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INVARIANT SUBSPACES OF COMPACT PERTURBATIONS OF LINEAR OPERATORS IN BANACH SPACES

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In [22] it was proved that if T is a self-adjoint operator on a complex Hilbert space H and K belongs to some Schatten-von Neumann ideal $\mathscr{C}_p$ of compact operators on H, $1 \leqslant p < \infty$, then $T + K$ has nontrivial invariant subspaces. This statement has been extended in [15] for T self-adjoint and K in the larger Macaev ideal $\mathscr{C}_\omega$ of compact operators and in [12] for T normal, the spectrum $\sigma(T)$ of T contained in a simple C^2 Jordan curve, and $K \in \mathscr{C}_\omega$.

The proofs of the above results are straightforward whenever $\sigma(T+K) \setminus \sigma(T) \neq \emptyset$; so one may consider, without loss of generality, only the case $\sigma(T+K) \subset \sigma(T)$. For T self-adjoint (respectively unitary) and $K \in \mathscr{C}_p$, $1 \leqslant p < +\infty$, with $\sigma(T+K) \subset \mathbf{R}$ (respectively $\sigma(T+K)$ on the unit circle), I. Colojoară and C. Foiaş proved in [4] that $T+K$ is $\mathfrak{A}$-scalar for some admissible function algebra $\mathfrak{A}$ on $\sigma(T+K)$; in particular, $T+K$ is decomposable. This result is extended by M. Radjabalipour and H. Radjavi in [18] as follows: if $\sigma(T)$ is contained in a closed C^2 Jordan curve Γ,

$$\|(\lambda - T)^{-1}\| \leqslant \frac{\text{constant}}{\mathrm{d}(\lambda, \Gamma)}$$

for $\lambda \notin \Gamma$, $K \in \mathscr{C}_\omega$ and $\sigma(T+K) \subset \Gamma$, then $T+K$ is decomposable.

For a complex Banach space X the operator ideals analogous to $\mathscr{C}_p$ and $\mathscr{C}_\omega$ may be defined in terms of the approximation numbers (see [17], Section 8.1). The main difficulty in extending the above results for operators on X consists in the absence of an appropriate determinant theory for operators on Banach spaces, which would enable the estimate of the norm of the resolvent of $T+K$ in terms of the norm of the resolvent of T and the approximation numbers of K. The main result of this paper (Theorem 3.1) consists just in a resolvent estimate of the above type. We reduce the general resolvent estimate problem to a finite-dimensional Hilbert space problem, where the determinant theory is available.

Once a convenient resolvent estimate is obtained for $T+K$, its decomposability follows by general criteria like [14; Th. 6], [12; Th. 2.2], [19; Th. 4.8]. The proofs of these criteria are based on an extension of [13; Lemma 27.1] (see [14; Lemma 2.2.1], [12; Lemma 2.3], [19; Lemma 4.3]). The proof of this extension — by our knowledges — has not been published and it is not obvious how the proof of [13; Lemma 27.1] may be extended. For this reason we prove two analytic tools (Propositions 4.1 and 4.4), which may present interest in themselves, and then we complete the proof of the most general of the above mentioned criteria ([19; Th. 4.8]), using the same arguments as those from [19].

We note that for operators with spectrum on a straight line, hence for operators with spectrum on an analytic curve, the assumptions of [19; Th. 4.8] lead to conclusions stronger than decomposability, namely the existence of a functional calculus with an appropriate algebra of ultradifferentiable functions (for the ideas we send to [2] and for details to [3; § 6]).

1. RESOLVENT ESTIMATES IN HILBERT SPACES

In this section we present some known results concerning compact operators and we prove a technical result which will be used later.

Let H be a complex Hilbert space, $\mathscr{L}(H)$ the C^*-algebra of all bounded linear operators on H and $\mathscr{C}(H)$ the two-sided ideal of $\mathscr{L}(H)$ consisting of all compact operators. For every $S \in \mathscr{C}(H)$ we denote by $\lambda_1(S), \lambda_2(S), \ldots$ the non-zero eigenvalues of S, taking account of their multiplicities and enumerating them, so that

$$|\lambda_1(S)| \geqslant |\lambda_2(S)| \geqslant$$

If the number $\nu(S)$ of the non-zero eigenvalues of S is finite, then for $j > \nu(S)$ we agree that

$$\lambda_j(S) = 0.$$

Let $S \in \mathscr{C}(H)$. Then we denote

$$\mu_j(S) = \lambda_j((S^*S)^{\frac{1}{2}}) \geqslant 0, \ j \geqslant 1.$$

Denoting by $\mathscr{F}_n(H)$ the set of all linear operators K on H with $\dim(KH) \leqslant n$, by a result of Dž. È. Allahverdiev (see [10; Ch. II, Th. 2.1] or [17; Th. 8.3.2]) we have

$$\mu_j(S) = \inf_{K \in \mathscr{F}_{j-1}(H)} \|S - K\|, \ j \geqslant 1. \tag{1.1}$$

The following theorem of H. Weyl (see [10; Ch. II, Th. 3.1]) establishes a connection between the numbers $\lambda_j(S)$ and $\mu_j(S)$: if $f : [0, +\infty) \to [0, +\infty)$, $f(0) = 0$, is an increasing function which becomes convex following the substitution $s = e^t$, then

$$\sum_{j=1}^{k} f(|\lambda_j(S)|) \leqslant \sum_{j=1}^{k} f(\mu_j(S)), \ k \geqslant 1.$$

In particular, for $f(s) = s$ we have

$$\sum_{j=1}^{k} |\lambda_j(S)| \leqslant \sum_{j=1}^{k} \mu_j(S),\ k \geqslant 1, \tag{1.2}$$

and for $f(s) = \ln(1+s)$ we have

$$\prod_{j=1}^{k} (1 + |\lambda_j(S)|) \leqslant \prod_{j=1}^{k} (1 + \mu_j(S)),\ k \geqslant 1. \tag{1.3}$$

Now we recall the definition of the Schatten-von Neumann ideals of compact operators:

$$\mathscr{C}_p(H) = \left\{S \in \mathscr{C}(H);\ \sum_{j=1}^{\infty} \mu_j(S)^p < +\infty\right\},\ 1 \leqslant p < +\infty,$$

$$\mathscr{C}_\infty(H) = \mathscr{C}(H).$$

Endowing $\mathscr{C}_p(H)$ with the norm $\|\cdot\|_p$ defined by

$$\|S\|_p = \left(\sum_{j=1}^{\infty} \mu_j(S)^p\right)^{1/p},\ 1 \leqslant p < +\infty,$$

$$\|S\|_\infty = \sup_{j \geqslant 1} \mu_j(S) = \|S\|,$$

makes it into a Banach space.

For $S \in \mathscr{C}_1(H)$, taking into account (1.2), one can define the determinant of $I - S$ by the formula

$$\det(I - S) = \prod_{j=1}^{\infty} (1 - \lambda_j(S)).$$

We note that 1 belongs to the spectrum $\sigma(S)$ of S if and only if $\det(I - S) = 0$. Following (1.3) we have

$$|\det(I - S)| \leqslant \prod_{j=1}^{\infty} (1 + \mu_j(S)) \leqslant e^{\|S\|_1}. \tag{1.4}$$

If $D \subset \mathbf{C}$ is open and the map $z \to S(z)$ from D to $\mathscr{C}_1(H)$ is analytic, then the function $z \to \det(I - S(z))$ from D to $\mathbf{C}$ is analytic (see [10; Ch. IV, § 1, Section 8]).

Variants of the next result, for which we refer to [10; Ch.V, Th.5.1], were obtained by several authors: if $S \in \mathscr{C}_1(H)$ and $1 \notin \sigma(S)$ then

$$\|(I - S)^{-1}\| \leqslant \frac{1}{|\det(I - S)|} \prod_{j=1}^{\infty} (1 + \mu_j(S)). \tag{1.5}$$

Using (1.4), (1.5) and standard function-theoretical arguments, in the same way as in [22], [4; Ch.6], [16], [12] and [18], we prove now a technical result concerning the majorization of the resolvents of operators from $\mathscr{C}_1(H)$, which depend analytically on a complex parameter. We include the used function theoretical tools in the following lemma:

1.1. LEMMA. *Let Γ be a closed C^2 Jordan curve in the complex plane* $\mathbf{C}$, *D the unbounded connected component of $\mathbf{C}\setminus\Gamma$ and let us denote*

$$\mathrm{d}(z, \Gamma) = \inf_{\zeta\in\Gamma} |z - \zeta|, \ z \in D.$$

Then there exists $c_\Gamma \geqslant 1$ such that for every $\varepsilon > 0$ and every analytic function

$$F : \{z \in D; \mathrm{d}(z, \Gamma) > \varepsilon\} \to \mathbf{C}$$

without zeros and satisfying $\lim_{|z|\to\infty} F(z) = 1$ we have

$$\sup_{\mathrm{d}(z,\Gamma)>2c_\Gamma^2\varepsilon} \ln\frac{1}{|F(z)|} \leqslant 2\left(1 + \frac{1}{c_\Gamma\varepsilon}\right)\sup_{\mathrm{d}(z,\Gamma)>2\varepsilon} \ln|F(z)|.$$

Proof. We denote

$$C = \{w \in \mathbf{C};\ |w| = 1\},$$

$$\Omega = \{w \in \mathbf{C};\ |w| > 1\}.$$

By the classical theorems of Riemann and Carathéodory, there exists a conformal map $\Phi\colon \Omega \to D$ such that $\Phi(\infty) = \infty$ and it can be extended to a homeomorphism of $\overline{\Omega} = \Omega \cup C$ onto $\overline{D} = D \cup \Gamma$. Now, by another classical result (see, for example, [21]), there exists $c_\Gamma \geqslant 1$ such that

$$c_\Gamma^{-1}|w_1 - w_2| \leqslant |\Phi(w_1) - \Phi(w_2)| \leqslant c_\Gamma|w_1 - w_2|,\ w_1, w_2 \in \Omega.$$

Hence

$$c_\Gamma^{-1}\mathrm{d}(w, C) \leqslant \mathrm{d}(\Phi(w), \Gamma) \leqslant c_\Gamma\mathrm{d}(w, C),\ w \in \Omega,$$

that is

$$c_\Gamma^{-1}(|w| - 1) \leqslant \mathrm{d}(\Phi(w), \Gamma) \leqslant c_\Gamma(|w| - 1),\ w \in \Omega.$$

We conclude that $\Psi : \{\zeta \in \mathbf{C}; |\zeta| < 1\} \to D \cup \{\infty\}$, defined by

$$\Psi(\zeta) = \Phi\left(\frac{1}{\zeta}\right).$$

is a conformal map and

$$(1.6)\qquad c_\Gamma^{-1}\frac{1-|\zeta|}{|\zeta|}\leqslant \mathrm{d}(\Psi(\zeta),\Gamma)\leqslant c_\Gamma\frac{1-|\zeta|}{|\zeta|},\quad 0\neq|\zeta|<1.$$

