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OPERATOR EXTREMAL PROBLEMS, EXPECTATION OPERATORS AND APPLICATIONS TO OPERATORS ON MULTIPLY CONNECTED DOMAINS

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INTRODUCTION

Let R denote a bounded, open, and connected subset of the complex plane whose boundary consists of $n + 1$ analytic nointersecting Jordan curves. Abrahamse and Douglas [4] have shown that bundle shifts on R are models for pure subnormal operators with spectrum contained in R and with normal spectrum (the spectrum of the minimal normal dilation) contained in ∂R. In the case where R is the unit disk D, such a bundle shift reduces to a unilateral shift of some multiplicity, the operator of multiplication by z on $H^2(D) \otimes \mathscr{K}$, where $H^2(D)$ is the standard Hardy space associated with the disk D and $\mathscr{K}$ is a separable Hilbert space. A bundle shift S_α is uniquely determined up to unitary equivalence by an element α of $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$ (homomorphisms from the fundamental group of R into the group of unitary operators $\mathscr{U}(\mathscr{K})$ on a separable Hilbert space $\mathscr{K}$). The dimension of $\mathscr{K}$ ($\dim \mathscr{K}$) is said to be the rank of S_α, and is the multiplicity of the minimal normal extension of S_α. The bundle shift S_α associated with α is realized as multiplication by z on a Hilbert space H^2_α of analytic cross-sections of an analytic vector bundle over R associated with α. An analytic cross-section of such a bundle equivalently can be viewed as a certain type of vector valued multiple-valued analytic function on R whose norm is single-valued. For two elements α and β of $\mathrm{Hom}\,(\pi_0(R), \mathscr{U}(\mathscr{K}))$ and $\mathrm{Hom}\,(\pi_0(R), \mathscr{U}(\mathscr{K}'))$ respectively, the space $H^\infty(\alpha, \beta)$ consists of those (multiple-valued) operator-valued multiplication operators which take H^2_α into H^2_β. An element Θ of $H^\infty(\alpha, \beta)$ is said to be inner if Θ maps H^2_β isometrically into H^2_α. When this is the case necessarily $\dim \mathscr{K}' \leqslant \dim \mathscr{K}$. A generalization of the well-known theorem of Beurling due to Abrahamse and Douglas is that any subspace $\mathfrak{M}$ contained in H^2_α and invariant under $\mathrm{Rat}(S_\alpha)$, the set of all operators $r(S_\alpha)$ where r is any rational function with poles off of the closure of R, is of the form ΘH^2_β for some β in $\mathrm{Hom}\,(\pi_0(R), \mathscr{K}')$, where Θ is an inner element of $H^\infty(\alpha, \beta)$.

In Section 2 of this paper, assuming $\dim \mathscr{K}' \leqslant \dim \mathscr{K}$ and that $\dim \mathscr{K}'$ is finite, we show that inner functions in $H^\infty(\alpha, \beta)$ arise as solutions of certain extremal problems. The idea is to adapt techniques of Widom [30], who studied similar extremal problems but only for scalar-valued functions, to an operator-valued setting. Other authors [6, 15, 16, 22] have studied such extremal problems for single-valued scalar-valued functions. An open question suggested by our result is whether the bundle shift S_α is universal in the sense that, if S_β is any other bundle shift with rank less than or equal to that of S_α, then, is there is a Rat(S_α)-invariant subspace $\mathfrak{M}$ such that the restriction $S_\alpha|\mathfrak{M}$ of S_α to $\mathfrak{M}$ is unitarily equivalent to S_β? Our result is that S_α contains any such S_β of finite rank in this sense.

Section 3 introduces an expectation operator defined on spaces of operator-valued functions on the covering space of R which generalizes the expectation operator of Earle and Marden [13]. This is of great utility for deducing a refinement of the result resting on a theorem of Grauert [17] and Bungart [9] that all bundle shifts of the same rank are similar [4]. We also use this expectation operator to deduce various invariant subspace theorems for spaces of operator-valued functions on R directly from the corresponding results for the disk. In particular we pick up an operator generalization of a theorem of Voichick [29] characterizing the closed ideals of $A(R)$, the space of (scalar) functions continuous on the closure of R and analytic on R. This result is needed to prove a result concerning approximation of a continuous function in $H^\infty(\alpha, \beta)$, by linear combinations of continuous inner functions in $H^\infty(\alpha, \beta)$. Applications of this machinery to commutants of operator models on multiply connected domains will appear in a forthcoming paper [7].

In Section 1 we give some invariant subspace theorems for spaces of operator functions on the unit disk which do not seem to appear explicitly in the literature. These are the results which are then generalized in Section 3 to the setting where R replaces the unit disk.

1. INVARIANT SUBSPACE THEOREMS ON THE DISK

Let $\mathscr{K}$ be a separable Hilbert space, $C^p(\mathscr{K})$ ($1 \leqslant p < \infty$) the Schatten p-class of operators on $\mathscr{K}$ with Schatten p-class norm [26]. For $p = \infty$, let $C^\infty(\mathscr{K})$ denote the compact operators on $\mathscr{K}$ with operator norm. Of course, if $\mathscr{K}$ is finite-dimensional, $C^p(\mathscr{K}) = \mathscr{L}(\mathscr{K})$ for all p and all norms are topologically equivalent, but the norms do affect the duality relations which we shall develop. For X any Banach space, let H^∞_X denote analytic X-valued functions on D uniformly bounded in X-norm, and for $1 \leqslant p < \infty$, let H^p_X denote analytic X-valued functions $F(z)$ on D such that $\|F\|_{H^p_X} = \sup_{0<r<1} \left\{ \int_0^{2\pi} \|F(r\,e^{i\theta})\|^p_X d\theta \right\}^{1/p} < \infty$. When convenient, we will identify an element of H^p_X with an element of L^p_X (weakly-measurable X-valued functions on the unit circle whose norm raised to the p-th power is integrable) via weak boundary values. It is known that under this correspondence H^p_X can be identified isometrically as a closed subspace of L^p_X. Finally we note that $H^p_{C^p(\mathscr{K})}$ is a $H^\infty_{\mathscr{L}(\mathscr{K})}$-module. We are interested in characterizing closed one-sided submodules. A function $B(z)$ in $H^\infty_{\mathscr{L}(\mathscr{K})}$

is said to be rigid if the weak boundary values $B(e^{i\theta})$, $0 \leqslant \theta < 2\pi$, are partial isometries a.e. (with respect to Lebesgue measure) having a fixed initial space.

THEOREM 1.1. a) *A non-zero closed (for $p=\infty$, weak $*$ closed) right $H^{\infty}_{\mathscr{L}(\mathscr{K})}$-submodule I of $H^{p}_{C^{p}(\mathscr{K})}$ is of the form $BH^{p}_{C^{p}(\mathscr{K})}$ where $B \in H^{\infty}_{\mathscr{L}(\mathscr{K})}$ is rigid.* b) *A non-zero closed (for $p=\infty$, weak $*$ closed) left $H^{\infty}_{\mathscr{L}(\mathscr{K})}$-submodule I of $H^{p}_{C^{p}(\mathscr{K})}$ is of the form $H^{p}_{C^{p}(\mathscr{K})}B$ where $B \in H^{\infty}_{\mathscr{L}(\mathscr{K})}$ is rigid.*

Proof: The case $p = 2$ is given by Sarason [25, p. 196—197]. The case $1 \leqslant p < 2$ can be reduced to the $p = 2$ case by adapting the proof of Helson [18, Lecture IV] for the scalar case. To do this, one must consider the case $\dim \mathscr{K} < \infty$ first, so that $C^{p}(\mathscr{K}) = C^{2}(\mathscr{K})$ and the norms are equivalent, and hence $H^{p}_{C^{p}(\mathscr{K})} \supset H^{2}_{C^{2}(\mathscr{K})}$; the case of $\dim \mathscr{K} = \infty$ can then be obtained from this case by approximation.

If $2 < p \leqslant \infty$ and q is chosen so that $\frac{1}{p} + \frac{1}{q} = 1$ ($q = 1$ if $p = \infty$), then $L^{q}_{C^{q}(\mathscr{K})}$ is the dual of $L^{p}_{C^{p}(\mathscr{K})}$, and $1 \leqslant q < 2$. This enables one to reduce the $2 < p \leqslant \infty$ case to the $1 \leqslant p < 2$ case as is done in Helson's argument.

For X a compact metric space, let $C_{C^{\infty}(\mathscr{K})}(X)$ be the space of continuous $C^{\infty}(\mathscr{K})$-valued functions on X. We will be particularly interested in the case where X is the unit circle, but the natural level of generality for the next theorem is with a general X. If $\dim \mathscr{K} = 1$, $C_{C^{\infty}(\mathscr{K})}(X)$ reduces to $C(X)$, the space of continuous complex-valued functions on X. It is well-known [20; p. 57] that a closed ideal of $C(X)$ is precisely the set of all $f(x)$ in $C(X)$ vanishing for all x in some closed subset Y of X. Our purpose here is to generalize this result to characterize the closed right ideals of $C_{C^{\infty}(\mathscr{K})}(X)$.

We will need a few facts concerning the C^*-algebra $C_{C^{\infty}(\mathscr{K})}(X)$. For general facts and definitions concerning C^*-algebras, see Sakai [24]. First of all the dual of $C_{C^{\infty}(\mathscr{K})}(X)$ can be identified with the space $M_{C^{1}(\mathscr{K})}(X)$ of regular Borel trace-class valued measures on X via the natural pairing [10, p. 337]. Therefore a state (positive linear functional of norm one) corresponds to a positive trace-class measure of total variation norm one; a pure state (extreme point of the convex set of states) is necessarily a trace-class measure of the form $\langle \cdot, e \rangle e\, \delta_x$, where e is a unit vector in $\mathscr{K}$, x is in X and δ_x is the unit point mass at x. By the correspondence between pure states and irreducible representations [24, Theorem 1.21.10], it follows that any irreducible representation of $C_{C^{\infty}(\mathscr{K})}(X)$ is unitarily equivalent to evaluation at some point x in X ($F \in C_{C^{\infty}(\mathscr{K})}(X) \to F(x) \in C^{\infty}(\mathscr{K})$); it follows immediately that $C_{C^{\infty}(\mathscr{K})}(X)$ is a Type I C^*-algebra; since X is a compact metric space and $\mathscr{K}$ is separable, we also have that $C_{C^{\infty}(\mathscr{K})}(X)$ is separable. Therefore the Stone-Weierstrass theorem for C^*-algebras of Sakai [24, p. 240] applies to $C_{C^{\infty}(\mathscr{K})}(X)$ and to its C^*-subalgebras.

