

SMOOTH FIELDS OF OPERATORS, A SMOOTH REDUCTION THEOREM, AND A DERIVATION FORMULA FOR CONSTANTS OF MOTION OF THE FREE HAMILTONIAN

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Communicated by Victor Nistor

ABSTRACT. We introduce a notion of smooth fields of operators following the notion of smooth fields of Hilbert spaces defined by L. Lempert and R. Szőke. We show that, if ∇ is the connection of a smooth field of Hilbert spaces, then $\widehat{\nabla} = [\nabla, \cdot]$ defines a connection on a suitable space of fields of operators. We prove a smooth version of the reduction theorem and we apply it to show that, if $h(q, p) = \|p\|^2$ and $\{u, h\} = 0$ then Weyl quantization maps u into an operator admitting a decomposition as a smooth field of operators over the interval $(0, \infty)$. Moreover, we prove an explicit formula to compute the derivatives of those fields of operators. We also introduce a notion of smooth field of C^* -algebras and we provide an explicit example.

KEYWORDS: *Smooth fields of Hilbert spaces and operators, connections, reduction theorem, constant of motion, Weyl quantization.*

MSC (2020): 47A56.

1. INTRODUCTION

Let $p : H \rightarrow \Lambda$ be a field of Hilbert spaces, i.e. p is a surjective map such that $\mathcal{H}(\lambda) := p^{-1}(\lambda)$ is a Hilbert space, and denote by $\langle \cdot, \cdot \rangle_\lambda$ the corresponding inner product. We denote by Γ the set of all sections of such field. For any pair of sections $\varphi, \psi \in \Gamma$, we set the function

$$h(\varphi, \psi)(\lambda) = \langle \varphi(\lambda), \psi(\lambda) \rangle_\lambda.$$

In order to obtain an interesting mathematical object, we should add further assumptions on a given field of Hilbert spaces. For instance, the notions of measurable and continuous field of Hilbert spaces were introduced by von Neumann [15] and Godement [8], respectively (see the Appendix A for details), and they were successfully studied and applied since then. However, a natural notion of

smooth field of Hilbert spaces was introduced only few years ago [12]. In this article, we are going to introduce a notion of smooth field of operators following [12], and we will provide an interesting family of examples. Let us explain briefly how we construct our examples. Recall that fields of operators appear naturally in operator theory: given a fix self-adjoint operator H (regarded as a quantum Hamiltonian), every self-adjoint operator commuting with H (i.e. a quantum constant of motion) admits a suitable decomposition as a (measurable) field of operators over the spectrum of H (measurable reduction theory). We are going to show a smooth version of the previous reduction theorem, and we are going to apply it to show that, under rather mild conditions, (Weyl) canonical quantization maps classical constant of motion of the classical free Hamiltonian $h(q, p) = \|p\|^2$ into operators commuting with the one parameter group of the Laplace operator and admitting a suitable decomposition as a smooth field of operators over the interval $(0, \infty)$. Moreover, we will also prove a remarkable formula to compute the derivatives of such fields of operators involving a sort of Poisson connection. Following our results, we will also introduce a notion of smooth field of C^* -algebras and we will give an example using Hilbert C^* -modules theory.

Let us provide a more detailed description of our constructions and results. The notions of measurable and continuous field of Hilbert spaces are defined in terms of certain space of sections Γ^0 satisfying certain properties (see Definitions A.1 and A.4). However, both notions can be described in terms of bundles or in terms of Hilbert C^* -modules. For instance, if Λ is a locally compact Hausdorff space and $p : H \rightarrow \Lambda$ is a continuous field of Hilbert spaces, then there is a unique suitable topology on H such that $p : H \rightarrow \Lambda$ is a continuous Hilbert bundle and Γ^0 is contained in the space of continuous sections $\Gamma(\Lambda)$ of $p : H \rightarrow \Lambda$. Moreover, the space of continuous sections vanishing at the infinity $\Gamma_0(\Lambda)$ is a Hilbert $C_0(\Lambda)$ -module, and conversely, every Hilbert $C_0(\Lambda)$ -module can be obtained in this way. There are analogous results in the measurable framework. In Appendix A we explain in more detail those equivalences in both cases.

A similar situation occurs in the C^* -algebraic framework. Indeed, upper semi-continuous fields of C^* -algebras over Λ are in one to one correspondence with upper semi-continuous C^* -bundles over Λ and with $C_0(\Lambda)$ -algebras, for any locally compact Hausdorff space Λ (for instance, see Theorem C.26 in [27]).

In the smooth framework, the canonical notion of vector bundle is well defined even if the fibers are Banach spaces (we call them Banach bundles, for instance, see Chapter III in [11] or the brief discussion in Subsection I.2.1 in [12]). However, we could not find in the literature a notion of smoothness for fields of Banach spaces in terms of certain space of sections such that, under reasonable conditions, it would determine a suitable underlying Banach bundle structure (as in the continuous or measurable framework). Nevertheless, for Hermitian bundles, the following notion of smooth field of Hilbert spaces was introduced and studied in [12].

DEFINITION 1.1. Let Λ be a finite dimensional smooth manifold and $\text{Vect}(\Lambda)$ the space of smooth vector fields on Λ . A smooth structure on a field of Hilbert spaces $H \rightarrow \Lambda$ is given by specifying a set of sections Γ^∞ , closed under addition and under multiplication by elements of $C^\infty(\Lambda)$, and a map $\nabla : \text{Vect}(\Lambda) \times \Gamma^\infty \rightarrow \Gamma^\infty$ such that, for $X, Y \in \text{Vect}(\Lambda)$, $a \in C^\infty(\Lambda)$ and $\varphi, \psi \in \Gamma^\infty$:

- (i) $\nabla_{X+Y} = \nabla_X + \nabla_Y$, $\nabla_{aX} = a\nabla_X$, $\nabla_X(a\varphi) = X(a)\varphi + a\nabla_X(\varphi)$;
- (ii) $h(\varphi, \psi) \in C^\infty(\Lambda)$ and $Xh(\varphi, \psi) = h(\nabla_X\varphi, \psi) + h(\varphi, \nabla_{\bar{X}}\psi)$;
- (iii) $\mathcal{H}^\infty(\lambda) := \{\varphi(\lambda) : \varphi \in \Gamma^\infty\}$ is dense in $\mathcal{H}(\lambda)$, for all $\lambda \in \Lambda$.

Conditions (ii) and (iii) implies that every smooth field of Hilbert spaces is continuous (see Definition A.4). Moreover, the corresponding space of continuous sections $\Gamma^0(\Lambda)$ is the closure of Γ^∞ with the Fréchet topology of local uniform convergence. Similarly, we can complete Γ^∞ to obtain the space of n -times differentiable sections $\Gamma^n(\Lambda)$ (using the seminorms given by equation (2.1)), for every $n \in \mathbb{N} \cup \{\infty\}$.

Our first goal is to show that the previous notion of smooth field of Hilbert spaces allows us to introduce a notion of smooth fields of operators and also to define a connection $\widehat{\nabla}$ on a suitable space of fields of operators satisfying properties analogous to (i) and (ii) in Definition 1.1. Formally, if such connection is applied on a field of operators $A = \{A(\lambda)\}$, where $A(\lambda)$ is an operator on $\mathcal{H}(\lambda)$ (with suitable domain), then we would expect that the Leibniz's identity holds, i.e.

$$\nabla_X(A\varphi) = \widehat{\nabla}_X(A)\varphi + A\nabla_X(\varphi), \quad \forall \varphi \in \Gamma^\infty.$$

Hence, the natural definition for such connection is $\widehat{\nabla}(A) = [\nabla_X, A] = \nabla_X A - A\nabla_X$. The first part of Section 2 is meant to construct suitable spaces of fields of operators A where such expression is well-defined, and $\widehat{\nabla}_X(A)$ is once again a field of operators. Precisely, we will prove that $\widehat{\nabla}_X(A) \in \mathfrak{A}^{n-1}$, for any vector field X and $A \in \mathfrak{A}^n$, where the space of fields of operators \mathfrak{A}^n is introduced in Definition 2.1, for any $n \in \mathbb{N}$ (see Theorem 2.3).

The simplest example of a smooth field of Hilbert spaces is the trivial case, i.e. we take each fiber constant $\mathcal{H}(\lambda) = V$, $\Gamma^\infty(\Lambda) = C^\infty(\Lambda, V)$ and $\nabla_X = X$. A more interesting case is when $H \rightarrow \Lambda$ admits a smooth trivialization.

DEFINITION 1.2. Let $p : H \rightarrow \Lambda$ be a smooth field of Hilbert spaces with connection ∇ . We say that $p : H \rightarrow \Lambda$ admits a smooth trivialization if there is a Hilbert space V and a map $T : H \rightarrow V$ such that $T_\lambda = T|_{\mathcal{H}(\lambda)} : \mathcal{H}(\lambda) \rightarrow V$ is unitary, $T(\Gamma^\infty) \subseteq C^\infty(\Lambda, V)$ and for each $X \in \text{Vect}(\Lambda)$ we have that

$$T\nabla_X\varphi = XT\varphi.$$

We say that T is full if in addition we have that $T(\Gamma^\infty(\Lambda)) = C^\infty(\Lambda, V)$.

Since each T_λ is unitary, T is continuous with respect to the locally uniform convergence topology, hence $T(\Gamma^n(\Lambda)) \subseteq C^n(\Lambda, V)$, for every $n \in \mathbb{N} \cup \{\infty\}$.

Therefore, T is full if and only if $T(\Gamma^\infty)$ is dense in $C^\infty(\Lambda, V)$, and in such case we have that $T(\Gamma^n(\Lambda)) = C^n(\Lambda, V)$, for every $n \in \mathbb{N} \cup \{\infty\}$.

More generally, we may suppose that the map T satisfies the identity

$$(1.1) \quad T\nabla_X\varphi = XT\varphi + \alpha(X)T\varphi,$$

where $\alpha : \text{Vect}(\Lambda) \rightarrow \text{End}(V)$ is a suitable map. For instance, if α is a 1-form (in particular, $\alpha(X) \in C^\infty(\Lambda)$), the couple (T, V) is called a projective trivialization (see Definition 2.4.1 in [12]). It turns out that every projective trivialization can be transformed into a smooth trivialization by taking the tensor product of the initial smooth field of Hilbert spaces with a suitable line bundle (see Subsection 2.4 in [12]).

Notice that, determining a Hermitian bundle structure will also require the unitary operators $T(\lambda)$ in Definition 1.2, but defined locally. In a forthcoming article, we are going to find the conditions that a local trivialization of a smooth field of Hilbert spaces must satisfy in order to determine a suitable Hermitian bundle structure. Some of the techniques that we will develop in this article will become useful for that purpose.

Regarding fields of operators, every trivialization (T, V) of $p : H \rightarrow \Lambda$ induces a sort of trivialization \widehat{T} for $\widehat{\nabla}$ on \mathfrak{A}^n . Let us be more precise. Essentially, \widehat{T} is defined by $\widehat{T}A(\lambda) = \widehat{A}(\lambda) = T_\lambda A(\lambda)T_\lambda^*$. Notice that each $\widehat{A}(\lambda)$ is an operator on V , but their corresponding domains might be different depending on λ . In order to overcome that technical difficulty, we will define a common domain V^∞ (see Definition 2.4). It turns out that \widehat{T} maps \mathfrak{A}^n into $C^n(\Lambda, L(V^\infty, V)_{*-s})$ and $\widehat{T}\widehat{\nabla}_X = X\widehat{T}$, where $L(V^\infty, V)_{*-s}$ denotes the space of linear operators from V^∞ to V with the $*$ -strong topology (see Theorem 2.5).

In Section 2, we will also consider the trivialisable case to give an additional motivation for our construction. Notice that the unitary operators $U_{\lambda_1, \lambda_2} = T_{\lambda_2}^* T_{\lambda_1} : \mathcal{H}(\lambda_1) \rightarrow \mathcal{H}(\lambda_2)$ play the role of the parallel transport of the connection ∇ . If $A = \{A(\lambda)\}$ is a field of operators, we would like to compare $A(\lambda)$ and $U_{\lambda, \lambda_0}^* A(\lambda_0) U_{\lambda, \lambda_0}$, for any $\lambda_0, \lambda \in \Lambda$ (both operators have domains in $\mathcal{H}(\lambda)$). We will show that those operators coincide if and only if we have that $\widehat{\nabla}_X(A) = 0$, for any vector field X (see Proposition 2.8). In such case we say that A is a horizontal field of operators. We will also give a weak estimation of the difference between $A(\lambda)$ and $U_{\lambda, \lambda_0}^* A(\lambda_0) U_{\lambda, \lambda_0}$ in the general case (see Proposition 2.7).

In Subsection 2.1, we will analyze fields of operators belonging to \mathfrak{A}^n as single operators acting on the direct integral $\mathcal{H} = \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$, where $d\eta$ is a fix volume form on Λ . Since the fields of operators that we are considering belong to some \mathfrak{A}^n , the natural domain in the direct integral is $\Gamma_2^\infty = \Gamma^\infty \cap \mathcal{H}$, and we will assume that Γ_2^∞ is dense in \mathcal{H} . We will obtain a sort of reduction theorem for our setting analogous to the well known result in the measurable framework [3, 15, 17]. More precisely, we will prove that, if $A : \Gamma_2^\infty \rightarrow \Gamma^0(\Lambda) \cap \mathcal{H}$ is an operator such that $A^*(\Gamma_2^\infty) \subset \Gamma^0(\Lambda) \cap \mathcal{H}$, then A can be decomposed

through the direct integral if and only if $fA = Af$ on Γ_2^∞ , for every $f \in C_c^\infty(\Lambda)$ (see Theorem 2.10). We will also introduce a space of fields of operators \mathfrak{A}_2^n such that $\mathfrak{A}_2^n \subseteq \mathfrak{A}^n$ and $\widehat{\nabla}_{X_1} \cdots \widehat{\nabla}_{X_k}(A)$ is an operator on \mathcal{H} (with domain Γ_2^∞), for every $A \in \mathfrak{A}_2^n$ and $X_1, \dots, X_k \in \text{Vect}(\Lambda)$, with $0 \leq k \leq n$ (see Definition 2.13).

In the trivialisable case, we will obtain two further remarkable properties that will become useful later. Let $X \in \text{Vect}(\Lambda)$ and denote by r_t the corresponding flow on Λ . We will show that the operator $-i(\nabla_X + \frac{1}{2}\text{div}(X))$ is self-adjoint on \mathcal{H} (see Proposition 2.15). If (T, V) is a trivialization, define $R_t = T^*r_t^*T$. Notice that, if $A \in \mathfrak{A}^n$, then the map $t \rightarrow \langle R_tAR_{-t}\varphi(\lambda), \psi(\lambda) \rangle_\lambda$ is differentiable for every $\lambda \in \Lambda$, and

$$\frac{d}{dt} \langle R_tAR_{-t}\varphi(\lambda), \psi(\lambda) \rangle_\lambda |_{t=s} = \langle R_s \widehat{\nabla}_X(A) R_{-s}\varphi(\lambda), \psi(\lambda) \rangle_\lambda.$$

We will prove that if Γ_2^∞ is invariant by R_t and A belongs \mathfrak{A}^1 , then for suitable $\varphi, \psi \in \Gamma_2^\infty$, the map $t \rightarrow \langle R_tAR_{-t}\varphi, \psi \rangle$ is differentiable, and the previous identity holds not only pointwise but also weakly (see Theorem 2.16).

In Section 3, we will study our main example. Let $n > 1$ and Q^2 be the self-adjoint operator on $L^2(\mathbb{R}^n)$ given by $[Q^2\varphi](q) = \|q\|^2\varphi(q)$. It is well known that the map $T : L^2(\mathbb{R}^n) \rightarrow \int_{(0,\infty)}^\oplus L^2(\mathbb{S}^{n-1})d\lambda$ given by $T\varphi(\lambda, z) = 2^{-1/2}\lambda^{(n-2)/4}\varphi(\sqrt{\lambda}z)$

is a spectral diagonalization of Q^2 . Moreover, T can be regarded as a trivialization of the field of Hilbert spaces $\mathcal{H}(\lambda) = L^2(\mathbb{S}_{\sqrt{\lambda}}^{n-1}, \mu_\lambda)$, where $\mu_\lambda = 2^{-1/2}\sqrt{\lambda}\eta_\lambda$ and η_λ is the canonical measure on the sphere $\mathbb{S}_{\sqrt{\lambda}}^{n-1}$. Identifying $\varphi \in C^\infty(\mathbb{R}^n)$ with the section given by $\varphi(\lambda) = \varphi|_{\mathbb{S}_{\sqrt{\lambda}}^{n-1}}$ and defining $\nabla_X = T^{-1}XT$ on $C_c^\infty(\mathbb{R}^n)$, we obtain a smooth structure on the latter field of Hilbert spaces, and by construction T is a full smooth trivialization. We will give an explicit expression of ∇_X in Proposition 3.1 applying a direct computation, and we will also obtain another expression in equation (3.2) using a more geometrical argument.

Recall that canonical reduction theory [3, 17] implies that if A is a self-adjoint operator A strongly commuting with Q^2 , then there is a field of self-adjoint operators $\{A(\lambda)\}$ such that $TA\varphi(\lambda) = A(\lambda)T\varphi(\lambda)$. The self-adjoint operators strongly commuting with Q^2 are sometimes called quantum constants of motion of Q^2 . Abusing the notation, we will also call constants of motion of Q^2 to the operators satisfying the condition ensuring decomposability according to our reduction theorem (without assuming self-adjointness, see Theorem 2.10).