Now let $\varepsilon>0$ and $F:\{z\in D;\ \mathrm{d}(z,\Gamma)>\varepsilon\}\to\mathbf{C}$ be an analytic function without zeros and satisfying $\lim\limits_{|z|\to\infty}F(z)=1$. Taking into account (1.6), we may define

$$G:\left\{\zeta\in\mathbf{C};\ |\zeta|<\frac{1}{1+c_\Gamma\varepsilon}\right\}\to\mathbf{C}$$

by

$$G(\zeta)=F(\Psi(\zeta)),\quad |\zeta|<\frac{1}{1+c_\Gamma\varepsilon}\cdot$$

Then G is analytic, has no zeros and satisfies $G(0)=1$. By the Carathéodory inequality we have

$$\sup_{|\zeta|<\frac{1}{1+2c_\Gamma\varepsilon}}\ln\frac{1}{|G(\zeta)|}\leqslant\frac{2\dfrac{1}{1+2c_\Gamma\varepsilon}}{\dfrac{1}{1+c_\Gamma\varepsilon}-\dfrac{1}{1+2c_\Gamma\varepsilon}}\sup_{|\zeta|<\frac{1}{1+2c_\Gamma\varepsilon}}\ln|G(\zeta)|.$$

Using again (1.6), we get

$$\{z\in D;\ \mathrm{d}(z,\Gamma)>2c_\Gamma^2\varepsilon\}\subset\Psi\left(\left\{\zeta\in\mathbf{C};\ |\zeta|<\frac{1}{1+2c_\Gamma\varepsilon}\right\}\right)\subset\{z\in D;\ \mathrm{d}(z,\Gamma)>2\varepsilon\},$$

so it follows that

$$\sup_{\mathrm{d}(z,\Gamma)>2c_\Gamma^2\varepsilon}\ln\frac{1}{|F(z)|}\leqslant\frac{2(1+c_\Gamma\varepsilon)}{c_\Gamma\varepsilon}\sup_{\mathrm{d}(z,\Gamma)>2\varepsilon}\ln|F(z)|.$$

q.e.d.

Now we prove the announced result concerning the majorization of the resolvents:

1.2. PROPOSITION. *Let Γ be a closed C^2 Jordan curve in* $\mathbf{C}$, *D the unbounded connected component of $\mathbf{C}\setminus\Gamma$ and $c_\Gamma\geqslant 1$ a constant satisfying the statement of Lemma* 1.1. *Then, for every $\varepsilon>0$, every complex Hilbert space H and every analytic map $z\to S(z)$ from $\{z\in D;\ \mathrm{d}(z,\Gamma)>\varepsilon\}$ to $\mathscr{C}_1(H)$ such that*

$$1\notin\sigma(S(z)),\quad z\in D,\ \mathrm{d}(z,\Gamma)>\varepsilon,$$

$$\lim_{|z|\to\infty}\|S(z)\|_1=0,$$

we have

$$\sup_{d(z,\Gamma)>2c_\Gamma^2\varepsilon} \ln\|(I-S(z))^{-1}\| \leqslant \left(3+\frac{2}{c_\Gamma\varepsilon}\right) \sup_{d(z,\Gamma)>2\varepsilon} \sum_{j=1}^{\infty} \ln(1+\mu_j(S(z))).$$

Proof. Applying Lemma 1.1 to $F(z)=\det(I-S(z))$, we get

$$\sup_{d(z,\Gamma)>2c_\Gamma^2\varepsilon} \ln|\det(I-S(z))| \leqslant 2\left(1+\frac{1}{c_\Gamma\varepsilon}\right) \sup_{d(z,\Gamma)>2\varepsilon} \ln|\det(I-S(z))|.$$

Using (1.4) and (1.5), our statement follows.

q.e.d.

2. RESOLVENT ESTIMATES IN BANACH SPACES

The aim of this section is to extend Proposition 1.2 for operators on Banach spaces.

Let X be a complex Banach space and $\mathscr{L}(X)$ the Banach algebra of all bounded linear operators on X. Denoting by $\mathscr{F}_n(X)$ the set of all $K\in\mathscr{L}(X)$ with $\dim(KX)\leqslant n$, for every $T\in\mathscr{L}(X)$ we define the approximation numbers

$$\alpha_j(T)=\inf_{K\in\mathscr{F}_{j-1}(X)}\|T-K\|,\ j\geqslant 1.$$

Clearly (see [17], Section 8.1)

(2.1) $$\alpha_j(ATB)\leqslant\|A\|\alpha_j(T)\|B\|,\quad T,A,B\in\mathscr{L}(X),\ j\geqslant 1.$$

We remark that, if X is a Hilbert space, then by (1.1) for every compact operator $T\in\mathscr{L}(X)$ we have $\alpha_j(T)=\mu_j(T), j\geqslant 1$.

2.1. LEMMA. *Let* $T\in\mathscr{L}(X)$ *be such that* $\dim(TX)<+\infty$ *and* $1\notin\sigma(T)$. *Let further* $P\in\mathscr{L}(X)$ *be a projection with* $\mathrm{Ker}(P)\subset\mathrm{Ker}(T)$ *and* $\dim(PX)<+\infty$. *Then, defining the compression* $T_P\in\mathscr{L}(PX)$ *by*

$$T_P(y)=PT(y),\ y\in PX,$$

we have

$$\alpha_j(T_P)\leqslant\|P\|\alpha_j(T),\ \ j\geqslant 1,$$
$$1\notin\sigma(T_P),$$
$$\|(I-T)^{-1}\|\leqslant 1+\|T\|\,\|P\|\,\|(I_P-T_P)^{-1}\|.$$

Proof. Let $j\geqslant 1$. For every $K\in\mathscr{F}_{j-1}(X)$ we have $K_P\in\mathscr{F}_{j-1}(PX)$, so

$$\alpha_j(T_P)\leqslant\|T_P-K_P\|\leqslant\|P\|\,\|T-K\|.$$

Consequently,

$$\alpha_j(T_P) \leqslant \|P\| \inf_{K \in \mathscr{F}_{j-1}(X)} \|T - K\| = \|P\| \alpha_j(T).$$

Since $(I - P)X = \mathrm{Ker}(P) \subset \mathrm{Ker}(T)$, we have

$$TP = T. \tag{2.2}$$

Using (2.2), for all $y \in PX$

$$\begin{aligned}(I_P - T_P)[(I - T)^{-1}]_P\, y &= P(I - T)^{-1} y - PTP(I - T)^{-1} y = \\ &= P(I - T)^{-1} y - PT(I - T)^{-1} y = \\ &= P(I - T)(I - T)^{-1} y = y,\end{aligned}$$

so

$$(I_P - T_P)[(I - T)^{-1}]_P = I_P.$$

It follows that $I_P - T_P$ is surjective, hence, PX being finite-dimensional, it is also injective. Thus $I_P - T_P$ is invertible, and

$$(I_P - T_P)^{-1} = [(I - T)^{-1}]_P. \tag{2.3}$$

In particular, $1 \notin \sigma(T_P)$.

Next, by (2.2) we have successively

$$I - P = (I - T)(I - P),$$

$$(I - T)^{-1}(I - P) = I - P,$$

$$P(I - T)^{-1}(I - P) = 0,$$

$$P(I - T)^{-1} = P(I - T)^{-1} P. \tag{2.4}$$

Using (2.2) and (2.4), we deduce

$$(I - T)^{-1} = I + T(I - T)^{-1} = I + TP(I - T)^{-1} P.$$

Therefore we have for all $x \in X$

$$\begin{aligned}\|(I - T)^{-1} x\| &\leqslant \|x\| + \|T\| \|P(I - T)^{-1} Px\| \leqslant \\ &\leqslant \|x\| + \|T\| \|[(I - T)^{-1}]_P\| \|Px\| \leqslant \\ &\leqslant (1 + \|T\| \|P\| \|[(I - T)^{-1}]_P\|) \|x\|\end{aligned}$$

and by (2.3) we conclude

$$\|(I - T)^{-1}\| \leqslant 1 + \|T\| \, \|P\| \, \|(I_P - T_P)^{-1}\|.$$

q.e.d.

In order to use Lemma 2.1, we need the following known result:

2.2. LEMMA. *If Y is a linear subspace of X with $n = \dim(Y) < +\infty$ then there exists a linear isomorphism $W : l_n^2 \to Y$ such that $\|W\| \leqslant \sqrt{n}$ and $\|W^{-1}\| \leqslant \sqrt{n}$.*

If Z is a closed linear subspace of X with $n = \dim(X/Z) < +\infty$ then for each $\delta > 0$ there exists a projection $P \in \mathscr{L}(X)$ such that $\operatorname{Ker}(P) = Z$ and $\|P\| \leqslant n + \delta$.

Proof. By the Auerbach theorem (see, for example, [17; 8.4.1, Lemma 1]) there exist $y_1, \ldots, y_n \in Y$ and $y_1^*, \ldots, y_n^* \in Y^*$ such that

$$\|y_j\| = 1, \ \|y_j^*\| = 1, \ 1 \leqslant j \leqslant n,$$

$$< y_j, y_k^* > = \delta_{jk}, \ 1 \leqslant j, k \leqslant n.$$

Defining $W : l_n^2 \to Y$ by

$$W(\{\alpha_j\}_{1 \leqslant j \leqslant n}) = \sum_{j=1}^{n} \alpha_j y_j,$$

W is a linear isomorphism and

$$W^{-1}(y) = \{< y, y_j^* >\}_{1 \leqslant j \leqslant n}.$$

Now an easy computation shows that $\|W\| \leqslant \sqrt{n}$ and $\|W^{-1}\| \leqslant \sqrt{n}$.

Applying the Auerbach theorem to X/Z, we deduce the existence of $x_1, \ldots, x_n \in X$ and $x_1^*, \ldots, x_n^* \in X^*$ such that

$$\|x_j\| \leqslant 1 + \frac{\delta}{n}, \ \|x_j^*\| = 1, \ 1 \leqslant j \leqslant n,$$

$$< x_j, x_k^* > = \delta_{jk}, \ 1 \leqslant j, k \leqslant n,$$

$$Z = \bigcap_{j=1}^{n} \operatorname{Ker}(x_j^*).$$

Defining $P \in \mathscr{L}(X)$ by

$$P(x) = \sum_{j=1}^{n} < x, x_j^* > x_j,$$

it is easy to verify that P is a projection, $\operatorname{Ker}(P) = Z$ and $\|P\| \leqslant n + \delta$.

q.e.d.