We are now ready to state our theorem. For T an operator, $[\operatorname{Ran} T]^{-}$ denotes the closure of the range of T and the symbol $\vee$ means closed span.

THEOREM 1.2. *Let I be a closed right ideal of $C_{C^{\infty}(\mathscr{K})}(X)$ and for each x in X, let*

$$P_I(x) = \vee \{[\operatorname{Ran} F(x)]^{-} \colon F \in I\}.$$

Then I is precisely the set of all functions $F(x)$ in $C_{C^\infty(\mathscr{K})}(X)$ such that

$$\operatorname{Ran} F(x) \subset P_I(x)$$

for each x in X.

Proof. Let J be the closed right ideal consisting of all F in $C_{C^\infty(\mathscr{K})}(X)$ such that $\operatorname{Ran} F(x) \subset P_I(x)$ for each x in X, and let J_+ be equal to $J \cap \mathscr{P}$ and I_+ equal to $I \cap \mathscr{P}$ where $\mathscr{P}$ consists of all positive elements in $C_{C^\infty(\mathscr{K})}(X)$. By [11, Lemma 2.9.3], J (respectively I) is the closed right ideal generated by J_+ (respectively I_+). Hence, to show that J is equal to I, it suffices to show that $J_+ = I_+$. Now let J_1 (respectively I_1) be the C^*-algebra generated by J_+ (respectively I_+); then $J_1 \cap \mathscr{P} = J_+$ and $I_1 \cap \mathscr{P} = I_+$. Therefore, it is sufficient to show that $J_1 = I_1$. By Sakai's Stone-Weierstrass theorem applied to J_1, it suffices to show, given any two distinct pure states p_1 and p_2 (one of them possibly zero) on J_1, there exists an F in I_+ such that $p_1(F) \neq p_2(F)$. By an argument similar to the discussion preceding the theorem, p_i $(i = 1, 2)$ arises from an element of $M_{C^1(\mathscr{K})}(X)$ of the form $\langle \cdot, e_i\rangle\, e_i \delta_{x_i}$ for some x_i in X and unit vector e_i in $P(x_i)$ (if p_i is zero, take e_i to be the zero vector). In any case, p_i is of the form $F \in J_1 \to \langle F(x_i)\, e_i, e_i\rangle$. If $x_1 \neq x_2$, it is clear that we can choose F in I_1 so that $p_1(F) \neq p_2(F)$. If $x_1 = x_2$ and $e_1 \neq 0$, since I is a right ideal and by the definition of $P(x_1)$, we can choose F in I so that the element $G = FF^*$ in I_+ has $G(x_1) = \langle \cdot, u_1\rangle u_1$ where $\|u_1 - e_1\| < \|e_2 - e_1\|$. Then G separates p_1 and p_2, and the theorem follows.

By taking adjoints we obtain the closed left ideals. For T an operator, ker T denotes the kernel or null space of T.

THEOREM 1.2′. *Let I be a closed left ideal of $C_{C^\infty(\mathscr{K})}(X)$ and for each x in X, let*

$$K_I(x) = \cap \{\operatorname{Ker} F(x) \colon F \in I\}.$$

Then I is precisely the set of all functions $F(x)$ in $C_{C^\infty(\mathscr{K})}(X)$ such that $\operatorname{Ker} F(x) \supset K_I(x)$ *for each x in X.*

Let $C_{C^\infty(\mathscr{K})}$ be the space of continuous $C^\infty(\mathscr{K})$-valued functions on the unit circle, and $A_{C^\infty(\mathscr{K})}$ the space of $C^\infty(\mathscr{K})$-valued functions continuous on the closure of the unit disk D, and analytic on D. If $\dim \mathscr{K}$ is equal to 1, $A_{C^\infty(\mathscr{K})}$ becomes the classical disc algebra, the ideal structure of which has been characterized by Rudin [23] and Beurling (unpublished). Our next concern is to generalize this to a characterization the closed right ideals of $A_{C^\infty(\mathscr{K})}$. First, as we noted above, the dual of $C_{C^\infty(\mathscr{K})}$ it $M_{C^1(\mathscr{K})}$, the space of $C^1(\mathscr{K})$-valued measures defined on the unit circle. Ryan has proved an F. and M. Riesz theorem in this setting: the annihilator of $A_{C^\infty(\mathscr{K})}$ in $M_{C^1(\mathscr{K})}$ is isometrically isomorphic to $H^1_{OC^1(\mathscr{K})}$ (functions in $H^1_{C^1(\mathscr{K})}$ vanishing at the origin). (See [21, p. 205].) The isometric injection of $H^1_{OC^1(\mathscr{K})}$ i to $M_{C^1(\mathscr{K})}$ is simply $F(e^{i\theta}) \to \frac{1}{2\pi} F(e^{i\theta})\, d\theta$. Furthermore, any measure μ in $M_{C^1(\mathscr{K})}$ has a Lebesgue decomposition $\mu = F(e^{i\theta})\, d\theta + \mu_{\text{sing}}$ where F is in $L^1_{C^1(\mathscr{K})}$ and μ_{sing} is in $M_{C^1(\mathscr{K})}$ and is singular with respect to Lebesgue measure. Finally, for any collection $\{B_\alpha: \alpha \in \mathfrak{A}\}$ of rigid functions in $H^\infty_{\mathscr{L}(\mathscr{K})}$, a greatest common left divisor always exists. For by the Beurling-Lax theorem [18], the subspace $\vee\{B_\alpha H^2_{\mathscr{K}}: \alpha \in \mathfrak{A}\}$ must be of the form $BH^2_{\mathscr{K}}$ for a rigid B. This B then is the greatest common left divisor of $\{B_\alpha: \alpha \in \mathfrak{A}\}$. The proof of the next theorem is an operator adaptation of the discussion in Hoffman's book [19, p. 85–87].

THEOREM 1.3. *Let I be nonzero closed right ideal in $A_{C^\infty(\mathscr{K})}$. For each z on ∂D, let*

$$P_I(z) = \vee\{[\mathrm{Ran}\ F(z)]^-: F \in I\},$$

and let B be the greatest common left divisor of all rigid functions dividing an F in I on the left. Then I is precisely the set of all functions of the form BG, where G ranges over all functions in $H^\infty_{C^\infty(\mathscr{K})}$ such that BG is in $A_{C^\infty(\mathscr{K})}$ and Ran $B(z)\,G(z) \subset P_I(z)$ *for each z on ∂D.*

Proof. Let J be the right ideal consisting of all elements F in $A_{C^\infty(\mathscr{K})}$ such that F has the form BG where G is in $H^\infty_{C^\infty(\mathscr{K})}$ and Ran $B(z)G(z) \subset P_I(z)$ for each z on ∂D. First we show that J is closed. For if $F_n = BG_n$ is in J for each n and converges to F, then, since $\|B^*(BG_n - BG_m)\| = \|BG_n - BG_m\|$, $\{B^*BG_n\}$ is a Cauchy sequence in $H^\infty_{C^\infty(\mathscr{K})}$ and converges to a G in $H^\infty_{C^\infty(\mathscr{K})}$. Since for each n and z, Ran $B(z)G_n(z) \subset P_I(z)$, it is trivial that Ran $B(z)G(z) \subset P_I(z)$, and hence $F = BG$ is in J as required. By definition, $I \subset J$. To show that $J \subset I$, by functional analysis, the problem is to show that any μ in $M_{C^1(\mathscr{K})}$ which annihilates I also annihilates J.

Hence let μ be any element of $M_{C^1(\mathscr{K})}$ which annihilates I:

$$\langle f, \mu\rangle = 0, \quad f \in I.$$

Fixing f in I, since I is a right ideal, we have

$$\int \mathrm{tr}(fF)\, d\mu = 0, \quad F \in A_{C^\infty(\mathscr{K})}.$$

This means that $d\mu f$ annihilates $A_{C^\infty(\mathscr{K})}$, and hence is of the form

$$d\mu\, f = \frac{1}{2\pi} H_f\, d\theta \quad \text{where} \quad H_f \in H^1_{OC^1(\mathscr{K})}.$$

Let

$$\mu = \frac{1}{2\pi}\varphi\, d\theta + \mu_{\mathrm{sing}}$$

be the Lebesgue decomposition for μ, where φ is in $L^1_{C^1(\mathscr{K})}$ and μ_{sing} is in $M_{C^1(\mathscr{K})}$ and is singular with respect to $d\theta$. From the above,

$$d\mu_{\mathrm{sing}} f = 0, \quad f \in I.$$

Let K be a set of measure zero carrying the singular measure μ_{sing}; by regularity there is an increasing sequence $\{K_n\}$ of closed sets of measure zero whose union is K. By an easy operator generalization of a theorem of Rudin [19, p. 81], $A_{C^\infty(\mathscr{K})}|K_n = C_{C^\infty(\mathscr{K})}(K_n)$. Hence $(I|K_n)^-$ is a right ideal of $C_{C^\infty(\mathscr{K})}(K_n)$. By Theorem 1.2 it follows that $(I|K_n)^-$ is equal to $\{F \in C_{C^\infty(\mathscr{K})}(K_n)\colon \text{Ran } F(z) \subset P_I(z) \text{ for } z \in K_n\}$. It follows that $(d\mu_{\text{sing}}Q|K_n)$ annihilates $C_{C^\infty(\mathscr{K})}(K_n)$, where $Q(z)$ is the projection onto $P_I(z)$, and hence is the zero measure. Since this holds for all K_n, it follows that $d\mu_{\text{sing}}Q$ is the zero measure, and hence annihilates J.