One of the most important manners to construct operators on $L^2(\mathbb{R}^n)$ is the so called canonical quantization \mathfrak{Op} (also called Weyl calculus or Weyl quantization, see [7, 26]). \mathfrak{Op} is meant to map classical observables (i.e. smooth functions on \mathbb{R}^{2n}) into quantum observables (i.e. self-adjoint operators on $L^2(\mathbb{R}^n)$) in a physically meaningful way, but in general we only know that $\mathfrak{Op}(u) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$, for any $u \in S'(\mathbb{R}^{2n})$. A classical constant of motion of $h(q, p) = \|q\|^2$ is a smooth function $u \in C^\infty(\mathbb{R}^{2n})$ such that $\{h, u\} = 0$, where $\{\cdot, \cdot\}$ is the canonical

Poisson bracket on \mathbb{R}^{2n} . Notice that $\mathfrak{Op}(h) = Q^2$. Under mild conditions, we will prove that Weyl quantization maps classical constants of motion of $h(q, p) = \|q\|^2$ into quantum constants of motion of Q^2 (see Theorem 3.6). The latter result is interesting on its own right and it does not depend on the smooth structure.

We will show that, if u is a classical constant of motion and $\mathfrak{Op}(u) \in \mathfrak{A}^1$ then, in certain sense, the derivative $\widehat{\nabla}_X(\mathfrak{Op}(u))$ is also of the form $\mathfrak{Op}(\widehat{u})$ and the function \widehat{u} is obtained in the following geometrical way. Let $\phi \in C^\infty(\mathbb{R}^n)$ given by $\phi(q) = \|q\|^2$. Clearly, the map $d\phi : T(\mathbb{R}^n \setminus \{0\}) \rightarrow T(0, \infty)$ is onto and at any point $q \in \mathbb{R}^n \setminus \{0\}$ its kernel is $T_q \mathbb{S}_{\|q\|^2}^{n-1}$. Therefore, for each $X \in \text{Vect}(0, \infty)$ there is a unique vector field \widetilde{X} on $\mathbb{R}^n \setminus \{0\}$ such that $\widetilde{X}(q) \in T_q^\perp \mathbb{S}_{\|q\|^2}^{n-1}$ and $d\phi(\widetilde{X}) = X$. Also, let \widehat{X} be the Hamiltonian lift of \widetilde{X} , i.e. \widehat{X} is the Hamiltonian vector field on \mathbb{R}^{2n} corresponding to the smooth function $h_{\widehat{X}}$ given by $h_{\widehat{X}}(q, p) = \langle \widetilde{X}(q), p \rangle$. One of our main results is the following derivation formula

$$\widehat{\nabla}_{X_0}(\mathfrak{Op}(u)) = \mathfrak{Op}(\widehat{X}_0(u)),$$

where X_0 is the vector field on $(0, \infty)$ given by $X_0(\lambda) = \lambda \frac{\partial}{\partial \lambda}$ (see Theorem 3.8). It is straightforward to show that $\widehat{X}_0(u)$ is also a classical constant of motion. The main reason why the previous formula holds at least for X_0 is that the flow of \widehat{X}_0 is linear ($\widehat{r}_t^0(q, p) = (e^t q, e^{-t} p)$). In fact, the proof of the previous formula essentially follows from the relation between Weyl quantization and the metaplectic representation, and Theorem 2.16 (the metaplectic representation is also key in the proof of Theorem 3.6). Since $\widehat{\nabla}_{aX_0} = a \widehat{\nabla}_{X_0}$ and X_0 is non-degenerate, we can explicitly compute $\widehat{\nabla}_X(\mathfrak{Op}(u))$ for any vector field X . Moreover, the right-hand side of the latter formula contains an interesting geometrical object. Indeed, the map $u \rightarrow \widehat{X}(u)$ defines an abelian Poisson connection on the Poisson algebra of classical constants of motion (see the discussion at the end of Subsection 3.1). In other words, our formula asserts that Weyl quantization exchanges a classical connection with a quantum connection at least for $X = X_0$.

We will obtain analogous results if we replace T by the spectral diagonalization $T \circ \mathcal{F}$ of the Laplacian $-\Delta$, where \mathcal{F} is the Fourier transform (see Corollary 3.12).

In Subsection 3.2, we will use the results in Subsection 3.1 to provide important and explicit examples of horizontal fields of operators (recall that $A \in \mathfrak{A}^1$ is horizontal if $\widehat{\nabla}_X(A) = 0$, for every $X \in \text{Vect}(0, \infty)$). Let $l_{i,j}$ be the classical angular momenta coordinates, i.e. $l_{i,j}(q, p) = q_i p_j - q_j p_i$ with $1 \leq i < j \leq n$. Also let $J : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{(n-1)/2}$ be the map given by $J(q, p) = (l_{1,2}(q, p), \dots, l_{n-1,n}(q, p))$. We will prove that $a \circ J$ is a classical constant motion of $h(q, p) = \|q\|^2$ and of $h_X(q, p) = \langle \widehat{X}(q), p \rangle$ as well, for all $X \in \text{Vect}(0, \infty)$. It is not difficult to show directly the latter result, but we prefer to show it interpreting J as a moment map and applying a more general result (see Proposition 3.14). In particular, Corollary 3.10 implies that $\mathfrak{Op}(a \circ J)$ is a horizontal field of operators under suitable

conditions (see also Corollary 3.15). It is well known that the quantum angular momenta coordinates $L_{i,j} = Q_i \frac{\partial}{\partial q_j} - Q_j \frac{\partial}{\partial q_i}$ and the total angular momentum operator $L^2 = \sum L_{i,j}^2$ are horizontal fields of operators, hence our result is a wide generalization of this fact.

In Section 4, we will study locally uniformly bounded fields of operators and we will introduce a notion of smoothness for fields of C^* -algebras. We denote by \mathfrak{A}_c^n the space of all maps $A : \Gamma^\infty \rightarrow \Gamma^n(\Lambda)$ which are continuous with respect to the Fréchet topology of $\Gamma^n(\Lambda)$. In order to characterize \mathfrak{A}_c^0 , we recall that $\Gamma_0^0(\Lambda)$ is a Hilbert $C_0(\Lambda)$ -module and we consider the corresponding space of adjointable operators \mathfrak{A}_0^0 (see Proposition 4.2). In Proposition 4.5, we describe how $\widehat{\nabla}$ relates consecutive spaces \mathfrak{A}_c^n and \mathfrak{A}_c^{n-1} . That result will lead us to introduce the spaces $\mathfrak{A}_{\text{lb}}^n$ (Definition 4.6) in such a way that $\widehat{\nabla}$ defines a sort of bounded connection on it (see Corollary 4.8). If (T, V) is a full projective trivialization, then $\mathfrak{A}_{\text{lb}}^n$ is isomorphic with $C_{\text{lb}}^n(\Lambda, B(V)_{*-st})$ (Proposition 4.9).

Inspired by the results of Sections 2 and 4, in Subsection 4.1 we define the notion of a smooth field of C^* -algebras (see Definition 4.12), but first we shall recall some facts of continuous fields of C^* -algebras. Reinterpreting some known results found in the literature, we will show in Proposition 4.11 that the space of compact operators $\mathbb{K}(\Gamma_0)$ on a Hilbert $C_0(\Lambda)$ -module Γ_0 , corresponding to a continuous field of Hilbert spaces $p : H \rightarrow \Lambda$, defines a continuous structure on the field of C^* -algebras $A(\lambda) = \mathbb{K}(\mathcal{H}(\lambda))$, and $\mathbb{K}(\Gamma_0)$ coincides with the corresponding spaces of continuous sections. That result will allow us to show that there exists a smooth structure on the field of compact operators associated to a smooth field of Hilbert spaces (Corollary 4.13).

Finally in Appendix A we summarize some well-known facts concerning measurable and continuous fields of Hilbert spaces, emphasizing that both notions have three equivalent ways to be introduced.

2. SMOOTH FIELDS OF OPERATORS

Throughout this article, we will denote by $H \rightarrow \Lambda$ a field of Hilbert spaces omitting the map p .

For the sake of self-containedness, let us recall the definition of the space $\Gamma^n(\Lambda)$ given in Subsection 3.1 in [12], $n \in \mathbb{N} \cup \{\infty\}$. The space $\Gamma^0(\Lambda)$ is the $C(\Lambda)$ -module of those sections of H that are locally uniform limits of a sequence in Γ^∞ . The space $\Gamma^1(\Lambda)$ is the $C^1(\Lambda)$ -module of those $\varphi \in \Gamma^0(\Lambda)$ for which there is a sequence $\varphi_j \in \Gamma^\infty$ such that $\varphi_j \rightarrow \varphi$ locally uniformly, and for every $X \in \text{Vec}(\Lambda)$, the sequence $\nabla_X \varphi_j$ converges locally uniformly. For such φ , we can define $\nabla_X \varphi = \lim \nabla_X \varphi_j$ (Lemma 3.1.2 in [12]). The space $\Gamma^n(\Lambda)$ is defined inductively: $\varphi \in \Gamma^n(\Lambda)$ if φ and $\nabla_X \varphi$ belongs to $\Gamma^{n-1}(\Lambda)$, for all $X \in \text{Vect}(\Lambda)$. Finally, $\Gamma^\infty(\Lambda) = \bigcap \Gamma^n(\Lambda)$. The spaces $\Gamma^n(\Lambda)$ and $\Gamma^\infty(\Lambda)$ are Fréchet spaces with

the seminorms defined by

$$(2.1) \quad \|\varphi\|_{C, X_1, \dots, X_m} = \sup\{\|\nabla_{X_1} \cdots \nabla_{X_m} \varphi(\lambda)\| : \lambda \in C\},$$

where $C \subseteq \Lambda$ is compact, $X_1, \dots, X_m \in \text{Vect}(\Lambda)$ and $m \leq n$ (we can take $X_1, \dots, X_m \in \mathcal{E}$, where $\mathcal{E} \subset \text{Vect}(\Lambda)$ is finite and generates the tangent plane at each $\lambda \in \Lambda$).

The space $\Gamma^0(\Lambda)$ is by construction the space of continuous sections of $H \rightarrow \Lambda$ regarded as a continuous Hilbert bundle (recall that every smooth field of Hilbert spaces is continuous). We denote by $\Gamma_0^0(\Lambda)$ the space of continuous sections vanishing at infinity. As mentioned in the introduction, the map $h : \Gamma_0^0(\Lambda) \times \Gamma_0^0(\Lambda) \rightarrow C_0(\Lambda)$ defines a Hilbert $C_0(\Lambda)$ -module structure on $\Gamma_0^0(\Lambda)$, this fact will become important in Section 4.

As we mentioned in the introduction, we will show that the commutator $\widehat{\nabla}_X(A) = [\nabla_X, A]$ defines a connection on a suitable space of fields of operators. The main purpose of this article is to study $\widehat{\nabla}$. We now introduce the basic sets of fields of operators that we will consider.

DEFINITION 2.1. Let $H \rightarrow \Lambda$ a smooth field of Hilbert spaces with connection $\nabla : \text{Vect}(\Lambda) \times \Gamma^\infty \rightarrow \Gamma^\infty$. We denote by \mathfrak{A}^n the space formed by the fields of operators $A = \{A(\lambda)\}$ such that:

- (i) the domain of $A(\lambda)$ and $A^*(\lambda)$ contains $\mathcal{H}^\infty(\lambda) = \{\varphi(\lambda) : \varphi \in \Gamma^\infty\}$;
- (ii) $A(\Gamma^\infty) \subseteq \Gamma^n$ and $A^*(\Gamma^\infty) \subseteq \Gamma^n$.

We say that $A = \{A(\lambda)\}$ is smooth if $A \in \mathfrak{A}^n$, for every $n \in \mathbb{N}$. We denote by \mathfrak{A}^∞ the space formed by the smooth fields of operators. We say that $A \in \mathfrak{A}^1$ is a horizontal field of operators if $\widehat{\nabla}_X(A) = 0$, for every $X \in \text{Vect}(\Lambda)$.

In particular, if $A \in \mathfrak{A}^n$ then each $A(\lambda)$ is closable. Moreover, it is clear that $\widehat{\nabla}_X(A)(\Gamma^\infty) \subseteq \Gamma^{n-1}$, but it is not so obvious that $\widehat{\nabla}_X(A)$ is given by a field of operators. The following result will be useful to show that claim.

LEMMA 2.2. Let $H \rightarrow \Lambda$ a smooth field of Hilbert spaces with connection ∇ . The following identification of quotient spaces holds:

- (i) $\Gamma^\infty / K^\infty(\lambda) \cong \mathcal{H}^\infty(\lambda)$, where $K^\infty(\lambda) := \{\varphi \in \Gamma^\infty : \varphi(\lambda) = 0\}$;
- (ii) $\Gamma^n(\Lambda) / K^n(\lambda) \cong \mathcal{H}^n(\lambda)$, where $K^n(\lambda) = \{\varphi \in \Gamma^n(\Lambda) : \varphi(\lambda) = 0\}$ and $\mathcal{H}^n(\lambda) := \{\varphi(\lambda) : \varphi \in \Gamma^n(\Lambda)\}$;
- (iii) $\Gamma_0^0(\Lambda) / K_0(\lambda) \cong \mathcal{H}(\lambda)$, where $K_0(\lambda) := \{\varphi \in \Gamma_0^0(\Lambda) : \varphi(\lambda) = 0\}$ and $\Gamma_0^0(\Lambda) / K_0(\lambda)$ is endowed with the canonical quotient norm.

Let $A : \Gamma^\infty \rightarrow \Gamma^n(\Lambda)$ be a linear operator. There is a field of operators $\{A(\lambda)\}$ such that the domain of $A(\lambda)$ contains $\mathcal{H}^\infty(\lambda)$ and $A\varphi(\lambda) = A(\lambda)\varphi(\lambda)$, for every $\lambda \in \Lambda$ and $\varphi \in \Gamma^\infty$, if and only if $A(K^\infty(\lambda)) \subseteq K^n(\lambda)$, for every $\lambda \in \Lambda$.

Proof. For each $\lambda \in \Lambda$, define $\rho_\lambda(\varphi) = \varphi(\lambda)$, for every section φ of $H \rightarrow \Lambda$. The restriction of ρ_λ defines an epimorphism from Γ^∞ onto $\mathcal{H}^\infty(\lambda)$ and its kernel is $K^\infty(\lambda)$, and this implies (i). The same argument shows (ii). Let $\varphi \in \Gamma_0^0(\Lambda)$ and

$\lambda \in \Lambda$. Clearly,

$$\|\varphi(\lambda)\| \leq \inf \left\{ \sup_{\mu \in \Lambda} \|(\varphi - \psi)(\mu)\| : \psi \in K_0(\lambda) \right\}.$$

In order to show that we actually have an equality, it is enough to prove that, for each $\varepsilon > 0$, there is $\psi \in K_0(\lambda)$ such that $\sup_{\mu \in \Lambda} \|(\varphi - \psi)(\mu)\| < \|\varphi(\lambda)\| + \varepsilon$. Let C be the compact set given by

$$C = \{\mu \in \Lambda : \|\varphi(\mu)\| \geq \|\varphi(\lambda)\| + \varepsilon\}.$$

If $f \in C_0(\Lambda)$ is such that $0 \leq f \leq 1$, $f(\lambda) = 0$ and $f|_C = 1$, then $\psi = f\varphi$ satisfies the required inequality. In particular, the subspace $\mathcal{H}_0(\lambda) = \{\varphi(\lambda) : \varphi \in \Gamma_0^0(\Lambda)\}$ is closed in $\mathcal{H}(\lambda)$. Since $\mathcal{H}^\infty(\lambda) \subset \mathcal{H}_0(\lambda)$, condition (iii) in Definition 1.1 implies (iii). The last claim of our lemma is a direct consequence of (i) and (ii). ■

THEOREM 2.3. *Let $H \rightarrow \Lambda$ a smooth field of Hilbert spaces with connection ∇ . For each $n \geq 1$, the map $\widehat{\nabla} : \text{Vect}(\Lambda) \times \mathfrak{A}^n \rightarrow \mathfrak{A}^{n-1}$ given by*

$$\widehat{\nabla}_X(A) = [\nabla_X, A]$$

is well-defined and satisfies the following properties for all $X, Y \in \text{Vect}(\Lambda)$, $a \in C^\infty(\Lambda)$:

- (i) $\widehat{\nabla}_{X+Y}(A) = \widehat{\nabla}_X(A) + \widehat{\nabla}_Y(A)$, $\widehat{\nabla}_{aX}(A) = a\widehat{\nabla}_X(A)$;
- (ii) $\widehat{\nabla}_X(aA) = X(a)A + a\widehat{\nabla}_X(A)$;
- (iii) $h(\widehat{\nabla}_X(A)\varphi, \psi) = h(\varphi, \widehat{\nabla}_{\overline{X}}(A^*)\psi)$, for every $\varphi, \psi \in \Gamma^\infty$;
- (iv) $\widehat{\nabla}_{\overline{X}}(A^*)(\lambda) \subseteq [\widehat{\nabla}_X(A)(\lambda)]^*$, for each $\lambda \in \Lambda$.