We note that the constants $\sqrt{n}$ and n in Lemma 2.2 are not the best possibles. For more exact estimates we refer the reader to [9] and [11].

Now we are able to prove our variant of Proposition 1.2 for Banach spaces:

2.3. PROPOSITION. *Let Γ be a closed C^2 Jordan curve, D the unbounded connected component of $\mathbf{C}\setminus\Gamma$ and $c_\Gamma \geqslant 1$ be a constant satisfying the statement of Lemma 1.1. Assume there exist an operator $K\in\mathscr{L}(X)$, a positive number ε and an analytic function $z\to T(z)$ from $\{z\in D;\, \mathrm{d}(z,\Gamma)>\varepsilon\}$ into $\mathscr{L}(X)$ such that* $\dim(KX)=n<\infty$ *and*

$$1\notin\sigma(T(z)K),\quad z\in D,\ \mathrm{d}(z,\Gamma)>\varepsilon, \tag{2.5}$$

$$\lim_{|z|\to\infty}\|T(z)\,x\|=0,\qquad x\in X. \tag{2.6}$$

Then

$$\sup_{\mathrm{d}(z,\Gamma)>2c_\Gamma^2\varepsilon}\ln\|(I-T(z)K)^{-1}\|\leqslant$$

$$\leqslant 1+\sup_{\mathrm{d}(z,\Gamma)>2c_\Gamma^2\varepsilon}\ln^+(\|T(z)\|\,\|K\|)+$$

$$+\,2\ln^+n+\left(3+\frac{2}{c_\Gamma\varepsilon}\right)n^2\sup_{\mathrm{d}(z,\Gamma)>2\varepsilon}\sum_{j=1}^{n}\ln(1+\|T(z)\|\,\alpha_j(K)),$$

where, as usual, $\ln^+\lambda=\ln\lambda$ *for* $\lambda\in[1,+\infty)$ *and* $\ln^+\lambda=0$ for $\lambda\in(-\infty,1]$.

Proof. Let $P\in\mathscr{L}(X)$ be an arbitrary projection such that $\mathrm{Ker}(P)=\mathrm{Ker}(K)$. Then $\dim(PX)=n$. Let further $W:l_n^2\to PX$ be an arbitrary linear isomorphism.

Similarly as in the statement of Lemma 2.1, for every $T\in\mathscr{L}(X)$ we define the compression $T_P\in\mathscr{L}(PX)$ by

$$T_P(y)=PT(y),\quad y\in PX.$$

Using this notation, we define an analytic mapp $z\to S(z)$ from $\{z\in D;\,\mathrm{d}(z,\ \Gamma)>\varepsilon\}$ to $\mathscr{L}(l_n{}^2)=\mathscr{C}_1(l_n{}^2)$ by the formula

$$S(z)=W^{-1}(T(z)K)_P\,W.$$

By (2.5) and by Lemma 2.1 we have

$$1\notin\sigma(S(z)),\quad z\in D,\quad \mathrm{d}(z,\Gamma)>\varepsilon,$$

and by (2.6)

$$\lim_{|z|\to\infty}\|S(z)\|_1=0.$$

Using Proposition 1.2, we conclude that

$$\sup_{\mathrm{d}(z,\Gamma)<2c_\Gamma^2\varepsilon}\ln^+\|(I-S(z))^{-1}\|\leqslant\left(3+\frac{2}{c_P\varepsilon}\right)\sup_{\mathrm{d}(z,\Gamma)>2\varepsilon}\sum_{j=1}^{n}\ln(1+\mu_j(S(z)). \tag{2.7}$$

Now, by Lemma 2.1 and by (2.1), we have, for all $z \in D$, $\mathrm{d}(z, \Gamma) > \varepsilon$ and $j \geqslant 1$

$$\mu_j(S(z)) = \alpha_j(S(z)) \leqslant$$
$$\leqslant \|W^{-1}\| \, \|W\| \, \alpha_j((T(z)K)_P) \leqslant$$
$$\leqslant \|W^{-1}\| \, \|W\| \, \|P\| \alpha_j(T(z)K) \leqslant$$
$$\leqslant \|W^{-1}\| \, \|W\| \, \|P\| \, \|T(z)\| \, \alpha_j(K).$$

On the other hand, again by Lemma 2.1, we have for all $z \in D$, $\mathrm{d}(z, \Gamma) > \varepsilon$

$$\ln\|(I - T(z)K)^{-1}\| \leqslant \ln[1 + \|T(z)\| \, \|K\| \, \|P\| \, \|(I_P - (T(z)K)_P)^{-1}\|] \leqslant$$
$$\leqslant \ln[1 + \|T(z)\| \, \|K\| \, \|W^{-1}\| \, \|W\| \, \|P\| \, \|(I - S(z))^{-1}\|] \leqslant$$
$$\leqslant 1 + \ln^+[\|T(z)\| \, \|K\| \, \|W^{-1}\| \, \|W\| \, \|P\| \, \|(I - S(z))^{-1}\|].$$

Denoting $d_{P,W} = \|W^{-1}\| \, \|W\| \, \|P\|$ and using (2.7), we deduce

$$\sup_{\mathrm{d}(z, \Gamma) > 2c_\Gamma^2 \varepsilon} \ln\|(I - T(z)K)^{-1}\| \leqslant$$
$$\leqslant 1 + \sup_{\mathrm{d}(z, \Gamma) > 2c_\Gamma^2 \varepsilon} \ln^+(\|T(z)\| \, \|K\| +$$
$$+ \ln^+ d_{P,W} + \left(3 + \frac{2}{c_\Gamma \varepsilon}\right) d_{P,W} \sup_{\mathrm{d}(z, \Gamma) > 2\varepsilon} \sum_{j=1}^{n} \ln(1 + \|T(z)\| \alpha_j(K)).$$

Finally, by Lemma 2.2, for every $\delta > 0$ the operators P and W can be chosen such that $d_{P,W} \leqslant n(n + \delta)$, so we conclude that in the above inequality $d_{P,W}$ may be replaced by n^2.

q.e.d.

3. RESOLVENT ESTIMATES FOR COMPACT PERTURBATIONS

In this section we prove our main result, extending Theorem 1 of [18] for operators on Banach spaces.

We note that if K is a bounded linear operator on a Banach space X then $\lim_{j \to \infty} \alpha_j(K) = 0$ means that K belongs to the norm-closure of the linear operators on X with finite-dimensional range. In this case we denote for each $t > 0$

$$n_K(t) = \text{the number of } \alpha_j(K) \text{ with } \alpha_j(K) \geqslant \frac{1}{t}\,.$$

3.1. THEOREM. *Let Γ be a closed C^2 Jordan curve, D the unbounded connected component of $\mathbf{C}\setminus\Gamma$ and $c_\Gamma \geqslant 1$ be a constant satisfying the statement of Lemma* 1.1. *Let further X be a complex Banach space, $T\in\mathscr{L}(X)$ with $\sigma(T)\subset\mathbf{C}\setminus D$ and $f:(0,+\infty)\to(0,+\infty)$ be an increasing function such that*

$$\|(\lambda-T)^{-1}\|\leqslant f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right),\qquad \lambda\in D.$$

Then, if K is in the norm-closure of the set of finite-rank operators on X and if $\sigma(T+K)\subset\mathbf{C}\setminus D$, we have

$$\ln^+\ln^+\|(\lambda-T-K)^{-1}\|\leqslant$$

$$\leqslant 8+\ln^+\ln^+(2(1+\|K\|))+\ln\left(1+\frac{c_\Gamma}{\mathrm{d}(\lambda,\Gamma)}\right)+\ln^+\ln^+f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)+$$

$$+3\ln^+\int_0^{2ef\left(\frac{c_\Gamma^2}{\mathrm{d}(\lambda,\Gamma)}\right)}\frac{n_K(t)}{t}\,\mathrm{d}t\leqslant$$

$$\leqslant 14+4\ln^+\ln^+(2(1+\|K\|))+\ln\left(1+\frac{c_\Gamma}{\mathrm{d}(\lambda,\Gamma)}\right)+4\ln^+\ln^+\mathrm{f}\left(\frac{2c_\Gamma^2}{\mathrm{d}(\lambda,\Gamma)}\right)+$$

$$+3\ln^+\mathrm{n}_K\left(2ef\left(\frac{2c^2}{\mathrm{d}(\lambda,\Gamma)}\right)\right),\qquad \lambda\in D.$$

Proof. Let $\lambda\in D$ be fixed. We define

$$\varepsilon=\frac{\mathrm{d}(\lambda,\Gamma)}{2c_\Gamma^2}>0, \tag{3.1}$$

$$n=n_K(2f(1/\varepsilon)). \tag{3.2}$$

Since $\alpha_{n+1}(K)>\dfrac{1}{2f(1/\varepsilon)}$, there exists $K_1\in\mathscr{L}(X)$ with $\dim(K_1X)\leqslant n$ such that for $K_2=K-K_1$ we have $\|K_2\|\leqslant\dfrac{1}{2f(1/\varepsilon)}$. Consequently, if $z\in D$ and $\mathrm{d}(z,\Gamma)>\varepsilon$, we have $\|(z-T)^{-1}K_2\|\leqslant\dfrac{1}{2}$, hence $(I-(z-T)^{-1}K_2)^{-1}$ exists and

$$\|(I-(z-T)^{-1}K_2)^{-1}\|\leqslant 2.$$

Since $\sigma(T)\subset \mathbf{C}\setminus D$ and $\sigma(T+K)\subset \mathbf{C}\setminus D$, it follows that if $z\in D$ and $\mathrm{d}(z, \Gamma)>\varepsilon$, then

$$\begin{aligned}&(z-T-K)^{-1}=\\ &=(I-(z-T)^{-1}K)^{-1}(z-T)^{-1}=\\ &=(I-(z-T)^{-1}K_2-(z-T)^{-1}K_1)^{-1}(z-T)^{-1}=\\ &=[I-(I-(z-T)^{-1}K_2)^{-1}(z-T)^{-1}K_1]^{-1}(I-(z-T)^{-1}K_2)^{-1}(z-T)^{-1}.\end{aligned}$$

Define

$$T(z)=(I-(z-T)^{-1}K_2)^{-1}(z-T)^{-1}$$

for $z\in D$ and $\mathrm{d}(z, \Gamma)>\varepsilon$. The map $z\mapsto T(z)$ is an operator-valued analytic function with the following properties:

$$1\notin\sigma(T(z)\,K_1),$$

$$\lim_{|z|\to\infty}\|T(z)\|=0$$

$$\|T(z)\|\leqslant 2f\left(\frac{1}{\mathrm{d}(z,\Gamma)}\right),$$

$$\|(z-T-K)^{-1}\|\leqslant 2f\left(\frac{1}{\mathrm{d}(z,\Gamma)}\right)\|(I-T(z)K_1)^{-1}\|.$$

Applying Proposition 2.3 and taking into account (3.1), we deduce

$$(3.3)\qquad\left\{\begin{aligned}&\|\ln(\lambda-T-K)^{-1}\|\leqslant\\ &\leqslant\ln\left(2f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\right)+1+\ln^+\left(2f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\|K_1\|\right)+\\ &+2\ln^+n+\left(3+\frac{2}{c_\Gamma\varepsilon}\right)n^2\sum_{j=1}^{n}\ln(1+2f(1/2\varepsilon)\alpha_j(K_1)).\end{aligned}\right.$$

We complete the proof by estimating appropriately the right side of the inequality (3.3).

