 2.1 in [41] for inhomogeneous spaces. Notice that the usual Calderón–Zygmund operators (the operators satisfying the condition (i) and (iii) in the following lemma) are not necessarily bounded on $\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Thus, we need to impose an extra decay at infinity on the kernel K.

LEMMA 5.1. *Assume that:*

(i) *the Fourier transform of K is bounded:* $\sup_{\xi\in\mathbb{R}^d} \|\widehat{K}(\xi)\|_{\mathcal{M}} < \infty$;

(ii) *K satisfies the size estimate at infinity: there exist C_1 and $\rho > 0$ such that*

$$\|K(s)\|_{\mathcal{M}} \leqslant \frac{C_1}{|s|^{d+\rho}}, \quad \forall\, |s| \geqslant 1;$$

(iii) *K has the Lipschitz regularity: there exist C_2 and $\gamma > 0$ such that*

$$\|K(s-t) - K(s)\|_{\mathcal{M}} \leqslant C_2 \frac{|t|^{\gamma}}{|s-t|^{d+\gamma}}, \quad \forall\, |s| > 2|t|.$$

Then K^{c} is bounded on $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ for $1 \leqslant p < \infty$ and from $\mathrm{bmo}^{\mathrm{c}}_0(\mathbb{R}^d, \mathcal{M})$ to $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

A similar statement also holds for K^{r} and the corresponding row spaces.

Proof. First suppose that K^{c} maps constant functions to zero. This amounts to requiring that $K^{\mathrm{c}}(\mathbb{1}_{\mathbb{R}^d}) = 0$. Let $Q \subset \mathbb{R}^d$ be a cube with $|Q| < 1$. Since the assumption of Lemma 2.1 in [41] are included in the ones of this lemma, we get

$$\Big\| \Big(\frac{1}{|Q|} \int_Q |K^{\mathrm{c}}(f) - K^{\mathrm{c}}(f)_Q|^2 \mathrm{d}t \Big)^{1/2} \Big\|_{\mathcal{M}} \lesssim \|f\|_{\mathrm{BMO}^{\mathrm{c}}} \lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}}.$$

Now let us focus on the cubes with side length 1. Let Q be a cube with $|Q| = 1$ and $\widetilde{Q} = 2Q$ be the cube concentric with Q and with side length 2. Decompose f as $f = f_1 + f_2$, where $f_1 = \mathbb{1}_{\widetilde{Q}} f$ and $f_2 = \mathbb{1}_{\mathbb{R}^d \setminus \widetilde{Q}} f$. Then $K^{\mathrm{c}}(f) = K^{\mathrm{c}}(f_1) + K^{\mathrm{c}}(f_2)$. We have

$$\Big\| \frac{1}{|Q|} \int_Q |K^{\mathrm{c}}(f)|^2 \mathrm{d}s \Big\|_{\mathcal{M}} \lesssim \Big\| \frac{1}{|Q|} \int_Q |K^{\mathrm{c}}(f_1)|^2 \mathrm{d}s \Big\|_{\mathcal{M}} + \Big\| \frac{1}{|Q|} \int_Q |K^{\mathrm{c}}(f_2)|^2 \mathrm{d}s \Big\|_{\mathcal{M}}.$$

The first term is easy to estimate. By assumption (i) and (1.10),

$$\begin{aligned} \Big\| \frac{1}{|Q|} \int_Q |K^{\mathrm{c}}(f_1)|^2 \mathrm{d}s \Big\|_{\mathcal{M}} &\leqslant \Big\| \frac{1}{|Q|} \int_{\mathbb{R}^d} |\widehat{K}(\xi) \widehat{f_1}(\xi)|^2 \mathrm{d}\xi \Big\|_{\mathcal{M}} \lesssim \Big\| \frac{1}{|Q|} \int_{\mathbb{R}^d} |\widehat{f_1}(\xi)|^2 \mathrm{d}\xi \Big\|_{\mathcal{M}} \\ &= \Big\| \frac{1}{|Q|} \int_{\widetilde{Q}} |f(s)|^2 \mathrm{d}s \Big\|_{\mathcal{M}} \lesssim \sup_{|Q|=1} \Big\| \frac{1}{|Q|} \int_Q |f(s)|^2 \mathrm{d}s \Big\|_{\mathcal{M}}. \end{aligned}$$

To estimate the second term, using assumption (ii) and (1.10) again, we have

$$\begin{aligned} |K^{\mathrm{c}}(f_2)(s)|^2 &= \Big| \int_{\mathbb{R}^d} K(s-t) f_2(t) \mathrm{d}t \Big|^2 = \Big| \int_{\mathbb{R}^d \setminus \widetilde{Q}} K(s-t) f(t) \mathrm{d}t \Big|^2 \\ &\leqslant \int_{\mathbb{R}^d \setminus \widetilde{Q}} \|K(s-t)\|_{\mathcal{M}} \mathrm{d}t \cdot \int_{\mathbb{R}^d \setminus \widetilde{Q}} \|K(s-t)\|_{\mathcal{M}}^{-1} |K(s-t) f(t)|^2 \mathrm{d}t \\ &\lesssim \int_{\mathbb{R}^d \setminus \widetilde{Q}} \|K(s-t)\|_{\mathcal{M}} |f(t)|^2 \mathrm{d}t \lesssim \int_{\mathbb{R}^d \setminus \widetilde{Q}} \frac{1}{|s-t|^{d+\rho}} |f(t)|^2 \mathrm{d}t. \end{aligned}$$

Set $\widetilde{Q}_m = \widetilde{Q} + 2m$ for every $m \in \mathbb{Z}^d$. Then $\mathbb{R}^d \setminus \widetilde{Q} = \bigcup_{m\neq 0} \widetilde{Q}_m$. Continuing the estimate of $|K^{\rm c}(f_2)(s)|^2$, for any $s \in Q$, we have

$$|K^{\rm c}(f_2)(s)|^2 \leqslant \sum_{m\neq 0} \int_{\widetilde{Q}_m} \frac{1}{|s-t|^{d+\rho}} |f(t)|^2 {\rm d}t \approx \sum_{m\neq 0} \frac{1}{|m|^{d+\rho}} \int_{\widetilde{Q}_m} |f(t)|^2 {\rm d}t \lesssim \|f\|_{{\rm bmo}^{\rm c}}.$$

Combining the previous estimates, it is easy to see that $K^{\rm c}$ is bounded from ${\rm bmo}^{\rm c}_0(\mathbb{R}^d, \mathcal{M})$ to ${\rm bmo}^{\rm c}(\mathbb{R}^d, \mathcal{M})$.

Now we illustrate that the additional requirement $K^{\rm c}(\mathbb{1}_{\mathbb{R}^d}) = 0$ is not needed. First, a similar argument as above ensures that for every compactly supported $f \in L_\infty(\mathcal{N})$, $\|K^{\rm c}(f)\|_{{\rm bmo}^{\rm c}} \lesssim \|f\|_\infty$. Then we follow the argument of Proposition II.5.15 in [8] to extend $K^{\rm c}$ on the whole $L_\infty(\mathcal{N})$, as

$$K^{\rm c}(f)(s) = \lim_j \Big[K^{\rm c}(f\mathbb{1}_{B_j})(s) - \int_{1<|t|\leqslant j} K(-t) f(t) {\rm d}t \Big], \quad \forall\, s \in \mathbb{R}^d,$$

where B_j is the ball centered at the origin with radius j. Let us show that the sequence on the right hand side converges pointwise in the norm $\|\cdot\|_{\mathcal{M}}$ and uniformly on any compact set $\Omega \subset \mathbb{R}^d$. To this end, we denote by g_j the j-th term of this sequence. Let l be the first natural number such that $l \geqslant 2 \sup_{s\in\Omega} |s|$. Then for $s \in \Omega$ and $j > l$, we have

$$g_j(s) = g_l(s) + \int_{l<|t|\leqslant j} (K(s-t) - K(-t)) f(t) {\rm d}t.$$

By assumption (iii), the integral on the right hand side is bounded by a bounded multiple of $\|f\|_\infty$, uniformly on $s \in \Omega$. This ensures the convergence of g_j, so $K^{\rm c}(f)$ is a well-defined function. Now we have to estimate the ${\rm bmo}^{\rm c}$-norm of $K^{\rm c}(f)$. Taking any cube $Q \subset \mathbb{R}^d$, by the uniform convergence of g_j on Q in $\mathcal{M}$, we have

$$\Big\| \Big(\int_Q |K^{\rm c}(f)(s) - (K^{\rm c}(f))_Q|^2 {\rm d}s \Big)^{1/2} \Big\|_{\mathcal{M}} = \lim_j \Big\| \Big(\int_Q |g_j(s) - (g_j)_Q|^2 {\rm d}s \Big)^{1/2} \Big\|_{\mathcal{M}}.$$

Similarly,

$$\Big\| \Big(\int_Q |K^{\rm c}(f)(s)|^2 {\rm d}s \Big)^{1/2} \Big\|_{\mathcal{M}} = \lim_j \Big\| \Big(\int_Q |g_j(s)|^2 {\rm d}s \Big)^{1/2} \Big\|_{\mathcal{M}}.$$

Hence, by the fact that g_j and $K^{\rm c}(f\mathbb{1}_{B_j})$ differ by a constant, we obtain

$$\|K^{\rm c}(f)\|_{{\rm bmo}^{\rm c}} = \lim_j \|g_j\|_{{\rm bmo}^{\rm c}} \lesssim \limsup_j \|K^{\rm c}(f\mathbb{1}_{B_j})\|_{{\rm bmo}^{\rm c}} + \|f\|_\infty \lesssim \|f\|_\infty.$$

Therefore, $K^{\rm c}$ defined above extends to a bounded operator from $L_\infty(\mathcal{N})$ to ${\rm bmo}^{\rm c}(\mathbb{R}^d, \mathcal{M})$. In particular, $K^{\rm c}(\mathbb{1}_{\mathbb{R}^d})$ determines a function in ${\rm bmo}^{\rm c}(\mathbb{R}^d, \mathcal{M})$.