Proof. We will show first properties (i), (ii) and (iii) and later that $\widehat{\nabla}_X(A)$ is given by a field of operators satisfying condition (i) in the Definition 2.1. The first equality in (i) is clear. Since A is given by a field of operators, we have that $aA = Aa$ and

$$\widehat{\nabla}_{aX}(A) = [\nabla_{aX}, A] = [a\nabla_X, A] = a\widehat{\nabla}_X(A).$$

A direct computation implies (ii). For (iii), a repeated application of (ii) in Definition 1.1. gives

$$\begin{aligned} Xh(A\varphi, \psi) &= h(\nabla_X A\varphi, \psi) + h(A\varphi, \nabla_{\overline{X}}\psi), \\ Xh(\varphi, A^*\psi) &= h(\nabla_X \varphi, A^*\psi) + h(\varphi, \nabla_{\overline{X}} A^*\psi), \end{aligned}$$

for any two sections $\varphi, \psi \in \Gamma^\infty$. Therefore,

$$\begin{aligned} h(A\nabla_X \varphi, \psi) + h(\varphi, \nabla_{\overline{X}} A^*\psi) &= h(\nabla_X A\varphi, \psi) + h(\varphi, A^*\nabla_{\overline{X}}\psi) \\ h(\widehat{\nabla}_X(A)\varphi, \psi) &= h(\varphi, \widehat{\nabla}_{\overline{X}}(A^*)\psi). \end{aligned}$$

The last identity implies that $\widehat{\nabla}_X(A)(K^\infty(\lambda)) \subseteq K^n(\lambda)$, and therefore Lemma 2.2 shows that $\widehat{\nabla}_X$ is well-defined.

Clearly, identity (iii) also implies (iv). ■

In order to obtain further properties of fields of operators belonging to \mathfrak{A}^n , we should impose additional conditions. For instance, in Section 4 we will consider locally uniformly bounded fields of operators belonging to \mathfrak{A}^n . In Subsection 2.1 we analyze a field of operators belonging to \mathfrak{A}^n as a single operator acting on a direct integral.

Let us assume that our smooth field of Hilbert spaces admits a projective trivialization $T : H \rightarrow V$. In such case $\widehat{A}(\lambda) = T_\lambda A(\lambda) T_\lambda^*$ is a field of operators on V . We would like to fix a common domain for those operators. A first candidate might be the space formed by all the vectors $f \in V$ such that $T^*f \in \Gamma^\infty$. However, that space may be trivial (for instance see the example in Section 3). Instead, we will consider the space where such condition holds locally.

DEFINITION 2.4. Let $T : H \rightarrow V$ be a projective trivialization of the smooth field of Hilbert spaces $H \rightarrow \Lambda$. Also, let $f \in V$. We say that $f \in V^\infty$ if for every $\lambda_0 \in \Lambda$, there are $U \subseteq \Lambda$ open and $\varphi \in \Gamma^\infty$ such that $\lambda_0 \in U$ and $T_\lambda^*f = \varphi(\lambda)$, for every $\lambda \in U$.

For instance, in the example of Section 3, we have that $V = L^2(\mathbb{S}^{n-1})$ and $V^\infty = C^\infty(\mathbb{S}^{n-1})$.

By definition, if $A \in \mathfrak{A}^n$ and $f \in V^\infty$, then $A(T^*f)$ is well-defined and belongs $\Gamma^n(\Lambda)$; therefore $\widehat{A}f \in C^n(\Lambda, V)$. The same happens with A^* . In other words, if $A \in \mathfrak{A}^n$ then $\widehat{A} \in C^n(\Lambda, L(V^\infty, V)_{*-s})$, where $L(V^\infty, V)_{*-s}$ is the space of linear operators from V^∞ to V with the $*$ -strong topology.

THEOREM 2.5. Let $T : H \rightarrow V$ be a projective trivialization of the smooth field of Hilbert spaces $H \rightarrow \Lambda$ with connection ∇ . The map $\widehat{T} : \mathfrak{A}^n \rightarrow C^n(\Lambda, L(V^\infty, V)_{*-s})$, given by

$$\widehat{T}A(\lambda) = \widehat{A}(\lambda) = T_\lambda A(\lambda) T_\lambda^*,$$

defines a trivialization of \mathfrak{A}^n with respect to the connection $\widehat{\nabla}$, i.e. the following identity holds:

$$\widehat{T}(\widehat{\nabla}_X A) = X\widehat{T}A.$$

Proof. Let $A \in \mathfrak{A}^n$, $X \in \text{Vect}(\Lambda)$ and $f \in V^\infty$. Since $\widehat{A}f \in C^n(\Lambda, V)$ and $Xf = 0$, we have that

$$\begin{aligned} \widehat{T}(\widehat{\nabla}_X A)(\lambda)f &= (T\nabla_X A T^*)(\lambda)f - (T A \nabla_X T^*)(\lambda)f \\ &= (X T A T^* + \alpha(X) T A T^*)(\lambda)f - (T A T^*)(T \nabla_X T^*)(\lambda)f \\ &= X \widehat{T}A(\lambda)f + \alpha(X) \widehat{T}A(\lambda)f - \widehat{T}A(\lambda)(Xf + \alpha(X)(\lambda)f) \\ &= X \widehat{T}A(\lambda)f + \alpha(X) \widehat{T}A(\lambda)f - \widehat{T}A \alpha(X)(\lambda)f = X \widehat{T}A(\lambda)f. \quad \blacksquare \end{aligned}$$

REMARK 2.6. Notice that if T is a trivialization and V^∞ is dense in V then T is full. Indeed, in such case every constant section $f \in V$ belongs to the closure of $T(\Gamma^\infty)$, and that implies that $T(\Gamma^\infty)$ is dense in $C^\infty(\Lambda, V)$.

Let us recall one of our motivation to consider the projectively trivializable case. The map $U_{\lambda_1, \lambda_2} = T_{\lambda_2}^* T_{\lambda_1} : \mathcal{H}(\lambda_1) \rightarrow \mathcal{H}(\lambda_2)$ is unitary and plays the role of the parallel transport of the connection ∇ . Notice that $U_{\lambda_2, \lambda_3} U_{\lambda_1, \lambda_2} = U_{\lambda_1, \lambda_3}$. If $A = \{A(\lambda)\}$ is a field of operators, we would like to compare $A(\lambda)$ and $U_{\lambda, \lambda_0}^* A(\lambda_0) U_{\lambda, \lambda_0}$, for any $\lambda_0, \lambda \in \Lambda$ (both operators have domains in $\mathcal{H}(\lambda)$). Assume that $A \in \mathfrak{A}^1$. Thus, the map $\lambda \rightarrow \langle \widehat{A}(\lambda) f, g \rangle_V$ belongs to $C^1(\Lambda)$, for every $f, g \in V^\infty$. Let $X \in \text{Vect}(\Lambda)$ and r_t its one parameter flow (integral curve). Therefore, if $\lambda = r_t(\lambda_0)$

$$\begin{aligned} \langle [A(\lambda) - U_{\lambda, \lambda_0}^* A(\lambda_0) U_{\lambda, \lambda_0}] T_\lambda^* f, T_\lambda^* g \rangle_\lambda &= \langle [\widehat{A}(\lambda) - \widehat{A}(\lambda_0)] f, g \rangle_V \\ &= \int_0^t \langle X \widehat{A}(r_s \lambda_0) f, g \rangle_V ds. \end{aligned}$$

The following result is a direct consequence of the previous identity.

PROPOSITION 2.7. *Let $T : H \rightarrow V$ be a projective trivialization of the smooth field of Hilbert spaces $H \rightarrow \Lambda$ with connection ∇ . Also, let $X \in \text{Vect}(\Lambda)$ and r_t its one parameter flow. Fix $\lambda_0 \in \Lambda$ and $\lambda_1 = r_t(\lambda_0)$, for some $t > 0$. If $A \in \mathfrak{A}^1$ and $v, w \in T_{\lambda_1}^*(V^\infty)$, then*

$$|\langle [A(\lambda_1) - U_{\lambda_1, \lambda_0}^* A(\lambda_0) U_{\lambda_1, \lambda_0}] v, w \rangle_{\lambda_1}| \leq t \sup_{\lambda \in \gamma[0, t]} |\langle \widehat{\nabla}_X A(\lambda)(U_{\lambda_1, \lambda} v), U_{\lambda_1, \lambda} w \rangle_\lambda|,$$

where $\gamma[0, t] = \{r_s(\lambda_0) : s \in [0, t]\}$ is the integral curve between λ_0 and λ_1 .

Theorem 2.5 implies that, if A is a horizontal section, then \widehat{A} is a constant field of operators on V^∞ , or equivalently $A(\lambda) = U_{\lambda, \lambda_0}^* A(\lambda_0) U_{\lambda, \lambda_0}$ on $T_\lambda^* V^\infty$. For the converse, we need to insure that V^∞ is large enough.

PROPOSITION 2.8. *Let $A \in \mathfrak{A}^1$. If V^∞ is dense in V and \widehat{A} is constant on V^∞ , then A is a horizontal section.*

Proof. Let us show first that $\widehat{A}X(T\varphi) = X(\widehat{A}T\varphi)$, for every $\varphi \in \Gamma^\infty$. Notice that, since $X(T\varphi) \in T\Gamma^\infty$ and $\widehat{A}T\varphi \in C^1(\Lambda, V)$, both sides of the previous identities are well-defined. Moreover, for each $f \in V^\infty$, we have that

$$\langle \widehat{A}X(T\varphi), f \rangle = \langle X(T\varphi), \widehat{A}^* f \rangle = X \langle (T\varphi), \widehat{A}^* f \rangle = \langle X(\widehat{A}T\varphi), f \rangle.$$

Finally, we have that

$$T(A\nabla_X \varphi) = \widehat{A}T\nabla_X \varphi = \widehat{A}(X + \alpha(X))T\varphi = (X + \alpha(X))\widehat{A}T\varphi = T(\nabla_X A\varphi). \quad \blacksquare$$

2.1. DIRECT INTEGRALS AND DECOMPOSABLE OPERATORS. Historically, fields of Hilbert spaces and operators emerged during the development of the reduction theory of von Neumann algebras [3, 15]. One key element of that theory was the notion of direct integral of measurable fields of Hilbert spaces. The canonical reduction theorem characterize bounded operators on the direct integral defined

by bounded measurable fields of operators. Such result was extended for (unbounded) closed operators in [17]. A similar reduction theory was developed in [22] in the continuous framework (but certain unnecessary topological condition was assumed). An interesting approach for the unbounded case was recently described in [16]. We will show an analogous reduction theorem within our smooth framework, and we will use it to construct examples of smooth fields of operators in Section 3.

In order to construct direct integrals, we need to endow $H \rightarrow \Lambda$ with a measurable structure, or equivalently, to consider a measurable Hilbert bundle structure on $H \rightarrow \Lambda$ (see Definition 2.4.8 in [25] and Appendix A). For the purposes of this article, we shall assume the following stronger assumption.

DEFINITION 2.9. Let $H \rightarrow \Lambda$ be a smooth field of Hilbert spaces. A weak smooth trivialization is a couple (V, T) , where V is a Hilbert space and $T : H \rightarrow V$ is a map such that $T|_{\mathcal{H}(\lambda)}$ is unitary for any $\lambda \in \Lambda$ and $T(\Gamma^\infty) \subseteq C^\infty(\Lambda, V)$.

Let η be a Borel measure on Λ (for instance, we can fix a density on Λ or a volume form $d\eta$). Recall that the direct integral $\mathcal{H} = \int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ is the Hilbert space of all measurable sections φ such that $\int_{\Lambda} \|\varphi(\lambda)\|_{\lambda}^2 d\eta(\lambda) < \infty$ (see Appendix A for details).

Clearly, $L^\infty(\Lambda, \eta)$ can be represented faithfully in \mathcal{H} (acting as constant operators on each fiber) and every measurable essentially uniformly bounded field of operators $A = \{A(\lambda)\}$ defines a bounded operator in \mathcal{H} given by $A\varphi(\lambda) := A(\lambda)\varphi(\lambda)$. Moreover, such operator belongs to the commutant of $L^\infty(\Lambda, \eta)$. Conversely, every bounded operator on \mathcal{H} belonging to the commutant of $L^\infty(\Lambda, \eta)$ can be decomposed as a measurable essentially uniformly bounded field of operators (for instance, see [2] or [3]). The extensions of the later result given in [17, 22] do not guarantee that the domain of each $A(\lambda)$ contains $\mathcal{H}^\infty(\lambda)$. We will overcome that issue in our framework. Let us consider the spaces:

$$\Gamma_2^\infty = \Gamma^\infty \cap \mathcal{H} = \left\{ \varphi \in \Gamma^\infty : \int_{\Lambda} \|\varphi(\lambda)\|_{\lambda}^2 d\eta(\lambda) < \infty \right\},$$

$$\Gamma_2^0(\Lambda) = \Gamma^0(\Lambda) \cap \mathcal{H} = \left\{ \varphi \in \Gamma^0(\Lambda) : \int_{\Lambda} \|\varphi(\lambda)\|_{\lambda}^2 d\eta(\lambda) < \infty \right\}.$$

Through the rest of this article we will assume that Γ_2^∞ is dense in \mathcal{H} (for instance, this holds true if T is full). Using a suitable bump function, it is straightforward to show that $\{\varphi(\lambda) : \varphi \in \Gamma_2^\infty\} = \mathcal{H}^\infty(\lambda)$ and $\{\varphi(\lambda) : \varphi \in \Gamma_2^0(\Lambda)\} = \mathcal{H}(\lambda)$. Therefore $\Gamma_2^\infty / K_2^\infty(\lambda) = \mathcal{H}^\infty(\lambda)$ and $\Gamma_2^0(\Lambda) / K_2^0(\lambda) = \mathcal{H}(\lambda)$, where $K_2^\infty(\lambda) = \{\varphi \in \Gamma_2^\infty : \varphi(\lambda) = 0\}$ and $K_2^0(\lambda) = \{\varphi \in \Gamma_2^0(\Lambda) : \varphi(\lambda) = 0\}$.

THEOREM 2.10. Let $H \rightarrow \Lambda$ be a smooth field of Hilbert spaces and $A : \Gamma_2^\infty \rightarrow \Gamma_2^0(\Lambda)$ be a linear operator. Assume that the domain of A^* contains Γ_2^∞ and $A^*(\Gamma_2^\infty) \subseteq$

$\Gamma_2^0(\Lambda)$. There is a field of operators $\{A(\lambda)\}$ such that the domain of $A(\lambda)$ contains $\mathcal{H}^\infty(\lambda)$ and $A\varphi(\lambda) = A(\lambda)\varphi(\lambda)$, for every $\varphi \in \Gamma_2^\infty$ and $\lambda \in \Lambda$, if and only if $fA = Af$ on Γ_2^∞ , for every $f \in C_c^\infty(\Lambda)$. In such case, A belongs to \mathfrak{A}^0 and moreover, A is locally uniformly bounded if and only if fA is bounded in \mathcal{H} , for each $f \in C_c^\infty(\Lambda)$.

Proof. Obviously if $A\varphi(\lambda) = A(\lambda)\varphi(\lambda)$, then clearly the required commutation holds. Let us prove the converse. If $f \in C_c^\infty(\Lambda)$, then

$$\int_{\Lambda} fh(A\varphi, \psi) d\eta = \langle fA\varphi, \psi \rangle = \langle f\varphi, A^*\psi \rangle = \int_{\Lambda} fh(\varphi, A^*\psi) d\eta.$$

Therefore $h(A\varphi, \psi) = h(\varphi, A^*\psi)$ almost everywhere. Since both functions are continuous, the latter identity holds everywhere. Hence, $AK_2^\infty(\lambda) \subseteq K_2(\lambda)$ and we can repeat the argument of Lemma 2.2 to show that there is a field of operators $\{A(\lambda)\}$ such that the domain of $A(\lambda)$ contains $\mathcal{H}^\infty(\lambda)$ and $A\varphi(\lambda) = A(\lambda)\varphi(\lambda)$, for every $\varphi \in \Gamma_2^\infty$ and $\lambda \in \Lambda$. In particular, A is well-defined on Γ^∞ . Moreover, if $\varphi \in \Gamma^\infty$, then φ locally coincides with sections belonging Γ_2^∞ , and therefore $A\varphi$ locally coincides with sections belonging to $\Gamma_2^0(\Lambda)$. The latter implies that $A(\Gamma^\infty) \subseteq \Gamma^0(\Lambda)$. The last claim is straightforward. ■

REMARK 2.11. The previous result depends only on the continuous structure of the field of Hilbert spaces. In other words, the same proof shows an analogous result if $H \rightarrow \Lambda$ is a continuous field of Hilbert spaces and Γ_2^∞ is replaced by any dense domain in \mathcal{H} invariant by multiplications of functions belonging to $C_c(\Lambda)$.

Reduction theory implies the existence of a diagonalization for every self-adjoint operator. More precisely, if H_0 is a self-adjoint operator on a Hilbert space \mathcal{H} , then there is a measurable field of Hilbert spaces $\{H(\lambda)\}$ over the spectrum $\sigma(H_0)$ of H_0 and a unitary operator $T : \mathcal{H} \rightarrow \int_{\sigma(H_0)}^\oplus \mathcal{H}(\lambda) d\eta(\lambda)$ such that

$Tf(H_0)\varphi(\lambda) = f(\lambda)T\varphi(\lambda)$ for every measurable function f on $\sigma(H_0)$, where $f(H_0)$ denotes the operator defined by the functional calculus and η is the so called scalar spectral measure of H_0 . Moreover, the self-adjoint operators A strongly commuting with H_0 (quantum constants of motion) admit a decomposition through T . The examples of smooth fields of Hilbert spaces, direct integral and smooth fields of operators that we will consider in Section 3 come from the diagonalization of certain self-adjoint operator.

COROLLARY 2.12. Assume $\Lambda \subset \mathbb{R}$ is open and let $H_0 = \int_{\Lambda}^\oplus \lambda d\eta(\lambda)$. If $A : \Gamma_2^\infty \rightarrow \Gamma_2^0$ is essentially self-adjoint and its closure \overline{A} strongly commutes with H_0 , then $A \in \mathfrak{A}_2^0$.