First, since $\|K_1\|\leqslant\|K\|+\|K_2\|\leqslant\|K\|+\dfrac{1}{2f(1/\varepsilon)}$ and $f\left(\dfrac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\leqslant f(1/\varepsilon)$, we have

$$\ln^+\left(2f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\|K_1\|\right)\leqslant\ln^+\left(1+2\|K\|f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\right)\leqslant$$

$$\leqslant 1+\ln^+\left(2\|K\|f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\right).$$

Using (3.3) we deduce that

$$(3.4)\quad \begin{cases} \ln\|(\lambda - T - K)^{-1}\| \leqslant \\ \leqslant 2 + 2\ln^+\left(2(1+\|K\|)f\left(\dfrac{1}{\mathrm{d}(\lambda, \Gamma)}\right)\right) + \\ + 2\ln^+ n + \left(3 + \dfrac{2}{c_\Gamma \varepsilon}\right) n^2 \sum_{j=1}^{n} \ln(1 + 2f(1/2\varepsilon)\alpha_j(K_1)). \end{cases}$$

Next, since

$$\alpha_j(K_1) \leqslant \alpha_j(K) + \|K_2\| \leqslant \alpha_j(K) + \frac{1}{2f(1/\varepsilon)}, \quad j \geqslant 1,$$

and $f(1/2\varepsilon) \leqslant f(1/\varepsilon)$, we have

$$\sum_{j=1}^{n} \ln(1 + 2f(1/2\varepsilon)\alpha_j(K_1)) \leqslant \sum_{j=1}^{n} \ln(2 + 2f(1/2\varepsilon)\alpha_j(K)) \leqslant$$

$$\leqslant n \ln 2 + \sum_{j=1}^{n} \ln(1 + f(1/\varepsilon)\alpha_j(K)).$$

On the other hand, taking into account (3.2) and integrating by parts, we get

$$\sum_{j=1}^{n} \ln(1 + f(1/\varepsilon)\alpha_j(K)) = \int_0^{2f(1/\varepsilon)} \ln\left(1 + \frac{f(1/\varepsilon)}{t}\right) \mathrm{d}n_K(t) =$$

$$= n_K(2f(1/\varepsilon)) \ln\frac{3}{2} + \int_0^{2f(1/\varepsilon)} \frac{n_K(t)}{t} \frac{f(1/\varepsilon)}{t + f(1/\varepsilon)} \mathrm{d}t \leqslant$$

$$\leqslant \int_{2f(1/\varepsilon)}^{3f(1/\varepsilon)} \frac{n_K(t)}{t} \mathrm{d}t + \int_0^{2f(1/\varepsilon)} \frac{n_K(t)}{t} \mathrm{d}t =$$

$$= \int_0^{3f(1/\varepsilon)} \frac{n_K(t)}{t} \mathrm{d}t.$$

Thus

$$\sum_{j=1}^{n} \ln(1 + 2f(1/2\varepsilon)\alpha_j(K_1)) \leqslant n \ln 2 + \int_0^{3f(1/\varepsilon)} \frac{n_K(t)}{t} \mathrm{d}t$$

and, using (3.4), we deduce

$$(3.5)\quad \begin{cases} \ln\|(\lambda - T - K)^{-1}\| \leqslant \\ \leqslant 2 + 2\ln^+\left(2(1 + \|K\|)f\left(\dfrac{1}{\mathrm{d}(\lambda, \Gamma))}\right)\right) + \\ + 2\ln^+ n + \left(3 + \dfrac{2}{c_\Gamma \varepsilon}\right) n^3 \ln 2 + \left(3 + \dfrac{2}{c_\Gamma \varepsilon}\right) n^2 \displaystyle\int_0^{3f(1/\varepsilon)} \frac{n_K(t)}{t} \mathrm{d}t. \end{cases}$$

Taking into account (3.2), we have

$$\ln^+ n \leqslant n \leqslant \int_{2f(1/\varepsilon)}^{2ef(1/\varepsilon)} \frac{n_K(t)}{t}\,\mathrm{d}t \leqslant \int_0^{2ef(1/\varepsilon)} \frac{n_K(t)}{t}\,\mathrm{d}t.$$

It follows from (3.5) that

$$\ln\|(\lambda - T - K)^{-1}\| \leqslant$$

$$\leqslant 2 + 2\ln^+\left(2(1+\|K\|)f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\right) + \left(8 + \frac{4}{c_\Gamma \varepsilon}\right)\left(\int_0^{2ef(1/\varepsilon)} \frac{n_K(t)}{t}\,\mathrm{d}t\right)^3 \leqslant$$

$$\leqslant 1 + [1 + 2\ln^+(2(1+\|K\|))]\left[1 + 2\ln^+ f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right)\right]\cdot$$

$$\cdot\left[1 + \left(8 + \frac{4}{c_\Gamma\varepsilon}\right)\left(\int_0^{2ef(1/\varepsilon)} \frac{n_K(t)}{t}\,\mathrm{d}t\right)^3\right],$$

hence

$$\ln^+\ln^+\|(\lambda - T - K)^{-1}\| \leqslant$$

$$\leqslant 1 + \ln^+[*] + \ln^+[**] + \ln^+[***] \leqslant$$

$$\leqslant 4 + \ln 32 + \ln^+\ln^+(2(1+\|K\|)) + \ln\left(1 + \frac{1}{2c_\Gamma\varepsilon}\right) + \ln^+\ln^+ f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right) +$$

$$+ 3\ln^+\int_0^{2ef(1/\varepsilon)} \frac{n_K(t)}{t}\,\mathrm{d}t.$$

By (3.1) we conclude that the first inequality in the statement holds.

Finally, for $s > \dfrac{1}{\alpha_1(K)} = \dfrac{1}{\|K\|}$ we have

$$\int_0^s \frac{n_K(t)}{t}\,\mathrm{d}t = \int_{1/\|K\|}^s \frac{n_K(t)}{t}\,\mathrm{d}t \leqslant n_K(s)\ln(\|K\|s),$$

so for all $s > 0$

$$\ln^+\int_0^s \frac{n_K(t)}{t}\,\mathrm{d}t \leqslant \ln^+\ln^+(\|K\|s) + \ln^+ n_K(s).$$

Therefore, for every $\lambda \in D$

$$\ln^+\int_0^{2ef\left(\frac{2c_\Gamma^2}{\mathrm{d}(\lambda,\Gamma)}\right)} \frac{n_K(t)}{t}\,\mathrm{d}t \leqslant \ln^+\ln^+\left(2e\|K\|f\left(\frac{2c_\Gamma^2}{\mathrm{d}(\lambda,\Gamma)}\right)\right) + \ln^+ n_K\left(2ef\left(\frac{2c_\Gamma^2}{\mathrm{d}(\lambda,\Gamma)}\right)\right)$$

Since

$$\ln^+\ln^+\left(2e\|K\| f\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right)\right) = \ln^+\left[1 + \ln^+(2\|K\|) + \ln^+ f\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right)\right] \leqslant$$

$$\leqslant \ln^+\left[(1 + \ln^+(2\|K\|))(1 + \ln^+ f\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right))\right] \leqslant$$

$$\leqslant 2 + \ln^+\ln^+(2\|K\|) + \ln^+\ln^+ f\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right),$$

it follows for all $\lambda \in D$

$$\ln^+ \int_0^{2ef\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right)} \frac{n_K(t)}{t}\, dt \leqslant 2 + \ln^+\ln^+(2(1 + \|K\|)) + \ln^+\ln^+ f\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right) +$$

$$+ \ln^+ n_K\left(2ef\left(\frac{2c_\Gamma^2}{d(\lambda, \Gamma)}\right)\right).$$

Using the above inequality, we get also the second inequality from the statement.

q.e.d.

The following consequence of Theorem 3.1 is particularly useful:

3.2. COROLLARY. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space, $T \in \mathscr{L}(X)$ and K in the norm-closure of finite-rank operators on X. Assume*

$$\sigma(T) \subset \Gamma, \quad \sigma(T + K) \subset \Gamma.$$

If there exists an increasing function $f : (0, +\infty) \to (0, +\infty)$ such that

$$\|(\lambda - T)^{-1}\| \leqslant f\left(\frac{1}{d(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

$$\int_1^{+\infty} \frac{\ln^+\ln^+ f(t)}{t^2}\, dt < +\infty,$$

$$\int_1^{+\infty} \frac{\ln^+ n_K(af(t) + b)}{t^2}\, dt < \infty, \quad a, b > 0,$$

then there exists an increasing function $g : (0, +\infty) \to (0, +\infty)$ such that

$$\|(\lambda - T - K)^{-1}\| \leqslant g\left(\frac{1}{d(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

$$\int_1^{+\infty} \frac{\ln^+\ln^+ g(t)}{t^2}\, dt < +\infty.$$

Proof. We may assume, without restricting the generality, that 0 belongs to the bounded component of $\mathbf{C}\setminus\Gamma$. We denote the bounded component of $\mathbf{C}\setminus\Gamma$ by D_0 and the unbounded component by D_∞.