Then for f and Q as above, we have $K^{\mathrm{c}}(f) = K^{\mathrm{c}}(f_1) + K^{\mathrm{c}}(f_2) + K^{\mathrm{c}}(\mathbb{1}_{\mathbb{R}^d}) f_{\widetilde{Q}}$, so

$$\begin{aligned}\|K^{\mathrm{c}}(f)\|_{\mathrm{bmo}^{\mathrm{c}}} &\leqslant \|K^{\mathrm{c}}(f_1)\|_{\mathrm{bmo}^{\mathrm{c}}} + \|K^{\mathrm{c}}(f_2)\|_{\mathrm{bmo}^{\mathrm{c}}} + \|K^{\mathrm{c}}(\mathbb{1}_{\mathbb{R}^d})\|_{\mathrm{bmo}^{\mathrm{c}}} \|f_{\widetilde{Q}}\|_{\mathcal{M}} \\ &\lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}} + \|f_{\widetilde{Q}}\|_{\mathcal{M}} \lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}}.\end{aligned}$$

Thus we have proved the $\mathrm{bmo}^{\mathrm{c}}$-boundedness of K^{c} in the general case.

By duality, the boundedness of K^{c} on $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M})$ is equivalent to that of its adjoint map $(K^{\mathrm{c}})^*$ on $\mathrm{bmo}^{\mathrm{c}}_0(\mathbb{R}^d, \mathcal{M})$. It is easy to see that $(K^{\mathrm{c}})^*$ is also a singular integral operator:

$$(K^{\mathrm{c}})^*(g) = \int_{\mathbb{R}^d} \widetilde{K}(s-t) g(t) \mathrm{d}t,$$

where $\widetilde{K}(s) = K^*(-s)$. Obviously, $\widetilde{K}$ also satisfies the same assumption as K, so $(K^{\mathrm{c}})^*$ is bounded on $\mathrm{bmo}^{\mathrm{c}}_0(\mathbb{R}^d, \mathcal{M})$. Thus we get the boundedness of K^{c} on $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M})$. Then, by the interpolation between $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ in Theorem 4.3, we get the boundedness of K^{c} on $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ for $1 < p < \infty$. The assertion is proved. ∎

REMARK 5.2. Under the assumption of the above lemma, $K^{\mathrm{c}}(\mathbb{1}_{\mathbb{R}^d})$ is a constant, so it is the zero element in $\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

A special case of Lemma 5.1 concerns the Hilbert-valued kernel K. Let H be a Hilbert space and $\mathsf{k} : \mathbb{R}^d \to H$ be a H-valued kernel. We view the Hilbert space as the column matrices in $B(H)$ with respect to a fixed orthonormal basis. Put $K(s) = \mathsf{k}(s) \otimes 1_{\mathcal{M}} \in B(H) \overline{\otimes} \mathcal{M}$. For nice functions $f : \mathbb{R}^d \to L_1(\mathcal{M}) + \mathcal{M}$, $K^{\mathrm{c}}(f)$ takes values in the column subspace of $L_1(B(H)\overline{\otimes}\mathcal{M}) + L_\infty(B(H)\overline{\otimes}\mathcal{M})$. Consequently,

$$\|K^{\mathrm{c}}(f)\|_{L_p(B(H)\overline{\otimes}\mathcal{N})} = \|K^{\mathrm{c}}(f)\|_{L_p(\mathcal{N}; H^{\mathrm{c}})}.$$

Since $\mathsf{k}(s) \otimes 1_{\mathcal{M}}$ commutes with $\mathcal{M}$, $K^{\mathrm{c}}(f) = K^{\mathrm{r}}(f)$ for $f \in L_2(\mathcal{N})$. Let us denote this common operator by k^{c}. Here the superscript c refers to the previous convention that H is identified with the column matrices in $B(H)$. Thus, Lemma 5.1 implies the following corollary.

COROLLARY 5.3. *Assume that:*

(i) $\sup_{\xi \in \mathbb{R}^d} \|\widehat{\mathsf{k}}(\xi)\|_H < \infty$;

(ii) $\|\mathsf{k}(s)\|_H \lesssim \frac{1}{|s|^{d+\rho}}$, $\forall\, |s| \geqslant 1$, *for some* $\rho > 0$;

(iii) $\|\mathsf{k}(s-t) - \mathsf{k}(s)\|_H \lesssim \frac{|t|^\gamma}{|s-t|^{d+\gamma}}$, $\forall\, |s| > 2|t|$, *for some* $\gamma > 0$.

Then the operator k^{c} *is bounded:*

(a) *from* $\mathrm{bmo}^{\alpha}_0(\mathbb{R}^d, \mathcal{M})$ *to* $\mathrm{bmo}^{\alpha}(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$, *where* $\alpha = c$, $\alpha = r$ *or we leave out* α;

(b) *and from* $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ *to* $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$ *for* $1 \leqslant p < \infty$.

Proof. Since $K^{\mathrm{c}}(f) = K^{\mathrm{r}}(f)$ on the subspace $L_p(\mathcal{N}) \subset L_p(B(H)\overline{\otimes}\mathcal{N})$, (i) follows immediately from Lemma 5.1. Let $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, H^{\mathrm{c}}\overline{\otimes}\mathcal{M})$ (respectively $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, H^{\mathrm{c}}\overline{\otimes}\mathcal{M})$) be the column subspace of $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$ (respectively $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$). Consider the adjoint operator of k^{c} which is denoted by $(\mathsf{k}^{\mathrm{c}})^*$. It admits the convolution kernel $\widetilde{K}(s) = \widetilde{\mathsf{k}}(s) \otimes 1_{\mathcal{M}}$, where $\widetilde{\mathsf{k}}(s) = \mathsf{k}(-s)^*$ (so it is a row matrix). Applying Lemma 5.1 to $(\mathsf{k}^{\mathrm{c}})^*$, we see that $(\mathsf{k}^{\mathrm{c}})^*$ is bounded from $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, H^{\mathrm{c}}\overline{\otimes}\mathcal{M})$ to $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Then k^{c} is bounded from $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M})$ to $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, H^{\mathrm{c}}\overline{\otimes}\mathcal{M})$, and thus bounded from $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M})$ into $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$. Interpolating this with the boundedness of k^{c} from $\mathrm{bmo}^{\mathrm{c}}_0(\mathbb{R}^d, \mathcal{M})$ to $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})$, we deduce the desired assertion in (ii). ∎

REMARK 5.4. Let $1 \leqslant p \leqslant 2$. Since $L_\infty(\mathcal{N}) \subseteq \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, we get $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M}) \subseteq L_1(\mathcal{N})$. By Theorem 4.3 and the fact that $\mathrm{h}^{\mathrm{c}}_2(\mathbb{R}^d, \mathcal{M}) = L_2(\mathcal{N})$, we have $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M}) \subseteq L_p(\mathcal{N})$. Then Corollary 5.3 ensures that, for any $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$:

$$\|\mathsf{k}^{\mathrm{c}}(f)\|_{L_p(\mathcal{N};H^{\mathrm{c}})} \lesssim \|\mathsf{k}^{\mathrm{c}}(f)\|_{\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, B(H)\overline{\otimes}\mathcal{M})} \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d,\mathcal{M})}.$$

6. GENERAL CHARACTERIZATIONS

Applying the operator-valued Calderón–Zygmund theory developed in the last section, we will show that the Poisson kernel in the square functions which are used to define $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ can be replaced by any reasonable test function. As an application, we are able to compare the operator-valued local Hardy spaces $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ defined in this paper with the operator-valued Hardy spaces $\mathcal{H}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ in [24]. We will use multi-index notation. For $m = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ and $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$, we set $s^m = s_1^{m_1} \cdots s_d^{m_d}$. Let $|m|_1 = m_1 + \cdots + m_d$ and $D^m = \frac{\partial^{m_1}}{\partial s_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial s_d^{m_d}}$.

6.1. GENERAL CHARACTERIZATIONS. Let Φ be a complex-valued infinitely differentiable function defined on $\mathbb{R}^d\backslash\{0\}$. Recall that $\widetilde{\Gamma} = \{(t, \varepsilon) \in \mathbb{R}^{d+1}_+ : |t| < \varepsilon < 1\}$ and $\Phi_\varepsilon(s) = \varepsilon^{-d}\Phi(\frac{s}{\varepsilon})$. For any $f \in L_1(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d) + L_\infty(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d)$, we define the local versions of the conic and radial square functions of f associated to Φ by

$$s^{\mathrm{c}}_\Phi(f)(s) = \Big(\iint\limits_{\widetilde{\Gamma}} |\Phi_\varepsilon * f(s+t)|^2 \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}} \Big)^{1/2}, \quad s \in \mathbb{R}^d,$$

$$g^{\mathrm{c}}_\Phi(f)(s) = \Big(\int\limits_0^1 |\Phi_\varepsilon * f(s)|^2 \frac{\mathrm{d}\varepsilon}{\varepsilon} \Big)^{1/2}, \quad s \in \mathbb{R}^d.$$

The function Φ that we use to characterize the operator-valued local Hardy spaces satisfies the following conditions:

(1) every $D^m\Phi$ with $0 \leqslant |m|_1 \leqslant d$ makes $f \mapsto s^{\mathrm{c}}_{D^m\Phi} f$ and $f \mapsto g^{\mathrm{c}}_{D^m\Phi} f$ Calderón–Zygmund singular integral operators in Corollary 5.3;

(2) there exist functions Ψ, ψ and ϕ such that

$$\widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} + \int_0^1 \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{\mathrm{d}\varepsilon}{\varepsilon} = 1, \quad \forall\, \xi \in \mathbb{R}^d; \tag{6.1}$$

(3) the above Ψ and ψ make $\mathrm{d}\mu_g = |\Psi_\varepsilon * g(s)|^2 \frac{\mathrm{d}\varepsilon \mathrm{d}s}{\varepsilon}$ and $\phi * g$ satisfy:

$$\max\left\{ \Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} \mathrm{d}\mu_g \Big\|_{q/2}^{1/2}, \|\psi * g\|_q \right\} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q} \quad \text{for } q > 2;$$

(4) the above ϕ makes $f \mapsto \phi * f$ a Calderón–Zygmund singular integral operator in Corollary 5.3.