Let $A = \{A(\lambda)\}$ be a field of operators such that the domain of each $A(\lambda)$ contains $\mathcal{H}^\infty(\lambda)$, $A(\Gamma_2^\infty) \subseteq \Gamma^1(\Lambda) \cap \mathcal{H}$ and the same properties hold for A^* . Since

every section in $\Gamma^1(\Lambda)$ locally coincides with sections belonging $\Gamma^1(\Lambda) \cap \mathcal{H}$, it follows that A belongs to \mathfrak{A}^1 . Therefore $\widehat{\nabla}_X(A)(\Gamma^\infty) \subseteq \Gamma^0(\Lambda)$, for each $X \in \text{Vect}(\Lambda)$. In order to show that the restriction of $\widehat{\nabla}_X(A)$ to Γ_2^∞ defines an operator on \mathcal{H} (i.e. $\widehat{\nabla}_X(A)(\Gamma_2^\infty) \subseteq \Gamma^0(\Lambda) \cap \mathcal{H}$), we need to make stronger assumptions.

DEFINITION 2.13. Assume that Γ_2^∞ is invariant by ∇ . We denote by $\Gamma_2^n(\Lambda)$ the space formed by all the sections $\varphi \in \Gamma^n(\Lambda)$ such that $\nabla_{X_1} \cdots \nabla_{X_k} \varphi \in \mathcal{H}$, for every $0 \leq k \leq n$ and $X_1, \dots, X_k \in \text{Vect}(\Lambda)$. We denote by \mathfrak{A}_2^n the space formed by all the fields of operators $A = \{A(\lambda)\}$ such that:

- (i) the domain of $A(\lambda)$ and $A^*(\lambda)$ contains $\mathcal{H}^\infty(\lambda) = \{\varphi(\lambda) : \varphi \in \Gamma^\infty\}$;
- (ii) $A(\Gamma_2^\infty) \subseteq \Gamma_2^n(\Lambda)$ and $A^*(\Gamma_2^\infty) \subseteq \Gamma_2^n(\Lambda)$.

REMARK 2.14. By definition we have the following properties:

- (i) $\Gamma_2^n(\Lambda) \subseteq \Gamma^n(\Lambda)$, $\Gamma_2^n(\Lambda) \subseteq \Gamma_2^{n-1}(\Lambda)$ and $\nabla_X(\Gamma_2^n(\Lambda)) \subseteq \Gamma_2^{n-1}(\Lambda)$, for all $X \in \text{Vect}(\Lambda)$;
- (ii) $\mathfrak{A}_2^n \subseteq \mathfrak{A}^n$, $\mathfrak{A}_2^n \subseteq \mathfrak{A}_2^{n-1}$ and $\widehat{\nabla}_X(\mathfrak{A}_2^n) \subseteq \mathfrak{A}_2^{n-1}$, for all $X \in \text{Vect}(\Lambda)$; compare the latter inclusions with Corollary 4.8.

Let us return to the trivialisable case and assume $d\eta$ is a volume form. Let X be a complete vector field on Λ and r_t its flow (i.e. $r_t(\lambda)$ is the integral curve of X passing through λ at $t = 0$). Also, let J_t be the Jacobian of r_t with respect to $d\eta$. Therefore, the map $W_t : L^2(\Lambda, V) \rightarrow L^2(\Lambda, V)$ given by

$$(2.2) \quad W_t f(\lambda) = \sqrt{|J_t|} f(r_t \lambda),$$

for $f \in L^2(\Lambda, V)$, is a unitary strongly continuous one parameter group. It is easy to show that the infinitesimal generator of W_t is

$$H_X = -i \left(X + \frac{1}{2} \text{div} X \right)$$

acting on $L^2(\Lambda, V)$. The following result is a consequence of the latter fact.

PROPOSITION 2.15. *Let $d\eta$ be a volume form on Λ and $H \rightarrow \Lambda$ be a trivialisable smooth field of Hilbert spaces. If X is a complete vector field on Λ , then the operator $-i \left(\nabla_X + \frac{1}{2} \text{div}(X) \right)$ is essentially self-adjoint on $\Gamma_2^\infty \subseteq \int_{\Lambda} \mathcal{H}(\lambda) d\eta(\lambda)$, where the divergence is computed with respect to $d\eta$.*

The domain of the self-adjoint extension of $-i \left(\nabla_X + \frac{1}{2} \text{div}(X) \right)$ is provided by Stone's theorem, more precisely, it is the space of sections $\varphi \in \int_{\Lambda} \mathcal{H}(\lambda) d\eta(\lambda)$ such that $T^* W_t T \varphi$ is strongly smooth (with respect to t).

Let $r_t^* : C^n(\Lambda, V) \rightarrow C^n(\Lambda, V)$ given by $[r_t^* f](\lambda) = f(r_t \lambda)$, for each $n \in \mathbb{N} \cup \{\infty\}$. Define $R_t = T^* r_t^* T$. If $A \in \mathfrak{A}^1$, then

$$\langle R_t A R_{-t} \varphi(\lambda), \psi(\lambda) \rangle_\lambda = \langle \widehat{A}(r_t \lambda) T \varphi(\lambda), T \psi(\lambda) \rangle_V.$$

Therefore,

$$\frac{d}{dt} \langle R_t A R_{-t} \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} |_{t=s} = \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle_{\lambda}.$$

The following result asserts that under suitable conditions the previous pointwise smoothness implies weakly smoothness in the direct integral. The latter turn to be the main tool to explicitly compute $\widehat{\nabla}_X(A)$ in the example of Section 3.

THEOREM 2.16. *Let $A \in \mathfrak{A}^1$, $X \in \text{Vect}(\Lambda)$ and r_t the one parameter flow of X . Define $R_t : \Gamma^\infty(\Lambda) \rightarrow \Gamma^\infty(\Lambda)$ by $R_t = T^* r_t^* T$, where $[r_t^* f](\lambda) = f(r_t(\lambda))$. Assume that $R_t(\Gamma^\infty) \subseteq \Gamma^\infty$ and let $\varphi, \psi \in \Gamma^\infty$ with compact support such that $\varphi(\lambda) \in T_\lambda^*(V^\infty)$, for every $\lambda \in \Lambda$. Then the map $t \rightarrow \langle R_t A R_{-t} \varphi, \psi \rangle$ is differentiable and*

$$\frac{d}{dt} \langle R_t A R_{-t} \varphi, \psi \rangle |_{t=s} = \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi, \psi \rangle.$$

Proof. It is enough to show that the map $s \rightarrow \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi, \psi \rangle$ is continuous and

$$\langle (R_t A R_{-t} - A) \varphi, \psi \rangle = \int_0^t \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi, \psi \rangle ds.$$

Since $A \in \mathfrak{A}^1$, the latter identity holds pointwise, i.e.

$$\langle (R_t A R_{-t} - A) \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} = \int_0^t \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} ds.$$

For a fix λ , since r_t is a diffeomorphism and $\varphi(\lambda) \in T_\lambda^*(V^\infty)$, the map

$$s \rightarrow \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle = \langle \widehat{\nabla}_X(A)(r_s(\lambda)) \varphi(\lambda), \psi(\lambda) \rangle$$

is continuous. For a fix s , since $\widehat{\nabla}_X(A) \in \mathfrak{A}^0$, the section $R_s \widehat{\nabla}_X(A) R_{-s} \varphi$ belongs to $\Gamma^0(\Lambda)$, thus the latter map is also continuous with respect to λ (so it is continuous and compactly supported on $[0, t] \times \Lambda$). In particular, the map $s \rightarrow \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi, \psi \rangle = \int_{\Lambda} \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} d\eta(\lambda)$ is also continuous. Moreover, Fubini's theorem implies that:

$$\begin{aligned} \langle (R_t A R_{-t} - A) \varphi, \psi \rangle &= \int_{\Lambda} \langle (R_t A R_{-t} - A) \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} d\eta(\lambda) \\ &= \int_{\Lambda} \int_0^t \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle ds d\eta(\lambda) \\ &= \int_0^t \int_{\Lambda} \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi(\lambda), \psi(\lambda) \rangle d\eta(\lambda) ds \end{aligned}$$

$$= \int_0^t \langle R_s \widehat{\nabla}_X(A) R_{-s} \varphi, \psi \rangle ds. \quad \blacksquare$$

3. AN IMPORTANT EXAMPLE AND CANONICAL QUANTIZATION

Let us consider the following simple but yet fundamental example: let Q^2 be the multiplication operator on $L^2(\mathbb{R}^n)$ corresponding to the function $\phi(q) = \|q\|^2$. As we previously mentioned, every self-adjoint operator admits a diagonalization. The map $T : L^2(\mathbb{R}^n) \rightarrow \int_{(0,\infty)}^{\oplus} L^2(\mathbb{S}^{n-1}) d\lambda$ given by

$$(3.1) \quad Tf(\lambda, z) = 2^{-1/2} \lambda^{(n-2)/4} f(\sqrt{\lambda}z)$$

is a diagonalization of Q^2 (for instance, see Lemma 3.6 in [24] for the more general case $\phi(q) = \alpha(\|q\|)$, and for an arbitrary submersion $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ see Theorem 5.2 in [1]).

Moreover, T can be regarded as a smooth trivialization considering $\mathcal{H}(\lambda) = L^2(\mathbb{S}^{n-1}_{\sqrt{\lambda}}, \mu_\lambda)$, where $\mu_\lambda = 2^{-1/2} \sqrt{\lambda} \eta_\lambda$ and η_λ is the canonical measure on $\mathbb{S}^{n-1}_{\sqrt{\lambda}}$. In particular, the restriction of T to $\mathcal{H}(\lambda)$ defines a unitary operator onto $L^2(\mathbb{S}^{n-1})$. The latter fact allows to identify $f \in C^\infty(\mathbb{R}^n)$ with a section of the field of Hilbert spaces $\{(0, \infty) \ni \lambda \rightarrow L^2(\mathbb{S}^{n-1}_{\sqrt{\lambda}}, \mu_\lambda)\}$ through the restriction $f(\lambda) = f|_{\mathbb{S}^{n-1}_{\sqrt{\lambda}}}$. Under that identification, the action of any $a \in C^\infty(\Lambda)$ on a section φ is given by $a\varphi(q) = a(\phi(q))\varphi(q)$.

We will use T to pullback the trivial smooth structure on the trivial field of Hilbert spaces $\widehat{H} = (0, \infty) \times L^2(\mathbb{S}^{n-1})$ into the field of Hilbert spaces $\{(0, \infty) \ni \lambda \rightarrow L^2(\mathbb{S}^{n-1}_{\sqrt{\lambda}}, \mu_\lambda)\}$, i.e. we will consider $\nabla_X = T^{-1}XT$. Taking derivatives in equation (3.1) we obtain the following result.

PROPOSITION 3.1. *The map $\nabla_X : C_c^\infty(\mathbb{R}^n) \rightarrow C_c^\infty(\mathbb{R}^n)$ given by*

$$\nabla_X(\varphi) = \widetilde{X}(\varphi) + \frac{n-2}{4} \phi^{-1} \widetilde{X}(\phi) \varphi$$

defines a smooth structure on the field of Hilbert spaces $\{(0, \infty) \ni \lambda \rightarrow L^2(\mathbb{S}^{n-1}_{\sqrt{\lambda}}, \mu_\lambda)\}$, where $\widetilde{X} = D\Psi(X)$ and $\Psi : (0, \infty) \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n \setminus \{0\}$ is the diffeomorphism given by $\Psi(\lambda, z) = \sqrt{\lambda}z$. Moreover, the map T defined by equation (3.1) is a trivialization of the latter smooth field.

REMARK 3.2. In the previous proposition one could define the connection over the sections space $C^\infty(\mathbb{R}^n)$, but we prefer to take $\Gamma^\infty = C_c^\infty(\mathbb{R}^n)$, because the fields of operators that we will consider later are defined on $C_c^\infty(\mathbb{R}^n)$. Moreover, we also have that $\Gamma_2^\infty = C_c^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n) \subset \Gamma^\infty(0, \infty)$.

REMARK 3.3. Notice that $d\phi(\tilde{X}) = X$, and \tilde{X} is the only vector field normal to each sphere satisfying that identity. Moreover, if \tilde{r}_t is the flow of \tilde{X} , then $\phi(\tilde{r}_t q) = r_t \phi(q)$.

Let us consider the operator $R_t = T^* r_t^* T$. A direct computation shows that

$$R_t \varphi(q) = a(t, q) \varphi(\tilde{r}_t q),$$

where

$$a(t, q) = \frac{n-2}{4} \phi(q)^{(2-n)/4} (r_t \circ \phi(q))^{(n-2)/4} = \frac{n-2}{4} \phi(q)^{(2-n)/4} (\phi \circ \tilde{r}_t(q))^{(n-2)/4}.$$

Notice that, if W_t is the one-parameter unitary group defined by equation (2.2), then $T^* W_t T \varphi(q) = \sqrt{(|J_t| \circ \phi)} a(t, q) \varphi \circ \tilde{r}_t(q)$. However, there is a unique one-parameter unitary group of that form and it is given by $\tilde{W}_t \varphi = \sqrt{|\tilde{J}_t|} \varphi \circ \tilde{r}_t$, where \tilde{J}_t is the Jacobian of \tilde{r}_t . Therefore,

$$a = \sqrt{|\tilde{J}_t|_{-t} \circ \phi}.$$

Moreover, since $\frac{d}{dt} R_t \varphi|_{t=0} = \nabla_X \varphi$, we obtain the following (geometrical) expression for our connection:

$$(3.2) \quad \nabla_X(\varphi) = \tilde{X}(\varphi) + \frac{1}{2}(\operatorname{div}(\tilde{X}) - \operatorname{div}(X) \circ \phi)\varphi.$$

The last expression defines a connection in a much general framework, but we will address that problem in a forthcoming article (for instance, the latter expression defines a connection if $\phi : M \rightarrow N$ is a submersion and \tilde{X} is the unique vector field normal to each $M_\lambda = \phi^{-1}(\lambda)$ such that $D\phi(\tilde{X}) = X$, where M and N are Riemannian manifolds).

Note that the operator $-i\left(\nabla_X + \frac{1}{2}\operatorname{div}(X) \circ \phi\right)$ considered in Proposition 2.15 coincides with $H_{\tilde{X}} = -i\left(\tilde{X} + \frac{1}{2}\operatorname{div}(\tilde{X})\right)$ and it is the infinitesimal generator of \tilde{W}_t .

In the next subsection we will compute $\hat{\nabla}_X(A)$ for suitable $A \in \mathfrak{A}^1$ and to do that we will need the following lemma.

LEMMA 3.4. *Let X_0 be the vector field given by $X_0(\lambda) = 2\lambda \frac{\partial}{\partial \lambda}$. Then $\tilde{X}_0(q) = \sum q_j \frac{\partial}{\partial q_j}$,*

$$(3.3) \quad R_t^0 \varphi(q) = e^{((n-2)/2)t} \varphi(e^t q)$$

and

$$\tilde{W}_t^0 \varphi(q) = e^{(n/2)t} \varphi(e^t q).$$

In particular, $\varphi \in C^\infty(\mathbb{R}^n)$ is a horizontal section if and only if

$$\varphi(\lambda q) = \lambda^{-(n-2)/2} \varphi(q), \quad \forall q \in \mathbb{R}^n, q \neq 0, \forall \lambda > 0.$$

Proof. Since $d\phi\left(\frac{\nabla\phi}{\|\nabla\phi\|^2}\right) = 1$ and $d\phi \circ D\Psi = I$, it follows that $D\Psi(X_0) = \tilde{X}_0 := \sum q_j \frac{\partial}{\partial q_j}$. Clearly $r_t^0 a(\lambda) = a(e^{2t}\lambda)$, for every $a \in C^\infty(0, \infty)$. The rest of the proof is a straightforward computation. ■

3.1. SMOOTH FIELDS OF OPERATORS COMING FROM CANONICAL QUANTIZATION. In this subsection we will construct fields of operators over the field of Hilbert spaces $\{(0, \infty) \ni \lambda \mapsto L^2(\mathbb{S}_{\sqrt{\lambda}}^{n-1}, \mu_\lambda)\}$ and look for conditions to guarantee that such fields are smooth, or even horizontal.

Recall that the self-adjoint operators on $L^2(\mathbb{R}^n)$ admitting a decomposition through T are the operators that strongly commute with Q^2 (i.e. the quantum constant of motion of Q^2). Such decomposition also holds for bounded (not necessarily self-adjoint) operators strongly commuting with Q^2 . Moreover, Theorem 2.10 also provides conditions to guarantee such decomposition within our smooth framework.

The operators that we shall consider arise from canonical quantization, i.e. they are of the form $\mathfrak{Op}(u)$, where u is a “reasonable function” on \mathbb{R}^{2n} and \mathfrak{Op} is the Weyl calculus (also called Weyl quantization) [7, 26]. Formally, $\mathfrak{Op}(u)$ is defined by

$$[\mathfrak{Op}(u)\varphi](x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u\left(\frac{x+y}{2}, \xi\right) e^{2\pi i(x-y)\xi} \varphi(y) d\xi dy.$$

In a certain sense, we will prove that Weyl calculus maps a classical constant of motion of $\phi(q) = \|q\|^2$ (where ϕ is seen as a function on \mathbb{R}^{2n} independent of the momentum variable) into a quantum constant of motion of Q^2 , so in that way we will obtain an important set of fields of operators where we might apply our connection $\widehat{\nabla}$.

