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CONNECTIONS BETWEEN AN OPERATOR AND A COMPACT OPERATOR THAT YIELD HYPERINVARIANT SUBSPACES

SCOTT BROWN

The algebra of all bounded, linear operators on a complex Banach space X will be denoted by $\mathscr{L}(X)$. Lomonosov's theorem [4] states that if an element $S \in \mathscr{L}(X)$ is not a multiple of the identity operator on X, and if S commutes with a nonzero compact operator $K \in \mathscr{L}(X)$, then S has a nontrivial hyperinvariant subspace. The theorem can be easily modified to include the case where S and K anticommute ($SK + KS = 0$). This and other variations on the hypotheses of the theorem are discussed in this note.

DEFINITION. For $S, K \in \mathscr{L}(X)$ and $\lambda \in \mathbf{C}$, S *λ-commutes with* K if

$$SK = \lambda KS.$$

A subalgebra, $\mathscr{A}$, of $\mathscr{L}(X)$ is called transitive if no nontrivial, closed subspaces of X are invariant under $\mathscr{A}$. The following is extracted from [4] in the Pearcy-Shields article [5] (Theorem 2, p. 222).

THEOREM 1. (Lomonosov). *If $\mathscr{A}$ is a transitive subalgebra of $\mathscr{L}(X)$, and if non-zero $K \in \mathscr{L}(X)$ is compact, then there exists $A \in \mathscr{A}$ such that* 1 *is an eigenvalue of AK.*

This can be used to prove a slight generalization of Lomonosov's theorem.

THEOREM 2. *Suppose $S \in \mathscr{L}(X)$ is not a multiple of the identity of $\mathscr{L}(X)$. Let $\mathscr{A}$ denote the commutant of S. If S λ-commutes with a nonzero, compact $K \in \mathscr{L}(X)$, then $\mathscr{A}$ has a nontrivial, invariant subspace.*

Proof. Case I: $|\lambda| \neq 1$. By Theorem 1 if $\mathscr{A}$ is transitive, then there exists $A \in \mathscr{A}$ such that AKS has eigenvalue 1. It is known (e.g. see [8], problem 2(b), p. 259) that for any two elements, a and b, of a Banach algebra $\mathscr{T}$, $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$ where $\sigma(x)$ denotes the spectrum of $x \in \mathscr{T}$. Hence $\sigma((AK)S) \cup \{0\} = \sigma(S(AK)) \cup \{0\}$, and so $\sigma(AKS) \cup \{0\} = \lambda\sigma(AKS) \cup \{0\}$ since $SAK = \lambda AKS$. This can only happen if $\sigma(AKS) = \{0\}$. This contradicts the fact that AKS has eigenvalue 1.

Case II: $|\lambda| \leq 1$. If $\mathscr{A}$ is transitive, then there exists $A \in \mathscr{A}$ such that the set $\mathscr{D} = \{x \in X: AKx = \beta x$ for some $\beta \in \mathbf{C}$, $|\beta| \geq 1\}$ contains more than just $0 \in X$. Let $\mathscr{F}$ be the subspace of X spanned by $\mathscr{D}$. Then $\mathscr{F} \neq 0$ and is finite dimensional. If $e \in \mathscr{D}$ then $AKe = \beta e$ for some $\beta \in \mathbf{C}$ with $|\beta| \geq 1$ and

$$\beta Se = SAKe = \lambda AKSe.$$

So, $Se \in \mathscr{D}$. This yields $S\mathscr{F} \subset \mathscr{F}$ and that S has an eigenvalue since $\mathscr{F}$ is finite dimensional. The eigenspace associated with this eigenvalue is invariant under $\mathscr{A}$.

In both cases, a contradiction to the supposed transitivity of $\mathscr{A}$ was achieved, thus completing the proof.