Fix the four functions Φ, Ψ, ϕ, ψ as above. The following is one of our main results in this section, which states that the functions Φ, ϕ satisfying the above four conditions give a general characterization for $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$.

THEOREM 6.1. *Let $1 \leqslant p < \infty$ and ϕ, Φ be as above. For any $f \in L_1(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d) + L_\infty(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d)$, $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ if and only if $s^{\mathrm{c}}_\Phi(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$ if and only if $g^{\mathrm{c}}_\Phi(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$. If this is the case, then*

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \approx \|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \approx \|g^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \tag{6.2}$$

with the relevant constants depending only on d, p, and the pairs (Φ, Ψ) and (ϕ, ψ).

One implication of the above theorem is an easy consequence of conditions (1) and (4), namely that

$$\|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}, \tag{6.3}$$

$$\|g^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}. \tag{6.4}$$

In order to prove the converse inequalities, we need the following lemma, which can be seen as a generalization of Lemma 3.10.

LEMMA 6.2. *Let $1 \leqslant p < 2$, q be its conjugate index and Φ, ϕ be the functions satisfying the above assumption. For $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$,*

$$\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s\Big| \lesssim (\|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p)\|g\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

Proof. The proof of this lemma is very similar to that of Lemma 3.10, we will just point out the necessary modifications to avoid duplication. We need two

auxiliary square functions associated with Φ. For $s \in \mathbb{R}^d$, $\varepsilon \in [0,1]$, we define

$$s^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon) = \Big(\int_{\varepsilon}^{1} \int_{B(s,r-\varepsilon/2)} |\Phi_r * f(t)|^2 \frac{\mathrm{d}t\mathrm{d}r}{r^{d+1}} \Big)^{1/2}, \tag{6.5}$$

$$\overline{s}^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon) = \Big(\int_{\varepsilon}^{1} \int_{B(s,r/2)} |\Phi_r * f(t)|^2 \frac{\mathrm{d}t\mathrm{d}r}{r^{d+1}} \Big)^{1/2}. \tag{6.6}$$

By assumption (ii) of Φ, we have

$$\begin{aligned}
&\tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s \\
&= \tau \int_{\mathbb{R}^d} \int_0^1 \Phi_\varepsilon * f(s) (\Psi_\varepsilon * g(s))^* \frac{\mathrm{d}s\mathrm{d}\varepsilon}{\varepsilon} + \tau \int_{\mathbb{R}^d} \phi * f(s) (\psi * g(s))^* \mathrm{d}s \\
&= \frac{2^d}{c_d} \tau \int_{\mathbb{R}^d} \int_0^1 \int_{B(s,\varepsilon/2)} \Phi_\varepsilon * f(t) s^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon)^{(p-2)/2} s^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon)^{(2-p)/2} (\Psi_\varepsilon * g(t))^* \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}} \mathrm{d}s \\
&\quad + \tau \int_{\mathbb{R}^d} \phi * f(s) (\psi * g(s))^* \mathrm{d}s \overset{\mathrm{def}}{=} \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Then by the Cauchy–Schwarz inequality,

$$\begin{aligned}
|\mathrm{I}|^2 &\lesssim \tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s,\varepsilon/2)} |\Phi_\varepsilon * f(t)|^2 \frac{\mathrm{d}t}{\varepsilon^{d+1}} \Big) \overline{s}^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon)^{p-2} \mathrm{d}\varepsilon\mathrm{d}s \\
&\quad \cdot \tau \int_{\mathbb{R}^d} \int_0^1 \Big(\int_{B(s,\varepsilon/2)} |\Psi_\varepsilon * g(t)|^2 \frac{\mathrm{d}t}{\varepsilon^{d+1}} \Big) s^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon)^{2-p} \mathrm{d}\varepsilon\mathrm{d}s \overset{\mathrm{def}}{=} A \cdot B.
\end{aligned}$$

Replacing $\varepsilon \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(f)$ and $\varepsilon \frac{\partial}{\partial \varepsilon} \mathrm{P}_\varepsilon(g)$ in the proof of Lemma 3.10 by $\Phi_\varepsilon * f$ and $\Psi_\varepsilon * g$, respectively and applying Lemma 6.7 and assumption (3) of Ψ and ψ, we get the estimates for the terms A and B:

$$A \lesssim \|s^{\mathrm{c}}_{\Phi}(f)\|_p^p \quad \text{and} \quad B \lesssim \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}_q} \|s^{\mathrm{c}}_{\Phi}(f)\|_p^{2-p}.$$

The term II is easy to deal with. By the Hölder inequality, Lemma 6.7 and assumption (3) again, we get

$$\Big| \tau \int_{\mathbb{R}^d} \phi * f(s) (\psi * g(s))^* \mathrm{d}s \Big| \leqslant \|\phi * f\|_p \|\psi * g\|_q \lesssim \|\phi * f\|_p \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

Combining the estimates for A, B and II, we finally get the desired inequality. ∎

We also need the radial version of Lemma 6.2. To this end, we need to majorize the radial square function by the conic one. When we consider the Poisson kernel, this result follows from the harmonicity of the Poisson integral (see Lemma 3.13). However, in the general case, the harmonicity is no longer available. To overcome this difficulty, a more sophisticated inequality has been developed in [41] to compare non-local radial and conic functions. Observe that the result given in Lemma 4.3 of [41] is a pointwise one, which also works for the local version of square functions if we consider integration over the interval $0 < \varepsilon < 1$. The following lemma is an obvious consequence of Lemma 4.3 in [41].

LEMMA 6.3. *Let $f \in L_1(\mathcal{M}; \mathrm{R}_d^{\mathrm{c}}) + L_\infty(\mathcal{M}; \mathrm{R}_d^{\mathrm{c}})$. Then*

$$g_\Phi^{\mathrm{c}}(f)(s)^2 \lesssim \sum_{|m|_1 \leqslant d} s_{D^m\Phi}^{\mathrm{c}}(f)(s)^2, \quad \forall\, s \in \mathbb{R}^d.$$

LEMMA 6.4. *Let $1 \leqslant p < 2$. For $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$,*

$$\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s\Big| \lesssim (\|g_\Phi^{\mathrm{c}}(f)\|_p + \|\phi * f\|_p)^{p/2} \|f\|_{\mathrm{h}_p^{\mathrm{c}}}^{1-p/2} \|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}.$$

Proof. This proof is similar to that of Lemma 6.2 and we keep the notation there. Let $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ with compact support (relative to the variable of $\mathbb{R}^d$). We assume that f is sufficiently nice so that all calculations below are legitimate. Now we need the radial version of $s_\Phi^{\mathrm{c}}(f)(s, \varepsilon)$,

$$g_\Phi^{\mathrm{c}}(f)(s, \varepsilon) = \Big(\int_\varepsilon^1 |\Phi_r * f(s)|^2 \frac{\mathrm{d}r}{r} \Big)^{1/2}$$

for $s \in \mathbb{R}^d$ and $0 \leqslant \varepsilon \leqslant 1$. By approximation, we can assume that $g_\Phi^{\mathrm{c}}(f)(s, \varepsilon)$ is invertible for every $(s, \varepsilon) \in S$. By (6.1), (1.12) and the Fubini theorem, we have

$$\begin{aligned}
&\Big|\tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s\Big|^2 \\
&\lesssim \tau \int_{\mathbb{R}^d} \int_0^1 |\Phi_\varepsilon * f(s)|^2 g_\Phi^{\mathrm{c}}(f)(s, \varepsilon)^{p-2} \frac{\mathrm{d}\varepsilon \mathrm{d}s}{\varepsilon} \cdot \tau \int_{\mathbb{R}^d} \int_0^1 |\Psi_\varepsilon * g(s)|^2 g_\Phi^{\mathrm{c}}(f)(s, \varepsilon)^{2-p} \frac{\mathrm{d}\varepsilon \mathrm{d}s}{\varepsilon} \\
&\quad + \Big|\tau \int_{\mathbb{R}^d} \phi * f(s) (\psi * g(s))^* \mathrm{d}s\Big|^2 \stackrel{\mathrm{def}}{=} A' B' + \mathrm{II}'.
\end{aligned}$$

II' is treated exactly in the same way as before,

$$\mathrm{II}' \lesssim \|\phi * f\|_p^2 \|\psi * g\|_q^2 \lesssim \|\phi * f\|_p^p \|f\|_{\mathrm{h}_p^{\mathrm{c}}}^{2-p} \|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}^2.$$

A' is also estimated similarly as in Lemma 6.2, we have $A' \lesssim \|g_\Phi^{\mathrm{c}}(f)\|_p^p$.

