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HERMITIAN LIFTINGS IN $B(\ell_p)$

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1. INTRODUCTION

Let X be a complex Banach space, $B(X)$ the space of bounded, linear operators on X and $C(X)$ the corresponding space of compact linear operators. The symbol $A(X)$ will denote the Calkin algebra $B(X)/C(X)$ and as is known both $B(X)$ and $A(X)$ are complex Banach algebras with unit e. For such unital Banach algebras B, set

$$S = \{f \in B^*: f(e) = 1 = \|f\|\}$$

and define the numerical range of $x \in B$ as

$$W(x) = \{f(x) : f \in S\}.$$

It was shown in ref. [11] that $W(x)$ is a nonempty, compact, convex set and that the radius of $W(x)$ is an equivalent norm on B. Hereafter $W(T)$ will denote the numerical range of $T \in B(X)$ and $W_e(T)$ the numerical range of $T + C(X)$ in $A(X)$ or, for short, the essential numerical range of T. In accordance with the terminology of [3] and [4], we say an element x of $B(X)$ is Hermitian if $W(x)$ is contained in the real line R; an element $x \in B(X)$ is said to be essentially Hermitian if $W_e(x) \subseteq R$. For more information on numerical ranges see refs. [3] and [4].

The main result of this paper is that if $X = \ell_p$, $1 < p < \infty$, then for each $T \in B(\ell_p)$ there exists a $K \in C(\ell)$ such that

$$W(T + K) = W_e(T).$$

In proving this result we answer a question of F. F. Bonsall [2] in the affirmative, namely: Is every essentially Hermitian element in $B(\ell_p)$, $1 < p < \infty$, a compact perturbation of a Hermitian element of $B(\ell_p)$?

2. ESSENTIALLY HERMITIAN ELEMENTS IN $B(\ell_p)$

The aim of this section is to prove the main result as stated in the introduction. The key to this is to establish the following fact: every essentially Hermitian element in $B(\ell_p)$, $1 < p < \infty$, is a compact perturbation of a Hermitian element of $B(\ell_p)$.

This result is trivial in $B(\ell_2)$. Indeed, an operator in $B(\ell_2)$ is essentially Hermitian if and only if its imaginary part is compact. It has been established by Tam [12, Theorem 2] that the Hermitian elements of $B(\ell_p)$, $1 \leqslant p < \infty$, $p \neq 2$, are precisely those operators which are real-diagonal with respect to the canonical basis $\{e_i\}_{i=1}^{\infty}$. Our main result is this: *The essentially Hermitian operators in* $B(\ell_p)$ *are exactly of the form real-diagonal plus compact for* $1 < p < \infty$, $p \neq 2$.

The proof will proceed in a series of lemmas. First note that every operator of the form real-diagonal plus compact is essentially Hermitian [3, p. 127] so the crux of the proof is to verify the converse. The basic idea is to show that for $p \neq 2$, any matrix in $B(\ell_p)$ at a fixed distance from its real-diagonal part must have a "large" imaginary part in its numerical range. In what follows, let

$\mathscr{P} = \{P$: P is a finite or co-finite rank projection onto a subspace of the form span $\{e_{n_i}\}\}$.

If $P \in \mathscr{P}$, $P^{\perp}$ is defined as $I - P$.

For $T \in B(\ell_p)$ define the *numerical radius*

$$r(T) = \sup\{|s| : s \in W(T)\}$$

and the *imaginary radius*

$$r_i(T) = \sup\{|\mathrm{Im}\, s| : s \in W(T)\}.$$

As will be shown, $r_i(T)$ is a measure of the distance of T to the Hermitian operators in $B(\ell_p)$.

If ψ is any unit vector in ℓ_p then ψ' will denote a linear functional in the dual ℓ_q of ℓ_p such that $\psi'(\psi) = 1 = \|\psi'\|_q$ (if $p > 1$, ψ' is unique). In coordinate notation, if

$$\psi = (\psi_1, \psi_2, \ldots)$$

then

$$\psi' = (\mathrm{sgn}\, \psi_1 |\psi_1|^{p-1}, \mathrm{sgn}\, \psi_2 |\psi_2|^{p-1}, \ldots)$$

where $\mathrm{sgn}\, \mu = e^{i\theta}$ if $\mu = \rho e^{i\theta}$. In Lemma 1, below, the following observation is used: If φ and ψ are unit vectors in ℓ_p having disjoint supports (always with respect to the canonical basis), and if c and d are complex scalars satisfying $|c|^p + |d|^p = 1$, then $c\varphi + d\psi$ is a unit vector in ℓ_p and

$$(c\varphi + d\psi)' = \mathrm{sgn}\, c\, |c|^{p-1} \varphi' + \mathrm{sgn}\, d\, |d|^{p-1} \psi'.$$

We wish to thank Professor W. B. Arveson for, among other things, pointing out to us the paper of Christensen [6] and sketching a proof of Lemma 6.