By Theorem 3.1 there exists an increasing function $g_\infty : (0, +\infty) \to (0, +\infty)$ such that

$$\|(\lambda - T - K)^{-1}\| \leqslant g_\infty\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \qquad \lambda \in D_\infty,$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ g_\infty(t)}{t^2}\, \mathrm{d}t < +\infty.$$

Now $\Gamma^{-1} = \{\lambda^{-1}; \lambda \in \Gamma\}$ is again a closed C^2 Jordan curve and $D_0^{-1} = \{\lambda^{-1}; 0 \neq \lambda \in D_0\}$ is the unbounded component of $\mathbf{C}\setminus\Gamma^{-1}$. We choose $r_0 > 0$ such that

$$\{\lambda \in \mathbf{C};\ |\lambda| \leqslant r_0\} \subset D_0 \subset \{\lambda \in \mathbf{C}; |\lambda| \leqslant r_0^{-1}\}.$$

Then

$$\{\mu \in \mathbf{C}; |\mu| \geqslant r_0^{-1}\} \subset D_0^{-1} \subset \{\mu \in \mathbf{C}; |\mu| \geqslant r_0\}.$$

Clearly,

$$\sigma(\mathrm{T}^{-1}) \subset \Gamma^{-1}, \qquad \sigma((T + K)^{-1}) \subset \Gamma^{-1}.$$

Since

$$(\mu - T^{-1})^{-1} = \mu^{-1} + \mu^{-2}(T - \mu^{-1})^{-1}, \qquad \mu \in D_0^{-1},$$

we get for all $\mu \in D_0^{-1}$, $|\mu| \leqslant r_0^{-1}$,

$$\|(\mu - T^{-1})^{-1}\| \leqslant r_0^{-1} + r_0^{-2} f\left(\frac{1}{\mathrm{d}(\mu^{-1}, \Gamma)}\right) \leqslant r_0^{-1} + r_0^{-2} f\left(\frac{r_0^{-2}}{\mathrm{d}(\mu, \Gamma^{-2})}\right).$$

Defining the increasing function f_0: $(0, +\infty) \to (0, +\infty)$ by

$$f_0(t) = \sup_{|\mu| \geqslant r_0^{-1}} \|(\mu - T^{-1})^{-1}\| + r_0^{-1} + r_0^{-2} f(r_0^{-2} t),$$

we have

$$\|(\mu - T^{-1})^{-1}\| \leqslant f_0\left(\frac{1}{\mathrm{d}(\mu, \Gamma^{-1})}\right), \quad \mu \in D_0^{-1},$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f_0(t)}{t^2}\, \mathrm{d}t < +\infty.$$

On the other hand, since

$$(T + K)^{-1} = T^{-1} - (T + K)^{-1} K T^{-1},$$

denoting $K_0 = -(T+K)^{-1}KT^{-1}$ and using (2.1) ,we have

$$(T+K)^{-1} = T^{-1} + K_0,$$

$$n_{K_0}(t) \leqslant n_K(\|(T+K)^{-1}\| \, \|T^{-1}\| t), \quad t > 0.$$

Consequently,

$$\int_0^{+\infty} \frac{\ln^+ n_{K_0}(af_0(t)+b)}{t^2}\, \mathrm{d}t < +\infty, \quad a, b > 0.$$

Applying again Theorem 3.1, we conclude that there exists an increasing function $h : (0, +\infty) \to (0, +\infty)$ such that

$$\|(\mu - (T+K)^{-1})^{-1}\| \leqslant h\left(\frac{1}{\mathrm{d}(\mu, \Gamma^{-1})}\right), \quad \mu \in D_0^{-1},$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ h(t)}{t^2}\, \mathrm{d}t < +\infty.$$

Finally, since

$$(\lambda - T - K)^{-1} = \lambda^{-1} + \lambda^{-2}((T+K)^{-1} - \lambda^{-1})^{-1}, \qquad 0 \neq \lambda \in D_0$$

we get for all $\lambda \in D_0$, $|\lambda| \geqslant r_0$,

$$\|(\lambda - T - K)^{-1}\| \leqslant r_0^{-1} + r_0^{-2} h\left(\frac{1}{\mathrm{d}(\lambda^{-1}, \Gamma^{-1})}\right) \leqslant r_0^{-1} + r_0^{-2} h\left(\frac{r_0^{-2}}{\mathrm{d}(\lambda, \Gamma)}\right).$$

Thus, defining the increasing function $g_0 : (0, +\infty) \to (0, +\infty)$ by

$$g_0(t) = \sup_{|\lambda| \leqslant r_0} \|(\lambda - T - K)^{-1}\| + r_0^{-1} + r_0^{-2} h(r_0^{-2} t),$$

we have

$$\|(\lambda - T - K)^{-1}\| \leqslant g_0\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in D_0,$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ g_0(t)}{t^2}\, \mathrm{d}t < +\infty.$$

Now our statement follows with $g = g_0 + g_\infty$.

q.e.d.

4. INVARIANT SUBSPACES OF COMPACT PERTURBATIONS

Using Theorem 3.1, all decomposability and invariant subspace results from [22], [4], [15], [12], [18] concerning compact perturbations of self-adjoint, unitary or normal operators with the spectrum on a closed C^2 Jordan curve can be extended to the Banach space setting. These extensions may be proved in the same way as in the above quoted papers (see also [19], Section 4), by only replacing the used resolvent estimate lemmas with Theorem 3.1 or Corollary 3.2. However, for the reason given in the introduction, we give here a self-contained exposition.

We begin with the following slight extension of a result of N. Sjöberg, for whose proof we adapted the reasonings from [5], Th. 3 and [3], Th. 1.12:

4.1. PROPOSITION. *Let Γ be a closed C^2 Jordan curve, K a closed subarc of Γ, N an open neighborhood of K and $g : (0, +\infty) \to (0, +\infty)$ a strictly increasing continuous function such that*

$$g(1) = 1, \quad \lim_{t\to+\infty} g(t) = +\infty,$$

$$\int_1^{+\infty} \frac{\ln g(t)}{t^2}\,\mathrm{d}t < +\infty.$$

Then there exists $c > 0$ such that for any subharmonic function $u : N \to \mathbf{R}$ with

$$u(\lambda) \leqslant g\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in N \setminus \Gamma$$

we have

$$u(\lambda) \leqslant c, \quad \lambda \in N, \quad \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, K).$$

Proof. Since Γ is a closed C^2 Jordan curve, there exists a C^1 homeomorphism h of some open neighborhood U of $C = \{z \in \mathbf{C};\ |z| = 1\}$ onto an open neighborhood V of Γ such that $h(C) = \Gamma$. Reducing U and V, if necessary, we may assume that the partial derivatives of h and h^{-1} are bounded on U and V, respectively. Hence there exist $c_1, c_2 > 0$ such that

$$|z_1 - z_2| \leqslant c_1 |h(z_1) - h(z_2)|, \quad z_1, z_2 \in U, \tag{4.1}$$

$$\text{area } h(S) \leqslant c_2 \text{ area } S, \quad S \text{ Borel subset of } U. \tag{4.2}$$

We define the sequence $1 = \alpha_1 < \alpha_2 < \dots$ by

$$g(\alpha_k) = e^{k-1}, \quad k \geqslant 1.$$

For each integer $n \geqslant 1$ we have

$$\sum_{k=1}^{n} 1/\alpha_k = 1/\alpha_1 + \sum_{k=2}^{n} (k-1)(1/\alpha_k - 1/\alpha_{k+1}) + (n-1)1/\alpha_{n+1} \leqslant$$

$$\leqslant 1 + \sum_{k=2}^{n} \int_{\alpha_k}^{\alpha_{k+1}} \frac{\ln g(t)}{t^2}\,\mathrm{d}t + \int_{\alpha_{n+1}}^{+\infty} \frac{\ln g(t)}{t^2}\,\mathrm{d}t =$$

$$= 1 + \int_{\alpha_2}^{+\infty} \frac{\ln g(t)}{t^2}\,\mathrm{d}t,$$

hence

$$\sum_{k=1}^{\infty} 1/\alpha_k < +\infty.$$

Let $\varepsilon > 0$ be such that

$$\{\lambda \in \mathbf{C};\ \mathrm{d}(\lambda, K) \leqslant 2\varepsilon\} \subset N \cap V. \tag{4.3}$$

Denoting

$$M = \frac{10\mathrm{e}^2 c_1^2 c_2}{\pi},$$

we choose an integer $p \geqslant 1$ for which

$$c_1(1 + M^2)1/\alpha_k < 1/4, \quad k \geqslant p, \tag{4.4}$$

and

$$M \sum_{k=p}^{\infty} 1/\alpha_k \leqslant \varepsilon. \tag{4.5}$$

We prove that for

$$c = \max\{\mathrm{e}^p,\ g(1/\varepsilon)\}$$

our statement holds.

Let $u : N \to \mathbf{R}$ be a subharmonic function such that

$$u(\lambda) \leqslant g\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in N \setminus \Gamma.$$

If $\lambda \in N$, $\mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, K) \geqslant \varepsilon$, then

$$u(\lambda) \leqslant g(1/\varepsilon) \leqslant 0,$$

so it is enough to show that for $\lambda \in N$, $\mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, K) < \varepsilon$ we have

$$u(\lambda) \leqslant \mathrm{e}^p.$$

For this purpose we assume that there exists $\lambda_0 \in N$ such that $d(\lambda_0, \Gamma)=d(\lambda_0, K)<\varepsilon$ and

$$u(\lambda_0) > e^p \tag{4.6}$$

and we look for a contradiction.

First, there exists $\lambda_1 \in N$ such that $|\lambda_0 - \lambda_1| \leqslant M/\alpha_p$ and

$$u(\lambda_1) > e^{p+1}.$$

Indeed, assuming the contrary and denoting

$$\Delta = \{\lambda \in \mathbf{C}; |\lambda_0 - \lambda| \leqslant M/\alpha_p\},$$

$$\Delta_1 = \Delta \cap \{\lambda \in \mathbf{C}; d(\lambda, \Gamma) < 1/\alpha_p\},$$

$$\Delta_2 = \Delta \cap \{\lambda \in \mathbf{C}; d(\lambda, \Gamma) \geqslant 1/\alpha_p\},$$

by (4.5) and (4.3)

$$\Delta \subset N \cap V,$$

so we should have

$$u(\lambda) \leqslant e^{p+1}, \quad \lambda \in \Delta_1,$$

$$u(\lambda) \leqslant g(\alpha_p) = e^{p-1}, \ \lambda \in \Delta_2.$$

By the subharmonicity of u it follows

$$u(\lambda_0) \leqslant \frac{1}{\text{area } \Delta} \int_\Delta u(\lambda) d\lambda =$$

$$= \frac{1}{\text{area } \Delta} \left(\int_{\Delta_1} u(\lambda) d\lambda + \int_{\Delta_2} u(\lambda) d\lambda \right) =$$

$$= \frac{\text{area } \Delta_1}{\text{area } \Delta} e^{p+1} e^{p-1}.$$

But, by (4.1) we have

$$h^{-1}(\Delta_1) \subset \{z \in \mathbf{C}; |h^{-1}(\lambda_0) - z| \leqslant c_1 M 1/\alpha_p\} \cap \{z \in \mathbf{C}; d(z, C) < c_1 1/\alpha_p\},$$

hence, taking into account (4.4), an easy geometric argument shows that $h^{-1}(\Delta_1)$ is contained in a rectangle with the lengths of the sides $2c_1 M.1/\alpha_p$ respectively $5c_1.1/\alpha_p$. Consequently,

$$\text{area } h^{-1}(\Delta_1) \leqslant 10 c_1^2 M\, 1/\alpha_p^2.$$

Using (4.2), we get

$$\frac{\text{area } \Delta_1}{\text{area } \Delta} \leqslant \frac{10c_1^2c_2M \cdot 1/\alpha_p^2}{\pi M^2 1/\alpha_p^2} = \frac{10c_1^2c_2}{M} = \frac{1}{e^2},$$

so we should have

$$u(\lambda_0) \leqslant 2e^{p-1},$$

in contradiction with (4.6).