REMARK. *If X is reflexive, then case* II *suffices. For if $SK = \lambda KS$ then $K^* S^* = \lambda S^* K^*$ where S^* and K^* denote the adjoints of S and K respectively. If $|\lambda| \geq 1$, case* II *can now be applied to S^* and K^*, since S^* $\frac{1}{\lambda}$-commutes with K^*. The annihilator of an invariant subspace of $\mathscr{A}^*$ is invariant under $\mathscr{A}$. (The annihilator, $\mathscr{S}$, of a subspace $\mathscr{S}^*$, of the dual of X is $\mathscr{S} = \{s \in X: s^*(s) = 0$ for all $s^* \in \mathscr{S}^*)\}$.*

Example. Let $\{\varphi_i\}_{i=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathscr{H}$. Let S be defined by $S\varphi_i = c_i\varphi_{i+1}$ $(i=1, 2, \ldots)$, $\{c_i\}_{i=1}^{\infty} \in \ell^{\infty}$. Let K be the compact operator defined by $K\varphi_i = \frac{1}{2^i}\varphi_i$, $i = 1, 2, \ldots$. Then S 2-commutes with $K(SK = 2KS)$.

A restatement of Lomonosov's theorem is that if $T, S \in \mathscr{L}(X)$ commute with one another, and S is not a multiple of the identity of $\mathscr{L}(X)$, and if a polynomial in T is compact and nonzero, then S has a nontrivial hyperinvariant subspace. This is also true if S λ-commutes with T, as will be shown using Corollary 1 below.

COROLLARY. 1. *If $\mathscr{S}$ is a finite dimensional subspace of $\mathscr{L}(X)$, $S \in \mathscr{L}(X)$ is not a multiple of the identity, $S\mathscr{S} = \mathscr{S}S$, and if $\mathscr{S}$ contains a nonzero compact operator K, then $\mathscr{A}$ (the commutant of S) has a nontrivial invariant subspace.*

Proof. If the operators $S_R: \mathscr{S} \to \mathscr{S}S$ given by $T \to TS$ for $T \in \mathscr{S}$ and $S_L: \mathscr{S} \to S\mathscr{S}$ given by $T \to ST$ for $T \in \mathscr{S}$ are not invertible, then there exists nonzero $B \in \mathscr{S}$ such that either SB or BS is zero. In this case S has a nontrivial hyperinvariant subspace.

If S_L and S_R are invertible, let $\mathscr{S}'$ be the subspace of $\mathscr{S}$ generated by $(S_L^{-1} \circ S_R)^i K$, $i = 1, 2, \ldots n =$ the dimension of $\mathscr{S}$. Then $\mathscr{S}'$ is invariant under $S_L^{-1} \circ S_R$, and hence there is a nonzero $B \in \mathscr{S}$, and $\lambda \in \mathbf{C}$ such that $(S_L^{-1} \circ S_R)\, B = \lambda B$, or rewriting this expression, $BS = \lambda SB$. Note, however, that for $i = 1, 2, \ldots, n$, the operator $S^n((S_L^{-1} \circ S_R)^i\, K) = S^{n-i}\, KS^i$ is compact, so $S^n B$ is compact. Theorem 2 can now be applied using the relation $(S^n B)S = \lambda S\, (S^n B)$, and the proof is complete.

Now if S λ-commutes with T and $\lambda_0 + \lambda_1 T + \lambda_2 T^2 + \ldots + \lambda_n T^n$ $(\lambda_i \in \mathbf{C}$, $i = 0, 1, 2, \ldots n)$ is compact and nonzero, then Corollary 1 can be applied letting $\mathscr{S}$

be the space spanned by T^i, $i = 0, 1, 2, \ldots n$. The result being that S has a nontrivial hyperinvariant subspace.

A proof similar to that of Corollary 1 can be given for:

COROLLARY 2. *Let* $S \in \mathscr{L}(X)$, *and let* $\mathscr{S}$ *be a finite dimensional subspace of the double commutant of* S *(the commutant of the commutant of* S*). If* $\mathscr{S}$ *contains an element that is not a multiple of the identity, and if there exists a nonzero, compact operator* $K \in \mathscr{L}(X)$ *such that* $K\mathscr{S} = \mathscr{S}K$, *then* S *has a nontrivial hyperinvariant subspace.*

The alteration of hypotheses can be continued even further.