Thus it remains only to show that $\frac{1}{2\pi}\,\varphi\, d\theta$ annihilates J. We will show that in fact φB is in $H^1_{OC^1(\mathscr{K})}$, and so $\frac{1}{2\pi}\,\varphi d\theta$ actually annihilates all of $BH^\infty_{\mathscr{L}(\mathscr{K})}$. Associated with each f in I is a $H^1_{OC^1(\mathscr{K})}$ function H_f such that $\varphi f = H_f$. Let $\mathfrak{M}$ be the weak-*closure in $H^\infty_{\mathscr{L}(\mathscr{K})}$ of I. Since I is a right ideal in $A_{C^\infty(\mathscr{K})}$, and $A_{C^\infty(\mathscr{K})}$ is weak-* dense in $H^\infty_{\mathscr{L}(\mathscr{K})}$, it follows that $\mathfrak{M}$ is a weak-*closed right ideal in $H^\infty_{\mathscr{L}(\mathscr{K})}$. By Theorem 1.1 a, $\mathfrak{M}$ is of the form $CH^\infty_{\mathscr{L}(\mathscr{K})}$ for some inner C. Since the greatest common left divisor of I is B, it follows that $\mathfrak{M}$ is equal to $BH^\infty_{\mathscr{L}(\mathscr{K})}$. Now note that if a net $\{f_\alpha\}$ converges weak-* to f in $H^\infty_{\mathscr{L}(\mathscr{K})}$, then, since φ is in $L^1_{C^1(K)}$, $\{\varphi f_\alpha\}$ converges to φf weakly in $L^1_{C^1(\mathscr{K})}$. Since by the original construction φf is in $H^1_{OC^1(\mathscr{K})}$ for all f in I, it follows that φf is in $H^1_{OC^1(\mathscr{K})}$ for all f in $\mathfrak{M} = BH^\infty_{\mathscr{L}(\mathscr{K})}$, and in particular, for f equal to B. Hence φB is in $H^1_{OC^1(\mathscr{K})}$, and hence $\frac{1}{2\pi}\,\varphi\, d\theta$ annihilates $BH^\infty_{\mathscr{L}(\mathscr{K})}$ as asserted.

2. OPERATOR EXTREMAL PROBLEMS

2.1 *Definitions and notations.* Let R be a bounded multiply-connected region in the complex plane as described in the introduction and let $\mathscr{K}$ be a separable Hilbert space. In this section we define more precisely the spaces of vector- and operator-valued functions over R alluded to in the introduction. The definitions here will differ slightly from those of [1, 8] and from the bundle definitions of [4, 5], in order to bring in the group structure of the fundamental group of R, but all these definitions are equivalent. Let $C_1, \ldots, C_n$ be n cuts in R such that, if C is the union of the cuts C_i, $R \setminus C$ is simply connected. Let $\pi_0(R)$ be the fundamental group for R, that is, the set of homotopy equivalence classes of closed curves in R; it is known that $\pi_0(R)$ is a free abelian group with n generators. If $\mathscr{K}$ is a separable Hilbert space, let $\mathscr{U}(\mathscr{K})$ be the group of unitary operators on $\mathscr{K}$. Then Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$ is the group of all group homomorphisms of $\pi_0(R)$ into $\mathscr{U}(\mathscr{K})$. If F is an operator or vector valued function analytic on $R \setminus C$ which has an analytic continuation across any of the cuts C_i (and thus F is a particular function element of a multi-valued function on R), and if A is an element of the fundamental group $\pi_0(R)$, for any z in R, define

$F \circ A(z)$ to be the analytic continuation of F, evaluated at $\gamma(1)$, along any closed curve $\{\gamma(t): 0 \leqslant t \leqslant 1\}$ in the equivalence class A which begins and ends at the point z. For $1 \leqslant p < \infty$ and α in Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$, define $H^p_{\mathscr{K}}(\alpha)$ to be the space of all $\mathscr{K}$-valued functions analytic on $R\setminus C$ but with analytic continuations along any curve in R, such that $\|F(z)\|^p$ has a harmonic majorant on R, and for all z in R and A in $\pi_0(R)$,

$$f \circ A(z) = \alpha(A) f(z). \tag{2.1}$$

It can be shown that the elements of $H^p_{\mathscr{K}}(\alpha)$ have well-defined nontangential boundary values on the boundary ∂R of R almost everywhere with respect to arc length measure [21]; specifying a measure m boundedly mutually absolutely continuous with respect to arc length measure on ∂R enables one to define a norm on $H^p_{\mathscr{K}}(\alpha)$ which makes it a Banach space (for $p = 2$, a Hilbert space), and isometrically isomorphic (via $f \to f^*$ where for a.e. z on ∂R, $f^*(z) = \lim_{w \to z} f(w)$ where w in R approaches z nontangentially) to a closed subspace of $L^p(m)$. In this section it is convenient to choose m equal to $(2\pi)^{-1}\,|dz|$, $(2\pi)^{-1}$ times arc length measure. For $p = \infty$, $H^\infty_{\mathscr{K}}(\alpha)$ is the space of bounded $\mathscr{K}$-valued functions analytic on $R\setminus C$, having analytic continuation along any curve in R, and satisfying (2.1).

Finally, for $1 \leqslant p \leqslant \infty$, $C^p(\mathscr{K})$ and $\mathscr{L}(\mathscr{K})$ are the spaces of operators as in Section 1. For α and β two elements of Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$ and $1 \leqslant q < \infty$, $H^q_{C^p(\mathscr{K})}(\alpha, \beta)$ denotes the space of $C^p(\mathscr{K})$-valued functions F analytic on $R\setminus C$, having analytic continuation along any curve in R, such that $\|F(z)\|^q_{C^p(K)}$ has a harmonic majorant, and such that

$$F \circ A(z) = \alpha(A)\, F(z)\, \beta(A)^* \tag{2.2}$$

for each A in $\pi_0(R)$ and z in R. For $q = \infty$, $H^\infty_{C^p(\mathscr{K})}(\alpha, \beta)$ is the space of $C^p(\mathscr{K})$-valued functions F analytic on $R\setminus C$, such that the analytic continuation along any curve satisfies (2.2), and such that F is uniformly bounded in $C^p(\mathscr{K})$-norm on R. The spaces $H^p_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ are defined analogously. With the use of an appropriate measure on the boundary (in this section, $(2\pi)^{-1}$ times arc length measure $|dz|$), all these spaces become Banach spaces, and $H^2_{C^2(\mathscr{K})}(\alpha, \beta)$ is a Hilbert space. The theorem of Grauert and Bungart (see [4]) implies that all these spaces are nontrivial. Finally for A a trace class operator, we wish to use the notation tr A for the trace of A (tr $A = \sum \langle Ae_i, e_i\rangle$ where $\{e_i\}$ is any orthonormal basis for $\mathscr{K}$) and Tr A for the trace-class norm of A (Tr $A = \mathrm{tr}(A^*A)^{\frac{1}{2}}$).

2.2 *The extremal problem.* With all these definitions and the notation established, we are now ready to set down the extremal problem of interest. Let A be a trace class operator on $\mathscr{K}$ with trivial kernel and dense range, and t an arbitrary but fixed point in R, and α and β two elements of Hom$(\pi_0(R), \mathscr{U}(\mathscr{K}))$. We wish to consider the problem

$$\text{(2.3)}\qquad \begin{aligned} &\text{Maximize:} && \operatorname{tr}\Theta(t)A \\ &\text{Subject to:} && \Theta \in H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta),\ \|\Theta\| \leqslant 1,\ 0 \leqslant \operatorname{tr}\Theta(t)A. \end{aligned}$$

A normal families argument, together with weak-* compactness of the unit ball of $\mathscr{L}(\mathscr{K})$, gives that the supremum of $\operatorname{tr}\Theta(t)A$ is achieved by some Θ_0 in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$, necessarily of norm 1.

In order to analyze this problem, we wish to exploit some duality and set up a dual extremal problem, as Widom [30] does for a similar class of scalar extremal problems. Note that if $g(z) = (z-t)^{-1}f(z)$ is an element of $(z-t)^{-1}H^1_{C^1(\mathscr{K})}(\beta,\alpha)$, where $f(t) = A$, and $\Theta(z)$ satisfies the constraints in (2.3), then $\operatorname{tr}(\Theta(z)\,g(z))$ is a single-valued meromorphic function on R, and by the residue theorem,

$$\operatorname{tr}\Theta(t)A = \frac{1}{2\pi \mathrm{i}}\int_{\partial R} \operatorname{tr}(\Theta(z)\,g(z))\,\mathrm{d}z.$$

Now $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$, via nontangential boundary values, can be considered as a closed subspace of $L^\infty_{\mathscr{L}(\mathscr{K})}(|\mathrm{d}z|)$, and inherits a weak-* topology from $L^\infty_{\mathscr{L}(\mathscr{K})}(|\mathrm{d}z|)$, with respect to which the functional $\Theta \to \operatorname{tr}\Theta(t)A$ is continuous. It is known [21] that $L^\infty_{\mathscr{L}(\mathscr{K})}(|\mathrm{d}z|)$ is the dual of $L^1_{C^1(\mathscr{K})}(|\mathrm{d}z|)$. It is convenient to use as the pairing establishing this duality the following:

$$\text{(2.4)}\qquad \langle F, f\rangle = (2\pi \mathrm{i})^{-1}\int \operatorname{tr}(F(z)f(z_0))\,\mathrm{d}z$$

for F in $L^\infty_{\mathscr{L}(\mathscr{K})}(|\mathrm{d}z|)$ and f in $L^1_{C^1(\mathscr{K})}(|\mathrm{d}z|)$. The annihilator of $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$ in $L^1_{C^1(\mathscr{K})}(|\mathrm{d}z|)$ via this duality is precisely $H^1_{C^1(\mathscr{K})}(\beta,\alpha)$. (The inclusion of $H^1_{C^1(\mathscr{K})}(\beta,\alpha)$ in the annihilator of $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$ is clear via Cauchy's theorem. Conversely, by using the Grauert-Bungart theorem, one can reduce the general case to the case where α and β both equal the identity element e of $\operatorname{Hom}(\pi_0(R),\mathscr{U}(\mathscr{K}))$. Then if $f(z)$ annihilates $H^\infty_{\mathscr{L}(\mathscr{K})}(=H^\infty_{\mathscr{L}(\mathscr{K})}(e,e))$, then $\langle f(z)u,u\rangle_{\mathscr{K}}$ annihilates H^∞ for every unit vector u in $\mathscr{K}$. By a result in [2], this implies that $\langle f(z)\,u,u\rangle$ is in H^1. Since u was arbitrary, $f(z)$ is in $H^1_{C^1(\mathscr{K})}$.) By a standard application of the Hahn-Banach theorem, the space $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$ can then be isometrically isomorphically represented as the dual of the quotient Banach space $L^1_{C^1(\mathscr{K})}(|\mathrm{d}z|)/H^1_{C^1(\mathscr{K})}(\beta,\alpha)$ by use of the pairing (2.2).