LEMMA 1. *Let $\mathscr{A} = \{A: A$ is a matrix with respect to the canonical basis in ℓ_p with only a finite number of nonzero entries$\}$. Then*

$$\sup_{P \in \mathscr{P}} \|PAP^{\perp}\|_p \leqslant c_p r_i(A) < \infty, \text{ for all } A \in \mathscr{A},$$

where $c_p^{-1} = \dfrac{1}{3p} \left|(p-1)^{(p-1)/p} - (p-1)^{1/p}\right|,\ p > 1,$ *and* $c_p^{-1} = 1/3,\ p = 1.$

Proof. Fix $A \in \mathscr{A}$ and define $\alpha = \sup_{P \in \mathscr{P}} \|PAP^{\perp}\|_p$. Since A has only finitely many nonzero entries there are unit vector φ and ψ and a projection $P \in \mathscr{P}$ so that $\langle PAP^{\perp}\varphi, \psi'\rangle = \alpha$. Moreover it follows that $P^{\perp}\varphi = \varphi$ and $P\psi = \psi$; and so, φ and ψ have disjoint supports. Set $\sigma = c\varphi + d\psi$ for complex numbers c and d such that $\|\sigma\|_p = |c|^p + |d|^p = 1$. Thus

$$\begin{aligned}
r_i(A) &\geqslant |\mathrm{Im}\ \langle A\sigma, \sigma'\rangle| \\
&= |\mathrm{Im}\ \{\langle PAP\sigma, \sigma'\rangle + \langle PAP^{\perp}\sigma, \sigma'\rangle + \langle P^{\perp}AP\sigma, \sigma'\rangle + \langle P^{\perp}AP^{\perp}\sigma, \sigma'\rangle\}| \\
&= |\mathrm{Im}\ \{\langle PAP\, d\psi, \operatorname{sgn} d|d|^{p-1}\psi'\rangle + \langle PAP^{\perp}c\varphi, \operatorname{sgn} d|d|^{p-1}\psi'\rangle \\
&\quad + \langle P^{\perp}AP^{\perp}d\psi, \operatorname{sgn} |c|^{p-1}\varphi'\rangle + \langle P^{\perp}AP^{\perp}c\varphi, \operatorname{sgn} c|c|^{p-1}\varphi'\rangle\}| \\
&\geqslant |\mathrm{Im}\ \{\langle PAP^{\perp}c\varphi, \operatorname{sgn} d|d|^{p-1}\psi'\rangle + \langle P^{\perp}APd\psi, \operatorname{sgn} c|c|^{p-1}\varphi'\rangle\}| - 2r_i(A) \\
&= |\mathrm{Im}\ \{c \operatorname{sgn} \bar{d}|d|^{p-1}\alpha + d \operatorname{sgn} \bar{c}|c|^{p-1}\beta\}| - 2r_i(A),
\end{aligned}$$

where $\beta = \langle P^{\perp}AP\psi, \varphi'\rangle$. So

$$r_i(A) \geqslant \frac{1}{3} |\mathrm{Im}\ \{c \operatorname{sgn} \bar{d}|d|^{p-1}\alpha + d \operatorname{sgn} \bar{c}|c|^{p-1}\beta\}|.$$

Select c and d to make $c \operatorname{sgn} \bar{d}$ purely imaginary. The maximum of $|c|\ |d|^{p-1}$, for $|c|^p + |d|^p = 1$, occurs at $d = ((p-1)/p)^{1/p}$. By construction $|\beta| \leqslant \alpha$; so,

$$\begin{aligned}
&|\mathrm{Im}\ \{c \operatorname{sgn} \bar{d}|d|^{p-1}\alpha + d \operatorname{sgn} \bar{c}|c|^{p-1}\beta\}| \\
&\geqslant \alpha \left| \left(\frac{1}{p}\right)^{1/p} \left(\frac{p-1}{p}\right)^{(p-1)/p} - \left(\frac{1}{p}\right)^{(p-1)/p} \left(\frac{p-1}{p}\right)^{1/p} \right| \\
&= 3\alpha c_p^{-1},
\end{aligned}$$

and the proof is complete for $p > 1$, $p \neq 2$. If $p = 1$, take $|c| = 1 - \varepsilon$ and $|d| = \varepsilon$ for sufficiently small ε to conclude $c_p^{-1} = 1/3$.