To estimate B', we notice that the proof of Lemma 1.3 in [41] also gives

$$g^{\mathrm{c}}_{\Phi}(f)(s,\varepsilon)^2 \lesssim \sum_{|m|_1\leqslant d} s^{\mathrm{c}}_{D^m\Phi}(f)(s,\varepsilon)^2,$$

where $s^{\mathrm{c}}_{D^m\Phi}(f)(s,\varepsilon)$ is defined by (6.5) with $D^m\Phi$ instead of Φ. Then by the above inequality, Lemma 6.7 and inequality (6.3) with $D^m\Phi$ instead of Φ, we obtain

$$\begin{aligned} B' &\lesssim \sum_{|m|_1\leqslant d} \tau \int_{\mathbb{R}^d}\int_0^1 |\Psi_\varepsilon * g(s)|^2 s^{\mathrm{c}}_{D^m\Phi}(f)(s,\varepsilon)^{2-p}\frac{\mathrm{d}\varepsilon \mathrm{d}s}{\varepsilon} \\ &\lesssim \sum_{|m|_1\leqslant d} \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}_q}\|s^{\mathrm{c}}_{D^m\Phi}(f)\|_p^{2-p} \lesssim \|g\|^2_{\mathrm{bmo}^{\mathrm{c}}_q}\|f\|^{2-p}_{\mathrm{h}^{\mathrm{c}}_p}. \end{aligned}$$

Therefore, we have the following which completes the proof:

$$\left|\tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s\right|^2 \lesssim (\|g^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p)^p \|f\|^{2-p}_{\mathrm{h}^{\mathrm{c}}_p}\|g\|^2_{\mathrm{bmo}^{\mathrm{c}}_q}. \quad \blacksquare$$

Proof of Theorem 6.1. From Lemmas 6.2, 6.4 and Theorem 3.11, we conclude that if $1 \leqslant p \leqslant 2$, we have

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \lesssim \|s^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p, \quad \|f\|_{\mathrm{h}^{\mathrm{c}}_p} \lesssim \|g^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p.$$

For the case $2 < p < \infty$, by Theorem 3.19, we can choose $g \in \mathrm{h}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$ (with q the conjugate index of p) with norm one such that

$$\begin{aligned} \|f\|_{\mathrm{h}^{\mathrm{c}}_p} &\approx \tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s \\ &= \tau\int_{\mathbb{R}^d}\int_0^1 \Phi_\varepsilon * f(s)\cdot(\Psi_\varepsilon * g(s))^*\frac{\mathrm{d}s\mathrm{d}\varepsilon}{\varepsilon} + \tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*\mathrm{d}s. \end{aligned}$$

Then by the Hölder inequality and (6.4) (applied to g, Ψ and q),

$$\begin{aligned} \|f\|_{\mathrm{h}^{\mathrm{c}}_p} &\lesssim \|g^{\mathrm{c}}_{\Phi}(f)\|_p\|g^{\mathrm{c}}_{\Psi}(g)\|_q + \|\phi * f\|_p\|\psi * g\|_q \\ &\lesssim (\|g^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p)\|g\|_{\mathrm{h}^{\mathrm{c}}_q} = \|g^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p. \end{aligned}$$

Similarly, we have

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \lesssim \|s^{\mathrm{c}}_{\Phi}(f)\|_p + \|\phi * f\|_p.$$

Therefore, combined with (6.4) and (6.3), we have proved the assertion. $\blacksquare$

The rest of this subsection is devoted to explaining how Theorem 6.1 generalizes the characterization of $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$.

Firstly and most naturally, we show how Theorem 6.1 covers the original definition of $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$. Let us take $\Phi = -2\pi I(\mathrm{P})$ and $\phi = \mathrm{P}$ for example. A

simple calculation shows that we can choose $\Psi = -8\pi I(\mathrm{P})$ and $\psi = 4\pi I(\mathrm{P}) + \mathrm{P}$ to fulfil (6.1). By the inverse Fourier transform formula, we have

$$\begin{aligned} -2\pi f * I(\mathrm{P})_\varepsilon(t) &= -2\pi \int \mathrm{e}^{2\pi \mathrm{i} t\cdot\xi} \widehat{f}(\xi)|\varepsilon\xi|\mathrm{e}^{-2\pi\varepsilon|\xi|}\mathrm{d}\xi \\ &= \varepsilon\frac{\partial}{\partial\varepsilon}\int \mathrm{e}^{2\pi \mathrm{i} t\cdot\xi}\widehat{f}(\xi)\mathrm{e}^{-2\pi\varepsilon|\xi|}\mathrm{d}\xi = \varepsilon\frac{\partial}{\partial\varepsilon}\mathrm{P}_\varepsilon(f)(t). \end{aligned}$$

So we return back to the original definition of $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$. Theorem 6.1 implies that

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \approx \|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \approx \|g^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p.$$

In particular, we have the following equivalent norm of $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$.

COROLLARY 6.5. *Let $1 \leqslant p < \infty$. Then for any $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$, we have*

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \approx \|g^{\mathrm{c}}(f)\|_p + \|\mathrm{P} * f\|_p.$$

Secondly, consider Φ to be a Schwartz function on $\mathbb{R}^d$ satisfying:

$$\begin{cases} \Phi \text{ is of vanishing mean;} \\ \Phi \text{ is nondegenerate in the sense of (1.5).} \end{cases} \tag{6.7}$$

Set $\Phi_\varepsilon(s) = \varepsilon^{-d}\Phi(\frac{s}{\varepsilon})$ for $\varepsilon > 0$. In the sequel, we will show that every Schwartz function satisfying (6.7) fulfils the four conditions in the beginning of this subsection. So they all can be used to characterize $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M})$.

It is a well-known elementary fact (ef. e.g. p. 186 of [36]) that there exists a Schwartz function Ψ of vanishing mean such that

$$\int_0^\infty \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{\mathrm{d}\varepsilon}{\varepsilon} = 1, \quad \forall\, \xi \in \mathbb{R}^d \setminus \{0\}. \tag{6.8}$$

LEMMA 6.6. $\int_0^1 \widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{\mathrm{d}\varepsilon}{\varepsilon}$ *is an infinitely differentiable function on $\mathbb{R}^d$ if we define its value at the origin as* 0.

Proof. To prove the assertion, it suffices to show that $\int_0^1 \widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{\mathrm{d}\varepsilon}{\varepsilon}$ is infinitely differentiable at the origin. Given $\varepsilon \in (0, 1]$, we expand $\widehat{\Phi}(\varepsilon\cdot)$ in the Taylor series at the origin

$$\widehat{\Phi}(\varepsilon\xi) = \sum_{|\gamma|_1 \leqslant N} D^\gamma\widehat{\Phi}(0)\frac{\varepsilon^{|\gamma|_1}\xi^\gamma}{\gamma!} + \sum_{|\gamma|_1 = N+1} R_\gamma(\varepsilon\xi)\,\xi^\gamma,$$

with the remainder of integral form equal to

$$R_\gamma(\varepsilon\xi) = \frac{(N+1)\varepsilon^{N+1}}{\gamma!}\int_0^1 (1-\theta)^N D^\gamma\widehat{\Phi}(\theta\varepsilon\xi)\mathrm{d}\theta\,.$$

Since $\widehat{\Phi}(0) = 0$, the above Taylor series implies that

$$\widehat{\Phi}(\varepsilon\xi) = \sum_{1\leqslant|\gamma|_1\leqslant N} D^\gamma\widehat{\Phi}(0)\frac{\varepsilon^{|\gamma|_1}\xi^\gamma}{\gamma!} + \sum_{|\gamma|_1=N+1} R_\gamma(\varepsilon\xi)\,\xi^\gamma.$$

Similarly, we have

$$\widehat{\Psi}(\varepsilon\xi) = \sum_{1\leqslant|\beta|_1\leqslant N} D^\beta\widehat{\Psi}(0)\frac{\varepsilon^{|\beta|_1}\xi^\beta}{\beta!} + \sum_{|\beta|_1=N+1} R'_\beta(\varepsilon\xi)\,\xi^\beta,$$

where R'_β is the integral form remainder of $\widehat{\Psi}$. Thus, both $\widehat{\Phi}(\varepsilon\xi)$ and $\widehat{\Psi}(\varepsilon\xi)$ contain only powers of ε with order at least 1. Therefore, the integral $\int_0^1 \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{\mathrm{d}\varepsilon}{\varepsilon}$ (and the integrals of arbitrary order derivatives of $\widehat{\Phi}(\varepsilon\xi)$ and $\widehat{\Psi}(\varepsilon\xi)$) converge uniformly for $\xi \in \mathbb{R}^d$ close to the origin. We then obtain that $\int_0^1 \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{\mathrm{d}\varepsilon}{\varepsilon}$ is infinitely differentiable at the origin $\xi = 0$. ∎

It follows immediately from Lemma 6.6 that $\int_1^\infty \widehat{\Phi}(\varepsilon\cdot)\overline{\widehat{\Psi}(\varepsilon\cdot)}\frac{\mathrm{d}\varepsilon}{\varepsilon}$ is a Schwartz function if we define its value at the origin by 1. Then we can find two other functions ϕ, ψ such that $\widehat{\phi}, \widehat{\psi} \in H_2^\sigma(\mathbb{R}^d)$, $\widehat{\phi}(0) > 0, \widehat{\psi}(0) > 0$ and

$$\widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} + \int_0^1 \widehat{\Phi}(\varepsilon\xi)\overline{\widehat{\Psi}(\varepsilon\xi)}\frac{\mathrm{d}\varepsilon}{\varepsilon} = 1, \quad \forall\,\xi \in \mathbb{R}^d. \tag{6.9}$$

Indeed, for $\beta > 0$ large enough, the function $(1 + |\cdot|^2)^{-\beta}$ belongs to $H_2^\sigma(\mathbb{R}^d)$. On the other hand, if $F \in \mathcal{S}(\mathbb{R}^d)$, the function $(1 + |\cdot|^2)^\beta F$ is still in $H_2^\sigma(\mathbb{R}^d)$. Thus we obtain (6.1).

Now let show that conditions (1) and (4) hold for Φ, ϕ satisfying (6.7). First, we deal with the case $1 \leqslant p \leqslant 2$. Let $H = L_2((0,1), \frac{\mathrm{d}\varepsilon}{\varepsilon})$. Define the kernel $\mathsf{k} : \mathbb{R}^d \to H$ by $\mathsf{k}(s) = \Phi_\cdot(s)$ with $\Phi_\cdot(s) : \varepsilon \mapsto \Phi_\varepsilon(s)$. Then it is easy to see that

$$\sup_{\xi\in\mathbb{R}^d} \|\widehat{\Phi}(\varepsilon\xi)\|_H < \infty, \quad \|\Phi_\varepsilon(s)\|_H \lesssim \frac{1}{|s|^{d+1}}, \quad \forall\, s \in \mathbb{R}^d \setminus \{0\}$$

and that

$$\|\nabla\Phi_\varepsilon(s)\|_H \lesssim \frac{1}{|s|^{d+1}}, \quad \forall\, s \in \mathbb{R}^d \setminus \{0\}.$$

Thus, k satisfies the assumption of Corollary 5.3. By Remark 5.4, we have, for any $1 \leqslant p \leqslant 2$,

$$\|\Phi_\varepsilon * f\|_{L_p(\mathcal{N};H^{\mathrm{c}})} = \|g^{\mathrm{c}}_\Phi(f)\|_p \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}.$$

The treatment of s^{c}_Φ is similar. In this case, we take the Hilbert space $H = L_2(\widetilde{\Gamma}, \frac{\mathrm{d}t\mathrm{d}\varepsilon}{\varepsilon^{d+1}})$. On the other hand, $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ implies $\phi \in L_1(\mathbb{R}^d)$, then $\|\phi *$

$f\|_{L_p(\mathcal{N})} \lesssim \|f\|_{L_p(\mathcal{N})} \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}$. Thus, combining the above estimates, we obtain

$$\|g^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}, \quad \|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \lesssim \|f\|_{\mathrm{h}^{\mathrm{c}}_p}.$$

Then, a simple duality argument using (6.1) and Theorem 3.19 gives the above inequalities for the case $p > 2$. Moreover, it is obvious that if we replace Φ by $D^m\Phi$, the above two inequalities still hold for any $1 \leqslant p < \infty$.