If L is the weak-* continuous linear functional

$$L\colon \Theta \to \operatorname{tr}\Theta(t)A$$

on $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha,\beta)$, then the norm of L is the extremal value sought in (2.3). By the above duality analysis, there is a $[g]$ in $L^1_{C^1(\mathscr{K})}/H^1_{C^1(\mathscr{K})}(\beta,\alpha)$ such that

$$\|L\| = \|[g]\| = \inf\{\|g+h\| : h \in H^1_{C^1(\mathscr{K})}(\beta,\alpha)\}.$$

We saw above that one way to obtain such a g is by considering any element g of $(z-t)^{-1}H^1_{C^1(\mathcal{K})}(\beta, \alpha)$ such that $\lim_{z\to t}(z-t)\,g(z) = A$. Now we know that any two such g's must differ by an element of $H^1_{C^1(\mathcal{K})}(\beta, \alpha)$. Hence the above construction is the only way to obtain such a g. All the above motivates the definition of the extremal problem dual to (2.3):

$$\text{(2.5)}\qquad \begin{aligned} &\text{Minimize:} && \|g\| = (2\pi)^{-1}\int_{\partial R} \operatorname{Tr} g(z)\,|dz| \\ &\text{Subject to:} && g \in (z-t)^{-1}H^1_{C^1(\mathcal{K})}(\beta, \alpha),\ \lim_{z\to t}(z-t)\,g(z) = A. \end{aligned}$$

A normal families argument (together with the compactness of the unit ball of $C^1(\mathcal{K})$ in the weak-* topology defined by the duality with $C^\infty(\mathcal{K})$) again implies that the extremal value is achieved by some g_0 satisfying the constraints in (2.5).

In a standard way, information can be obtained concerning a solution Θ_0 of (2.3) and a solution g_0 of (2.5) by considering both together. By multiplying Θ_0 by a scalar, we can assume that $\operatorname{tr}\Theta_0(t)A$ is positive. By the discussion above, $\|L\| = \operatorname{tr}\Theta_0(t)A = \|g_0\|$. But also,

$$\text{(2.6)}\qquad \begin{aligned} \operatorname{tr}\Theta_0(t)A &= (2\pi i)^{-1}\int \operatorname{tr}(\Theta_0(z)\,g_0(z))\,dz \leqslant \\ &\leqslant (2\pi)^{-1}\int |\operatorname{tr}\Theta_0(z)\,g_0(z)|\,|dz| \leqslant \\ &\leqslant (2\pi)^{-1}\int \operatorname{Tr}\Theta_0(z)\,g_0(z)\,|dz| \leqslant \\ &\leqslant (2\pi)^{-1}\int \operatorname{Tr} g_0(z)\,|dz| = \|g_0\|, \end{aligned}$$

and hence there is equality throughout. As we shall see, this will give us much information concerning Θ_0 and g_0.

First, from the second equality in the chain of equalities (2.6), we deduce

$$\text{(2.7)}\qquad (2\pi i)^{-1}\operatorname{tr}(\Theta_0(z)\,g_0(z))\,\frac{dz}{|dz|} \geqslant 0 \quad \text{a.e. on } \partial R$$

where $\dfrac{dz}{|dz|}$ is the Radon-Nikodym derivative of dz with respect to $|dz|$ on ∂R. The next equality in (2.6) gives

$$|\operatorname{tr}(\Theta_0(z)\,g_0(z))| = \operatorname{Tr}(\Theta_0(z)\,g_0(z)) \quad \text{a.e. on } \partial R.$$

This, combined with (2.7) gives

$$(2.8) \qquad (2\pi i)^{-1}\operatorname{tr}(\Theta_0(z)\,g_0(z))\frac{dz}{|dz|} = (2\pi)^{-1}\operatorname{Tr}(\Theta_0(z)\,g_0(z)) \quad \text{a.e. on } \partial R.$$

Finally, the next to last equality in the chain (2.6) gives

$$(2.9) \qquad \operatorname{Tr}\Theta_0(z)\,g_0(z) = \operatorname{Tr} g_0(z) \quad \text{a.e. on } \partial R.$$

Most of our detailed analysis of (2.3) and (2.5) will depend on the assumption that $\mathscr{K}$ is finite dimensional. In particular, we shall show that, if $\mathscr{K}$ is finite dimensional, the solution Θ_0 of (2.3) is analytic on $\bar{R}\setminus\bar{C}$ and is inner, that is $\Theta_0(z)$ is unitary for each z on ∂R. First we need some easy lemmas.

LEMMA 2.1. *If* $0 \leqslant A \leqslant B$ *are trace-class operators such that* $\operatorname{tr} A = \operatorname{tr} B$, *then* $A = B$.

Proof. From $\operatorname{tr} A = \operatorname{tr} B$ we have, for any orthonormal basis $\{e_n\}$, $\sum_k \langle Ae_k, e_k\rangle = \sum_k \langle Be_k, e_k\rangle$. But from $0 \leqslant A \leqslant B$ we have

$$0 \leqslant \langle Ae_k, e_k\rangle \leqslant \langle Be_k, e_k\rangle$$

for each k. It follows that $\langle Au, u\rangle = \langle Bu, u\rangle$ for any unit vector u, and hence $A = B$.

LEMMA 2.2. *If* A *is a trace-class operator such that* $\operatorname{tr} A = \operatorname{Tr} A$, *then* A *is positive.*

Proof. Let A have polar decomposition $A = U|A|$, and let $\{e_k\}$ be a complete orthonormal set of eigenvectors for $|A|$ with corresponding eigenvalues $\{s_k\}$. Then

$$\operatorname{tr} A = \sum \langle U|A|e_k, e_k\rangle \leqslant \sum |\langle |A|e_k, U^*e_k\rangle| \leqslant \sum s_k|\langle e_k, U^*e_k\rangle| \leqslant \sum s_k = \operatorname{Tr} A,$$

and hence there is equality throughout. This forces, for $s_k \neq 0$, $\langle e_k, U^*e_k\rangle = \langle e_k, e_k\rangle$. Equality in the Schwarz inequality forces $Ue_k = e_k$ for all such k, or U is the identity on Ran $|A|$. Hence $A = |A|$ and A is positive.

THEOREM 2.3. *If* Θ_0 *and* g_0 *are the solutions of* (2.3) *and* (2.5) *respectively, then* $\Theta_0 g_0$ *and* g_0 *have analytic continuations across* $\partial R\setminus(\partial R \cap \bar{C})$. *If* $\dim \mathscr{K} < \infty$, *then* Θ_0 *is inner, has analytic continuation across* $\partial R\setminus(\partial R \cap \bar{C})$, *and is the unique solution of* (2.3) *satisfying* $\operatorname{tr}\Theta_0(t)A \geqslant 0$.

Proof. The following analytic continuation argument is an adaptation of that of Gamelin [15, Section 8] used for the scalar case. It suffices to show that the analytic continuation occurs across any arc of $\partial R\setminus(\partial R \cap \bar{C})$. By a conformal mapping, since ∂R consists of analytic Jordan curves, we may assume that the arc is an interval J on the real line $\left(\text{so } \frac{dz}{|dz|} = 1 \text{ on } J\right)$, and that $\Theta_0 g_0$, Θ_0 and g_0 are analytic on an open set W_+ in the upper half plane whose boundary includes J. Define H_+ on W_+ by

$$H_+(z) = (2\pi i)^{-1}\Theta_0(z)\, g_0(z), \quad z \in W_+,$$

and H_- on $W_- = \{z : \bar{z} \in W_+\}$ by

$$H_-(z) = H_+(\bar{z})^*, \quad z \in W_-.$$

Now by equations (2.8) and Lemma 2.2, $(2\pi i)^{-1}\Theta_0(z)g_0(z) \geqslant 0$ on J, and hence $H_+(z)$ and $H_-(z)$ both approach this function on J as z approaches a point in J from W_+ or W_- respectively. Applying Morera's theorem to the scalar function $\langle H_+(z)u, u\rangle$ where u is any unit vector and the Lebesgue dominated convergence theorem, one can conclude that $\langle H_-(z)u, u\rangle$ is an analytic continuation of $\langle H_+(z)u, u\rangle$ across J. Since u was arbitrary, we conclude that $\Theta_0(z)g_0(z)$ continues analytically across J.

Now from equation (2.9) together with Lemma 2.1, we have

$$g_0(z)^*\Theta_0(z)^*\Theta_0(z)g_0(z) = g_0(z)^*g_0(z) \quad \text{for a.e. } z \text{ on } \partial R.$$

It follows that $\Theta_0(z)$ is isometric on $\operatorname{Ran} g_0(z)$ and also $\Theta_0(z)^*\Theta_0(z)g_0(z) = g_0(z)$ for a.e. z on ∂R. Therefore $\Theta_0(\bar{z})^*(\Theta_0(z)g_0(z))$ is an analytic continuation of $g_0(z)$ to W_-, such that the boundary values along J from W_+ match up with those from W_-. Since $\langle g_0(z)e, e\rangle$ is integrable along J for each unit vector e, it follows as above that g has an analytic continuation across J.