Now, by induction, we find a sequence $\lambda_0, \lambda_1, \ldots \in N$ such that for all $j \geqslant 0$

$$|\lambda_j - \lambda_{j+1}| \leqslant M\, 1/\alpha_{p+j}, \quad u(\lambda_j) > e^{p+j},$$

so u is not bounded above on the compact set

$$\{\lambda \in \mathbf{C};\, |\lambda_0 - \lambda| \leqslant M \sum_{k=p}^{\infty} 1/\alpha_k\} \subset N,$$

in contradiction with the subharmonicity of u.

q.e.d.

Using Proposition 4.1, it is easy to deduce the following extension of [13], Th. XLIII:

4.2. COROLLARY. *Let Γ be a closed C^2 Jordan curve, K a closed subarc of Γ, N an open neighborhood of K and $f : (0, +\infty) \to (0, +\infty)$ an increasing function such that*

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f(t)}{t^2}\, dt < +\infty.$$

Then there exists $d > 0$ such that for any complex Banach space X and any analytic mapping $F : N \to X$ with

$$\|F(\lambda)\| \leqslant f\left(\frac{2}{d(\lambda, \Gamma)}\right), \quad \lambda \in N \setminus \Gamma$$

we have

$$\|F(\lambda)\| \leqslant d, \quad \lambda \in N,\, d(\lambda, \Gamma) = d(\lambda, K).$$

Proof. We define the increasing function $f_0 : (0, +\infty) \to (0, +\infty)$ by

$$f_0(t) = \ln(1 + f(t)^2).$$

Next we define the strictly increasing continuous function $g\colon (0, +\infty) \to (0, +\infty)$ by

$$g(t) = \begin{cases} \dfrac{1}{2}(t+1), & t \leqslant 1, \\ \left(\displaystyle\int_1^2 f_0(s)\,\mathrm{d}s\right)^{-1} t \displaystyle\int_t^{t+1} f_0(s)\,\mathrm{d}s, & t \geqslant 1. \end{cases}$$

Then

$$g(1) = 1, \ \lim_{t \to +\infty} g(t) = +\infty,$$

$$\int_1^{+\infty} \frac{\ln g(t)}{t^2}\,\mathrm{d}t < +\infty$$

and

$$f_0(t) \leqslant 2\int_1^2 f_0(s)\,\mathrm{d}s\, g(t), \qquad t \in (0, +\infty).$$

Let $c > 0$ be the constant associated in Proposition 4.1 to Γ, K, N and g and we put

$$d = \mathrm{e}^{c\int_1^2 f_0(s)\,\mathrm{d}s}.$$

Now let $F : N \to X$ be an analytic mapping such that

$$\|F(\lambda)\| \leqslant f\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \qquad \lambda \in N \setminus \Gamma.$$

Let also $x^* \in C^*$, $\|x^*\| \leqslant 1$, be arbitrary. We consider the continuous subharmonic function $F_{x^*}\colon N \to \mathbf{R}$ defined by

$$F_{x^*}(\lambda) = \frac{1}{2\int_1^2 f_0(s)\,\mathrm{d}s} \ln(1 + |\langle F(\lambda),\ x^* \rangle|^2).$$

Then

$$F_{x^*}(\lambda) \leqslant g\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in N \setminus \Gamma,$$

and so, by Proposition 4.1, for $\lambda \in N$ and $\mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\mu, K)$ we have

$$F_{x^*}(\lambda) \leqslant c,$$

$$|\langle F(\lambda), x^* \rangle| \leqslant \left(\mathrm{e}^{2c\int_1^2 f_0(s)\mathrm{d}s} - 1\right)^{\frac{1}{2}} \leqslant d.$$

Since this is true for any x^* with $\|x^*\| \leqslant 1$, the proof is complete.

q.e.d.

Now let X be a complex Banach space and $T\in\mathscr{L}(X)$. Following [8] (see also [4], Ch. 1 or [19], Section 1), for each closed set $S\subset \mathbf{C}$ we define the linear subspace of X

$$X_T(S)=\left\{x\in X;\ \begin{array}{l}\text{there exists an analytic mapping } F_x\colon \mathbf{C}\setminus C\to X\\ \text{with}(\lambda-T)F_x(\lambda)=x \text{ for all } \lambda\in\mathbf{C}\setminus S\end{array}\right\}.$$

If the interior of $\sigma(T)$ is empty, then

$$X_T(S)=\left\{x\in X;\ \begin{array}{l}\text{the map } \lambda\to(\lambda-T)^{-1}x \text{ from } \mathbf{C}\setminus\sigma(T) \text{ to } X\\ \text{can be extended analytically to } \mathbf{C}\setminus S.\end{array}\right\},$$

4.3. LEMMA *Let Γ be a closed C^2 Jordan curve, X a complex Banach space and $T\in\mathscr{L}(X)$, $\sigma(T)\subset\Gamma$, such that there exists an increasing function $f:(0,+\infty)\to$ $\to(0,+\infty)$ with*

$$\|(\lambda-T)^{-1}\|\leqslant f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right),\qquad \lambda\in\mathbf{C}\setminus\Gamma,$$

$$\int_1^{+\infty}\frac{\ln^+\ln^+f(t)}{t^2}\,\mathrm{d}t<+\infty.$$

Then for each closed subarc J of Γ the linear subspace $X_T(J)$ of X is closed.

Proof. For $J=\Gamma$ our statement is trivial, so we may restrict ourselves to the case $J\neq\Gamma$. Let $\{x_n\}_{n\geqslant1}\subset X_T(J)$ be a sequence and $x_0\in X$ such that $\lim\limits_{n\to\infty}\|x_n-x_0\|=0$.

Let $\lambda_0\in\Gamma\setminus J$ be arbitrary. We choose a closed subarc K of Γ such that λ_0 belongs to the interior of K and $K\cap J=\emptyset$. Denoting $N=\mathbf{C}\setminus J$, there exists an open neighborhood V of λ_0 with

$$V\subset N\cap\{\lambda\in\mathbf{C};\ \mathrm{d}(\lambda,\Gamma)=\mathrm{d}(\lambda,K)\}.$$

Let $d>0$ be the constant associated in Corollary 4.2 to Γ, K, N and f.

Since for every $m>n\geqslant1$ the mapping

$$\lambda\mapsto\|x_n-x_m\|^{-1}(\lambda-T)^{-1}(x_n-x_m)$$

from $\mathbf{C}\setminus\Gamma$ in X, can be extended analytically to N and

$$\|\,\|x_n-x_m\|^{-1}(\lambda-T)^{-1}(x_n-x_m)\|\leqslant f\left(\frac{1}{\mathrm{d}(\lambda,\Gamma)}\right),\ \lambda\in\mathbf{C}\setminus\Gamma,$$

by Corollary 4.2 we have

$$\|(\lambda-T)^{-1}(x_n-x_m)\|\leqslant d\|x_n-x_m\|,\qquad \lambda\in V\setminus\Gamma.$$

Consequently, the analytic extensions of the mappings $\lambda \mapsto (\lambda - T)^{-1}x_n$ to $\mathbf{C}\setminus J$ converge uniformly on V to some analytic mapping $F : V \to X$. For $\lambda \in V\setminus\Gamma$ we have $F(\lambda) = (\lambda - T)^{-1}x_0$, so we conclude that $\mathbf{C}\setminus\Gamma \ni \lambda \to (\lambda - T)^{-1}x_0 \in X$ can be extended analytically to $(\mathbf{C}\setminus\Gamma) \cup V$.

Since $\lambda_0 \in \Gamma\setminus J$ is arbitrary, it follows that $\mathbf{C}\setminus\Gamma \ni \lambda \mapsto (\lambda - T)^{-1}x_0 \in X$ can be extended analytically to $\mathbf{C}\setminus J$, that is $x_0 \in X_T(J)$.

q.e.d.

Next we prove the solvability of a generalized Dirichlet problem, extending Lemma II of [1]:

4.4. Proposition. *Let Γ be a closed C^2 Jordan curve, $J \neq \Gamma$ be a closed subarc of Γ with end-points λ_1, λ_2 and $g: (0, +\infty) \to (0, +\infty)$ be a strictly increasing continuous function such that*

$$g(1) = 1, \lim_{t \to +\infty} g(t) = +\infty,$$

$$\int_1^{+\infty} \frac{\ln g(t)}{t^2} \mathrm{d}t < +\infty.$$

Then there exists an open set D and a continuous function

$$u : \bar{D} \to [0, +\infty]$$

such that ∂D is a rectifiable closed Jordan curve and

$$J\setminus\{\lambda_1, \lambda_2\} \subset D \subset \{\lambda \in \mathbf{C};\, \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, J)\},$$

$$\{\lambda \in \bar{D};\, u(\lambda) = +\infty\} = \{\lambda_1, \lambda_2\},$$

u is harmonic on D,

$$u(\lambda) = g\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in \partial D.$$

Proof. We define the sequence $1 = t_1 < t_2 < \ldots$ by

$$g(t_k) = \mathrm{e}^{k-1},\ k \geqslant 1.$$

By arguments already used in the proof of Proposition 4.1, we have

$$\sum_{k=1}^{\infty} 1/t_k < +\infty.$$

Denoting

$$s_k = \frac{1}{(\sum_{j \geqslant k} 1/t_j)^{\frac{1}{2}} - (\sum_{j \geqslant k+1} 1/t_j)^{\frac{1}{2}}}, \quad k \geqslant 1,$$

we get a sequence $s_1, s_2, \ldots > 0$ with

(4.7)
$$\frac{s_k}{t_k} \searrow 0 \text{ when } k \to \infty,$$

$$\sum_{k=1}^{\infty} 1/s_k < +\infty.$$

We denote

$$\sigma_k = \sum_{j \geqslant k} 1/s_j, \quad k \geqslant 1$$

and we define the continuous increasing function $\varphi : [0, \sigma_1] \to [0, 2/t_1]$ by

$$\varphi(0) = 0,$$

$$\varphi(\sigma_k) = 2/t_k, \quad k \geqslant 1,$$

φ is linear on $[\sigma_{k+1}, \sigma_k]$ for each $k \geqslant 1$.