A classical constant of motion of a classical Hamiltonian $h \in C^\infty(\mathbb{R}^{2n})$ is a smooth function $u \in C^\infty(\mathbb{R}^{2n})$ such that $\{h, u\} = 0$, where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on \mathbb{R}^{2n} . It is well-known that u is a classical constant of motion if and only if $u \circ \alpha_t = u$, where α_t is the Hamiltonian flow of h . It is straightforward to check that, if $h(q, p) = \phi(q)$, then $\alpha_t(q, p) = (q, p + t\nabla\phi(q)) = (q, p + 2tq)$. Note that in this case α_t is linear.

Weyl calculus is meant to map real smooth functions on \mathbb{R}^{2n} into self-adjoint operators on $L^2(\mathbb{R}^n)$, but this is not always the case. Indeed, Weyl calculus is a continuous isomorphism $\mathfrak{Op} : S'(\mathbb{R}^{2n}) \rightarrow \mathcal{B}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$, where $S(\mathbb{R}^m)$ is the Schwartz space endowed with its canonical locally convex topology, $S'(\mathbb{R}^m)$ is the topological dual of $S(\mathbb{R}^m)$, i.e the space of tempered distributions and $\mathcal{B}(S(\mathbb{R}^n), S'(\mathbb{R}^n))$ is endowed with the strong operator topology.

One of the main properties of Weyl calculus is the following identity: if $\varphi, \psi \in S(\mathbb{R}^n)$, then

$$\langle \mathfrak{Op}(u)\varphi, \psi \rangle = \langle \varphi, \mathfrak{Op}(\bar{u})\psi \rangle,$$

where the complex conjugation of distributions is defined by $\bar{u}(\varphi) = \overline{u(\bar{\varphi})}$ (Proposition 2.6 in [7]).

The following property of \mathfrak{Dp} is our main tool in this section. Let $\mathrm{Sp}(2n)$ be the symplectic group, i.e. the group of linear symplectomorphisms on \mathbb{R}^{2n} ; there is a map $m : \mathrm{Sp}(2n) \rightarrow \mathcal{U}(L^2(\mathbb{R}^n))$ (called the metaplectic representation) such that for any $S \in \mathrm{Sp}(2n)$ and every $u \in S'(\mathbb{R}^{2n})$, we have that

$$(3.4) \quad \mathfrak{Dp}(u \circ S^*) = m(S)\mathfrak{Dp}(u)m(S)^{-1}.$$

For a detailed presentation of the metaplectic representation see Chapter 4 in [7]. In particular, see Theorem 2.15 for the previous identity.

Note that equation (3.4) makes sense because the pull back by S^* maps $S(\mathbb{R}^{2n})$ into itself and it can be extended to an isomorphism from $S'(\mathbb{R}^{2n})$ into itself.

DEFINITION 3.5. Let $h \in C^\infty(\mathbb{R}^{2n})$ and assume that its Hamiltonian flow α_t is linear and defined for any $t \in \mathbb{R}$. We say that $u \in S'(\mathbb{R}^{2n})$ is a tempered constant of motion if $\alpha_t^* u = u$, for all $t \in \mathbb{R}$.

THEOREM 3.6. Let u be a tempered constant of motion of $h(q, p) = \|q\|^2$.

- (i) $[\mathfrak{Dp}(u), e^{itQ^2}] = 0$ on $S(\mathbb{R}^n)$, for all $t \in \mathbb{R}$.
- (ii) If $\mathfrak{Dp}(u)$ sends $S(\mathbb{R}^n)$ into itself, then $[\mathfrak{Dp}(u), Q^2] = 0$ on $S(\mathbb{R}^n)$.
- (iii) If $\mathfrak{Dp}(u)$ is bounded or essentially self-adjoint on $S(\mathbb{R}^n)$, then it strongly commutes with Q^2 .
- (iv) $\mathfrak{Dp}(u) \in \mathfrak{A}_2^0$ if and only if $\mathfrak{Dp}(u)(C_c^\infty(\mathbb{R}^n)) \subseteq \Gamma_2^0(0, \infty)$ and $\mathfrak{Dp}(\bar{u})(C_c^\infty(\mathbb{R}^n)) \subseteq \Gamma_2^0(0, \infty)$.

Proof. Equation 4.25 in [7] implies that $m(\alpha_t^*) = e^{2itQ^2}$. Therefore, equation (3.4) implies that

$$\mathfrak{Dp}(u) = e^{itQ^2} \mathfrak{Dp}(u) e^{-itQ^2}$$

and this shows (i). When $\mathfrak{Dp}(u)$ sends $S(\mathbb{R}^n)$ into itself, taking strong derivatives in the previous equality we obtain (ii). Clearly, (i) implies the bounded case in (iii). If $\mathfrak{Dp}(u)$ is essentially self-adjoint on $S(\mathbb{R}^n)$, let \mathfrak{D} be the domain of $\overline{\mathfrak{Dp}(u)}$ (the closure of $\mathfrak{Dp}(u)$). We will prove that $e^{itQ^2}\mathfrak{D} \subseteq \mathfrak{D}$ and $e^{itQ^2}\overline{\mathfrak{Dp}(u)} = \overline{\mathfrak{Dp}(u)}e^{itQ^2}$ on \mathfrak{D} . Let $f \in \mathfrak{D}$. Then, there is a sequence $f_n \in S(\mathbb{R}^n)$ such that $f_n \rightarrow f$ and $\mathfrak{Dp}(u)f_n \rightarrow \overline{\mathfrak{Dp}(u)}f$. Thus, $e^{itQ^2}f_n \rightarrow e^{itQ^2}f$ and

$$\mathfrak{Dp}(u)e^{itQ^2}f_n = e^{itQ^2}\mathfrak{Dp}(u)f_n \rightarrow e^{itQ^2}\overline{\mathfrak{Dp}(u)}f.$$

Since $\overline{\mathfrak{Dp}(u)}$ is closed and $g_n := e^{itQ^2}f_n$ is convergent in the graph topology, we have that $e^{itQ^2}f \in \mathfrak{D}$ and

$$\overline{\mathfrak{Dp}(u)}e^{itQ^2}f = e^{itQ^2}\overline{\mathfrak{Dp}(u)}f.$$

This implies that $\overline{\mathfrak{Dp}(u)}$ and Q^2 strongly commute.

Let us show (iv). According to Theorem 2.10, we only need to prove that $a(Q^2)\mathfrak{Dp}(u) = \mathfrak{Dp}(u)a(Q^2)$ on $C_c^\infty(\mathbb{R}^n)$, for every $a \in C_c^\infty(0, \infty)$. Let \hat{a} be the Fourier transform of a . For each $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$, the Fubini's theorem implies that

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{a}(t)\langle e^{itQ^2}\mathfrak{Dp}(u)\varphi, \psi \rangle dt &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \hat{a}(t)e^{it\lambda} dt \right) \langle [\mathfrak{Dp}(u)\varphi](\lambda), \psi(\lambda) \rangle_\lambda d\lambda \\ &= \langle a(Q^2)\mathfrak{Dp}(u)\varphi, \psi \rangle. \end{aligned}$$

Using (i) and repeating the same argument, we obtain that

$$\int_{-\infty}^{\infty} \hat{a}(t)\langle e^{itQ^2}\mathfrak{Dp}(u)\varphi, \psi \rangle dt = \langle a(Q^2)\varphi, \mathfrak{Dp}(\bar{u})\psi \rangle = \langle \mathfrak{Dp}(u)a(Q^2)\varphi, \psi \rangle. \quad \blacksquare$$

REMARK 3.7. It is well-known that if u belongs to a global Hörmander class $S_{\rho,\delta}^m(\mathbb{R}^{2n})$, then $\mathfrak{Dp}(u)(S(\mathbb{R}^n)) \subseteq S(\mathbb{R}^n)$ (for instance, see Theorem 2.21 in [7]). Therefore, if u is a constant of motion of $h(q, p) = \|q\|^2$ belonging to a Hörmander class $S_{\rho,\delta}^m(\mathbb{R}^{2n})$, then $\mathfrak{Dp}(u) \in \mathfrak{A}_2^0 \cap \mathfrak{A}^\infty$. Similarly, if u belongs to a local Hörmander class (see Definition 1.1 in [23]), it is not difficult to prove that $\mathfrak{Dp}(u)(C_c^\infty(\mathbb{R}^n)) \subseteq S(\mathbb{R}^n)$, for instance repeating the proof of Theorem 2.21 in [7]. Therefore, if u is a tempered constant of motion of $h(q, p) = \|q\|^2$ belonging to a local Hörmander class, then $\mathfrak{Dp}(u) \in \mathfrak{A}_2^0 \cap \mathfrak{A}^\infty$.

The main result of this section is a formula to compute $\widehat{\nabla}_X \mathfrak{Dp}(u)$ (see Theorem 3.8 below), but in order to understand it, we need to recall some well-known construction in symplectic geometry. For $\tilde{X} \in \text{Vect}(\mathbb{R}^n)$, we define $h_{\tilde{X}} \in C^\infty(\mathbb{R}^{2n})$ by

$$(3.5) \quad h_{\tilde{X}}(q, p) = \langle \tilde{X}(q), p \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the duality between the tangent and the cotangent plane at q . Equivalently, if $\tilde{X} = \sum_j^n a_j(q) \frac{\partial}{\partial p_j}$, then $h_{\tilde{X}}(q, p) = \sum_j^n a_j(q) p_j$.

We will denote by \widehat{X} the Hamiltonian vector field corresponding to $h_{\tilde{X}}$, i.e. $\widehat{X}(u) = \{h_{\tilde{X}}, u\}$. Recall that the flow of \tilde{X} is denoted by \tilde{r}_t (see Remark 3.3). It is easy to show the flow \widehat{r}_t of \widehat{X} is the canonical lift of \tilde{r}_t , i.e. $\widehat{r}_t(q, p) = (\tilde{r}_t(q), \tilde{r}_{-t}^*(p))$.

It is also well-known that $\mathfrak{Dp}(h_{\tilde{X}}) = H_{\tilde{X}} = -i\left(\tilde{X} + \frac{1}{2}\text{div}(\tilde{X})\right)$. In particular, if $\tilde{X} = D\Psi(X)$, then $\mathfrak{Dp}(h_{\tilde{X}}) = -i\left(\widehat{\nabla}_X + \frac{1}{2}\text{div}(X)\right)$.

When $X_0(\lambda) = 2\lambda \frac{d}{d\lambda}$, we have that $h_0(q, p) := h_{\tilde{X}_0}(q, p) = \sum q_j p_j$ and $\widehat{r}_t^0(q, p) = (e^t q, e^{-t} p)$. In particular, \widehat{r}_t^0 is linear. Moreover, equation (4.24) in [7] implies that

$$m(\widehat{r}_t^0) = \widehat{W}_t^0.$$

The latter identity allows us to compute $\tilde{W}_t^0 \mathfrak{Dp}(u) \tilde{W}_{-t}^0$ using the metaplectic representation, and it is the main reason why we are considering the vector field X_0 . Furthermore, since $e^t R_t^0 = \tilde{W}_t^0$ (Lemma 3.4), we obtain the identity

$$R_t^0 \mathfrak{Dp}(u) R_{-t}^0 = \mathfrak{Dp}((\hat{r}_t^0)^* u).$$

Using a Taylor expansion, it is straightforward to show that the limit $\lim_{t \rightarrow 0} \frac{1}{t} (u \circ \hat{r}_t^0 - u) = \{h_0, u\}$ holds in $S'(\mathbb{R}^{2n})$. Since $\mathfrak{Dp} : S'(\mathbb{R}^{2n}) \rightarrow B(S(\mathbb{R}^n), S'(\mathbb{R}^n))$ is continuous, Theorem 2.16 implies the following remarkable result.

THEOREM 3.8. *If u is a tempered constant of motion of $h(q, p) = \|q\|^2$, $X = aX_0$ and $\mathfrak{Dp}(u) \in \mathfrak{A}^1$, then*

$$(3.6) \quad \widehat{\nabla}_X \mathfrak{Dp}(u) = a(Q^2) \mathfrak{Dp}(\widehat{X}_0(u)).$$

In particular, if $\mathfrak{Dp}(u)$ and $\mathfrak{Dp}(\widehat{X}_0(u))$ belong to \mathfrak{A}_2^0 , then $\mathfrak{Dp}(u) \in \mathfrak{A}_2^1$.

REMARK 3.9. Since $\{h, h_0\} = 2h$, the Jacobi identity implies that $\{h_0, u\} = \widehat{X}_0(u)$ is a tempered constant of motion of h .

COROLLARY 3.10. *If u is a tempered constant of motion of h , then $\mathfrak{Dp}(u)$ is horizontal if and only if u is also a tempered constant of motion of h_0 and $\mathfrak{Dp}(u) \in \mathfrak{A}_2^0$.*

Proof. If $\mathfrak{Dp}(u)$ is horizontal, Theorem 3.8 implies that $\mathfrak{Dp}(\widehat{X}_0(u)) = 0$. Since \mathfrak{Dp} is faithful, u is a constant of motion of h_0 . The converse is a direct consequence of Lemma 5.1.1. in [12]. ■

COROLLARY 3.11. *If u is a tempered constant of motion of $h(q, p) = \|q\|^2$ belonging to a local Hörmander class $S_{\rho, \delta}^m(\mathbb{R}^{2n})$, then $\mathfrak{Dp}(u)$ belongs to \mathfrak{A}_2^0 .*

Proof. We already note that $\mathfrak{Dp}(u)$ belongs to $\mathfrak{A}_2^0 \cap \mathfrak{A}^\infty$. Since $\{h_0, u\} \in S_{\rho, \delta}^{m'}(\mathbb{R}^{2n})$ with $m' = m + \max(1 - \rho, \delta)$, Theorems 3.6 and 3.8 imply our result. ■

The following result is analogue to Theorem 3.8, but taking $h(q, p) = \|p\|^2$. The corresponding operator is the Laplacian $H = -\Delta$, which is fundamental for physical applications, and its diagonalization \tilde{T} is obtained after composing T defined by equation (3.1) with the Fourier transform \mathcal{F} . In order to make \tilde{T} a trivialization, we define the connection by $\nabla_X = \tilde{T}^* X \tilde{T}$. Also notice that the metaplectic representation m maps the symplectic matrix \mathfrak{J} into the Fourier transform \mathcal{F} . Moreover, $h \circ \mathfrak{J} = \tilde{h}$ and $h_0 \circ \mathfrak{J} = -h_0$, where $\tilde{h}(q, p) = \|q\|^2$. Since \mathfrak{J} is a symplectomorphism, u is a constant of motion of h if and only if $u \circ \mathfrak{J}$ is a constant of motion of \tilde{h} .

COROLLARY 3.12. *If u is a tempered constant of motion of $h(q, p) = \|p\|^2$, $X = aX_0$ and $\mathfrak{Dp}(u) \in \mathfrak{A}^1$, then*

$$\widehat{\nabla}_X \mathfrak{Dp}(u) = a(-\Delta) \mathfrak{Dp}(\widehat{X}_0(u)).$$

In particular, if $\mathfrak{Dp}(u)$ and $\mathfrak{Dp}(\widehat{X}_0(u))$ belong to \mathfrak{A}_2^0 , then $\mathfrak{Dp}(u) \in \mathfrak{A}_2^1$.

REMARK 3.13. Through the rest of this article we will only consider fields of operators coming from Weyl quantization. However, there are other ways to construct operators admitting a decomposition through \tilde{T} . For example, if S is the scattering operator corresponding to a suitable Schrödinger operator, then S is unitary and it strongly commutes with $-\Delta$, therefore it can be decomposed through \tilde{T} (for instance, see [29]). For the moment, we do not know if our notion of smooth fields of operators and our results can be applied in scattering theory, and we look forward to study this problem in the future.

Let us return to the case $H = Q^2$. When $X = X_0$, formula (3.6) becomes

$$(3.7) \quad \widehat{\nabla}_{X_0} \mathfrak{Op}(u) = \mathfrak{Op}(\widehat{X}_0(u)).$$

We do not know if the latter equation holds true when we replace X_0 by an arbitrary vector field X . Such identity is equivalent to

$$a(Q^2) \mathfrak{Op}(\widehat{X}_0(u)) = \mathfrak{Op}((a \circ \phi) \widehat{X}_0(u)),$$

where $\phi(q) = \|q\|^2$. It is well-known that \mathfrak{Op} is not a multiplicative homomorphism. However, because of the particular type of symbols that we are considering, the latter identity might still hold true. Notice that for an arbitrary vector field X , the Hamiltonian flow \widehat{r}_t is not necessarily linear, so we cannot use the metaplectic representation. Nevertheless, in order to estimate the difference between the right and left hand of equation (3.7) for an arbitrary X , we might use semi-classical theory, introducing Planck’s constant dependence and applying Egorov’s theorem (for instance, see Theorem 11.1 in [30]). We expect to address the latter problem in the future.

The following observation relates our results and the problem discussed in the previous paragraph with deformation quantization. Let \mathcal{A} be the Poisson algebra of constants of motion of $h(p, q) = \|q\|^2$. Then, the map $\widetilde{\nabla} : \text{Vect}((0, \infty)) \times \mathcal{A} \mapsto \mathcal{A}$, given by $\widetilde{\nabla}_X(u) = \widehat{X}(u)$ satisfies the following properties:

- (i) $\widetilde{\nabla}_X(\{u, v\}) = \{\widetilde{\nabla}_X u, v\} + \{u, \widetilde{\nabla}_X v\};$
- (ii) $\widetilde{\nabla}_X \widetilde{\nabla}_Y - \widetilde{\nabla}_Y \widetilde{\nabla}_X = \widetilde{\nabla}_{[X, Y]}.$

In other words, $\widetilde{\nabla}$ defines a sort of abelian Poisson connection.