Now assume that $\dim \mathscr{K} < \infty$. Then $\det g$ is a function meromorphic on $\bar{R}$ such that the only pole on $\bar{R}\setminus\bar{C}$ is a pole at t with principal part equal to $(z-t)^{-\dim \mathscr{K}} \det A \neq 0$. Hence $\det g$ cannot be identically zero, and therefore is nonzero on $\partial R\setminus(\partial R \cap \bar{C})$ except for isolated points. Hence $\operatorname{Ran} g(z)$ is equal to the whole space $\mathscr{K}$ except at isolated points for z on ∂R. By a previous remark, $\Theta_0(z)$ is isometric on $\operatorname{Ran} g(z)$ for a.e. z on ∂R, and so in fact Θ_0 is inner.

It remains to verify that Θ_0 has an analytic continuation across J if $\dim \mathscr{K} < \infty$. Since $\Theta_0(z)g_0(z)$ and $g_0(z)$ have such analytic continuations, the formula

$$\Theta_0(z) = (\Theta_0(z)\, g_0(z))\, g_0(z)^{-1}$$

gives a meromoprhic continuation of $\Theta_0(z)$ into $W_+ \cup J \cup W_-$. Since $\Theta_0(z)$ is unitary for a.e. z on ∂R, no poles can occur on J nor can poles accumulate on J. Thus Θ_0 is actually analytic on a neighborhood of J.

Finally the condition that $\Theta_0(z)$ be unitary and that $(2\pi i)^{-1}\,\Theta_0(z)\, g_0(z)\,\dfrac{dz}{|dz|}$ be a positive operator for each z on ∂R uniquely determines $\Theta_0(z)$ for all such z, and hence Θ_0 is uniquely determined.

THEOREM 2.4. *Suppose* $\dim \mathscr{K} < \infty$, *and* $\mathscr{K}_0$ *is the solution of* (2.3). *Then* $\dim H^2_{\mathscr{K}}(\alpha) \ominus \Theta_0 H^2_{\mathscr{K}}(\beta) \leqslant n\,(\dim \mathscr{K})$ *where* $n+1$ *is the number of components of* ∂R.

Proof. By Theorem 2.3, Θ_0 is an inner element of $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ extends to be analytic on a neighborhood of $\bar{R}\setminus\bar{C}$. For any such Θ_0, an inductive argument gives that

$$\dim [H^2_{\mathscr{K}}(\alpha) \ominus \Theta_0 H^2_{\mathscr{K}}(\beta)] = N_0(\det \Theta)$$

where $N_0(\det \Theta)$ is the number of zeroes (counted according to multiplicity) of $\det \Theta$ on R. Let $1_{\mathscr{K}}$ denote the identity operator on $\mathscr{K}$. As noted once before, equation (2.8) and Lemma 2.2 imply that

$$(2\pi i)^{-1}\, \Theta_0(z)\, g_0(z) \left(\frac{dz}{|dz|}\, 1_{\mathscr{K}}\right)$$

is a positive operator for each z on ∂R, and hence also

$$\det \left((2\pi i)^{-1}\, \Theta_0(z)\, g_0(z) \left(\frac{dz}{|dz|}\, 1_{\mathscr{K}}\right)\right) \geqslant 0.$$

But then

$$\Delta_{\arg} \det (\Theta_0(z)) + \Delta_{\arg} \det (g_0(z)) + \Delta_{\arg} \det \left(\frac{dz}{|dz|}\, 1_{\mathscr{K}}\right) = 0$$

where $\Delta_{\arg}$ means the change in argument along ∂R. By definition, $\Delta_{\arg} \det\left(\frac{dz}{|dz|}\, 1_{\mathscr{K}}\right) = \Delta_{\arg} \left(\frac{dz}{|dz|}\right)^{\dim \mathscr{K}} = (1-n)\dim \mathscr{K}$. Also $\det \Theta_0(z)$ has no poles in $\bar{R}$ and $\det g_0(z)$ has a pole at t of order $\dim \mathscr{K}$, and no other poles in $\bar{R}$. Hence, by the argument principle,

$$N_0(\det \Theta_0(z)) + N_0(\det g_0(z)) = n \dim \mathscr{K}.$$

(If $\det h_0(z)$ has zeroes on ∂R, these are counted with one-half their multiplicities. Then the above formula is still correct.) Hence

$$0 \leqslant N_0(\det \Theta_0(z)) \leqslant n \dim \mathscr{K}$$

and the theorem follows.

The above result, for the case that $\dim \mathscr{K} = 1$, has been obtained by Tumarkin and Havinson [28].

The next result says that, in some sense, there are sufficiently many inner functions in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$.

THEOREM 2.5. *Assume* $\dim \mathscr{K} < \infty$. *For any point t in R,*

$$\vee \{\operatorname{Ran} \Theta(t) \colon \Theta \in H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta),\ \Theta \text{ inner},$$
$$\dim H^2_{\mathscr{K}}(\alpha) \ominus \Theta H^2_{\mathscr{K}}(\beta) \leqslant n(\dim \mathscr{K})\} = \mathscr{K}.$$

Proof. Let Θ_0 be the extremal function for the extremal problem (2.3) at the point t in R. For any A in $\pi_0(R)$, $\Theta_0 \circ A$ is also an inner element of $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$, and satisfies the dimension condition in the statement of the theorem since Θ_0 does by Theorem 2.4. Since $\Theta_0 \circ A = \alpha(A)\, \Theta_0 \beta(A)^*$, it follows that

$$\vee \{\text{Ran } \Theta_0 \circ A(t) \colon A \in \pi_0(R)\} = \mathscr{L}$$

where $\mathscr{L}$ is the smallest subspace of $\mathscr{K}$ invariant for all $\alpha(A)$, A in $\pi_0(R)$, and containing Ran $\Theta_0(t)$. It follows that the projection $P_{\mathscr{L}}$ onto $\mathscr{L}$ commutes with all $\alpha(A)$, A in $\pi_0(R)$, and hence $P_{\mathscr{L}}\Theta_0$ is in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$. Since Ran $\Theta_0(t) \subset \mathscr{L}$, we have that $P_{\mathscr{L}}\Theta_0$ is an extremal function for (2.3). By uniqueness of the extremal function (Theorem 2.3), it follows that $\mathscr{L} = \mathscr{K}$ and the theorem follows.

2.3 *Existence of inner functions: general case.* In Section 2.2 we considered the space $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ for a fixed Hilbert space $\mathscr{K}$. We now wish to generalize this to two Hilbert spaces $\mathscr{K}$ and $\mathscr{K}'$ as follows: Let α be an element of Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$, and β of Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}'))$, and let $\mathscr{L}(\mathscr{K}', \mathscr{K})$ be the space of bounded linear operators mapping $\mathscr{K}'$ into $\mathscr{K}$. Then $H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta)$ is the space of those functions analytic on $R\setminus C$ taking values in $\mathscr{L}(\mathscr{K}', \mathscr{K})$ such that, for any A in $\pi_0(R)$, $F \circ A = \alpha(A)\, F\beta(A)^*$. Of course this generalization of $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ only has significance if $\mathscr{K}$ and $\mathscr{K}'$ are of different dimensions. An element Θ of $H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta)$ is said to be inner if $\Theta(z)$ is isometric for a.e. z on ∂R. (If such a Θ exists, of course $\dim \mathscr{K}' \leqslant \dim \mathscr{K}$.) Subspaces of $H^2_{\mathscr{K}}(\alpha)$ invariant under multiplication by rational functions with poles off of R have been characterized by Abrahamse and Douglas [4] as being of the form $\Theta H^2_{\mathscr{K}}(\beta)$ for some inner Θ in $H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta)$. For such a Θ, the restriction of the operator S_α (multiplication by z on $H^2_{\mathscr{K}}(\alpha)$) to ΘH^2_β is unitarily equivalent to S_β. Thus the following general theorem implies that the bundle shift S_α contains a copy of any bundle shift S_β whose rank is finite and is less than or equal to that of S_α.

THEOREM 2.6. *Let $\mathscr{K}$ and $\mathscr{K}'$ be separable Hilbert spaces with* $\dim \mathscr{K}' \leqslant \dim \mathscr{K}$, *and* $\dim \mathscr{K}' < \infty$. *Then, for any elements α in* Hom $(\pi_0(R), \mathscr{K})$ *and β in* Hom $(\pi_0(R), \mathscr{K}')$ *there exist an inner function in $H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta)$.*

Proof. The case where $\dim \mathscr{K}' = \dim \mathscr{K} < \infty$ is contained in Theorem 2.3. Let A be an operator from $\mathscr{K}$ into $\mathscr{K}'$ having range equal to all of $\mathscr{K}'$. We consider the problem

(2.3′)
$$\begin{aligned} &\text{Maximize:} \quad \operatorname{tr} \Theta(t) A \\ &\text{Subject to:} \quad \Theta \in H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta), \quad \|\Theta\|_\infty \leqslant 1,\ 0 \leqslant \operatorname{tr} \Theta(t) A. \end{aligned}$$

We note that since $\dim \mathscr{K}' < \infty$, the functional $\Theta \to \operatorname{tr} \Theta(t) A$ is a weak-* continuous linear functional on $H^\infty_{\mathscr{L}(\mathscr{K}', \mathscr{K})}(\alpha, \beta)$. The dual problem associated with (2.3′) is

(2.5′)
$$\begin{aligned} &\text{Minimize:} \quad \|g\| = \frac{1}{2\pi} \int_{\partial R} \operatorname{Tr} g(z)\, |dz| \\ &\text{Subject to:} \quad g \in (z-t)^{-1} H^1_{C^1(\mathscr{K}, \mathscr{K}')}(\beta, \alpha),\ \lim_{z \to t} (z-t)\, g(z) = A. \end{aligned}$$