REMARK. *Lemma 1 together with Tam's result* [12, *Theorem* 1] *that if A is Hermitian on $B(\ell_p)$ then so is PAP, for any $P \in \mathscr{P}$, establishes that Hermitian elements of $B(\ell_p)$, $1 \leqslant p < \infty$, $p \neq 2$, must be diagonal with respect to the canonical basis.*

The key to the remainder of the proof lies in showing that

$$\sup_{P \in \mathscr{P}} \|PAP^{\perp}\| \geqslant C \operatorname{dist}(A, \mathscr{D})$$

where $\mathscr{D}$ denotes the diagonal operators relative to the canonical basis $\{e_i\}_{i=1}^{\infty}$ and $\operatorname{dist}(A, \mathscr{D}) = \inf\{\|A - D\| : D \in \mathscr{D}\}$. The proof of this inequality is quite technical so the reader may wish to temporarily assume its validity and proceed to the remainder of the proof of the theorem starting with Lemma 7.

Recall that the diagonal operators $\mathscr{D}$ viewed as a subspace of $B(\ell_p)$ for any $1 \leqslant p \leqslant \infty$ is isometrically isomorphic to ℓ_∞. Thus, $\mathscr{D}$ is a C^*-algebra with the $*$ operation being complex conjugation. An element u of $\mathscr{D}$ for which $u^*u = 1$, is called unitary. Recall further that for $1 < p < \infty$, $B(\ell_p)$ is a dual space and, as such, is equipped with a weak star topology [8]. For a given operator $A \in B(\ell_p)$, $1 < p < \infty$, let $K(A)$ denote the weak star closed convex subset of $B(l_p)$ generated by $\{u^*Au | \, u$ a unitary element of $\mathscr{D}\}$. It is the aim here to show that the diagonal of such an operator A relative to the canonical basis, denoted by $\operatorname{diag} A$, is an element of $K(A)$. The above inequality will then be derived rather easily. The necessary preliminary lemmas now follow.

LEMMA 2. *Let $w_{k,n} \equiv e^{i\frac{2\pi k}{n}}$ denote the n roots of unity. Then for any positive integer $j < n$,*

$$\sum_{k=0}^{n-1} w_{jk,n} = 0.$$

Proof. Note that

$$\sum_{k=0}^{n-1} e^{i\frac{2\pi jk}{n}} = \sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z}$$

for $z = e^{i\frac{2\pi j}{n}}$. However $j < n$ implies $z = e^{i\frac{2\pi j}{n}} \neq 1$ with $z^n = 1$ and the conclusion follows.

In what follows, P_n will denote the projection onto the first n coordinate vectors $e_1, \ldots, e_n$.

LEMMA 3. *For each $A \in B(\ell_p)$ and fixed n, there exist n unitary operators in $\mathscr{D}$, $u_1, \ldots, u_n$ such that*

$$P_n\left(\sum_{k=1}^{n} \frac{1}{n} u_k^* A u_k\right) P_n = P_n(\operatorname{diag} A) P_n.$$

Proof. Let $\alpha_{1,k} = 0$, $k = 1, \ldots, n$ and define $\alpha_{r+1,k} - \alpha_{r,k} = \dfrac{2\pi k}{n}$, $r = 1, \ldots, \ldots, n-1$. For $r > n$, and $k = 1, \ldots, n$, $\alpha_{r,k}$ may assume any real value. For $A \equiv (a_{ij})_{i,j=1}^{\infty}$ and the unitary diagonal operators $u_k \equiv (e^{i\alpha_{j,k}})_{j=1}^{\infty}$, the (r, s) entry of $u_k^* A u_k$ is given by $e^{i(\alpha_{s,k} - \alpha_{r,k})} a_{r,s}$ while the (r, s) entry of $\sum_{k=1}^{n} u_k^* A u_k$ is $\sum_{k=1}^{n} e^{i(\alpha_{s,k} - \alpha_{r,k})} a_{r,s}$. Thus for $r = 1, \ldots, n-1$