In the end, it remains to check the condition (3) for Ψ, ψ obtained in (6.8) and (6.9). This can be done by showing a Carleson measure characterization of $\mathrm{bmo}^{\mathrm{c}}_q$ by general test functions. The proof of the following lemma has the same pattern with that of Lemma 2.5, so is left to the reader.

LEMMA 6.7. *Let $2 < q \leqslant \infty$, $g \in \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{d}\mu_g = |\Psi_\varepsilon * g(s)|^2 \frac{\mathrm{d}s\mathrm{d}\varepsilon}{\varepsilon}$. Then $\mathrm{d}\mu_g$ is an $\mathcal{M}$-valued q-Carleson measure on the strip $\mathbb{R}^d \times (0,1)$. Furthermore, let ψ be any function on $\mathbb{R}^d$ such that*

$$\widehat{\psi} \in H^\sigma_2(\mathbb{R}^d) \quad \textit{with } \sigma > \frac{d}{2}. \tag{6.10}$$

We have

$$\max\Big\{\Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d\\ |Q|<1}}{}^{+} \frac{1}{|Q|}\int_{T(Q)} \mathrm{d}\mu_g\Big\|^{1/2}_{q/2},\ \|\psi * g\|_q\Big\} \lesssim \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

REMARK 6.8. It is worthwhile to note that, if Ψ and ψ are determined by (6.8) and (6.9), the opposite of the above lemma is also true. This can be deduced by a similar argument as that of Corollary 3.12; we omit the details.

By the discussion above, we deduce the following corollary from Theorem 6.1.

COROLLARY 6.9. *Let Φ be the Schwartz function on $\mathbb{R}^d$ satisfying* (6.7) *and ϕ be the function given by* (6.9). *Then for any $1 \leqslant p < \infty$, we have*

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \approx \|s^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \approx \|g^{\mathrm{c}}_\Phi(f)\|_p + \|\phi * f\|_p \tag{6.11}$$

with the relevant constants depending only on d, p, Φ and ϕ.

6.2. DISCRETE CHARACTERIZATIONS. In this subsection, we give a discrete characterization for operator-valued local Hardy spaces. To this end, we need some modifications of the four conditions in the beginning of last subsection. The square functions $s^{\mathrm{c}}_\Phi(f)$ and $g^{\mathrm{c}}_\Phi(f)$ can be discretized as follows:

$$g^{\mathrm{c},D}_\Phi(f)(s) = \Big(\sum_{j\geqslant 1} |\Phi_j * f(s)|^2\Big)^{1/2},$$

$$s^{\mathrm{c},D}_\Phi(f)(s) = \Big(\sum_{j\geqslant 1} 2^{dj} \int_{B(s,2^{-j})} |\Phi_j * f(t)|^2 \mathrm{d}t\Big)^{1/2}.$$

Here Φ_j is the inverse Fourier transform of $\Phi(2^{-j}\cdot)$. This time, to get a resolvent of the unit on $\mathbb{R}^d$, we need to assume that Φ, Ψ, ϕ, ψ satisfy

$$\sum_{j=1}^{\infty} \widehat{\Phi}(2^{-j}\xi)\,\overline{\widehat{\Psi}(2^{-j}\xi)} + \widehat{\phi}(\xi)\overline{\widehat{\psi}(\xi)} = 1, \quad \forall\, \xi \in \mathbb{R}^d. \tag{6.12}$$

In brief, the complex-valued infinitely differentiable function Φ considered in this subsection satisfies:

(1) every $D^m\Phi$ with $0 \leqslant |m|_1 \leqslant d$ makes $f \mapsto s^{c,D}_{D^m\Phi} f$ and $f \mapsto g^{c,D}_{D^m\Phi} f$ Calderón–Zygmund singular integral operators in Corollary 5.3;

(2) there exist functions Ψ, ψ and ϕ that fulfil (6.12);

(3) the above Ψ and ψ make $\mathrm{d}\mu_f^D = \sum\limits_{j\geqslant 1} |\Psi_j * f(s)|^2 \mathrm{d}s \times \mathrm{d}\delta_{2^{-j}}(\varepsilon)$ (with $\delta_{2^{-j}}(\varepsilon)$ the unit Dirac mass at the point 2^{-j}) and $\phi * f$ satisfy:

$$\max\left\{ \Big\| \sup_{\substack{s\in Q\subset\mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_{T(Q)} r\mathrm{d}\mu_f^D \Big\|_{q/2}^{1/2}, \|\psi * f\|_q \right\} \lesssim \|f\|_{\mathrm{bmo}^{\mathrm{c}}_q} \quad \text{for } q > 2;$$

(4) the above ϕ makes $f \mapsto \phi * f$ a Calderón–Zygmund singular integral operator in Corollary 5.3.

REMARK 6.10. Any Schwartz function that has vanishing mean and is non-degenerate in the sense of (1.5) satisfies all the four conditions above.

The following discrete version of Theorem 6.1 will play a crucial role in the study of operator-valued Triebel–Lizorkin spaces on $\mathbb{R}^d$ in our forthcoming paper [39]. Now we fix the pairs (Φ, Ψ) and (ϕ, ψ) satisfying the above four conditions.

THEOREM 6.11. *Let $1 \leqslant p < \infty$. Then for any $f \in L_1(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d) + L_\infty(\mathcal{M}; \mathrm{R}^{\mathrm{c}}_d)$, $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ if and only if $s^{c,D}_\Phi(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$ if and only if $g^{c,D}_\Phi(f) \in L_p(\mathcal{N})$ and $\phi * f \in L_p(\mathcal{N})$. Moreover,*

$$\|f\|_{\mathrm{h}^{\mathrm{c}}_p} \approx \|s^{c,D}_\Phi(f)\|_p + \|\phi * f\|_p \approx \|g^{c,D}_\Phi(f)\|_p + \|\phi * f\|_p$$

with the relevant constants depending only on d, p, and the pairs (Φ, Ψ) and (ϕ, ψ).

The following paragraphs are devoted to the proof of Theorem 6.11 which is similar to that of Theorem 6.1. We will just indicate the necessary modifications. We first prove the discrete counterparts of Lemmas 6.2 and 6.4.

LEMMA 6.12. *Let $1 \leqslant p < 2$ and q be the conjugate index of p. For any $f \in \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$ and $g \in \mathrm{bmo}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$,*

$$\left| \tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s \right| \lesssim \left(\|s^{c,D}_\Phi(f)\|_p + \|\phi * f\|_p \right) \|g\|_{\mathrm{bmo}^{\mathrm{c}}_q}.$$

Proof. First, note that by (6.12), we have

$$\tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s = \tau\int_{\mathbb{R}^d}\sum_{j\geqslant 1}\Phi_j * f(s)(\Psi_j * g(s))^*\,\mathrm{d}s + \tau\int_{\mathbb{R}^d}\phi * f(s)(\psi * g(s))^*\mathrm{d}s.$$

The second term on the right hand side of the above formula is exactly the same as the corresponding term II in the proof of Lemma 6.2. Now we need the discrete versions of s^{c}_{Φ} and $\overline{s}^{\mathrm{c}}_{\Phi}$: for $j\geqslant 1$, $s\in\mathbb{R}^d$, let

$$s^{\mathrm{c},D}_{\Phi}(f)(s,j) = \Big(\sum_{1\leqslant k\leqslant j} 2^{dk}\int\limits_{B(s,2^{-k}-2^{-j-1})}|\Phi_j * f(t)|^2\mathrm{d}t\Big)^{1/2}$$

$$\overline{s}^{\mathrm{c},D}_{\Phi}(f)(s,j) = \Big(\sum_{1\leqslant k\leqslant j} 2^{dk}\int\limits_{B(s,2^{-k-1})}|\Phi_j * f(t)|^2\mathrm{d}t\Big)^{1/2}.$$

Denote $s^{\mathrm{c},D}_{\Phi}(f)(s,j)$ and $\overline{s}^{\mathrm{c},D}_{\Phi}(f)(s,j)$ simply by $s(s,j)$ and $\overline{s}(s,j)$, respectively. By approximation, we may assume that $s(s,j)$ and $\overline{s}(s,j)$ are invertible for every $s\in\mathbb{R}^d$ and $j\geqslant 1$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\Big|\tau\int_{\mathbb{R}^d}\sum_{j\geqslant 1}\Phi_j * f(s)(\Psi_j * g(s))^*\,\mathrm{d}s\Big|^2\\
&\quad= \Big|\frac{2^d}{c_d}\,\tau\int_{\mathbb{R}^d}\sum_j 2^{dj}\int\limits_{B(s,2^{-j-1})}\Phi_j * f(t)(\Psi_j * g(t))^*\,\mathrm{d}t\,\mathrm{d}s\Big|^2\\
&\quad\lesssim \tau\int_{\mathbb{R}^d}\sum_j s(s,j)^{p-2}\Big(2^{dj}\int\limits_{B(s,2^{-j-1}))}|\Phi_j * f(t)|^2\,\mathrm{d}t\Big)\mathrm{d}s\\
&\qquad\cdot\tau\int_{\mathbb{R}^d}\sum_j s(s,j)^{2-p}\Big(2^{dj}\int\limits_{B(s,2^{-j-1})}|\Psi_j * g(t)|^2\,\mathrm{d}t\Big)\mathrm{d}s \overset{\mathrm{def}}{=} \mathrm{A}\cdot\mathrm{B}.
\end{aligned}$$