We note that the theorem of Grauert implies that the space $H^{\infty}_{\mathcal{L}(\mathcal{K}',\mathcal{K})}(\alpha,\beta)$ is nontrivial, and that the class of g satisfying the constraints in (2.5′) is nonempty. For, by Grauert's theorem, there exists a boundedly invertible E_1 in $H^{\infty}_{\mathcal{L}(\mathcal{K})}(\alpha, e_{\mathcal{K}})$ and boundedly invertible E_2 in $H^{\infty}_{\mathcal{L}(\mathcal{K}')}(e_{\mathcal{K}'}, \beta)$, where $e_{\mathcal{K}}$ and $e_{\mathcal{K}'}$ are the identity elements of $\mathrm{Hom}(\pi_0(R,)\ \mathcal{U}(\mathcal{K}))$ and $\mathrm{Hom}(\pi_0(R), \mathcal{U}(\mathcal{K}'))$, respectively. Then $(z-t)^{-1}E_2(z)^{-1}E_2(t)A\ E_1(t)\,E_1(z)^{-1}$ is an element of $(z-t)^{-1}H^{\infty}_{\mathcal{L}(\mathcal{K},\mathcal{K}')}(\beta, \alpha)$ satisfying the constraints of (2.5′). If Θ_0 is a solution of (2.3′) and g_0 a solution of (2.5′), the proof that $\Theta_0 g_0$ and g_0 have analytic continuations and that $\Theta_0(z)$ is isometric on $\mathrm{Ran}\, g_0(z)$ for z on ∂R is exactly as in the proof of Theorem 2.3. Since the residue of g_0 at t has full range and $\mathcal{K}'$ is finite-dimensional, we claim that $g_0(z)$ has full range for every z in $\bar{R}$ with the exception of some isolated points. To see this, let $\{e_j\}_{j=1}^{\dim \mathcal{K}'}$ be a basis for $\mathcal{K}'$, and $\{Ax_j\}_{j=1}^{\dim \mathcal{K}}$ be a basis for Ran A $(=\mathcal{K}')$. Then $g(z)$ has full range at any z where the determinant of the analytic matrix $[\langle (z-t)g(z)x_i, e_j\rangle]_{i,j=1}^{\dim \mathcal{K}'}$ is non-zero. Since the determinant is non-zero at z equal to t, this analytic function is nonzero and hence has only isolated zeros. Hence Θ_0 is an inner element of $H^{\infty}_{\mathcal{L}(\mathcal{K}',\mathcal{K})}(\alpha, \beta)$ as asserted.

Theorems 2.3 and 2.6 of course leave open the following natural question.

QUESTION 1: Given two elements α and β in $\mathrm{Hom}(\pi_0(R), \mathcal{K})$ where $\mathcal{K}$ is an infinite-dimensional separable Hilbert space, does there exist an inner function Θ_0 in $H^{\infty}_{\mathcal{L}(\mathcal{K})}(\alpha, \beta)$?

It can be shown that the answer is yes if R is an annulus.

REMARK. Most of the results of this section are true with a measure on ∂R more general than $|dz|$ to define $\|g\|_1$ in (2.5) and with a relaxation of the assumption that the components of ∂R be analytic Jordan curves, by a simple adaptation of Widom's techniques [31]. Also Widom shows how to derive more specific information concerning the location of the zeroes of det $\Theta_0(z)$ and of det $\Theta_0(z)g_0(z)$. These refinements however are not central to the point of view of this paper.

3. EXPECTATION OPERATORS

3.1 *The universal covering surface.* The universal covering surface for a region R is conformally equivalent to the open unit disk D; a sketch of the construction of the universal covering surface and of the basic facts concerning this construction needed here may be found in [4, Section 2.1]. In summary, the construction produces the following:

1. A group G of linear fractional transformations that map the disk onto the disk.
2. An open G-invariant subset Ω of ∂D with $\mu(\partial D \setminus \Omega) = 0$ (μ is normalized linear Lebesgue measure).
3. A simply connected open G-invariant subset D' containing $D \cup \Omega$.
4. An open set R' containing the closure of R.

5. A holomorphic covering map π from D' onto R' such that $\pi(D) = R$ and $\pi(\Omega) = \partial R$, and G is the group of all linear fractional transformations A having the property that $\pi \circ A = \pi$.

The set $\partial D \setminus \Omega$ is the limit set of the Fuchsian group G, and is the set of accumulations points of orbits of G, that is sets of the form $\{Az: A \in G\}$ for z in D. Let us choose a fundamental polygon $\mathscr{R}$ for G in D, that is, a connected subset of D such that every orbit of G contains precisely one point of $\mathscr{R}$. The closure $\bar{\mathscr{R}}$ of $\mathscr{R}$ in the plane meets ∂D in a set of $n + 1$ arcs each of which corresponds to a boundary contour of R via the covering map π. We note π induces a bijection between $\mathscr{R}$ and R.

An element $\tilde{A}$ of the fundamental group $\pi_0(R)$ of R lifts via π to an element A of G in a canonical way: for z in D, define $A(z)$ to be $\pi^{-1} \circ \gamma(1)$ where $\pi^{-1} \circ \gamma(0) = z$ and $\pi^{-1} \circ \gamma$ is the analytic continuation of π^{-1} along any closed curve $\gamma = \{\gamma(t): 0 \leqslant t \leqslant 1\}$ whose equivalence class in $\pi_0(R)$ is $\tilde{A}$ and which has $\gamma(0) = \gamma(1) = z$. Conversely, any A in G induces an $\tilde{A}$ in $\pi_0(R)$ in a similar way. This correspondence between $\pi_0(R)$ and G induces a correspondence $\tilde{\alpha} \to \alpha$ between Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$ and Hom$(G, \mathscr{U}(\mathscr{K}))$, the group of group homomorphisms from G into the group of unitary operators on a Hilbert space $\mathscr{K}$.

To establish some conformal invariance for our spaces of functions over the region R, let us choose a fixed but arbitrary point t in R, and let m be the uniquely determined measure on ∂R such that

$$\int_{\partial R} f(z)\, \mathrm{d}m(z) = f(t)$$

for every function f continuous on $\bar{R}$ and harmonic on R. The covering map π can be chosen to that $\pi(0) = t$. When this is done, π lifts the measure m on ∂R to normalized linear Lebesgue measure on ∂D: $\mu \circ \pi^{-1} = m$ ([2, p 124]). In the sequel let it be understood that the measure m (rather than $(2\pi)^{-1}\, |\mathrm{d}z|$) is to be used to define the norms on the spaces $H^p_{C^p(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ for $1 \leqslant p < \infty$, and $\tilde{\alpha}$ and $\tilde{\beta}$ elements of Hom $(\pi_0(R), \mathscr{K})$.

Now let α and β be any two elements of Hom $(G, \mathscr{U}(\mathscr{K}))$, and define $H^p_{C^p(\mathscr{K})}(\alpha, \beta)$ $(1 \leqslant p \leqslant \infty)$ to be the space of all $H^p_{C^p(\mathscr{K})}$ functions F on D (see Section 1) such that

$$F \circ A(z) = \alpha(A)\, F(z)\, \beta(A)^* \quad \text{for all } A \text{ in } G.$$

A similar condition defines $H^p_{\mathscr{K}}(\alpha)$ as a subspace of $H^p_{\mathscr{K}}$. If $\tilde{\alpha}$ and $\tilde{\beta}$ are the elements of Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$ corresponding to α and β respectively via the correspondence mentioned above, we assert that $H^p_{C^p(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ and $H^p_{C^p(\mathscr{K})}(\alpha, \beta)$ are isomorphic in the following canonical way: any $\tilde{F}$ in $H^p_{C^p(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ can be thought of, via π, as being defined on $\mathscr{R}$, and then extends, via analytic continuation, to D to produce an element of F in $H^p_{C^p(\mathscr{K})}(\alpha, \beta)$. Conversely, any F in $H^p_{C^p(\mathscr{K})}(\alpha, \beta)$, restricted to $\mathscr{R}$, induces via

π a $\tilde{F}$ in $H^p_{C^p(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$. The fact that the correspondence is isometric is due to the correspondence between m and μ mentioned above.

We will need another space of operator functions over R which has not yet been formally defined. For $\tilde{\alpha}$ and $\tilde{\beta}$ in Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}))$ let $A_{C^\infty(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ be those functions in $H^\infty_{C^\infty(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ which are continuous on $\bar{R}\setminus(\partial R \cap \bar{C})$. When viewed via π as functions on $\mathscr{R}$, and then via analytic continuation as functions on D, a function $\tilde{F}$ in $A_{C^\infty(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ induces a function F in $H^\infty_{C^\infty(\mathscr{K})}(\alpha, \beta)$ which is continuous on $\bar{\mathscr{R}}$, or equivalently on $\bar{D}\setminus(\partial D\setminus\Omega)$, and conversely, any such F arises from an $\tilde{F}$ in $A_{C^\infty(\mathscr{K})}(\tilde{\alpha}, \tilde{\beta})$ in this way. Let $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$ denote the space of all such functions F on the disk. Finally let $A_{C^\infty(\mathscr{K})}(G)$ be the space of functions in $H^\infty_{C^\infty(\mathscr{K})}$ on D continuous on $\bar{\mathscr{R}}$.

3.2 *Definition and properties of the expectation operators.* Earle and Marden [13] have constructed a projection operator which maps $H^p_{\mathbf{C}}$ boundedly onto $H^p_{\mathbf{C}}(e)$ ($\mathbf{C}$ is the complex numbers, e the identity homomorphism), and which has certain additional properties associated with expectation operators. Other expectation operators have appeared in the literature ([12, 14]) in a similar context, but that discussed in [13] seems to have the best properties and is the basis for our extension to spaces of operator-valued functions. The construction is based on the facts that

i) $\sum_{A\in G}|A'(z)|^2$ converges uniformly for z in a compact subset of D', (in particular, on $\bar{\mathscr{R}}$) and

ii) there is a polynomial $p(z)$ such that the Poincaré series

$$\theta p(z) = \sum_{A\in G} p \circ A(z)\, A'(z)^2 \geqslant m^{-1} > 0 \quad \text{for } z \in \mathscr{R}.$$

We will use these two facts to construct a bounded projection of $H^\infty_{\mathscr{L}(\mathscr{K})}$ onto $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ with expectation-like properties.