Considering on Γ the counter clockwise orientation, we assume that λ_1 is the end of J in the negative direction and λ_2 is the end of J in the positive direction. We denote by Δ_1 the positively oriented tangent half-line to Γ in λ_1 and for each $\lambda \in \Delta_1$ let $\Delta_1(\lambda)$ be the line perpendicular to Δ_1 in λ. Similarly, we denote by Δ_2 the negatively oriented tangent half-line to Γ in λ_2 and for each $\lambda \in \Delta_2$ let $\Delta_2(\lambda)$ be the line perpendicular to Δ_2 in λ.

There exists $0 < a \leqslant \sigma_1$ such that for $j = 1, 2$ the following statement holds: for every $\lambda \in \Delta_j$ with $|\lambda - \lambda_j| \leqslant a$, we have $\Delta_j(\lambda) \cap J \neq \emptyset$ and, denoting by $\mu_j(\lambda)$ the nearest point of $\Delta_j(\lambda) \cap J$ to λ, the mapping

$$\{\lambda \in \Delta_j;\ |\lambda - \lambda_j| \leqslant a\} \ni \lambda \mapsto \mu_j(\lambda) \in J$$

is continuous. We denote by $\mu_j^+(\lambda)$ (respectively $\mu_j^-(\lambda)$) that point of $\Delta_j(\lambda)$ for which the direction of the oriented segment starting from $\mu_j(\lambda)$ and ending in $\mu_j^+(\lambda)$ (respectively $\mu_j^-(\lambda)$) coincides with the direction of the inner (respectively outer) normal of Γ in λ_j and the length of this segment is equal to $\varphi(|\lambda - \lambda_j|)$.

Let us denote by Ω^+ and Ω^- the bounded and the unbounded components of $\mathbf{C} \setminus \Gamma$ respectively. There exists $0 < b \leqslant a$ such that for $j = 1, 2$ the following statement holds: for every $\lambda \in \Delta_j$ with $|\lambda - \lambda_j| \leqslant b$, we have

$$\{\varepsilon\mu_j(\lambda) + (1 - \varepsilon)\mu_j^+(\lambda);\ 0 \leqslant \varepsilon < 1\} \subset \{\lambda \in \Omega^+;\ \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, J)\},$$

$$\{\varepsilon\mu_j(\lambda) + (1 - \varepsilon)\mu_j^-(\lambda);\ 0 \leqslant \varepsilon < 1\} \subset \{\lambda \in \Omega^-;\ \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, J)\},$$

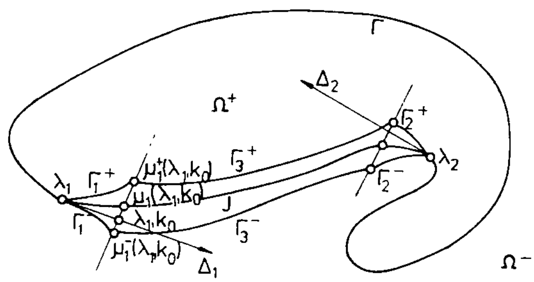
(4.8)
$$\begin{cases} \varphi(|\lambda - \lambda_j|) \leqslant 2\mathrm{d}(\mu_j^+(\lambda), J) = 2\mathrm{d}(\mu_j^+(\lambda), \Gamma), \\ \varphi(|\lambda - \lambda_j) \leqslant 2\mathrm{d}(\mu_j^-(\lambda), J) = 2\mathrm{d}(\mu_j^-(\lambda), \Gamma). \end{cases}$$

Now we choose $k_0 \geqslant 1$ such that

$$\frac{t_{k_0}}{s_{k_0}} \geqslant 16, \tag{4.9}$$

$$\sigma_{k_0} \leqslant b.$$

For $j = 1, 2$ and $k \geqslant k_0$ we define $\lambda_{j,k} \in \Delta_j$ by $|\lambda_{j,k} - \lambda_j| = \sigma_k$. Then $\lambda \mapsto \mu_j^+(\lambda)$ (respectively $\lambda \mapsto \mu_j^-(\lambda)$), defined on the segment joining λ_j and λ_{j,k_0}, parametrizes a rectifiable arc Γ_j^+ (respectively Γ_j^-). We connect $\mu_1^+(\lambda_{1,k_0})$ with $\mu_2^+(\lambda_{2,k_0})$ (respectively $\mu_1^-(\lambda_{1,k_0})$ with $\mu_2^-(\lambda_{2,k_0})$) by a rectifiable arc Γ_3^+ (respectively Γ_3^-) such that $\Gamma_0 = \Gamma_1^- \cup \Gamma_3^- \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3^+ \cup \Gamma_1^+$ is a closed Jordan curve and the bounded component D of $\mathbf{C} \setminus \Gamma_0$ is contained in $\{\lambda \in \mathbf{C};\, \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, J)\}$. So we have the following picture:



By the construction

$$J \setminus \{\lambda_1, \lambda_2\} \subset D \subset \{\lambda \in \mathbf{C};\, \mathrm{d}(\lambda, \Gamma) = \mathrm{d}(\lambda, J)\}$$

and $\partial D = \Gamma_1^- \cup \Gamma_3^- \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3^+ \cup \Gamma_1^+$ is a rectifiable closed Jordan curve. It remains to verify that the generalized Dirichlet problem in the statement is solvable.

Let Ψ be a conformal map of D onto $\left\{w \in \mathbf{C};\, |\mathrm{Im}\, w| < \frac{\pi}{2}\right\}$ such that λ_1 corresponds to $-\infty$, λ_2 corresponds to $+\infty$ and the counter clockwise orientation of ∂D corresponds to the counter clockwise orientation of the boundary of $\left\{w \in \mathbf{C};\, |\mathrm{Im}\, w| < \frac{\pi}{2}\right\}$. Let next Θ be the conformal map of $\left\{w \in \mathbf{C};\, |\mathrm{Im}\, w| < \frac{\pi}{2}\right\}$ onto $\{z \in \mathbf{C};\, \mathrm{Re}\, z > 0\}$ defined by

$$\Theta(w) = \mathrm{e}^w.$$

Denoting $\Phi = \Theta \circ \Psi$, it is enough to prove that

$$\int_{-\infty}^{+\infty} \frac{g\left(\dfrac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t), \Gamma)}\right)}{1 + t^2}\, \mathrm{d}t < +\infty. \tag{4.10}$$

Indeed, if (4.10) holds then the Poisson integral of $(\mathrm{i}t)\mapsto g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)$ in the right half-plane defines a continuous positive function v on $\{z\in\mathbf{C};\ \mathrm{Re}\, z\geqslant 0\}$, which takes the value $+\infty$ only in 0 and ∞, is harmonic on $\{z\in\mathbf{C};\ \mathrm{Re}\, z>0\}$ and whose boundary value in it is $g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)$. Then $u=v\circ\Phi$ satisfies the requirements of the statement.

For each $k\geqslant k_0$ we denote

$$\mathrm{i}t_k^+=\Phi(\mu_1^+(\lambda_{1,k})),\ \mathrm{i}t_k^-=\Phi(\mu_1^-(\lambda_{1,k})),$$

$$\mathrm{i}s_k^+=\Phi(\mu_2^+(\lambda_{2,k})),\ \mathrm{i}s_k^-=\Phi(\mu_2^-(\lambda_{2,k})).$$

Then

$$\ldots<s_{k_0+1}^-<s_{k_0}^-<t_{k_0}^-<t_{k_0+1}^-<\ldots<0<\ldots<t_{k_0+1}^+<t_{k_0}^+<s_{k_0}^+<s_{k_0+1}^+<\ldots$$

and

$$\lim_{k\to\infty}t_k^+=\lim_{k\to\infty}t_k^-=0,$$

$$\lim_{k\to\infty}s_k^+=+\infty,\ \lim_{k\to\infty}s_k^-=-\infty.$$

Since $t\mapsto g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)$ is bounded on each compact interval which doesn't contain 0, in order to verify (4.10) it is enough to prove the following inequalities:

$$(4.11)\quad \int_0^{t_{k_0}^+} g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)\mathrm{d}t<+\infty,\ \int_{t_{k_0}^-}^0 g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)\mathrm{d}t<+\infty,$$

$$(4.12)\quad \int_{s_{k_0}^+}^{+\infty}\frac{1}{t^2}g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)\mathrm{d}t<+\infty,\ \int_{-\infty}^{s_{k_0}^-}\frac{1}{t^2}g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)\mathrm{d}t<+\infty.$$

If $k\geqslant k_0$ and $t\in[t_{k+1}^+,t_{k_0}^+]$ then $\Phi^{-1}(\mathrm{i}t)$ belongs to the subarc of Γ_1^+ between $\mu_1^+(\lambda_{1,k+1})$ and $\mu_1^+(\lambda_{1,k_0})$, hence by (4.8)

$$\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\leqslant 2\sup_{\lambda\in[\lambda_{1,k+1},\lambda_{1,k_0}]}\frac{1}{\varphi(|\lambda-\lambda_1|)}=$$

$$=2\frac{1}{\varphi(|\lambda_{1,k+1}-\lambda_1|)}=2\frac{1}{\varphi(\sigma_{k+1})}=t_{k+1}.$$

Therefore

$$(4.13)\qquad g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t),\Gamma)}\right)\leqslant g(t_{k+1})=\mathrm{e}^k,\ t\in[t_{k+1}^+,\ t_{k_0}^+].$$

On the other hand, taking in account (4.9), we have for all $k \geqslant k_0$

$$\int_{\sigma_{k+1}}^{\sigma_{k_0}} \frac{ds}{2\varphi(s)} = \sum_{q=k_0}^{k} \int_{\sigma_{q+1}}^{\sigma_q} \frac{ds}{2\varphi(s)} \geqslant \sum_{q=k_0}^{k} \frac{1}{4} \frac{t_q}{s_q} \geqslant 4.$$

Using a theorem of L. V. Ahlfors about the behaviour of conformal maps in the neighborhood of beak portions of the boundary (see [7], Ch. V, Th. 6.1), it follows for $k \geqslant k_0$

$$\Theta^{-1}(\mathrm{i}t_{k_0}^+) - \Theta^{-1}(\mathrm{i}t_{k+1}^+) \geqslant \pi \int_{s_{k+1}}^{\sigma_{k_0}} \frac{ds}{2\varphi(s)} - 4\pi,$$

hence, denoting

(4.14) $$x_k = \pi \sum_{q=k_0}^{k} \frac{1}{4} \frac{t_q}{s_q},$$

we have

$$\Theta^{-1}(\mathrm{i}t_{k_0}^+) - \Theta^{-1}(\mathrm{i}t_{k+1}^+) \geqslant x_k - 4\pi,$$

$$t_{k+1}^+ \leqslant t_{k_0}^+ \, \mathrm{e}^{-x_k + 4\pi}.$$

By (4.13) we deduce

$$g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t), \Gamma)}\right) \leqslant \mathrm{e}^k, \quad t \in [t_{k_0}^+ \mathrm{e}^{-x_k + 4\pi}, t_{k_0}^+].$$

Consequently,

$$\int_0^{t_{k_0}^+} g\left(\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t), \Gamma)}\right) \mathrm{d}t \leqslant \mathrm{e}^{k_0} t_{k_0}^+ + \sum_{k=k_0}^{\infty} \mathrm{e}^{k+1} t_{k_0}^+ \mathrm{e}^{-x_k + 4\pi} =$$

$$= \mathrm{e}^{k_0} t_{k_0}^+ + \mathrm{e}^{4\pi} t_{k_0}^+ \sum_{k=k_0}^{\infty} \left(\mathrm{e}^{\frac{x_k}{k+1} - 1}\right)^{-k-1} < +\infty,$$

because by (4.7)

$$\lim_{k \to \infty} \frac{x_k}{k+1} = +\infty.$$

Thus the first inequality from (4.11) is verified. The proof of the second inequality from (4.11) is completely similar.