The term A is less easy to estimate than the corresponding term A in the proof of Lemma 6.2. To deal with it we simply set $\overline{s}_j = \overline{s}(s,j)$ and $\overline{s} = \overline{s}(s,+\infty) \leqslant s^{\mathrm{c},D}(f)(s)$. Then

$$\begin{aligned}
\mathrm{A} &= \tau\int_{\mathbb{R}^d}\sum_{j\geqslant 1} s_j^{p-2}(\overline{s}_j^2-\overline{s}_{j-1}^2)\mathrm{d}s \leqslant \tau\int_{\mathbb{R}^d}\sum_j \overline{s}_j^{p-2}(\overline{s}_j^2-\overline{s}_{j-1}^2)\mathrm{d}s\\
&= \tau\int_{\mathbb{R}^d}\sum_j(\overline{s}_j-\overline{s}_{j-1})\mathrm{d}s + \tau\int_{\mathbb{R}^d}\sum_j \overline{s}_j^{p-2}\overline{s}_{j-1}(\overline{s}_j-\overline{s}_{j-1})\mathrm{d}s,
\end{aligned}$$

where $\overline{s}_0 = 0$. Since $1\leqslant p<2$, $\overline{s}_j^{p-1}\leqslant s^{p-1}$, we have

$$\tau\int_{\mathbb{R}^d}\sum_j \overline{s}_j^{p-1}(\overline{s}_j-\overline{s}_{j-1})\mathrm{d}s \lesssim \tau\int_{\mathbb{R}^d}\overline{s}^p\mathrm{d}s \leqslant \|s^{\mathrm{c},D}(f)\|_p^p.$$

On the other hand,

$$\tau \int_{\mathbb{R}^d} \sum_j \overline{s}_j^{p-2} \overline{s}_{j-1} (\overline{s}_j - \overline{s}_{j-1}) \mathrm{d}s$$
$$= \tau \int_{\mathbb{R}^d} \sum_j \overline{s}^{(1-p)/2} \overline{s}_j^{p-2} \overline{s}_{j-1} \overline{s}^{(1-p)/2} \overline{s}^{(p-1)/2} (\overline{s}_j - \overline{s}_{j-1}) \overline{s}^{(p-1)/2} \mathrm{d}s,$$

since $\overline{s}_j \geqslant \overline{s}_{j-1}$ for any $j \geqslant 1$, we have $\overline{s}^{(1-p)/2} \overline{s}_j^{p-2} \overline{s}_{j-1} \overline{s}^{(1-p)/2} \leqslant 1$. Thus, by the Hölder inequality,

$$\tau \int_{\mathbb{R}^d} \sum_j \overline{s}_j^{p-2} \overline{s}_{j-1} (\overline{s}_j - \overline{s}_{j-1}) \mathrm{d}s \leqslant \tau \int_{\mathbb{R}^d} \sum_j \overline{s}^{(p-1)/2} (\overline{s}_j - \overline{s}_{j-1}) \overline{s}^{(p-1)/2} \mathrm{d}s$$
$$= \tau \int_{\mathbb{R}^d} \overline{s}^p \mathrm{d}s \leqslant \| s_\Phi^{\mathrm{c},D}(f) \|_p^p.$$

Combining the preceding inequalities, we get the desired estimate of A:

$$\mathrm{A} \leqslant 2 \| s_\Phi^{\mathrm{c},D}(f) \|_p^p.$$

The estimate of the term B is, however, almost identical to that of B in the proof of Lemma 6.2. There are only two minor differences. The first one concerns the square function $\mathbb{S}^{\mathrm{c}}(f)(s,j)$ in (3.8): it is now replaced by

$$\mathbb{S}^{\mathrm{c}}(f)(s,j) = \Big(\sum_{1 \leqslant k \leqslant j} 2^{dk} \int_{B(c_{m,j}, 2^{-k})} |\Phi_j * f(t)|^2 \, \mathrm{d}t \Big)^{1/2} \quad \text{if } s \in Q_{m,j}.$$

Then we have $s(s,j) \leqslant \mathbb{S}^{\mathrm{c}}(f)(s,j)$. The second difference is about the Carleson characterization of $\mathrm{bmo}_q^{\mathrm{c}}$; we now use its discrete analogue, namely, $\mathrm{d}\mu_g^D$. Apart from these two differences, the remainder of the argument is identical to that in the proof of Lemma 6.2. ∎

LEMMA 6.13. *Let $1 \leqslant p < 2$ and $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \cap L_2(\mathcal{N})$, $g \in \mathrm{bmo}_q^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Then*

$$\Big| \tau \int_{\mathbb{R}^d} f(s) g^*(s) \mathrm{d}s \Big| \lesssim (\| g_\Phi^{\mathrm{c},D}(f) \|_p + \| \phi * f \|_p)^{p/2} \, \| f \|_{\mathrm{h}_p^{\mathrm{c}}}^{1-p/2} \, \| g \|_{\mathrm{bmo}_q^{\mathrm{c}}}.$$

Proof. We use the truncated version of $g_\Phi^{\mathrm{c},D}(f)$:

$$g_\Phi^{\mathrm{c},D}(f)(s,j) = \Big(\sum_{k \leqslant j} |\Phi_k * f(s)|^2 \Big)^{1/2}.$$

The proof of Lemma 4.3 in [41] is easily adapted to the present setting to ensure

$$g_\Phi^{\mathrm{c},D}(f)(s,j)^2 \lesssim \sum_{|m|_1 \leqslant d} s_{D^m \Phi}^{\mathrm{c},D}(f)(s,j)^2.$$

Then

$$\Big|\tau \int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s|^2 \leqslant \mathrm{I}' \cdot \mathrm{II}' + |\tau \int \phi * f(s)(\psi * g(s))^* \mathrm{d}s\Big|,$$

where

$$\mathrm{I}' = \tau \int_{\mathbb{R}^d} \sum_j g_\Phi^{\mathrm{c},D}(f)(s,j)^{p-2} |\Phi_j * f(s)|^2 \mathrm{d}s\,,$$

$$\mathrm{II}' = \tau \int_{\mathbb{R}^d} \sum_j g_\Phi^{\mathrm{c},D}(f)(s,j)^{2-p} |\Psi_j * g(s)|^2 \mathrm{d}s\,.$$

Both terms I' and II' are estimated exactly as before, so we have

$$\mathrm{I}' \leqslant 2\|g_\Phi^{\mathrm{c}}(f)\|_p^p \quad \text{and} \quad \mathrm{II}' \lesssim \|f\|_{\mathrm{h}_p^{\mathrm{c}}}^{2-p}\, \|g\|_{\mathrm{bmo}_q^{\mathrm{c}}}^2\,.$$

This gives the announced assertion. ∎

Armed with the previous two lemmas and the Calderón–Zygmund theory in Section 5, we can prove Theorem 6.11 in the same way as Theorem 6.1. Details are left to the reader.

We also include a discrete Carleson measure characterization of $\mathrm{bmo}_q^{\mathrm{c}}$ by general test functions. Much as the characterization in Lemma 6.7 and Remark 6.8, it is a byproduct of the proof of Theorem 6.11.

COROLLARY 6.14. *Let $2 < q \leqslant \infty$, ψ and Ψ be given in* (6.12). *Assume further*

$$\widehat{\psi} \in H_2^\sigma(\mathbb{R}^d) \quad \textit{with } \sigma > \frac{d}{2}.$$

Then for every $g \in \mathrm{bmo}_q^{\mathrm{c}}$, we have

$$\|g\|_{\mathrm{bmo}_q^{\mathrm{c}}} \approx \|\psi * g\|_q + \Big\| \sup_{\substack{s \in Q \subset \mathbb{R}^d \\ |Q|<1}}{}^{+} \frac{1}{|Q|} \int_Q \sum_{j \geqslant -\log_2(l(Q))} |\Psi_j * g(s)|^2 \mathrm{d}s \Big\|_{q/2}^{1/2}.$$

6.3. THE RELATION BETWEEN $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M})$ AND $\mathcal{H}_p(\mathbb{R}^d, \mathcal{M})$. Due to the noncommutativity, for any $1 < p < \infty$ and $p \neq 2$, the column operator-valued local Hardy space $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and the column operator-valued Hardy space $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ are not equivalent. On the other hand, if we consider the mixture spaces $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M})$ and $\mathcal{H}_p(\mathbb{R}^d, \mathcal{M})$, then we will have the same situation as in the classical case.

Since $\|\mathrm{P} * f\|_p \lesssim \|f\|_p \lesssim \|f\|_{\mathcal{H}_p^{\mathrm{c}}}$ for any $1 \leqslant p \leqslant 2$, we deduce the inclusion

$$\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \quad \text{for } 1 \leqslant p \leqslant 2. \tag{6.13}$$

Then by the duality obtained in Theorem 3.19, we have

$$\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \subset \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \quad \text{for } 2 < p < \infty. \tag{6.14}$$

However, we see from the following proposition that we do not have the inverse inclusion of (6.13) nor (6.14).

PROPOSITION 6.15. *Let ϕ be a function on $\mathbb{R}^d$ such that $\widehat{\phi}(0) \geqslant 0$ and $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ with $\sigma > \frac{d}{2}$. Let $2 < p < \infty$. If for any $f \in \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$,*

$$\|\phi * f\|_p \lesssim \|f\|_{\mathcal{H}_p^{\mathrm{c}}}, \tag{6.15}$$

then we must have $\widehat{\phi}(0) = 0$.