THEOREM 3.1. *Let α and β be any two elements of* Hom $(G, \mathscr{U}(\mathscr{K}))$. *There is a bounded projection mapping $E_{\alpha,\beta}$ of $H^\infty_{\mathscr{L}(\mathscr{K})}$ onto $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ (respectively, of $A_{C^\infty(\mathscr{K})}(G)$ onto $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$), with bound independent of α and β, such that: if $F \in H^\infty_{\mathscr{L}(\mathscr{K})}$, $w \in D$ and $\gamma \in$* Hom $(G, \mathscr{U}(\mathscr{K}))$ *are such that $F(Aw) = \alpha(A)\, F(w)\, \gamma(A)^*$ for all $A \in G$, then, for any $H \in H^\infty_{\mathscr{L}(\mathscr{K})}$,*

$$E_{\alpha,\beta}(FH)\,(w) = F(w)\, E_{\gamma,\beta}(H)\,(w). \tag{3.1}$$

In particular, if $F \in H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \gamma)$, then

$$E_{\alpha,\beta}(FH) = FE_{\gamma,\beta}(H). \tag{3.2}$$

Proof. For a function F in $H^\infty_{\mathscr{L}(\mathscr{K})}$, elements α and β of Hom $(G, \mathscr{U}(\mathscr{K}))$, define $E_{\alpha,\beta}(F)$ by

$$E_{\alpha,\beta}(F)\,(z) = \sum_{A\in G} p \circ A(z)\, A'(z)^2 \alpha(A)^* F \circ A(z)\, \beta(A)/\theta p(z) \tag{3.3}$$

where p and θp are as in Fact ii). By Fact i), the numerator in (3.3) converges uniformly and absolutely on compact subsets of D. By Fact ii), the denominator is an analytic function bounded above and below on $\mathscr{R}$, and hence $E_{\alpha,\beta}(F)(z)$ is analytic on $\mathscr{R}$. Since $\theta_p(B(z)) = \theta_p(z)B'(z)^2$ for $B \in G$, a formal computation, justified by the absolute convergence of the numerator of (3.3) in operator norm, gives

$$E_{\alpha,\beta}(F)\,(Az) = \alpha(A)\; E_{\alpha,\beta}(F)\,(z)\;\beta(A)^* \quad \text{for all } A \text{ in } G;$$

hence (3.1) follows and also $E_{\alpha,\beta}(F)(z)$ is analytic on all of D. To compute norms, set

$$m^{-1} = \inf\{|\theta p(z)| : z \in \mathscr{R}\} > 0,$$

$$M = \sup\{\sum_{A\in G}|A'(z)|^2 : z \in \mathscr{R}\} < \infty.$$

Then

$$\|E_{\alpha,\beta}(F)\|_\infty = \sup\{\|E_{\alpha,\beta}F(z)\| : z \in D\} =$$

$$= \sup\{\|E_{\alpha,\beta}F(z)\| : z \in \mathscr{R}\} \leqslant Mm\|F\|_\infty\|p\|_\infty,$$

or

$$\|E_{\alpha,\beta}\| \leqslant Mm\|p\|_\infty$$

for all α and β in Hom $(G, \mathscr{U}(\mathscr{K}))$.

It remains only to verify that $E_{\alpha,\beta}$ takes $A_{C^\infty(\mathscr{K})}(G)$ into $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$. Since $\sum_{A\in G} A'(z)^2$ converges uniformly and absolutely on $\mathscr{R}$, this is clear.

3.3 *An application to outer functions.* It follows from the theorem of Grauert [17] and Bungart [8] that, for any α and β in Hom $(G, \mathscr{U}(\mathscr{K}))$, there is a function $F_{\alpha,\beta}$ in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta) \cap (H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha))^{-1}$. We prove a refinement of this result, at least for a finite-dimensional auxiliary space $\mathscr{K}$.

THEOREM 3.2. *Suppose that* $\dim \mathscr{K}$ *is finite. Then for each pair* (α, β) *in* Hom$(G, \mathscr{U}(\mathscr{K}))^2$ *(the Cartesian product of* Hom $(G, \mathscr{U}(\mathscr{K}))$ *with itself), there is associated a* $F_{\alpha,\beta}$ *in*

$$H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta) \cap (H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha))^{-1}, \quad \textit{such that}$$

$$\sup\{\|F_{\alpha,\beta}\|, \|F^{-1}_{\alpha,\beta}\| : (\alpha, \beta) \in \text{Hom}\,(G, \mathscr{U}(\mathscr{K}))^2\} < \infty. \tag{3.4}$$

First we need some lemmas.

LEMMA 3.3. *Suppose* $\dim \mathscr{K}$ *is finite. If* $\mathscr{U}(\mathscr{K})$ *is given the topology induced by the operator norm of* $\mathscr{L}(\mathscr{K})$, *and* Hom $(G, \mathscr{U}(\mathscr{K}))$ *is given the topology of uniform convergence on finite subsets of* G, *then* Hom $(G, \mathscr{U}(\mathscr{K}))$ *is compact.*

Proof. The group G is the free group on n generators, say $A_1, A_2, \ldots, A_n$. Hence, if α is an element of Hom $(G, \mathscr{U}(\mathscr{K}))$, α is completely determined by the n unitary operators $\alpha(A_1), \ldots, \alpha(A_n)$. Conversely, any ordered n-tuple of unitary operators arises in this way. Hence the correspondence $\alpha \to (\alpha(A_1), \ldots, \alpha(A_n))$ induces a bijection between $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))$ and $\mathscr{U}(\mathscr{K})^n$ (the Cartesian product of $\mathscr{U}(\mathscr{K})$ with itself n times). Since multiplication is jointly continuous in the norm topology on $\mathscr{L}(\mathscr{K})$, one can see that the topology of uniform convergence on finite subsets on $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))$ is carried over to the Cartesian product topology on $\mathscr{U}(\mathscr{K})^n$ induced by the operator norm topology on $\mathscr{U}(\mathscr{K})$. Since $\dim \mathscr{K}$ is finite, this latter topology is compact.

LEMMA 3.4. *Let F be an element of $H^\infty_{\mathscr{L}(\mathscr{K})}$. Then the mapping from* $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$ *into $H^\infty_{\mathscr{L}(\mathscr{K})}$ given by $(\alpha, \beta) \to E_{\alpha,\beta}(F)$ is continuous. (The topology on* $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$ *is the product topology induced by the topology of uniform convergence on finite subsets on* $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))$).

Proof. Choose $\varepsilon > 0$. By Fact i), there is a finite subset $\mathscr{F}$ of G such that

$$\sum_{A \in G \setminus \mathscr{F}} |A'(z)|^2 < \varepsilon/4\|F\|\,\|p\|m \quad \text{for } z \in \mathscr{R}.$$

Then, for z in $\mathscr{R}$, (α, β) and (γ, β) in $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$,

$$\|E_{\alpha,\beta}F(z) - E_{\gamma,\delta}F(z)\| \leqslant$$

$$\leqslant \Big\| \sum_{A \in \mathscr{F}} A'(z)^2 p \circ A(z) \{\alpha(A)^* F \circ A(z)\, \beta(A) - \gamma(A)^* F \circ A(z)\, \delta(A)\}/\theta p(z) \Big\| +$$

$$+ \varepsilon/2 \leqslant M\|p\|\, m\|F\| \sup_{A \in \mathscr{F}} \{\|\alpha(A) - \gamma(A)\| + \|\beta(A) - \delta(A)\|\} + \varepsilon/2.$$

Since $\mathscr{F}$ is a finite set, it follows that, for all (γ, δ) in a sufficiently small neighborhood n $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$ of (α, β),

$$\sup_{A \in \mathscr{F}} [\{\|\alpha(A) - \gamma(A)\| + \|\beta(A) - \delta(A)\|\} < \varepsilon/2M\,\|p\|m\|F\|.$$

The lemma follows.

Proof of Theorem 3.2. By Grauert's theorem [17], for each (α, β) in $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$ there is a $G_{\alpha,\beta}$ in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta) \cap (H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha))^{-1}$. By Lemma 3.4, there is a neighbourhood $U(\alpha, \beta)$ of (α, β) in $\mathrm{Hom}(G, \mathscr{U}(\mathscr{K}))^2$ such that, for all (γ, δ) in $U(\alpha, \beta)$,

$$\sup \{\|E_{\gamma,\delta}G_{\alpha,\beta}(z) - G_{\alpha,\beta}(z)\| : z \in \mathscr{R}\} < \frac{1}{2} \inf \{\|G^{-1}_{\alpha,\beta}(z)\|^{-1} : z \in \mathscr{R}\}.$$

By Theorem 3.1, $E_{\gamma,\delta}G_{\alpha,\beta}$ is in $H^\infty_{\mathscr{L}(\mathscr{K})}(\gamma, \delta)$ and $\|E_{\gamma,\delta}G_{\alpha,\beta}\| \leqslant Mm\|p\|\,\|G_{\alpha,\beta}\|$. By the Neumann series expansion, the above estimate implies that $E_{\gamma,\delta}G_{\alpha,\beta}(z)$ is invertible for all (γ, δ) in $U(\alpha, \beta)$ and

$$\sup \{\|E_{\gamma,\delta}G_{\alpha,\beta}(z)^{-1}\| : z \in \mathscr{R}\} < 2\|G^{-1}_{\alpha,\beta}\|.$$

Since $E_{\gamma,\delta}G_{\alpha,\beta}$ is in $H^{\infty}_{\mathscr{L}(\mathscr{K})}(\gamma, \delta)$, this estimate holds for z in D as well. Hence we have shown that for each (α, β) in Hom $(G, \mathscr{U}(\mathscr{K}))^2$, there is a neighborhood $U(\alpha, \beta)$ of (α, β) for which the conclusion of the theorem holds. By Lemma 3.3, if $\dim \mathscr{K}$ is finite, then Hom$(G, \mathscr{U}(\mathscr{K}))^2$ is compact, and hence finitely many of the neighbourhoods $U(\alpha, \beta)$ cover Hom $(G, \mathscr{U}(\mathscr{K}))$. The theorem follows.

As in Section 2, the case where $\mathscr{K}$ is infinite-dimensional remains mysterious, unless R is an annulus.

QUESTION 2. Does the conclusion of Theorem 2.2 hold if $\dim \mathscr{K} = \infty$? In particular, is the supremum expression bounded uniformly with respect to $\dim \mathscr{K} < \infty$?