THEOREM 3. *Let* X *be a reflexive Banach space, and let* $\lambda, \theta \in \mathbf{C}$ *be such that* $|\lambda| \neq 1$ *and* $\dfrac{|\theta| - 1}{|\lambda| - 1} \geqslant 0$. *If nonzero* $B, S \in \mathscr{L}(X)$ *are such that* **B** λ*-commutes with* S, *and* B θ*-commutes with a nonzero compact* $K \in \mathscr{L}(X)$, *then* S *has a nontrivial invariant subspace.*

Proof. The remark following Theorem 2 shows that it can be assumed that $|\lambda| < 1$, and so $|\theta| \leqslant 1$. By Theorem 1 if S has no nontrivial invariant subspaces, then there are $\lambda_i \in \mathbf{C}$, $i = 1, 2, \ldots, n$, such that

$$(\lambda_1 S + \lambda_2 S^2 + \ldots + \lambda_n S^n) Kx = x$$

for some $x \in X$, $x \neq 0$. Thus, for $m = 0, 1, 2, \ldots$,

$$B^m(\lambda_1 S + \lambda_2 S^2 + \ldots + \lambda_n S^n) Kx = B^m x =$$

$$[(\lambda_1 \lambda^m \theta^m S + \lambda_2 \lambda^{2m} \theta^m S^2 + \ldots + \lambda_n \lambda^{nm} \theta^m S^n) K] B^m x.$$

For each $m = 0, 1, 2, \ldots$ let T_m be the operator given by the expression inside the square brackets directly above. Then $\|T_m\| \to 0$ as $m \to \infty$, since $|\lambda| < 1$ and $|\theta| \leqslant 1$. However $T_m B^m x = B^m x$ for $m = 0, 1, 2, \ldots$, so for some $m_0 = 1, 2, 3, \ldots$, $B^{m_0} x = 0$. So B has a nontrivial nullspace, and this is invariant for S. By contradiction then, S must have a nontrivial invariant subspace, and the proof is done.

It is tempting to try to extend the theorems above to the case where $SK - \lambda KS$ (K compact) is a rank one operator on X, $\lambda \in \mathbf{C}$. This has already been done in the case $\lambda = 1$ by Daughtry [1] for invariant subspaces and extended to show hyperinvariant subspaces for S by Kim, Pearcy, and Shields [2]. Note, however, that if $\sigma(S) \cap \sigma(\lambda S) = \emptyset$, then given any rank one operator, C, there exists a compact operator K such that $SK - \lambda KS = C$ by Theorem 3.1 *i*) of [7]. (Part *iii*) of the theorem can be used to show that the K involved is actually compact). This suggests that at the very least the additional hypothesis $\sigma(S) \cup \sigma(\lambda S) \neq \emptyset$ is needed.

Both Daughtry and Kim, Pearcy, and Shields made use of Lomonosov's theorem in obtaining their results. A subsequent paper [3] by Kim, Pearcy, and Shields gives examples of applications of these results. It is perhaps of interest then, that when restricted to the Hilbert space case, these examples also provide applications of Theorem 4 below which does not depend on Lomonosov's theorem. The rank one commutators presented in Theorems 2, 3, and 4 of [3] employ compact operators that are in actuality trace class operators if the space involved is a Hilbert space. It if for this reason that the examples are also covered by Theorem 4.

Let $\mathscr{H}$ be a complex Hilbert space. The Banach space $\mathscr{L}(\mathscr{H})$ can be identified in a natural way with the dual of the space of trace class operators, $\mathscr{C}_1$, with trace norm. (Ringrose [6] provides an account of this and related properties of $\mathscr{C}_1$). Any $B \in \mathscr{L}(\mathscr{H})$ can be evidenced as a linear functional on $\mathscr{C}_1$ by $[B](C) = \text{trace}\ (BC)$ for $C \in \mathscr{C}_1$. Now for $A \in \mathscr{L}(\mathscr{H})$, let $\delta_A\colon \mathscr{C}_1 \to \mathscr{C}_1$ be the bounded operator defined by $\delta_A(K) = AK - KA$ for all $K \in \mathscr{C}_1$.