Proof. We prove the assertion by contradiction. Suppose that there exists ϕ such that $\widehat{\phi}(0) > 0$, $\widehat{\phi} \in H_2^\sigma(\mathbb{R}^d)$ and (6.15) holds for any $f \in \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Since both $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $L_p(\mathcal{N})$ are homogeneous spaces, we have, for any $\varepsilon > 0$,

$$\|\phi * f(\varepsilon\cdot)\|_p = \|(\phi_\varepsilon * f)(\varepsilon\cdot)\|_p = \varepsilon^{-d/p}\|\phi_\varepsilon * f\|_p \quad \text{and} \quad \|f(\varepsilon\cdot)\|_{\mathcal{H}_p^{\mathrm{c}}} = \varepsilon^{-d/p}\|f\|_{\mathcal{H}_p^{\mathrm{c}}}.$$

This implies that

$$\|\phi_\varepsilon * f\|_p \lesssim \|f\|_{\mathcal{H}_p^{\mathrm{c}}}, \tag{6.16}$$

for any $\varepsilon > 0$ with the relevant constant independent of ε. Now we consider a function $f \in L_p(\mathcal{N})$ which takes values in $\mathcal{S}_{\mathcal{M}}^+$ and such that supp $\widehat{f}$ is compact, i.e. there exists a positive real number N such that supp $\widehat{f} \subset \{\xi \in \mathbb{R}^d : |\xi| \leqslant N\}$. Since $\widehat{\phi}(0) > 0$, we can find $\varepsilon_0 > 0$ and $c > 0$ such that $\widehat{\phi}(\varepsilon_0\xi) \geqslant c$ whenever $|\xi| \leqslant N$. Thus, in this case, $\|\phi_\varepsilon * f\|_p \geqslant c\|f\|_p$. Then by (6.16), we have

$$\|f\|_p \lesssim \|f\|_{\mathcal{H}_p^{\mathrm{c}}},$$

which leads to a contradiction when $p > 2$. Therefore, $\widehat{\phi}(0) = 0$. ∎

By the definition of the $\mathrm{h}_p^{\mathrm{c}}$-norm and the duality in Theorem 3.19, we get the following result.

COROLLARY 6.16. *Let $1 \leqslant p < \infty$ and $p \neq 2$; $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ are not equivalent.*

Although $\mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ do not coincide when $p \neq 2$, for those functions whose Fourier transforms vanish at the origin, their $\mathrm{h}_p^{\mathrm{c}}$-norms and $\mathcal{H}_p^{\mathrm{c}}$-norms are still equivalent.

THEOREM 6.17. *Let $\phi \in \mathcal{S}$ such that $\int_{\mathbb{R}^d} \phi(s)\mathrm{d}s = 1$.*

(i) *If $1 \leqslant p \leqslant 2$ and $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, then $f - \phi * f \in \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\|f - \phi * f\|_{\mathcal{H}_p^{\mathrm{c}}} \lesssim \|f\|_{\mathrm{h}_p^{\mathrm{c}}}$.*

(ii) *If $2 < p < \infty$ and $f \in \mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, then $f - \phi * f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\|f - \phi * f\|_{\mathrm{h}_p^{\mathrm{c}}} \lesssim \|f\|_{\mathcal{H}_p^{\mathrm{c}}}$.*

Proof. (i) Let $f \in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and Φ be a nondegenerate Schwartz function with vanishing mean. By the general characterization of $\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ in Lemma 1.1, $\|f - \phi * f\|_{\mathcal{H}_p^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})} \approx \|G_\Phi^{\mathrm{c}}(f - \phi * f)\|_p$. Let us split $\|G_\Phi^{\mathrm{c}}(f - \phi * f)\|_p$

into two parts:

$$\begin{aligned}
&\|G_\Phi^{\mathrm{c}}(f-\phi*f)\|_p\\
&\quad\lesssim \Big\|\Big(\int_0^1 |\Phi_\varepsilon*(f-\phi*f)|^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\Big\|_p+\Big\|\Big(\int_1^\infty |\Phi_\varepsilon*(f-\phi*f)|^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\Big\|_p\\
&\quad= \Big\|\Big(\int_0^1 |\Phi_\varepsilon*(f-\phi*f)|^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\Big\|_p+\Big\|\Big(\int_1^\infty |(\Phi_\varepsilon-\Phi_\varepsilon*\phi)*f|^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\Big\|_p.
\end{aligned}$$

In order to estimate the first term in the last equality, we notice that $\phi*f\in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$, thus we have $f-\phi*f\in \mathrm{h}_p^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. Then by Theorem 6.1, this term can be majorized from above by $\|f\|_{\mathrm{h}_p^{\mathrm{c}}}$.

To deal with the second term, we express it as a Calderón–Zygmund operator with Hilbert-valued kernel. Let $H=L_2((1,+\infty),\frac{\mathrm{d}\varepsilon}{\varepsilon})$ and define the kernel $\mathsf{k}:\mathbb{R}^d\to H$ by $\mathsf{k}(s)=\Phi_\cdot(s)-\Phi_\cdot*\phi(s)$ ($\Phi_\cdot(s)$ being the function $\varepsilon\mapsto\Phi_\varepsilon(s)$). Now we prove that k satisfies the hypotheses of Corollary 5.3. The condition (i) of that corollary is easy to verify. So we only check the conditions (ii) and (iii) there. By the fact that $\int_{\mathbb{R}^d}\phi(s)\mathrm{d}s=1$ and the mean value theorem, we have

$$\begin{aligned}
|(\Phi_\varepsilon-\Phi_\varepsilon*\phi)(s)|&=\Big|\int_{\mathbb{R}^d}[\Phi_\varepsilon(s)-\Phi_\varepsilon(s-t)]\phi(t)\mathrm{d}t\Big|\\
&\leqslant\int_{\mathbb{R}^d}|t|\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\Big|\nabla\Phi\Big(\frac{s-\theta t}{\varepsilon}\Big)\Big||\phi(t)|\mathrm{d}t.
\end{aligned}$$

Then we split the last integral into two parts:

$$\begin{aligned}
\Big\|(\Phi_\cdot-\Phi_\cdot*\phi)(s)\Big\|_H&\lesssim\Big(\int_1^\infty\Big(\int_{|t|<|s|/2}|t|\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\Big|\nabla\Phi\Big(\frac{s-\theta t}{\varepsilon}\Big)\Big||\phi(t)|\mathrm{d}t\Big)^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\\
&\quad+\Big(\int_1^\infty\Big(\int_{|t|>|s|/2}|t|\frac{1}{\varepsilon^{d+1}}\sup_{0<\theta<1}\Big|\nabla\Phi\Big(\frac{s-\theta t}{\varepsilon}\Big)\Big||\phi(t)|\mathrm{d}t\Big)^2\frac{\mathrm{d}\varepsilon}{\varepsilon}\Big)^{1/2}\\
&\overset{\mathrm{def}}{=}\mathrm{I}+\mathrm{II}.
\end{aligned}$$

If $|t|<\frac{|s|}{2}$, we have $|s-\theta t|\geqslant\frac{|s|}{2}$, thus $|\nabla\Phi(\frac{s-\theta t}{\varepsilon})|\lesssim\frac{\varepsilon^{d+1/2}}{|s|^{d+1/2}}$ for any $0\leqslant\theta\leqslant1$. Then

$$\mathrm{I}\lesssim\Big(\int_1^\infty\frac{1}{\varepsilon^2}\mathrm{d}\varepsilon\Big)^{1/2}\frac{1}{|s|^{d+1/2}}\lesssim\frac{1}{|s|^{d+1/2}}.$$

When $|t| > \frac{|s|}{2}$, since $\phi \in \mathcal{S}$, we have $\int\limits_{|t|>|s|/2} |t||\phi(t)|\mathrm{d}t \lesssim \frac{1}{|s|^{d+1/2}}$. Hence

$$\mathrm{II} \lesssim \Big(\int\limits_1^\infty \frac{1}{\varepsilon^{2d+2}} \frac{\mathrm{d}\varepsilon}{\varepsilon} \Big)^{1/2} \cdot \frac{1}{|s|^{d+1/2}} \lesssim \frac{1}{|s|^{d+1/2}}.$$

The estimates of I and II imply

$$\|(\Phi_\varepsilon - \Phi_\varepsilon * \phi)(s)\|_H \lesssim \frac{1}{|s|^{d+1/2}}.$$

In a similar way, we obtain

$$\|\nabla(\Phi_\varepsilon - \Phi_\varepsilon * \phi)(s)\|_H \lesssim \frac{1}{|s|^{d+1}}.$$

Thus, it follows from Corollary 5.3 that $\Big\| \Big(\int\limits_1^\infty |(\Phi_\varepsilon - \Phi_\varepsilon * \phi) * f|^2 \frac{\mathrm{d}\varepsilon}{\varepsilon} \Big)^{1/2} \Big\|_p$ is also majorized from above by $\|f\|_{\mathrm{h}^{\mathrm{c}}_p}$.

(ii) The case $p > 2$ can be deduced from the duality between $\mathrm{h}^{\mathrm{c}}_p$ and $\mathrm{h}^{\mathrm{c}}_q$ (Theorem 3.19) and that between $\mathcal{H}^{\mathrm{c}}_p$ and $\mathcal{H}^{\mathrm{c}}_q$ (q being the conjugate index of p). There exists $g \in \mathrm{h}^{\mathrm{c}}_q(\mathbb{R}^d, \mathcal{M})$ with norm one such that we have the following which completes the proof:

$$\begin{aligned} \|f - \phi * f\|_{\mathrm{h}^{\mathrm{c}}_p} &= \Big| \tau \int_{\mathbb{R}^d} (f - \phi * f)(s) g^*(s) \mathrm{d}s \Big| = \Big| \tau \int_{\mathbb{R}^d} f(s)(g^* - \phi * g^*)(s) \mathrm{d}s \Big| \\ &\leqslant \|f\|_{\mathcal{H}^{\mathrm{c}}_p} \|g - \overline{\phi} * g\|_{\mathcal{H}^{\mathrm{c}}_q} \lesssim \|f\|_{\mathcal{H}^{\mathrm{c}}_p} \|g\|_{\mathrm{h}^{\mathrm{c}}_q} = \|f\|_{\mathcal{H}^{\mathrm{c}}_p}. \quad \blacksquare \end{aligned}$$

From the interpolation result of mixture local Hardy spaces in Proposition 4.4, we deduce the equivalence between mixture local Hardy spaces and L_p-spaces.