3.4 *An application to invariant subspace theorems.* We now use the expectation operators $E_{\alpha,\beta}$ to replace the underlying region D with the region R in the invariant subspace theorems arrived at in Section 1. For α and β in Hom $(G, \mathscr{U}(\mathscr{K}))$ and F on in $H^{\infty}_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$, if B is a rigid function in $H^{\infty}_{\mathscr{L}(\mathscr{K})}$ (not necessarily of full range) which divides F on the left, then necessarily there is a γ in Hom $(G, \mathscr{U}(\mathscr{K}))$ such that B is in $H^{\infty}_{\mathscr{L}(\mathscr{K})}(\alpha, \gamma)$. Also the greatest common left rigid divisor of F is in $H^{\infty}_{\mathscr{L}(\mathscr{K})}(\alpha, \gamma)$ for some γ in Hom $(G, \mathscr{U}(\mathscr{K}))$ (See [4, Section 3]). The expectation operator approach to the proof of the following theorem is similar to that of Theorem 12 of Abrahamse and Douglas [4]. The scalar case of this result (with α and β equal to the identity) is due to Voichick [29] who used different techniques.

THEOREM 3.5. *Let α and β be elements of* Hom $(G, \mathscr{U}(\mathscr{K}))$ *and let I be a non zero closed $A_{C^{\infty}(\mathscr{K})}(\beta, \beta)$-right submodule of $A_{C^{\infty}(\mathscr{K})}(\alpha, \beta)$. Let B in $H^{\infty}_{\mathscr{L}(\mathscr{K})}(\alpha, \gamma)$ (for some γ in* Hom $(G, \mathscr{U}(\mathscr{K}))$ *be the greatest common left rigid divisor of elements of I, and, for z on $\partial\mathscr{R} \cap \partial D$, let $P_I(z) = \vee \{[\text{Ran } F(z)]^{-}\colon F \in I\}$. Then I is precisely the right submodule of all functions of the form BG where G runs through all functions in $H^{\infty}_{C^{\infty}(\mathscr{K})}(\gamma, \beta)$ such that BG is in $A_{C^{\infty}(\mathscr{K})}(\alpha, \beta)$ and* Ran $B(z)\, G(z) \subset P_I(z)$ *for all z on $\partial\mathscr{R} \cap \partial D$.*

Proof. By a theorem of Fatou [19, p. 80], there exists a (scalar-valued) function f analytic on D and continuous on $\bar{D}$ such that f vanishes precisely on $\partial D \setminus \Omega$. Then $f \cdot I$ is contained in $A_{C^{\infty}(\mathscr{K})}$. Let $\tilde{I}$ be the smallest closed right ideal of $A_{C^{\infty}(\mathscr{K})}$ containing $f \cdot I$. Clearly,

$$f \cdot I \subset \tilde{I} \cap f A_{C^{\infty}(\mathscr{K})}(\alpha, \beta). \tag{3.5}$$

We claim that the containment is actually equality, as follows. Any element H of $\tilde{I}$ is a uniform limit of functions of the form $f\left(\sum_{i=1}^{m} F_i G_i\right)$ where F_i is in I and G_i is in $A_{C^{\infty}(\mathscr{K})}$. Let us assume that H is also in $f A_{C^{\infty}(\mathscr{K})}(\alpha, \beta)$. If we restrict z to $\mathscr{R}$, then, since f is bounded away from zero on $\mathscr{R}$,

$$f(z)^{-1} H(z) = \lim \sum_{i=1}^{m} F_i(z)\, G_i(z)$$

uniformly for z in $\mathscr{R}$. Since H is in $fA_{C^\infty(\mathscr{K})}(\alpha, \beta)$, $f^{-1}H|\mathscr{R}$ is in $A_{C^\infty(\mathscr{K})}(\alpha, \beta)|\mathscr{R}$. Hence, for all z in $\mathscr{R}$,

$$f(z)^{-1}H(z) = E_{\alpha,\beta}(f^{-1}H|\mathscr{R})(z) =$$

$$= \lim \sum_{i=1}^{m} E_{\alpha,\beta}(F_i G_i)(z) =$$

$$= \lim \sum_{i=1}^{m} F_i(z)\, E_{\beta,\beta} G_i(z)$$

by properties catalogued in Theorem 3.1. Since G_i is in $A_{C^\infty(\mathscr{K})}$, $E_{\beta,\beta}G_i$ is in $A_{C^\infty(\mathscr{K})}(\beta, \beta)$; since I is a closed $A_{C^\infty(\mathscr{K})}(\beta, \beta)$-right submodule of $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$, it follows that H is in $f \cdot I$, as claimed.

Now by Theorem 1.3, $\tilde{I}$ consists of all functions of the form BG for some rigid function B, where G ranges over all functions in $H^\infty_{C^\infty(\mathscr{K})}$ such that BG is in $A_{C^\infty(\mathscr{K})}$ and $\operatorname{Ran} B(z)\, G(z) \subset P_{\tilde{I}}(z)$ for each z on ∂D. The theorem follows from this representation for $\tilde{I}$ upon using the equality (3.2) and restricting z to $\bar{\mathscr{R}}$.

A similar characterization holds for $A^\infty_{C^\infty(\mathscr{K})}(\alpha, \alpha)$-left submodules of $A^\infty_{C^\infty(\mathscr{K})}(\alpha, \beta)$, and for closed submodules of $H^p_{C^p(\mathscr{K})}(\alpha, \beta)$ (the generalization of Theorem 1.1), the statements of which we leave to the reader.

We now have all the pieces needed for an approximation result.

THEOREM 3.6. *Assume that* $\dim K$ *is finite, and* α *and* β *are in* $\operatorname{Hom}(G, \mathscr{U}(\mathscr{K}))$. *Then any element of* $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$ *can be approximated uniformly by a finite linear combination of inner functions* $B_1, \ldots, B_m$ *in* $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$.

LEMMA 3.7. *Assume* $\dim \mathscr{K} < \infty$ *and* α *and* β *are as above. Then*

$$g.c.d.\ \{\Theta : \Theta \in A_{C^\infty(\mathscr{K})}(\alpha, \beta),\ \Theta \text{ inner}\} = 1_{\mathscr{K}}$$

(g.c.d. = greatest common (left rigid) divisor).

Sketch of the Proof. Let B be the greatest common divisor of the indicated set. Then necessarily B is in $A_{C^\infty(\mathscr{K})}(\alpha, \gamma)$ for some γ. Also the inner functions in Theorem 2.5 are in $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$, and hence $B(z)$ is invertible for every z in $\mathscr{R}$. This forces $B(z)$ to equal $1_{\mathscr{K}}$.

Proof of Theorem 3.6. We organize the proof into three cases of increasing generality.

Case 1: $\dim \mathscr{K} = 1$ and $\alpha = \beta = e$ (the identity homomorphism). It follows by a result of Stout [27, Theorem II.1 and IV.1] and an argument in [3, p. 325] that there are three inner functions φ, ψ and χ analytic on $\bar{R}$ such that the uniform closure of all polynomials $p(\varphi, \psi, \chi)$ in φ, ψ, and χ is precisely $A(R)$, the algebra of all functions analytic on R and continuous on $\bar{R}$. Case 1 follows via a use of the correspondence between $A(R)$ and $A_C(e, e)$.

Let J denote the uniform closure of all finite linear combinations of inner functions in $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$.

Case 2: $\alpha = \beta = e$, $\dim \mathscr{K} < \infty$. Case 1 implies that J contains any element of $A_{C^\infty(\mathscr{K})}(e, e)$ of the form

$$fP_{\{x\}} + (1_{\mathscr{K}} - P_{\{x\}})$$

where f is any function in $A_C(e, e)$, x is any unit vector, and $P_{\{x\}}$ is the projection onto the span of the vector x. Since elements of this form span $A_{C^\infty(\mathscr{K})}(e, e)$, Case 2 follows.

Case 3: α and β arbitrary. If Θ, X and Ψ are inner functions in $A_{C^\infty(\mathscr{K})}(\alpha, e)$, $A_{C^\infty(\mathscr{K})}(e, e)$ and $A_{C^\infty(\mathscr{K})}(e, \beta)$ respectively, then $\Theta X \Psi$ is an inner function in $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$. Hence, by Case 2, J contains $\Theta A_{C^\infty(\mathscr{K})}(e, e)\Psi$ for any such Θ and Ψ. Now if J_1 is the closed span of $\{\Theta A_{C^\infty(\mathscr{K})}(e, e) \colon \Theta \text{ inner}, \Theta \in A_{C^\infty(\mathscr{K})}(\alpha, e)\}$, then J_1 is a closed $A_{C^\infty(\mathscr{K})}(e, e)$-right submodule of $A_{C^\infty(\mathscr{K})}(\alpha, e)$. Since any inner Θ in $A_{C^\infty(\mathscr{K})}(\alpha, e)$ is unitary on $\partial\mathscr{R} \cap \partial D$, and by Lemma 3.7, the g.c.d. of such Θ is $1_{\mathscr{K}}$, Theorem 3.5 implies that J_1 is all of $A_{C^\infty(\mathscr{K})}(\alpha, e)$. It now follows that J contains

$$\{A_{C^\infty(\mathscr{K})}(\alpha, e)\, \Psi : \Psi \text{ inner in } A_{C^\infty(\mathscr{K})}(e, \beta)\}.$$

By a similar argument, applying Theorem 3.5 stated for $A_{C^\infty(\mathscr{K})}(\alpha, \alpha)$-left submodules of $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$, we conclude that J must be all of $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$. The Theorem follows.

4. CONCLUDING REMARKS

As mentioned in the introduction, the machinery developed in this paper (particularly Section 3) is used in [7] to prove results for operator models on finitely connected domains (see [5]). Specifically, the "uniform similarity" result (Theorem 3.2) is used to prove the Sz-Nagy-Foiaş lifting theorem for models of finite rank in this setting, and this together with the approximation result (Theorem 3.6) are the key ingredients to get an analogue of Muhly's characterization [21] of the compact operators in the commutant of a contraction. Removing the finite dimension restrictions in the result of this paper (Questions 1 and 2) would enable one to prove the above results for models not necessarily of finite rank. To accomplish this, however, new techniques appear to be needed.

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