We can generalize the previous construction as follows. Let M be a Riemannian manifold and $\phi \in C^\infty(M)$ regular. For each $X \in \text{Vect}(\phi(M))$, let \widetilde{X} be the unique vector field normal to each $M_\lambda := \phi^{-1}(\lambda)$ such that $D\phi(\widetilde{X}) = X$, and let \widehat{X} be the Hamiltonian vector field of $h_{\widetilde{X}}$ on T^*M . If \mathcal{A} is the Poisson algebra of constants of motion of $h(q, p) = \phi(q)$, then the map $\widetilde{\nabla} : \text{Vect}(\phi(M)) \times \mathcal{A} \mapsto \mathcal{A}$, given by $\widetilde{\nabla}_X(u) = \widehat{X}(u)$ is well-defined and satisfies (i) and (ii) above as well.

Let α_t be the Hamiltonian flow of $h(q, p) = \phi(q)$. For each $\lambda \in \phi(M)$, let $\widehat{\Sigma}_\lambda = h^{-1}(\lambda)$ be the constant energy submanifold (do not confuse it with M_λ above) and denote by Σ_λ the orbit space $\widehat{\Sigma}_\lambda / \alpha$ endowed with the symplectic structure obtained after applying Marsden–Weinstein–Meyer reduction [13, 14]

(one of the authors showed in [1] that $\Sigma_\lambda = T^*M_\lambda$). For each $u \in \mathcal{A}$, we can define $u_\lambda \in C^\infty(\Sigma_\lambda)$ by $u_\lambda([\sigma]) = u(\sigma)$, where $\sigma \in \Sigma_\lambda$ and $[\sigma]$ denotes the orbit σ . Then, every $u \in \mathcal{A}$ can be regarded as a section of a field of Poisson algebras over $\phi(M)$ with fibers $C^\infty(\Sigma_\lambda)$. The relation between symplectic connections and deformation quantization has been successfully studied, specially since Fedosov's paper [6]. Fedosov showed that, if Σ is a symplectic manifold then the Poisson algebra $C^\infty(\Sigma)$ admits a star product. In order to do so, Fedosov used an abelian symplectic connection to glue the canonical Moyal product on each tangent plane. In our framework, if a star product \star_h^λ is given on each $C^\infty(\Sigma_\lambda)$, we would like to adapt Fedosov's ideas to use our Poisson connection to glue those star products into a single one defined on \mathcal{A} . If the latter construction works, then we might wonder if there is a relation between such star product and the canonical Moyal product restricted to \mathcal{A} . An analogous question was formulated within the framework of canonical quantization and Wigner transforms in [1]. Both questions can be interpreted as problems analogue to the commutation of reduction and quantization problem in geometric quantization theory [9, 28]. We shall leave the latter problem open as well.

3.2. FUNCTIONS OF ANGULAR MOMENTA AS HORIZONTAL CONSTANTS OF MOTION. Let us construct some examples of constants of motion. For future reference, we will assume for a while that ϕ is any smooth function on a smooth manifold M .

Let G be a Lie group acting on M . Such action induces the Lie algebra homomorphism $\zeta : \mathfrak{g} \rightarrow \text{Vect}(M)$ given by

$$\zeta(X)(q) = \frac{d}{dt}(\exp(tX) \cdot q)|_{t=0},$$

where \mathfrak{g} is the Lie algebra corresponding to G and $X \in \mathfrak{g}$. Also let $\mathcal{J} : T^*M \rightarrow \mathfrak{g}^*$ be the induced moment map given by

$$\mathcal{J}((q, p))(X) = \langle \zeta(X)(q), p \rangle = h_{\zeta(X)}(q, p),$$

where $\langle \cdot, \cdot \rangle$ implements the duality between T_qM and T_q^*M .

If we endow \mathfrak{g}^* with the coadjoint action of G and the negative of the canonical Lie–Poisson structure, and we lift the action of G to T^*M , then \mathcal{J} turns to be an equivariant Poisson map.

PROPOSITION 3.14. *Let G be a Lie group acting on M and \mathcal{J} be the moment map defined above.*

(i) *If $\phi(g \cdot q) = \phi(q)$, for each $q \in M$ and $g \in G$, then $a \circ \mathcal{J}$ is constant of motion of $h(q, p) = \phi(q)$, for any $a \in C^\infty(\mathfrak{g}^*)$.*

(ii) *If \tilde{Y} is a vector field on M such that $[\tilde{Y}, \zeta(X)] = 0$ for every $X \in \mathfrak{g}$, then $a \circ \mathcal{J}$ is constant of motion of $h_{\tilde{Y}}$, for any $a \in C^\infty(\mathfrak{g}^*)$.*

Proof. Suppose that $\phi(g \cdot q) = \phi(q)$, for each $g \in G$ and $q \in M$. Thus, $\zeta(X)$ is tangent to each M_λ . Moreover, for each $t \in \mathbb{R}$ and $(q, p) \in T^*M$, since $\nabla\phi$ is

normal to each M_λ , we have that

$$\begin{aligned} [\mathcal{J} \circ \alpha_t(q, p)](X) &= [\mathcal{J}(q, \text{td}\phi(q) + p)](X) = \langle \zeta(X)(q), \text{td}\phi(q) + p \rangle \\ &= \langle \zeta(X)(q), p \rangle = \mathcal{J}(q, p)(X). \end{aligned}$$

In other words $\mathcal{J} \circ \alpha_t = \mathcal{J}$ and this implies (i). For the second part, we shall prove that $\mathcal{J} \circ \widehat{r}_t = \mathcal{J}$, where \widehat{r}_t is the Hamiltonian flow of $h_{\zeta(X)}$. Notice that $[\mathcal{J} \circ \widehat{r}_t(q, p)](X) = (h_{\zeta(X)} \circ \widehat{r}_t)(q, p)$, therefore $\mathcal{J} \circ \widehat{r}_t = \mathcal{J}$ if and only if each $h_{\zeta(X)}$ is a constant of motion of $h_{\zeta(X)}$. Our result follows from the identity $\{h_{\zeta(X)}, h_{\zeta(X)}\} = h_{[\zeta(X), \zeta(X)]}$ (equation II 3.11 in [10]). ■

Let us return to our example $\phi(x) = \|x\|^2$. Also, let $G = O(n)$ be the orthogonal group acting canonically on \mathbb{R}^n . Then clearly condition (i) in the previous proposition is satisfied. Since $\widehat{r}_t^0(q) = e^t q$ and rotations commute with dilations, condition (ii) is satisfied for $\widetilde{Y} = \widetilde{X}$, for any $X \in \text{Vect}(0, \infty)$.

COROLLARY 3.15. *Let $O(n)$ act on \mathbb{R}^n canonically and $\mathcal{J} : \mathbb{R}^{2n} \rightarrow \mathfrak{so}(n)^*$ the corresponding moment map. Then $a \circ \mathcal{J}$ is a classical constant of motion of $h(q, p) = \|q\|^2$ and of $h_{\widetilde{X}}$, for any $X \in \text{Vect}(0, \infty)$ and $a \in C^\infty(\mathfrak{so}(n)^*)$. In particular, if $a \circ \mathcal{J}$ is a tempered distribution belonging to some local Hörmander class $S_{\rho, \delta}^m(\mathbb{R}^{2n})$, then $\mathfrak{Dp}(a \circ \mathcal{J})$ is a horizontal field of operators.*

REMARK 3.16. Let $l_{i,j}(q, p) = q_i p_j - q_j p_i$ and $L_{ij} = \mathfrak{Dp}(l_{ij}) = q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i}$. Then $l_{ij} = h_{\zeta(X_{ij})}$, where X_{ij} is the element of the canonical basis of $\mathfrak{so}(n)$ corresponding to the infinitesimal generator of the clockwise rotation on the plane in \mathbb{R}^n generated by e_i and e_j . The functions $l_{i,j}$ and the operator L_{ij} are called the classic and quantum angular momenta coordinates, respectively. Notice that defining a polynomial of the family of operators L_{ij} is not at all trivial, because the operators L_{ij} do not commute (they come from a representation of $\mathfrak{so}(n)$, so they satisfy the same commutation relations than the corresponding vectors X_{ij}). A similar situation occurs if we consider the canonical family of positions and momenta operators Q_j and P_j . For example, for the polynomial $u(q, p) = q_i p_j$, one might choose to define $u(Q, P) = Q_i P_j$ or, instead one might choose to define $u(Q, P) = P_j Q_i$. This is sometimes called the canonical ordering problem. Weyl calculus provide a symmetric solution for this problem if we define $u(Q, P) = \mathfrak{Dp}(u)$. For instance, if $u(q, p) = q_i p_j$, then $u(Q, P) = \frac{1}{2}(Q_i P_j + P_j Q_i)$ is symmetric (notice that neither $Q_i P_i$ or $P_i Q_i$ are symmetric). Our result suggest that Weyl calculus is also a convenient solution for the angular momenta ordering problem.

REMARK 3.17. The spectral analysis of the operators L_{ij} and $L^2 = \sum_{i < j} L_{ij}^2$ are usually presented in any course on quantum mechanics. L^2 is called the total angular momentum operator. Using polar coordinates, one can show that the spectra of those operators coincide with the spectra of their restrictions to

$C^\infty(\mathbb{S}^{n-1}) \subset L^2(\mathbb{S}^{n-1})$. Our previous result implies that the same conclusion holds for a much larger class of operators. For instance, the same conclusion holds if $\mathfrak{Op}(a \circ \mathcal{J})$ is a horizontal field of operators and $\mathfrak{Op}(a \circ \mathcal{J})$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^n)$. Moreover, if $\mathfrak{Op}(a \circ \mathcal{J})$ is a horizontal field of operators and the restriction of $\mathfrak{Op}(a \circ \mathcal{J})$ to $C^\infty(\mathbb{S}^{n-1})$ is essentially self-adjoint, then $\mathfrak{Op}(a \circ \mathcal{J})$ is essentially self-adjoint $C_c^\infty(\mathbb{R}^n)$. The previous claims follow from Theorem 3 in [17].

4. LOCALLY UNIFORMLY BOUNDED SMOOTH FIELDS OF OPERATORS AND SMOOTH FIELDS OF C^* -ALGEBRAS

In this section we will study how the smoothness of a field of operators $A \in \mathfrak{A}^n$ interacts with the continuity of A as an operator on $\Gamma^n(\Lambda)$. The discussion will lead us to introduce a notion of smoothness for fields of C^* -algebras.

It will be useful to consider the Hilbert $C_0(\Lambda)$ -module $\Gamma_0^0(\Lambda)$. Recall that an operator $A : \Gamma_0^0(\Lambda) \rightarrow \Gamma_0^0(\Lambda)$ is called adjointable if and only if there is another operator $A^* : \Gamma_0^0(\Lambda) \rightarrow \Gamma_0^0(\Lambda)$ such that $h(A\varphi, \psi) = h(\varphi, A^*\psi)$, for any $\varphi, \psi \in \Gamma_0^0(\Lambda)$. It is well-known that, in contrast with operators on Hilbert spaces, the continuity of A does not guarantee that A is adjointable. On the other hand, if A is adjointable then A is continuous (for instance, see Lemma 2.18 in [20]). Moreover, the space of all adjointable operators form a C^* -algebra [20, Proposition 2.21].

Another important property is that every adjointable operator is given by a field of operators. Indeed, if A is an adjointable operator, then clearly $A(K_0(\lambda)) \subseteq K_0(\lambda)$. Therefore, by Lemma 2.2 there is an operator $A(\lambda)$ on $\mathcal{H}(\lambda)$ such that $A\varphi(\lambda) = A(\lambda)\varphi(\lambda)$, for all $\varphi \in \Gamma_0^0(\Lambda)$ and $\lambda \in \Lambda$.

Let us introduce one of the spaces of fields of operators that we will consider in this subsection.

DEFINITION 4.1. For each $n \in \mathbb{N} \cup \{\infty\}$, we denote by \mathfrak{A}_c^n the space formed by the fields of operators $A \in \mathfrak{A}^n$ such that $A : \Gamma^\infty \rightarrow \Gamma^n(\Lambda)$ is continuous with respect to the Fréchet topology of $\Gamma^n(\Lambda)$. We denote by \mathfrak{A}_c^0 the space of adjointable operators on $\Gamma_0^0(\Lambda)$.

In particular, if A belongs to \mathfrak{A}_c^n , then A can be extended to $\Gamma^n(\Lambda)$; we will also denote by A such extension. Let us apply our previous discussion to characterize \mathfrak{A}_c^0 .

PROPOSITION 4.2. *Let $A : \Gamma^\infty \rightarrow \Gamma^0(\Lambda)$ be a linear operator. The following statements are equivalent:*

- (i) $A \in \mathfrak{A}_c^0$;
- (ii) $A \in \mathfrak{A}^0$ and it is a locally uniformly bounded field of operators;
- (iii) A extends to $\Gamma^0(\Lambda)$ and its extension is adjointable.

REMARK 4.3. We are abusing the notation in part (iii) of the previous statement, because we have defined adjointability on $\Gamma_0^0(\Lambda)$, but the corresponding definition is the same on $\Gamma^0(\Lambda)$.

Proof. Let us show that (i) implies (ii). Since A is a field of operators, $A|_{\Gamma^0(C)}$ is a well-defined and continuous operator on $\Gamma^0(C)$, for every compact set $C \subseteq \Lambda$. Moreover, $AK_0(\lambda) \subseteq K_0(\lambda)$ and Lemma 2.2 implies that

$$\begin{aligned} \|A(\lambda)\varphi(\lambda)\|_\lambda &= \inf\{\|A\varphi - \psi\| : \psi \in K_0(\lambda)\} \\ &\leq \inf\{\|A(\varphi - \psi)\| : \psi \in K_0(\lambda)\} \leq \|A\| \|\varphi(\lambda)\|_\lambda, \end{aligned}$$

for all $\varphi \in \Gamma^0(C)$. Since $\mathcal{H}(\lambda) = \{\varphi(\lambda) : \varphi \in \Gamma^0(C)\}$, the latter inequality implies (ii).

Clearly (ii) implies (iii) and (ii) implies (i). In order to show that (iii) implies (i), notice that since A is adjointable, $\varphi|_C = 0$ implies that $A\varphi|_C = 0$, for every compact $C \subseteq \Lambda$. Therefore the map $A|_C : \Gamma^0(C) \rightarrow \Gamma^0(C)$ given by $A|_C(\varphi|_C) = A(\varphi)|_C$ is well-defined and also adjointable. Since $\Gamma^0(C)$ is a Hilbert module, $A|_C$ is a bounded field of operators and clearly this implies (i). ■

REMARK 4.4. Let $A \in \mathfrak{A}^0$ and $C \subseteq \Lambda$ compact. Even if $A|_C$ is not an uniformly bounded field of operators, it is a densely defined operator on $\Gamma^0(C)$. Moreover, it is easy to show that $A|_C$ is closable on $\Gamma^0(C)$. Therefore $A|_C$ is a semiregular operator on $\Gamma^0(C)$ as defined in [18] (i.e. a closable densely defined operator on a Hilbert C^* -module with an adjoint also densely defined). We would like to obtain conditions to guarantee that $A|_C$ is also regular and to study how regularity is related with self-adjointness of such operators but acting on a direct integral, as explained in Subsection 2.1. However, we will not consider the latter problems in this article.

Let us look for conditions to guarantee that a given field of operators A belongs \mathfrak{A}_c^n . Let $\varphi_j, \varphi \in \Gamma^\infty$ such that $\varphi_j \rightarrow \varphi$ in $\Gamma^n(\Lambda)$. Then $A\varphi_j \rightarrow A\varphi$ in $\Gamma^n(\Lambda)$ if and only if

- (i) $A\varphi_j \rightarrow A\varphi$ in $\Gamma^{n-1}(\Lambda)$;
- (ii) $\nabla_X A\varphi_j = A\nabla_X\varphi_j + \widehat{\nabla}_X(A)\varphi_j \rightarrow \nabla_X A\varphi$ in $\Gamma^{n-1}(\Lambda)$, for every $X \in \text{Vect}(\Lambda)$.

Since $\varphi_j \rightarrow \varphi$ in $\Gamma^{n-1}(\Lambda)$ and $\nabla_X\varphi_j \rightarrow \nabla_X\varphi$ in $\Gamma^{n-1}(\Lambda)$, for every $X \in \text{Vect}(\Lambda)$, we obtain the following result.

PROPOSITION 4.5. *Let $A \in \mathfrak{A}^n$.*

(i) *Assume that $A \in \mathfrak{A}_c^{n-1}$. Then $A \in \mathfrak{A}_c^n$ if and only if $\widehat{\nabla}_X(A) : \Gamma^\infty \rightarrow \Gamma^{n-1}(\Lambda)$ is continuous when Γ^∞ is endowed with the $\Gamma^n(\Lambda)$ -topology.*

(ii) *If $A \in \mathfrak{A}_c^{n-1}$ and $\widehat{\nabla}_X(A) \in \mathfrak{A}_c^{n-1}$, then $A \in \mathfrak{A}_c^n$.*

In particular, if $\widehat{\nabla}_{X_1} \cdots \widehat{\nabla}_{X_k} A \in \mathfrak{A}_c^0$, for every $X_1, \dots, X_k \in \text{Vect}(\Lambda)$ and $0 \leq k \leq n$, then $A \in \mathfrak{A}_c^n$.

Notice that $\mathfrak{A}_c^n \not\subseteq \mathfrak{A}_c^{n-1}$ and $\widehat{\nabla}_X(\mathfrak{A}_c^n) \not\subseteq \mathfrak{A}_c^{n-1}$, for any non-zero $X \in \text{Vect}(\Lambda)$. The previous result allows us to introduce the following subspace of \mathfrak{A}_c^n , where the corresponding inclusions become true.