THEOREM 4. *The trace norm closure of the range of* δ_A *(hereafter denoted* $\overline{R(\delta_A)}$*) contains a rank one operator iff* A *has a nontrivial hyperinvariant subspace.*

Proof. Suppose a rank one operator is in $\overline{\mathscr{R}(\delta_A)}$. There exist $x, y \in \mathscr{H}$ such that this rank one operator equals the operator $x \otimes y$ given by $(x \otimes y)\, z = \langle z, y\rangle x$ for all $z \in \mathscr{H}$. The adjoint of δ_A is the operator $(\delta_A)^*\colon \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ given by $(\delta_A)^*(B) = BA - AB$ for all $B \in \mathscr{L}(\mathscr{H})$. (This is an elementary consequence of the fact that trace $(AK) =$ trace (KA) for $K \in \mathscr{C}_1$). The annihilator of the null space of $(\delta_A)^*$ is $\overline{\mathscr{R}(\delta_A)}$. In other words, if $B \in \mathscr{L}(\mathscr{H})$ commutes with A, then trace $(BC)=0$ for all $C \in \overline{\mathscr{R}(\delta_A)}$. So in particular, trace $(B(x \otimes y)) = \langle Bx, y\rangle = 0$ if B commutes with A. The element x is noncyclic for the commutant of A.

If the commutant of A has a nontrivial invariant subspace, then choose $0 \neq x \in \mathscr{H}$ from this subspace and $0 \neq y \in \mathscr{H}$ from its orthogonal complement. Then $x \otimes y \in \mathscr{C}_1$, and since trace $(B(x \otimes y)) = \langle Bx, y\rangle = 0$ for all B in the null space of $(\delta_A)^*$, $x \otimes y \in \overline{\mathscr{R}(\delta_A)}$.

REMARK. *Let* $\mathscr{A}$ *denote the commutant of* $A \in \mathscr{L}(\mathscr{H})$. *Correspondences analogous to those of Theorem 4 hold for the invariant subspaces of the algebra* $\mathscr{A} \oplus \mathscr{A} \oplus \ldots \oplus \mathscr{A}$ *(n times) and the operators of rank* $\leqslant n$ *in* $\overline{\mathscr{R}(\delta_A)}$, *for any* $n = 1, 2, \ldots, \infty$.

Theorem 4 is paranthetical to the main focus of discussion, namely Lomonosov technique, although it does offer a restatement of Lomonosov's theorem. That is, if the range of δ_A is not equal to $\{0\} \subset \mathscr{C}_1$ and is not weak* dense in $\mathscr{C}_1$ (viewing $\mathscr{C}_1$ as the dual of the space of compact operators on $\mathscr{H}$, then $\overline{\mathscr{R}(\delta_A)}$ contains a rank one operator.

In summation, the Lomonosov method can be used to yield invariant subspace results for an operator $S \in \mathscr{L}(X)$ provided certain relations exist between S and some compact operator $K \in \mathscr{L}(X)$. Examples of this are given by Theorems 2 and 3 above and by the rank one commutator results of [1] and [2]. Theorem 2 suggests the possibility that other polynomial identities in the variables S and K may provide hyperinvariant subspaces for S. The vague and general question arises as to what is the crucial connection between an operator S and a compact operator (or operators) that provide an invariant subspace assertion for S. Using still different techniques, but still depending on the compactness of some of the operators involved, Theorems 4 and 9 of [5] give further examples of such phenomena. It may be necessary to catalogue many more results of this type before uncovering the most useful and applicable relations between an operator and a compact operator yielding the desired invariant subspace results.

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The reasoning in Theorem 4 and related discussion originates in part from Theorem 3 of [9].

Addendum. Theorem 2 has also been announced by H. W. Kim and C. Pearcy (Notices Amer. Math. Society, June, 1978, p. A-431).

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SCOTT BROWN, *University of California at Santa Barbara, U.S.A.*

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