PROPOSITION 6.18. *For any* $1 < p < \infty$, $\mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = \mathcal{H}_p(\mathbb{R}^d, \mathcal{M}) = L_p(\mathcal{N})$ *with equivalent norms.*

Proof. It is known that $\mathcal{H}_p(\mathbb{R}^d, \mathcal{M}) = L_p(\mathcal{N})$ with equivalent norms. One can see Corollary 5.4 of [24] for more details. One the other hand, since $L_\infty(\mathcal{N}) \subset \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, by duality, we get $\mathrm{h}^{\mathrm{c}}_1(\mathbb{R}^d, \mathcal{M}) \subset L_1(\mathcal{N})$. Combining (2.2) and the interpolation result in Theorem 4.3, we deduce that $\mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M}) \subset L_p(\mathcal{N})$ for any $1 < p \leqslant 2$ and $L_p(\mathcal{N}) \subset \mathrm{h}^{\mathrm{c}}_p(\mathbb{R}^d, \mathcal{M})$ for any $2 < p < \infty$. Similarly, we also have $\mathrm{h}^{\mathrm{r}}_p(\mathbb{R}^d, \mathcal{M}) \subset L_p(\mathcal{N})$ for any $1 < p \leqslant 2$ and $L_p(\mathcal{N}) \subset \mathrm{h}^{\mathrm{r}}_p(\mathbb{R}^d, \mathcal{M})$ for any $2 < p < \infty$. Combined with (6.13) and (6.14), we get

$$\mathcal{H}_p(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) \subset L_p(\mathcal{N}) \quad \text{for } 1 < p \leqslant 2, \quad \text{and} \tag{6.17}$$

$$L_p(\mathcal{N}) \subset \mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) \subset \mathcal{H}_p(\mathbb{R}^d, \mathcal{M}) \quad \text{for } 2 < p < \infty. \tag{6.18}$$

Then (6.17), (6.18) and Corollary 5.4 of [24] imply the following which completes the proof:

$$\mathrm{h}_p(\mathbb{R}^d, \mathcal{M}) = \mathcal{H}_p(\mathbb{R}^d, \mathcal{M}) = L_p(\mathcal{N}) \quad \text{for } 1 < p < \infty. \quad \blacksquare$$

7. THE ATOMIC DECOMPOSITION

In this section, we give the atomic decomposition of $\mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. The atomic decomposition of $\mathcal{H}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ studied in [24] and the characterizations obtained in the last section will be the main tools for us.

DEFINITION 7.1. Let Q be a cube in $\mathbb{R}^d$ with $|Q| \leqslant 1$. If $|Q| = 1$, an $\mathrm{h}_1^{\mathrm{c}}$-atom associated with Q is a function $a \in L_1(\mathcal{M}; L_2^{\mathrm{c}}(\mathbb{R}^d))$ such that:

(i) $\operatorname{supp} a \subset Q$;

(ii) $\tau\Big(\int\limits_Q |a(s)|^2 \mathrm{d}s\Big)^{1/2} \leqslant |Q|^{-1/2}$.

If $|Q| < 1$, we assume additionally:

(iii) $\int\limits_Q a(s)\mathrm{d}s = 0$.

Let $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ be the space of all f admitting a representation of the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the a_j's are $\mathrm{h}_1^{\mathrm{c}}$-atoms and $\lambda_j \in \mathbb{C}$ such that $\sum\limits_{j=1}^{\infty} |\lambda_j| < \infty$. The above series converges in the sense of distribution. We equip $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ with the following norm:

$$\|f\|_{\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}} = \inf\Big\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j;\ a_j\text{'s are } \mathrm{h}_1^{\mathrm{c}}\text{-atoms}, \lambda_j \in \mathbb{C}\Big\}.$$

Similarly, we define the row version $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{r}}(\mathbb{R}^d, \mathcal{M})$. Then we set

$$\mathrm{h}_{1,\mathrm{at}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) + \mathrm{h}_{1,\mathrm{at}}^{\mathrm{r}}(\mathbb{R}^d, \mathcal{M}).$$

THEOREM 7.2. *We have* $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) = \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ *with equivalent norms.*

Proof. First, we show the inclusion $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. To this end, it suffices to prove that for any atom a in Definition 7.1, we have

$$\|a\|_{\mathrm{h}_1^{\mathrm{c}}} \lesssim 1. \tag{7.1}$$

Recall that the atomic decomposition of $\mathcal{H}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ has been considered in [24]. An $\mathcal{H}_1^{\mathrm{c}}$-atom is a function $b \in L_1(\mathcal{M}; L_2^{\mathrm{c}}(\mathbb{R}^d))$ such that, for some cube Q,

(1) $\operatorname{supp} b \subset Q$;

(2) $\int_Q b(s)\mathrm{d}s = 0$;

(3) $\tau\Big(\int_Q |b(s)|^2\mathrm{d}s\Big)^{1/2} \leqslant |Q|^{-1/2}$.

If a is supported in Q with $|Q| < 1$, then a is also an $\mathcal{H}_1^{\mathrm{c}}$-atom, so $\|a\|_{\mathrm{h}_1^{\mathrm{c}}} \lesssim \|a\|_{\mathcal{H}_1^{\mathrm{c}}} \lesssim 1$. Now assume that the supporting cube Q of a is of side length one. We use the discrete characterization obtained in Theorem 6.11, i.e.

$$\|a\|_{\mathrm{h}_1^{\mathrm{c}}} \approx \Big\|\Big(\sum_{j=1}^{\infty} |\Phi_j * a|^2\Big)^{1/2}\Big\|_1 + \|\phi * a\|_1.$$

Apart from the assumption on Φ and ϕ in Theorem 6.1, we may take Φ and ϕ satisfying

$$\operatorname{supp}\Phi, \operatorname{supp}\phi \subset B_1 = \{s \in \mathbb{R}^d : |s| \leqslant 1\}.$$

Then

$$\operatorname{supp}\phi * a \subset 3Q \quad \text{and} \quad \operatorname{supp}\Phi_\varepsilon * a \subset 3Q \quad \text{for any } 0 < \varepsilon < 1.$$

By the Cauchy–Schwarz inequality we have

$$\|\phi * a\|_1 \leqslant \int_{3Q} \Big(\int_Q |\phi(t-s)|^2\mathrm{d}s\Big)^{1/2} \cdot \tau\Big(\int |a(s)|^2\mathrm{d}s\Big)^{1/2}\mathrm{d}t \lesssim 1.$$

Similarly,

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^{\infty} |\Phi_j * a|^2\Big)^{1/2}\Big\|_1 &= \tau\int_{3Q}\Big(\sum_{j=1}^{\infty} |\Phi_j * a(s)|^2\Big)^{1/2}\mathrm{d}s \lesssim \tau\Big(\int_{3Q}\sum_{j=1}^{\infty} |\Phi_j * a(s)|^2\mathrm{d}s\Big)^{1/2} \\ &= \tau\Big(\int_{\mathbb{R}^d}\sum_{j=1}^{\infty} |\widehat{\Phi}(2^{-j}\xi)\widehat{a}(\xi)|^2\mathrm{d}\xi\Big)^{1/2} \leqslant \tau\Big(\int |a(s)|^2\mathrm{d}s\Big)^{1/2} \leqslant 1.\end{aligned}$$

Therefore, $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \subset \mathrm{h}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$.

Now we turn to proving the reverse inclusion. Observe that $\mathcal{H}_1^{\mathrm{c}}$-atoms are also $\mathrm{h}_1^{\mathrm{c}}$-atoms. Then by the duality between $\mathcal{H}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and $\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and the atomic decomposition of $\mathcal{H}_1^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$, every continuous functional ℓ on $\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ corresponds to a function g lying in $\mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$. Moreover, since $L_1(\mathcal{M}; L_2^{\mathrm{c}}(Q)) \subset \mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ for any cube Q with side length one, ℓ induces a continuous functional on $L_1(\mathcal{M}; L_2^{\mathrm{c}}(Q))$ with norm less than or equal to $\|\ell\|_{(\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}})^*}$. Thus, the function g satisfies the condition that

$$g \in \mathrm{BMO}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}) \quad \text{and} \quad \sup_{Q\subset\mathbb{R}^d,\,|Q|=1} \|g|_Q\|_{L_\infty(\mathcal{M};L_2^{\mathrm{c}}(Q))} \leqslant \|\ell\|_{(\mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}})^*}. \tag{7.2}$$

Consequently, $g \in \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M})$ and

$$\ell(f) = \tau\int_{\mathbb{R}^d} f(s)g^*(s)\mathrm{d}s, \quad \forall\, f \in \mathrm{h}_{1,\mathrm{at}}^{\mathrm{c}}(\mathbb{R}^d, \mathcal{M}).$$

Thus, $\mathrm{h}^{\mathrm{c}}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M})^* \subset \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$. On the other hand, by the previous result, $\mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}^{\mathrm{c}}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M})^*$. Thus, $\mathrm{h}^{\mathrm{c}}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M})^* = \mathrm{bmo}^{\mathrm{c}}(\mathbb{R}^d,\mathcal{M})$ with equivalent norms. Since $\mathrm{h}^{\mathrm{c}}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M}) \subset \mathrm{h}^{\mathrm{c}}_{1}(\mathbb{R}^d,\mathcal{M})$ densely, we deduce that $\mathrm{h}^{\mathrm{c}}_{1,\mathrm{at}}(\mathbb{R}^d,\mathcal{M}) = \mathrm{h}^{\mathrm{c}}_{1}(\mathbb{R}^d,\mathcal{M})$ with equivalent norms. ∎

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RUNLIAN XIA, LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ DE FRANCHE-COMTÉ, 25030 BESANÇON CEDEX, FRANCE, *and* INSTITUTO DE CIENCIAS MATEMÁTICAS, 28049 MADRID, SPAIN

E-mail address: runlian91@gmail.com

XIAO XIONG, INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN, 150001, CHINA *and* DEPARTMENT OF MATHEMATICS, HARBIN INSTITUTE OF TECHNOLOGY, HARBIN, 150001, CHINA

E-mail address: xxiong@hit.edu.cn

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