DEFINITION 4.6. For each $n \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathfrak{A}_{\text{lb}}^n$ the space formed by the fields of operators $A \in \mathfrak{A}^n$ such that the field of operators $\widehat{\nabla}_{X_1} \cdots \widehat{\nabla}_{X_k} A$ is locally uniformly bounded, for any $X_1, \dots, X_k \in \text{Vect}(\Lambda)$ and $0 \leq k \leq n$.

We could use statements (i) or (iii) in Proposition 4.2 to define $\mathfrak{A}_{\text{lb}}^n$, instead of (ii).

We can endow $\mathfrak{A}_{\text{lb}}^n$ with a family of seminorms, just as it was done for $\Gamma^n(\Lambda)$. More precisely, we define

$$\|A\|_{C, X_1, \dots, X_m} = \sup\{\|\widehat{\nabla}_{X_1} \cdots \widehat{\nabla}_{X_m} A(\lambda)\| : \lambda \in C\},$$

where $C \subseteq \Lambda$ is compact, $X_1, \dots, X_m \in \text{Vect}(\Lambda)$ and $m \leq n$. Clearly, $\mathfrak{A}_{\text{lb}}^n$ becomes a Fréchet space with the latter family of seminorms.

REMARK 4.7. We can also put the natural direct limit topology on the space $\mathfrak{A}_{\text{lb}}^n$ (or $\Gamma^n(\Lambda)$), but we will not consider it in this article.

COROLLARY 4.8. For any $n \geq 1$ and $X \in \text{Vect}(\Lambda)$, $\mathfrak{A}_{\text{lb}}^n \subseteq \mathfrak{A}_{\text{lb}}^{n-1}$, $\widehat{\nabla}_X(\mathfrak{A}_{\text{lb}}^n) \subseteq \mathfrak{A}_{\text{lb}}^{n-1}$ and the map $\widehat{\nabla}_X : \mathfrak{A}_{\text{lb}}^n \rightarrow \mathfrak{A}_{\text{lb}}^{n-1}$ is continuous.

Also notice that if $A, B \in \mathfrak{A}_{\text{lb}}^n$, then $AB \in \mathfrak{A}_{\text{lb}}^n$ and the following identity holds:

$$(4.1) \quad \widehat{\nabla}_X(AB) = \widehat{\nabla}_X(A)B + A\widehat{\nabla}_X(B).$$

In the full projectively trivializable case, we obtain stronger properties of the map \widehat{T} , defined in Theorem 2.5, if we restrict it to $\mathfrak{A}_{\text{lb}}^n$. For instance, since each operator $\widehat{A}(\lambda) = T_\lambda A(\lambda) T_\lambda^*$ is bounded and $T^*f \in \Gamma^\infty(\Lambda) \subset \Gamma^n(\Lambda)$, for any $A \in \mathfrak{A}_{\text{lb}}^n$ and $f \in V$, we no longer need to consider the common domain V^∞ defined in Definition 2.4. Recall that $B(V)_{*s}$ denotes the space of bounded operators on V endowed with the $*$ -strong topology. We also denote by $C_{\text{lb}}^n(\Lambda, B(V)_{*s})$ the space of n -times differentiable functions from Λ to $B(V)_{*s}$ such that any derivative of order lower or equal to n is continuous and locally bounded.

PROPOSITION 4.9. Let $T : H \rightarrow V$ be a full projective trivialization of the smooth field of Hilbert spaces $H \rightarrow \Lambda$. The map \widehat{T} , defined by $\widehat{T}A(\lambda) = T_\lambda A(\lambda) T_\lambda^*$, is a local isometry of $\mathfrak{A}_{\text{lb}}^n$ onto $C_{\text{lb}}^n(\Lambda, B(V)_{*s})$.

Proof. The same proof of Theorem 2.5 and the previous comment show that \widehat{T} is a well-defined local isometry. It is enough to show that if $\widehat{A} \in C_{\text{lb}}^n(\Lambda, B(V)_{*s})$, then $A = T^* \widehat{A} T \in \mathfrak{A}_{\text{lb}}^n$. Equivalently, it is enough to prove that $\widehat{A}(C^\infty(\Lambda, V)) \subseteq C^n(\Lambda, V)$. The case $n = 0$ follows from the inequality

$$\|\widehat{A}\varphi(\lambda) - \widehat{A}\varphi(\lambda_0)\| \leq \|[\widehat{A}(\lambda) - \widehat{A}(\lambda_0)]\varphi(\lambda_0)\| + \|\widehat{A}(\lambda)\|\|\varphi(\lambda) - \varphi(\lambda_0)\|.$$

Similarly, we can use the canonical proof of Leibniz’s multiplication formula to show that $\widehat{A}\varphi \in C^n(\Lambda, V)$, for $n > 0$ and every $\varphi \in C^\infty(\Lambda, V)$ (or Lemma 5.1.1 in [12]). ■

4.1. SMOOTH FIELDS OF C^* -ALGEBRAS. Recall that one of our aims is to propose a definition of a smooth field of C^* -algebras. Let us recall the definition of a (upper semi-)continuous field of C^* -algebras.

DEFINITION 4.10. Let Λ be a locally compact Hausdorff space and $p : \mathcal{A} \rightarrow \Lambda$ be a field of C^* -algebras (i.e. p is a surjection such that $\mathfrak{A}(\lambda) := p^{-1}(\lambda)$ is a C^* -algebra, for any $\lambda \in \Lambda$). A (upper semi-)continuous structure on $p : \mathcal{A} \rightarrow \Lambda$ is given by specifying a $*$ -algebra of sections \mathfrak{A} , closed under multiplication by elements of $C(\Lambda)$, and such that:

- (i) the map $\lambda \mapsto \|A(\lambda)\|$ is (upper semi-)continuous, for all $A \in \mathfrak{A}$;
- (ii) $\{A(\lambda) : A \in \mathfrak{A}\}$ is dense in $\mathfrak{A}(\lambda)$, for all $\lambda \in \Lambda$.

As we mentioned in the introduction, the previous definition of (upper semi-)continuous field of C^* -algebras is equivalent to two other notions: the notion of (upper semi-)continuous C^* -bundle (see Definition C.16 and Theorem C.25 in [27]), and the notion of $C_0(\Lambda)$ -algebra (see Definition C.1 in [27]). Both notions are interesting and have their own advantages, but we will not require them in what follows. The notion of (upper semi-)continuous field of C^* -algebras allows to describe the continuous structure in terms of a space of sections, and this is the perspective that we are following in this article (recall the definition of smooth fields of Hilbert spaces and compare with the Definition 4.12 of smooth field of C^* -algebras that we will propose latter).

If $A \in \mathfrak{A}_{\text{lb}}^0$, by definition the maps $\lambda \mapsto \|A(\lambda)\varphi(\lambda)\|$ and $\lambda \mapsto \|A^*(\lambda)\varphi(\lambda)\|$ are continuous, for every $\varphi \in \Gamma^0(\Lambda)$. However, the map $\lambda \mapsto \|A(\lambda)\|$ is not necessarily (upper semi-)continuous. Therefore, $\mathfrak{A}_{\text{lb}}^0$ does not define an (upper semi-)continuous structure on the field of C^* -algebras $\mathfrak{A}(\lambda) = \{A(\lambda) : A \in \mathfrak{A}_{\text{lb}}^0\}$. Notice that \mathfrak{A}_0^0 is a $C_0(\Lambda)$ -algebra, thus $\mathfrak{A}_{\text{lb}}^0$ defines an upper semi-continuous structure but on the field of C^* -algebras $\widetilde{\mathfrak{A}}(\lambda) = \mathfrak{A}_0^0/I_\lambda\mathfrak{A}_0^0$ (and generically this algebra does not coincide with $\mathfrak{A}(\lambda)$).

There are subalgebras \mathfrak{A} of $\mathfrak{A}_{\text{lb}}^0$ such that the map $\lambda \mapsto \|A(\lambda)\|$ is continuous, for any $A \in \mathfrak{A}$. Indeed, we will show that the space of compact operators $\mathbb{K}_0(\Lambda) := \mathbb{K}(\Gamma_0^0(\Lambda))$ on the Hilbert module $\Gamma_0^0(\Lambda)$ satisfies such property. Let us recall the construction of the C^* -algebra $\mathbb{K}_0(\Lambda)$. For each $\varphi, \psi \in \Gamma_0^0(\Lambda)$, we define the adjointable operator $|\varphi\rangle\langle\psi|$ by

$$|\varphi\rangle\langle\psi|(\lambda)(\phi(\lambda)) = \langle\psi(\lambda), \phi(\lambda)\rangle\varphi(\lambda),$$

where the inner product is taken on $\mathcal{H}(\lambda)$. It is straightforward to show that $|\varphi\rangle\langle\psi|^* = |\psi\rangle\langle\varphi|$ and $\| |\varphi\rangle\langle\psi|(\lambda) \| = \|\varphi(\lambda)\|\|\psi(\lambda)\|$. The space of compact operators $\mathbb{K}_0(\Lambda)$ is the closure span of the set of operators of the form $|\varphi\rangle\langle\psi|$, with $\varphi, \psi \in \Gamma_0^0(\Lambda)$.

The following proposition can be found in the discussion of Subsection 10.7.2 of [4], but we write it within the framework of Hilbert modules.

PROPOSITION 4.11. *Let $H \rightarrow \Lambda$ be a smooth field of Hilbert spaces and let $p : \mathcal{K} \rightarrow \Lambda$ be the field of C^* -algebras with fibers $\mathbb{K}(\mathcal{H}(\lambda))$, where $\mathbb{K}(\mathcal{H}(\lambda))$ is the C^* -algebra of compact operators on the Hilbert space $\mathcal{H}(\lambda)$. Identifying $A \in \mathbb{K}_0(\Lambda)$ with the section $\lambda \rightarrow A(\lambda)$ defines a continuous structure on $p : \mathcal{K} \rightarrow \Lambda$, and $\mathbb{K}_0(\Lambda)$ becomes the corresponding space of continuous sections vanishing at infinity. In particular, the map $\lambda \rightarrow \|A(\lambda)\|$ is continuous and vanishing at infinity, for every $A \in \mathbb{K}_0(\Lambda)$.*

Proof. To prove this, we have to specify a $*$ -algebra \mathfrak{A} of sections that satisfies (i) and (ii) in Definition 4.10. We take \mathfrak{A} to be the set of linear combinations of operators of the form $|\varphi\rangle\langle\psi|$, with $\varphi, \psi \in \Gamma_0^0(\Lambda)$.

Let V be a Hilbert space and $\varphi_1, \dots, \varphi_{2n} : X \rightarrow V$ be fields of vectors, where X is some topological space. Assume that $\lim_{x \rightarrow x_0} \langle \varphi_i(x), \varphi_j(x) \rangle = \langle \psi_i, \psi_j \rangle$, where $\psi_1, \dots, \psi_{2n} \in V$. Lemma 3.5.6 in [4] implies that

$$\lim_{x \rightarrow x_0} \left(\|\varphi_1(x)\rangle\langle\varphi_2(x)| + \dots + \|\varphi_{2n-1}(x)\rangle\langle\varphi_{2n}(x)| \right) = \left(\|\psi_1\rangle\langle\psi_2| + \dots + \|\psi_{2n-1}\rangle\langle\psi_{2n}| \right).$$

In particular, if $\langle \varphi_i, \varphi_j \rangle = \langle \psi_i, \psi_j \rangle$, then

$$\|\varphi_1\rangle\langle\varphi_2| + \dots + \|\varphi_{2n-1}\rangle\langle\varphi_{2n}| = \|\psi_1\rangle\langle\psi_2| + \dots + \|\psi_{2n-1}\rangle\langle\psi_{2n}|.$$

Therefore, the map $V^{2n} \ni (\psi_1, \dots, \psi_{2n}) \rightarrow \|\psi_1\rangle\langle\psi_2| + \dots + \|\psi_{2n-1}\rangle\langle\psi_{2n}|$ is actually a continuous function on the inner products $\langle \psi_i, \psi_j \rangle$. Thus, \mathfrak{A} satisfies (i) and clearly it also satisfies (ii). Moreover, since \mathfrak{A} is closed under multiplication by elements of $C_0(\Lambda)$, the $*$ -algebra \mathfrak{A} is dense in the set of continuous sections vanishing at infinity of the corresponding C^* -bundle. By definition, the closure of \mathfrak{A} is $\mathbb{K}_0(\Lambda)$, and this finishes the proof. ■

In view of Theorem 2.3, Proposition 4.5 and equation (4.1), we define a smooth field of C^* -algebras as follows.

DEFINITION 4.12. Let Λ be a smooth manifold and $p : \mathcal{A} \rightarrow \Lambda$ be a field of C^* -algebras. A smooth structure on $\mathcal{A} \rightarrow \Lambda$ is given by specifying a $*$ -algebra of sections \mathfrak{A}^∞ closed under multiplication by elements of $C^\infty(\Lambda)$, and a map $\widehat{\nabla} : \text{Vect}(\Lambda) \times \mathfrak{A}^\infty \rightarrow \mathfrak{A}^\infty$ such that for $X, Y \in \text{Vect}(\Lambda)$, $A, B \in \mathfrak{A}^\infty$ and $a \in C^\infty(\Lambda)$

- (i) $\widehat{\nabla}_{X+Y}(A) = \widehat{\nabla}_X(A) + \widehat{\nabla}_Y(A)$ and $\widehat{\nabla}_{aX}(A) = a\nabla_X(A)$;
- (ii) $\widehat{\nabla}_X(aA) = X(a)A + a\widehat{\nabla}_X(A)$ and $\widehat{\nabla}_X(AB) = \widehat{\nabla}_X(A)B + A\widehat{\nabla}_X(B)$;
- (iii) $(\widehat{\nabla}_X(A))^* = \widehat{\nabla}_{\overline{X}}(A^*)$;
- (iv) for each $m \in \mathbb{N}$ and $X_1, \dots, X_m \in \text{Vect}(\Lambda)$, the map $\lambda \mapsto \|\widehat{\nabla}_{X_1} \cdots \widehat{\nabla}_{X_m} A(\lambda)\|$ is continuous;
- (v) $\mathfrak{A}^\infty(\lambda) = \{A(\lambda) : A \in \mathfrak{A}^\infty\}$ is dense in $\mathfrak{A}(\lambda)$, for all $\lambda \in \Lambda$.

Let us return to the smooth fields of Hilbert spaces framework. A straightforward computation shows that, if $\varphi, \psi \in \Gamma^n(\Lambda)$ then $|\varphi\rangle\langle\psi| \in \mathfrak{A}_{\text{fb}}^n$ and the

following natural identity holds:

$$\widehat{\nabla}_X(|\varphi\rangle\langle\psi|) = |\nabla_X\varphi\rangle\langle\psi| + |\varphi\rangle\langle\nabla_X\psi|.$$

The latter fact and Proposition 4.11 implies the following result.

COROLLARY 4.13. *Let $H \rightarrow \Lambda$ be a smooth field of Hilbert spaces with connection ∇ and let $\mathcal{K} \rightarrow \Lambda$ be the field of C^* -algebras with fibers $\mathbb{K}(\mathcal{H}(\lambda))$. The space of sections $\mathbb{K}^\infty = \text{span}\{|\varphi\rangle\langle\psi| : \varphi, \psi \in \Gamma^\infty\}$ together with the connection $\widehat{\nabla} = [\nabla, \cdot]$ makes $\mathcal{K} \rightarrow \Lambda$ a smooth field of C^* -algebras.*

Let $\varphi, \psi \in C_0(\Lambda, V)$, where V is some Hilbert space. A straightforward computation shows that

$$\| |\varphi\rangle\langle\psi|(\lambda) - |\varphi\rangle\langle\psi|(\lambda_0) \| \leq \| \varphi(\lambda) - \varphi(\lambda_0) \| \| \psi(\lambda) \| + \| \psi(\lambda) - \psi(\lambda_0) \| \| \varphi(\lambda_0) \|.$$

Therefore, any finite rank operator on Γ_0 belongs to $C_0(\Lambda, \mathbb{K}(V))$, where $\mathbb{K}(V)$ is endowed with the norm operator topology. The following result holds within the more general framework of continuous fields of Hilbert spaces. We use the notion of continuous trivialization of Dixmier and Douady (Definitions 2 and 3 in [5]).

PROPOSITION 4.14. *Let $H \rightarrow \Lambda$ be a continuous field of Hilbert spaces and Γ_0 be the corresponding space of continuous sections vanishing at infinity. Also, let $T : H \rightarrow V$ be a continuous trivialization. The map $\widehat{T} : \mathbb{K}(\Gamma_0) \rightarrow C_0(\Lambda, \mathbb{K}(V))$ given by $\widehat{T}(A)(\lambda) = T_\lambda A(\lambda) T_\lambda^*$ is an isomorphism of C^* -algebras.*

Proof. Notice that $\widehat{T}(|\varphi\rangle\langle\psi|) = |T(\varphi)\rangle\langle T(\psi)|$. Therefore $\widehat{T}(A) \in C_0(\Lambda, \mathbb{K}(V))$, for any finite rank operator A on Γ_0 . Since \widehat{T} is an isometry, \widehat{T} is well-defined. Moreover, since $T(\Gamma_0) = C_0(\Lambda, V)$ (fullness is included in the definition of continuous trivialization), $\widehat{T}(\mathbb{K}(\Gamma_0))(\lambda) = \mathbb{K}(V)$. Therefore, Proposition C.24 in [27] implies that $\widehat{T}(\mathbb{K}(\Gamma_0(\Lambda)))$ is dense in $C_0(\Lambda, \mathbb{K}(V))$ and so \widehat{T} is surjective. ■

APPENDIX A. MEASURABLE AND CONTINUOUS FIELDS OF HILBERT SPACES

In this appendix we recall the definitions of measurable and continuous fields of Hilbert spaces over a space Λ . Each notion admits at least three equivalent ways to introduce them. Let us discuss some of their main characteristics in order to understand their relation.