$$\sum_{k=1}^{n} e^{i(\alpha_{r+1,k} - \alpha_{r,k})} = \sum_{k=1}^{n} e^{i\frac{2\pi k}{n}} = 0,$$

and thus the $(r, r+1)$ entry of $\sum_{k=1}^{n} \dfrac{1}{n} u_k^* A u_k$ is zero. More generally for $s - r = j$, $0 < j < n$, j an integer

$$\begin{aligned}\alpha_{s,k} - \alpha_{r,k} &= (\alpha_{s,k} - \alpha_{s-1,k}) + (\alpha_{s-1,k} - \alpha_{s-2,k}) + \ldots + (\alpha_{r+1,k} - \alpha_{r,k}) \\ &= \frac{2\pi k}{n} \cdot j.\end{aligned}$$

It follows from Lemma 2 that

$$\sum_{k=1}^{n} e^{i(\alpha_{s,k} - \alpha_{r,k})} = \sum_{k=1}^{n} e^{i\frac{2\pi jk}{n}} = 0,$$

and for such r and s the (r, s) entry of $\sum_{k=1}^{n} \dfrac{1}{n} u_k^* A u_k$ is zero. The case $s - r = j$, $0 > j > -n$ follows analogously and this completes the proof of the lemma.

In ref. [8, Thms. 3.1, 3.2], Hennefeld has shown that for a Banach space X with an unconditionally monotone, shrinking basis, $B(X) = C(X)^{**}$ where $(\cdot)^{**}$ denotes the second dual space of $(\cdot)$. Moreover, this correspondence is given by

$$A(f) = \sum_{j=1}^{\infty} f(R_j)$$

where $A \in B(X)$, $f \in C(X)^*$ and R_j is the j^{th} row of the matrix A. It was also established that elements in $C(X)^*$ corresponded to infinite matrices in the following manner: if $f \in C(X)^*$ and E_{ij} is the compact operator which takes e_j to e_i and is zero elsewhere, then $f \cong (f(E_{ij}))_{i,j=1}^{\infty}$. In addition, it was established [8, Prop. 3.2] that, under this correspondence, the finite matrices are dense in $C(X)^*$ (here finite means only a finite number of nonzero entries). Of course, ℓ_p, $1 < p < \infty$, satisfies the above assumptions on X.

Recall that a sequence $\{y_n^*\}$ in a dual space Y^* (where Y^* is dual to Y) is weak star convergent to $y^* \in Y^*$ if and only if $y_n^*(y)$ converges to $y^*(y)$ for every $y \in Y$.

LEMMA 4. *The operator* $\operatorname{diag}(A)$ *is an element of* $K(A)$ *for every* $A \in B(\ell_p)$, $1 < p < \infty$.

Proof. By Lemma 3, it suffices to show that for given $A \in B(\ell_p)$, the operators $\sum_{k=1}^{n} \frac{1}{n} u_k^* A u_k$ converge to $\operatorname{diag} A$ in the weak star topology. Thus it must be established that

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{n} u_k^* A u_k \right) \varphi = (\operatorname{diag} A)\varphi$$

for every $\varphi \in C(\ell_p)^*$. In the light of the discussion preceding the lemma, φ may be assumed to be a finite matrix. Thus if $\varphi(E_{ij}) = 0$ for all $i, j > N$, then clearly for $n \geqslant N$,

$$\varphi \left(\sum_{k=1}^{n} \frac{1}{n} u_k^* A u_k \right) = \varphi(\operatorname{diag} A).$$

DEFINITION. *For a given operator* $A \in B(\ell_p)$,

$$\|D_A|\mathscr{D}\| \equiv \sup \{\|SA - AS\| : S \in \mathscr{D},\ \|S\| = 1\}.$$

The next lemma is similar to [6, Theorem 2.3].

LEMMA 5. *For a given* $A \in B(\ell_p)$, $\|A - \operatorname{diag} A\| \leqslant \|D_A|\mathscr{D}\|$.

Proof. The proof follows from the inequalities

$$\begin{aligned}
\|\operatorname{diag} A - A\| &\leqslant \limsup_{n\to\infty} \left\| \sum_{k=1}^{n} \frac{1}{n} u_k^* A u_k - A \right\| \\
&\leqslant \limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \|u_k^* A u_k - A\| \\
&= \limsup_{n\to\infty} \max_{1\leqslant k\leqslant n} \|A u_k - u_k A\| \\
&\leqslant \|D_A|\mathscr{D}\|,
\end{aligned}$$

where the second inequality follows from the weak star lower semicontinuity of the norm.

The next result is due to W. B. Arveson [1].

LEMMA 6. $\|D_A|\mathscr{D}\| \leqslant 8 \sup_{P \in \mathscr{P}} \|PAP^{\perp}\|$.

Proof. For any $D \in \mathscr{D}$, $\|D\| \leqslant 1$, write

$$D = D_1 - D_2 + \mathrm{i}(D_3 - D_4)$$

where $0 \leqslant D_i \leqslant 1$. This decomposition implies that

$$\sup\{\|DA - AD\| : D \in \mathscr{D},\ \|D\| \leqslant 1\} \leqslant 4 \sup\{\|DA - AD\| : D \in \mathscr{D},\ 0 \leqslant D \leqslant 1\}.$$

It is easy to see that any $D \in \mathscr{D}$ can be norm approximated by an element of $\mathscr{D}$ with only a finite number of values in its range. Such D, in turn may be expressed as a finite convex combination $\sum_{j=1}^{n} \lambda_j P_j$ where $0 < \lambda_j < 1$, $\sum_{j=1}^{n} \lambda_j = 1$ and P_j is a projection in $\mathscr{D}$. For instance, if the operator D attains the values $0 \leqslant a_0 \leqslant \ldots \ldots \leqslant a_n = 1$ on the sets $E_0, \ldots, E_n$ and if

$$F_i = \bigcup_{j=i}^{n} E_j, \quad i = 0, 1, \ldots, n,$$

then

$$D = a_0 P_{F_0} + (a_1 - a_0) P_{F_1} + \ldots + (a_n - a_{n-1}) P_{F_n}$$

where P_E denotes the projection onto the set E. Thus

$$\begin{aligned} &\sup\{\|DA - AD\| : D \in \mathscr{D},\ 0 \leqslant D \leqslant 1\} \\ &\leqslant \sup\{\|PA - AP\| : P \in \mathscr{D},\ P^2 = P\} \\ &\leqslant \sup\{\|PA - AP\| : P \in \mathscr{P}\} + \varepsilon \\ &\leqslant 2 \sup\{\|PAP^{\perp}\| : P \in \mathscr{P}\} + \varepsilon. \end{aligned}$$

For any projection P, $\|PA - AP\|$ may be approximated by $\|P'A - AP'\|$ where P' has finite rank. This fact insures the validity of the next to last inequality above. Since ε is arbitrary, the conclusion follows.

LEMMA 7. *Let $A \in B(\ell_p)$, $1 < p < \infty$, $p \neq 2$, be any matrix with only a finite number n of nonzero entries. Then*

$$\|A - \operatorname{diag} A\| \leqslant \frac{8 r_i(A)}{c_p}.$$

Proof. Lemmas 1, 5 and 6 imply that

$$r_i(A) \geqslant c_p \sup \{\|PAP^{\perp}\| : P \in \mathscr{P}\}$$

$$\geqslant \frac{c_p}{8} \sup \{\|DA - AD\| : D \in \mathscr{D},\ \|D\| \leqslant 1\}$$

$$\geqslant \frac{c_p}{8} \|A - \operatorname{diag} A\|$$

Thus

$$\|A - \operatorname{diag} A\| \leqslant \frac{8r_i(A)}{c_p}$$

and is independent of n.

Recall that an operator $T \in B(\ell_p)$ is called real-diagonal if T is a diagonal matrix with respect to the canonical basis $\{e_i\}_{i=1}^{\infty}$ with only real entries. An operator $T \equiv (A_n)_{n=1}^{\infty}$ will be called block diagonal if it is block diagonal with finite rank blocks with respect to the canonical basis.

LEMMA 8. *Let T be an essentially Hermitian block diagonal operator in $B(\ell_p)$, $p \neq 1, 2, \infty$. Then T is of the form $D + K$ where D is a real-diagonal operator and K is compact.*

Proof. Set $T = \{T_n\}_{n=1}^{\infty}$, $A_n = T_n - \operatorname{diag} T_n$. Since obviously both diag T and $T - \operatorname{diag} T$ are essentially Hermitian, we may suppose that $D = \operatorname{diag} T$ is Hermitian and $r_i(T_n) \to 0$. Since by Lemma 7 we have

$$\|A_n\| = \|T_n - \operatorname{diag} T_n\| \leqslant \frac{8r_i(T_n)}{c_p},$$

it follows that

$$T - D = \{A_n\}_{n=1}^{\infty}$$

is compact.