The first (historic) definition of a measurable field of Hilbert spaces describes the measurable structure in terms of a space of sections. In fact, the latter approach is used throughout this article.

DEFINITION A.1. Let Λ be a measurable space. A measurable structure on a field of Hilbert spaces $p : H \rightarrow \Lambda$ is given by specifying a linear space of sections Γ possessing the following properties:

- (i) the function $\lambda \rightarrow \| \varphi(\lambda) \|_\lambda$ is measurable, for every $\varphi \in \Gamma$;

(ii) if ψ is a section such that, for every $\varphi \in \Gamma$, the function $\lambda \rightarrow \langle \varphi(\lambda), \psi(\lambda) \rangle_\lambda$ is measurable, then $\psi \in \Gamma$;

(iii) there exists a sequence (φ_j) of elements of Γ such that $\text{span}\{\varphi_j(\lambda) : j \in \mathbb{N}\}$ is dense in $\mathcal{H}(\lambda)$, for every $\lambda \in \Lambda$.

The elements of Γ are called measurable sections. The sequence $(\varphi_1, \varphi_2, \dots)$ is called a base of measurability or a measurable field of bases. The set $\{\lambda \in \Lambda : \mathcal{H}(\lambda) \neq 0\}$ is called the support of the field.

The second possible way to introduce measurability into fields of Hilbert spaces is given by the notion of a measurable Hilbert bundle (Definition A.2 below or Definition 2.4.8. in [25]). Let us explain how we meet that notion beginning with a measurable field of Hilbert spaces.

Starting from the sequence (φ_j) we can obtain a local field of measurable orthonormal bases. More precisely, there is a measurable partition $\Lambda = \bigcup_{n=1}^{n=\infty} \Lambda_n$ and a sequence $\varphi_j : \Lambda \rightarrow H$ such that, $\dim \mathcal{H}(\lambda) = n$ for all $\lambda \in \Lambda_n$, $\varphi_j(\lambda) = 0$ for all $j > \dim \mathcal{H}(\lambda)$ and $\{\varphi_j(\lambda) : 1 \leq j \leq \dim \mathcal{H}(\lambda)\}$ is a orthogonal basis of $\mathcal{H}(\lambda)$. Up to certain technicalities, the main idea is just to apply Gram–Schmidt orthogonalization process. Details can be found in part II Chapter 1, Lemma 2.1 and Proposition 4.1 in [3]. Since choosing a pointwise orthonormal basis on the fiber $\mathcal{H}(\lambda)$ is equivalent to fix an unitary operator from $\mathcal{H}(\lambda)$ to $l^2(\mathbb{N})$, then the partition induces maps $T_n : H|_{\Lambda_n} \rightarrow \mathcal{H}_n$ such that \mathcal{H}_n is a Hilbert space and $T_n|_{\mathcal{H}(\lambda)}$ is unitary for all $\lambda \in \Lambda_n$. Moreover, the map $T : H \rightarrow \bigsqcup \mathcal{H}_n$ defined by $T(x) = T_n(x)$, for every $x \in H$ with $p(x) \in \Lambda_n$, can be interpreted as a sort of measurable trivialization. Indeed, φ is measurable if and only if $T(\varphi)$ is weakly measurable. Therefore H is equivalent to the bundle $\bigcup \Lambda_n \times \mathcal{H}_n$, and we have met the following definition of measurable Hilbert bundle.

DEFINITION A.2. Let Λ be a measurable space. A (separable) measurable Hilbert bundle (MHB) over Λ is a disjoint union $H = \bigcup \Lambda_n \times \mathcal{H}_n$ where $\{\Lambda_n\}$ is a measurable partition of Λ and \mathcal{H}_n is a Hilbert space of dimension n , with $0 < n \leq \infty$.

Conversely, if T is a trivialization $T : H \rightarrow \bigsqcup \mathcal{H}_n$ as before, we can define the space Γ as the set of all the sections φ of H such that $T\varphi$ is weakly measurable. Clearly, Γ satisfies (i), (ii) and (iii) of Definition A.1.

A much richer structure is obtained if a fixed measure η on Λ is considered. Notice that the notion of measurable Hilbert bundle in [25] requires Λ to be a σ -finite measure space. Moreover, the third way to describe measurable fields of Hilbert spaces also requires the latter additional assumption and it is given within the context of Hilbert modules over $L^\infty(\Lambda, \eta)$. If \mathcal{E} is such Hilbert module, we can define the dual module \mathcal{E}^* in a canonical way, i.e. taking the bounded linear maps from the module to $L^\infty(\Lambda, \eta)$. When every element $f \in \mathcal{E}^*$ admits a representation of the form $f(\phi) = \langle \phi, \psi \rangle$ for some $\psi \in \mathcal{E}$, the module is called

self-dual. If Λ is a σ -finite measure space and $H \rightarrow \Lambda$ is a measurable Hilbert bundle, it is not difficult to show that the space of weakly measurable essentially bounded sections of a measurable bundle is a self-dual Hilbert $L^\infty(\Lambda)$ -module (Propositions 9.2.3 in [25]). Conversely, every self-dual weakly separable module is obtained in the latter way (Theorem 9.2.4 in [25]). In summary, there are two equivalent ways to introduce measurability for fields of Hilbert spaces over a measurable space Λ :

- (i) measurable fields of Hilbert spaces;
- (ii) measurable Hilbert bundles.

If in addition we assume (Λ, η) is a σ -finite measure space the latter notions are equivalent to the following:

- (iii) self-dual weakly separable Hilbert $L^\infty(\Lambda, \eta)$ -modules.

One of the main reasons to consider measurable fields of Hilbert spaces is that it allow us to introduce direct integrals (we require them in Subsection 2.1).

DEFINITION A.3. Let (Λ, η) be a measure space and $H \rightarrow \Lambda$ be a measurable field of Hilbert spaces. Up to quotient by the space of measurable sections vanishing almost everywhere, the direct integral $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ is the space of square integrable measurable vector fields, i.e.

$$\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda) := \left\{ \varphi \in \Gamma : \int_{\Lambda} \|\varphi(\lambda)\|_{\lambda}^2 d\eta(\lambda) < \infty \right\}.$$

It is well-known (for instance, see part II, Chapter I, Proposition 5 in [3]) that $\int_{\Lambda}^{\oplus} \mathcal{H}(\lambda) d\eta(\lambda)$ is a Hilbert space with the inner product

$$\langle \varphi, \psi \rangle := \int_{\Lambda} \langle \varphi(\lambda), \psi(\lambda) \rangle_{\lambda} d\eta(\lambda).$$

In the topological framework there are also three equivalent ways to introduce continuity for fields of Hilbert spaces. Assume that Λ is a Hausdorff locally compact space. The first historical definition given by Godement [8] of a continuous field of Hilbert spaces is the following.

DEFINITION A.4. Let Λ be a locally compact space and $H \rightarrow \Lambda$ a field of Hilbert spaces. A continuous structure is given by specifying a linear space of sections Γ possessing the following properties:

- (i) the set $\{\varphi(\lambda) : \varphi \in \Gamma\}$ is dense in $\mathcal{H}(\lambda)$, for every $\lambda \in \Lambda$;
- (ii) the function $\lambda \rightarrow \|\varphi(\lambda)\|$ is continuous, for every $\varphi \in \Gamma$.

Latter Dixmier and Douady in [4, 5] added the following condition:

- (iii) if φ is a section and for every $\lambda_0 \in \Lambda$ and every $\varepsilon > 0$ there exists $\varphi' \in \Gamma$ such that $\|\varphi(\lambda) - \varphi'(\lambda)\| \leq \varepsilon$ for every λ in some neighborhood (depending on ε) of λ_0 , then $\varphi \in \Gamma$.

They proved that if Γ satisfies conditions (i) and (ii) then there exists a unique space of sections $\tilde{\Gamma}$ satisfying (i), (ii) and (iii) such that $\Gamma \subseteq \tilde{\Gamma}$ (Proposition 3 in [5] or Proposition 10.2.3. in [4]). Moreover, they also show that if Γ satisfies (i), (ii) and (iii) then the following properties hold:

- (iv) Γ is a $C(\Lambda)$ -module;
- (v) the set $\{\varphi(\lambda) : \varphi \in \Gamma\}$ equals $\mathcal{H}(\lambda)$, for every $\lambda \in \Lambda$.

Note that if Γ satisfies (i), (ii) and (iv) then Γ is dense in $\tilde{\Gamma}$ with respect to the local uniform convergence topology.

Now we consider the definition of a continuous Hilbert bundle. The first definition of this structure can be found in [21]. We follow [25] as reference as we did in the measurable case.

DEFINITION A.5. Let Λ be a compact Hausdorff space. A covering space of Λ is a topological space H together with a continuous open surjection $p : H \rightarrow \Lambda$. A continuous Hilbert bundle over Λ is then a covering space H such that $\mathcal{H}(\lambda) = p^{-1}(\lambda)$ is equipped with a Hilbert space structure for each $\lambda \in \Lambda$, and satisfying the following conditions:

- (i) the map $x \rightarrow \|x\|$ is continuous from H to \mathbb{R} ;
- (ii) the map $(x_1, x_2) \rightarrow x_1 + x_2$ is continuous from $H \times H$ to H ;
- (iii) the map $x \rightarrow ax$ is continuous from H to H for every $a \in \mathbb{C}$; and
- (iv) for any neighborhood \mathcal{O} of the origin of $\mathcal{H}(\lambda)$ in H there exists a neighborhood \mathcal{O}' of λ in Λ and an $\varepsilon > 0$ such that

$$\{x \in H : p(x) \in \mathcal{O}' \text{ and } \|x\| < \varepsilon\} \subset \mathcal{O}.$$

A section of H is a function $\varphi : \Lambda \rightarrow H$ such that $\varphi(\lambda) \in \mathcal{H}(\lambda)$, for all $\lambda \in \Lambda$. The set of all continuous sections of H is denoted $\Gamma(H)$.

Clearly, if $H \rightarrow \Lambda$ is a continuous Hilbert bundle, then $\Gamma(H)$ satisfies the conditions (i), (ii) and (iii) of Definition A.4. Conversely, if Γ satisfies (i) and (ii) of Definition A.4 then there is a topology on H such that $H \rightarrow \Lambda$ is a continuous Hilbert bundle and $\Gamma \subset \Gamma(H)$ (in fact $\tilde{\Gamma} = \Gamma(H)$).

The third way to describe continuous fields of Hilbert spaces is through the notion of Hilbert $C_0(\Lambda)$ -module. In fact, the space of continuous sections vanishing at infinity $\Gamma_0(H)$ of a continuous Hilbert bundle is clearly a Hilbert $C_0(\Lambda)$ -module. Conversely, every Hilbert $C_0(\Lambda)$ -module Γ_0 corresponds to the space of continuous sections vanishing at infinity of a Hilbert bundle. Indeed, for $\lambda \in \Lambda$, let $I_\lambda = \{f \in C_0(\Lambda) : f(\lambda) = 0\}$. It is easy to show that the quotient $\mathcal{H}(\lambda) = \Gamma_0 / I_\lambda \Gamma_0$ is a Hilbert space. Therefore, Γ_0 can be regarded as a space of sections by defining $\varphi(\lambda) = \pi_\lambda(\varphi)$ for every $\varphi \in \Gamma_0$, where $\pi_\lambda : \Gamma_0 \rightarrow \Gamma_0 / I_\lambda \Gamma_0$ is the canonical projection. Moreover, the map $\lambda \rightarrow \|\varphi(\lambda)\|$ is clearly continuous, then Γ_0 defines a continuous field of Hilbert spaces (see Definition A.4). The latter equivalence was first noticed in [21]. If we want to pass directly from Hilbert

$C_0(\Lambda)$ -modules to continuous Hilbert bundles over Λ , we can apply Proposition 9.15 in [25] in the compact case. In the general case and use a one-point compactification, as explained in [19].

Summarizing, there are three equivalent ways to introduce the notion of continuity for fields of Hilbert spaces:

- (i) continuous fields of Hilbert spaces;
- (ii) continuous Hilbert bundles;
- (iii) Hilbert $C_0(\Lambda)$ -modules.

The reader should notice the similarities between these three descriptions and the analogous three descriptions given for the measurable framework.

In Section 4 a similar discussion can be found for the case of continuous fields of C^* -algebras.

REFERENCES

- [1] F. BELMONTE, Canonical quantization of constants of motion, *Rev. Math. Phys.* **32**(2020), 2050030 (35 pages).
- [2] M.S. BIRMAN, M.Z. SOLOMYAK, *Spectral Theory of Self-adjoint Operators in Hilbert Spaces*, Math. Appl. (Soviet Ser.), vol. 5, Reidel Publ. Co., Dordrecht 1987.
- [3] J. DIXMIER, *Von Neumann Algebras*, North-Holland Math. Library, vol. 27, North-Holland Publ. Cmp., Amsterdam-New York-Oxford 1981.
- [4] J. DIXMIER, *C^* -Algebras*, North-Holland Math. Library, vol. 15, North-Holland, Amsterdam 1983.
- [5] J. DIXMIER, A. DOUADY, Champs continus d'espaces Hilbertiens et de C^* -algèbres, *Bull. Soc. Math. France* **91**(1963), 227–284.
- [6] B. FEDOSOV, A simple geometrical construction of deformation quantization, *J. Differential Geom.* **40**(1994), 213–238.
- [7] G.B. FOLLAND, *Harmonic Analysis in Phase Space*, Ann. Math. Stud., vol. 122, Princeton Univ. Press, Princeton, NJ 1989.
- [8] R. GODEMENT, Sur la theorie des representations unitaires, *Ann. of Math.* **53**(1951), 68–124.
- [9] V. GUILLEMIN, S. STERNBERG, Geometric quantization and multiplicities of group representations, *Invent. Math.* **67**(1982), 515–538.
- [10] N.P. LANDSMAN, *Mathematical Topics between Classical and Quantum Mechanics*, Springer Monogr. Math., Springer-Verlag, New York 1999.
- [11] S. LANG, *Fundamentals of Differential Geometry*, Grad. Texts Math., vol. 191, Springer-Verlag, New York 1999.
- [12] L. LEMPERT, R. SZÖKE, Direct images, fields of Hilbert spaces, and geometric quantization, *Comm. Math. Phys.* **327**(2014), 49–99.
- [13] J.E. MARSDEN, A. WEINSTEIN, Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5**(1974), 121–130.
- [14] K.R. MEYER, Symmetries and integrals in mechanics, in *Dynamical Systems (Proc. Sympos., Univ. Bahia, Salvador, 1971)*, Academic Press, New York 1973, pp. 259–272.

- [15] J. VON NEUMANN, On rings of operators. Reduction theory, *Ann. of Math.* **50**(1949), 401–485.
- [16] A.C.S. NG ABRAHAM, Direct integrals of strongly continuous operator semigroups, *J. Math. Anal. Appl.* **489**(2020), No. 2, Article ID 124176, 20 p.
- [17] A.E. NUSSBAUM, Reduction theory for unbounded closed operators in Hilbert spaces, *Duke Math. J.* **31**(1964), 33–44.
- [18] A. PAL, Regular operators on Hilbert C^* -modules, *J. Operator Theory* **42**(1999), 331–350.
- [19] N.C. PHILLIPS, N. WEAVER, Modules with norms which take values in a C^* -algebra, *Pacific J. Math.* **180**(1998), 163–181.
- [20] I. RAEBURN, D. WILLIAMS, *Morita Equivalence and Continuous-Trace C^* -Algebras*, Math. Surveys Monogr., vol. 60, Amer. Math. Soc., Providence, RI 1998.
- [21] N. TAKAHASHI, A duality between Hilbert modules and fields of Hilbert spaces, *Rev. Colomb. Mat.* **13**(1979), 93–120.
- [22] H. TAKEMOTO, Decomposable operators in continuous fields of Hilbert spaces, *Tohoku Math. J.* **27**(1975), 413–435.
- [23] M. TAYLOR, *Pseudodifferential Operators*, Princeton Math. Ser., vol. 34, Princeton Univ. Press, Princeton 1981.
- [24] R. TIEDRA DE ALDECOA, Time delay for dispersive systems in quantum scattering theory, *Rev. Math. Phys.* **21**(2009), 675–708.
- [25] N. WEAVER, *Mathematical Quantization*, Stud. Adv. Math., Chapman & Hall/CRC, Boca Raton, FL 2001.
- [26] H. WEYL, Quantenmechanik und Gruppentheorie, *Z. Physik* **46**(1927), 1–46.
- [27] D. WILLIAMS, *Crossed Products of C^* -Algebras*, Math. Surveys Monogr., vol. 134, Amer. Math. Soc., Providence, RI 2007.
- [28] N.M.J. WOODHOUSE, *Geometric Quantization*, second edition, Clarendon Press, Oxford 1992.
- [29] D. YAFAEV, *Mathematical Scattering Theory: General Theory*, Transl. Math. Monogr., vol. 105, Amer. Math. Soc., Providence, RI 1992.
- [30] M. ZWORSKI, *Semi-Classical Analysis*, Grad. Stud. Math., vol. 138, Amer. Math. Soc., Providence, RI 2012.

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Received March 16, 2022.