THEOREM 1. *Let $1 < p < \infty$, $p \neq 2$. Then $T \in B(\ell_p)$ is an essentially Hermitian operator on $B(\ell_p)$ if and only if T has the form $T = D + K$ where D is real-diagonal and K is compact.*

Proof. Assume the contrary holds. That is, $T_0 = T -$ (real-diagonal part of T) is an essentially Hermitian non-compact operator. Then there is a sequence of unit vectors $\{\Phi_i\}$ tending weakly to zero for which $\|T\Phi_i\|_p \geqslant \varepsilon$, for all i and some $\varepsilon > 0$. From this sequence, using standard methods, construct a sequence $\{\psi_j\}$ of unit vectors with mutually disjoint finite supports V_j for which $\|P_{V_j} T \psi_j\|_p \geqslant \varepsilon/2$ for all j, where P_{V_j} is the projection to the span of V_j. Then the operator $T_1 = \sum_i P_{V_i} T_0 P_{V_i}$ is a non-compact essentially Hermitian (cf. [12, Thm. 1]) block diagonal operator which contradicts the hypothesis of Lemma 8. This contradiction in turn proves the theorem.

THEOREM 2. *Let* $1 < p < \infty$. *Then for each* $T \in B(\ell_p)$ *there exists a* $K \in C(\ell_p)$ *such that*

$$W(T + K) = W_e(T).$$

Proof. Note that by [11, Theorem 1], $W_e(T)$ is convex. Now suppose that the interior of $W_e(T)$ is nonempty. Then by [7, Theorem 1], there exists a $K \in B(\ell_p)$ such that $W(T + K) = W_e(T)$. Now assume that the interior of $W_e(T)$ is empty. The case $p = 2$ is already treated in ref. [7, Theorem 3.2]. For $p \neq 2$, by suitable translation by αI and rotation we may assume $W_e(T) \subset R$. By Theorem 1, $T = D + K$ where D is real-diagonal. After perturbing D by a compact operator K_1, we may assume that the new diagonal operator $(a_{ii})_{i=1}^{\infty}$ has the property

$$\max |a_{ii}| = \lim \sup |a_{ii}|$$

and

$$\min a_{ii} = \lim \inf a_{ii}.$$

follows that $T - K - K_1$ is a Hermitian operator in $B(\ell_p)$ satisfying

$$W(T - K - K_1) = W_e(T).$$

3. REMARKS

As mentioned previously, the numerical radius provides an equivalent norm in a complex Banach algebra with unit. It might be suspected that the distance of an element x to the Hermitian elements of a space is related to the size of the imaginary part of its numerical range. R. R. Smith has pointed out the following example which indicates that this conjecture is not, in general, true.

Example. Let $X = A(D)$ be the disc algebra of analytic functions continuous on the boundary. It is easy to establish that for each element $f \in A(D)$,

$$W(f) = \text{conv}\,\{f(z): z \in D\}.$$

Using arguments similar to those in ref. [7, Example 4.1] or using the Riemann mapping theorem, one may construct a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \in A(D)$, so that

$$W(f_n) = \text{conv}\,\{0, 1, i/n\}.$$

However since analytic maps are open, it may be immediately, verified that the only Hermitian elements of $A(D)$ are the real constant maps. Thus dist(f_n, Hermitian elements of $A(D)$) is greater than or equal to 1/2. This completes the example.

On the other hand, the spaces $B(\ell_p)$ and $B(\ell_p)/C(\ell_p)$ are examples where the maximum of the imaginary part of the numerical range provides an indication of the distance of an element x to the corresponding set of Hermitian elements. This latter property seems crucial in answering any Hermitian lifting property, and the results of Section 2 carry over to a wide class of Orlicz spaces, where this distance can be estimated.

We close this paper with a question.

Stampfli [10] has shown that for any operator $T \in B(\ell_2)$, there exists a compact operator K so that

$$\sigma(T+K) = \sigma_W(T),$$

where $\sigma(S)$ denotes the spectrum of S and $\sigma_W(S)$ designates the Weyl spectrum of S. This result is the spectral analogue of Theorem 2. Our question is the following: does Stampfli's result extend to $B(\ell_p)$, $1 < p < \infty$?

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