Finally, we sketch the proof of the first inequality from (4.12), the proof of the second one being again similar.

If $k \geqslant k_0$ and $t \in [s_{k_0}^+, s_{k+1}^+]$ then $\Phi^{-1}(\mathrm{i}t)$ belongs to the subarc of Γ_2^+ between $\mu_2^+(\lambda_{2,k_0})$ and $\mu_2^+(\lambda_{2,k+1})$, hence by (4.8)

$$\frac{1}{\mathrm{d}(\Phi^{-1}(\mathrm{i}t), \Gamma)} \leqslant t_{k+1}.$$

Therefore

$$g\left(\frac{1}{d(\Phi^{-1}(it), \Gamma)}\right) \leqslant e^k, \ t \in [s_{k_0}^+, s_{k+1}^+].$$

On the other hand, using again Ahlfors' theorem, we get for $k \geqslant k_0$

$$\Theta^{-1}(is_{k+1}^+) - \Theta^{-1}(is_{k_0}^+) \geqslant x_k - 4\pi,$$

$$s_{k+1}^+ \geqslant s_{k_0}^+ e^{x_k - 4\pi},$$

where x_k is defined by (4.14). Thus

$$g\left(\frac{1}{d(\Phi^{-1}(it), \Gamma)}\right) \leqslant e^k, \ t \in [s_{k_0}^+, s_{k_0}^+ e^{x_k - 4\pi}].$$

Consequently,

$$\int_{s_{k_0}^+}^{+\infty} \frac{1}{t^2} g\left(\frac{1}{d(\Phi^{-1}(it), \Gamma)}\right) dt \leqslant \frac{1}{(s_{k_0}^+)^2} e^{k_0} s_{k_0}^+ e^{x_{k_0} - 4\pi} + \sum_{k=k_0}^{\infty} e^{k+1} \int_{s_{k_0}^+ e^{x_k - 4\pi}}^{+\infty} \frac{dt}{t^2} =$$

$$= e^{k_0 + x_{k_0} - 4\pi} (s_{k_0}^+)^{-1} + e^{4\pi} (s_{k_0}^+)^{-1} \sum_{k=k_0}^{\infty} \left(e^{\frac{x_k}{k+1} - 1}\right)^{-k-1} < +\infty.$$

q.e.d.

We note that, choosing in the proof of Proposition 4.4 the function φ and the arcs Γ_3^+, Γ_3^- appropriately, one can construct D such that ∂D is of class C^2 in all points excepting λ_1 and λ_2.

We note also that, using the maximum principle for subharmonic functions, it is easy to derive Proposition 4.1 from Proposition 4.4.

Using Proposition 4.4, we prove a weakened variant of [14], Lemma 2.2.1, [12], Lemma 2.3 and [19], Lemma 4.3:

4.5. COROLLARY. *Let Γ be a closed C^2 Jordan curve, $J \neq \Gamma$ be a closed subarc of Γ with end-points λ_1, λ_2 and $f : (0, +\infty) \to (0, +\infty)$ be an increasing function such that*

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f(t)}{t^2} dt < +\infty.$$

Then there exists an open set D and an analytic function

$$\Phi : D \to \mathbf{C}$$

such that ∂D is a rectifiable closed Jordan curve and

$$J \setminus \{\lambda_1, \lambda_2\} \subset D \subset \{\lambda \in \mathbf{C};\ d(\lambda; \Gamma) = d(\lambda; J)\},$$

$$0 \neq |\Phi(\lambda)| \leqslant 1, \ \lambda \in D,$$

$$\lim_{D \ni \mu \to \lambda} |\Phi(\mu)| \leqslant f\left(\frac{1}{d(\lambda, \Gamma)}\right), \quad \lambda \in \partial D.$$

Proof. Similarly as in the proof of Corollary 4.2, we define the strictly increasing continuous function $g : (0, +\infty) \to (0, +\infty)$ by

$$g(t) = \begin{cases} \frac{1}{2}(t+1) & , t \leqslant 1, \\ \left(\int_1^2 \ln(1+f(s))\,\mathrm{d}s\right)^{-1} t \int_t^{t+1} \ln(1+f(s))\,\mathrm{d}s, & t \geqslant 1. \end{cases}$$

Then

$$g(1) = 1, \quad \lim_{t \to +\infty} g(t) = +\infty,$$

$$\int_1^{+\infty} \frac{\ln g(t)}{t^2}\,\mathrm{d}t < +\infty$$

and

$$\ln(1+f(t)) \leqslant c\, g(t), \quad t \in (0, +\infty),$$

where $c = 2\int_1^2 \ln(1+f(s))\,\mathrm{d}s$.

Let D and u be such that the requirements of the statement of Proposition 4.4 hold with the above g. Since D is simply connected, there is an analytic function Ψ on D such that $\mathrm{Re}\,\Psi = u$. It is easy to verify that $\Phi = \mathrm{e}^{-c\Psi}$ satisfies the conditions of the statement.

q.e.d.

Now we return to the study of the spectral subspaces $X_T(J)$. Taking in account Remark 1 following Theorem 1 of [23], we deduce the following lemma:

4.6. Lemma. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space and $T \in \mathscr{L}(X)$, $\sigma(T) \subset \Gamma$, such that there exists an increasing function $f : (0, +\infty) \to (0, +\infty)$ with*

$$\|(\lambda - T)^{-1}\| \leqslant f\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f(t)}{t^2}\,\mathrm{d}t < +\infty.$$

Then for each closed subarc J of Γ, such that the interior of J intersects $\sigma(T)$, we have

$$X_T(J) \neq \{0\}.$$

Proof. Clearly, it is enough to consider the case $J \neq \Gamma$. Let λ_1, λ_2 be the endpoints of J.

By Corollary 4.5 there exists an open set D, whose intersection with Γ is $J\setminus\{\lambda_1, \lambda_2\}$ and whose boundary ∂D is a rectifiable closed Jordan curve, and an analytic function Φ on D such that

$$0 \neq |\Phi(\lambda)| \leqslant 1, \quad \lambda \in D,$$

$$\lim_{D\ni\mu\to\lambda} \|\Phi(\mu)(\mu - T)^{-1}\| \leqslant 1, \ \lambda \in \partial D\setminus\{\lambda_1, \lambda_2\}.$$

By [6], Th. 10.3 Φ has non-tangential limit $\Phi(\lambda)$ in almost every $\lambda \in \partial D$ with respect to arc length and by [6], Th. 10.4

$$\frac{1}{2\pi i}\int_{\partial D} \frac{\Phi(\lambda)}{\lambda-\mu}\,d\lambda = \begin{cases}\Phi(\mu), & \mu \in D,\\ 0, & \mu \notin \bar{D}.\end{cases} \tag{4.15}$$

One can consider the integral

$$A = \frac{1}{2\pi i}\int_{\partial D} \Phi(\lambda)(\lambda - T)^{-1}d\lambda$$

and, using the resolvent identity and (4.15), we have

$$(\mu - T)^{-1}A = \frac{1}{2\pi i}\int_{\partial D} \frac{\Phi(\lambda)}{\mu - \lambda}(\lambda - T)^{-1}d\lambda, \quad \mu \in \mathbf{C}\setminus(\bar{D} \cup \Gamma).$$

Thus $AX \subset X_T(J)$, so it is enough to prove that $AX \neq \{0\}$.

Let λ_0 be in the intersection of the interior of J with $\sigma(T)$. Since λ_0 is a boundary point of $\sigma(T)$, there exists $x_0 \in X$ such that

$$\|x_0\| = 1,$$

$$\|(\lambda_0 - T)x_0\| < |\Phi(\lambda_0)|\left\|\frac{1}{2\pi i}\int_{\partial D}\frac{\Phi(\lambda)}{\lambda - \lambda_0}(\lambda - T)^{-1}d\lambda\right\|^{-1}.$$

Using (4.15), it follows

$$\|Ax_0\| = \left\|\frac{1}{2\pi i}\int_{\partial D}\frac{\Phi(\lambda)}{\lambda - \lambda_0}\,d\lambda\, x_0 - \frac{1}{2\pi i}\int_{\partial D}\frac{\Phi(\lambda)}{\lambda - \lambda_0}(\lambda - T)^{-1}(\lambda_0 - T)x_0\,d\lambda\right\| \geqslant$$

$$\geqslant |\Phi(\lambda_0)| - \left\|\frac{1}{2\pi i}\int_{\partial D}\frac{\Phi(\lambda)}{\lambda - \lambda_0}(\lambda - T)^{-1}d\lambda\right\|\cdot\|(\lambda_0 - T)x_0\| > 0.$$

Consequently, $AX \neq \{0\}$.

q.e.d.

Following [8], a closed invariant subspace Y of $T \in \mathscr{L}(X)$ is called spectral maximal subspace of T if it includes every closed invariant subspace Z of T with $\sigma(T|Z) \subset \sigma(T|Y)$. The operator $T \in \mathscr{L}(X)$ is called decomposable if for every finite open covering $G_1, \ldots, G_n$ of $\sigma(T)$ there exist spectral maximal subspaces $Y_1, \ldots, Y_n$ of T such that $\sigma(T|Y_j) \subset G_j$ for all $1 \leqslant j \leqslant n$ and $X = \sum_{j=1}^{n} Y_j$. For several characterizations of decomposable operators we send to [19].

Now we are able to prove [19; Th. 4.8], essentially in the same way as that one suggested in [19]:

4.7. THEOREM. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space and $T \in \mathscr{L}(X)$, $\sigma(T) \subset \Gamma$, such that there exists an increasing function $f : (0, +\infty) \to (0, +\infty)$ with*

$$\|(\lambda - T)^{-1}\| \leqslant f\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f(t)}{t^2} \, \mathrm{d}t < +\infty.$$

Then T is decomposable.

Proof. Let J be a closed subarc of Γ. Then, by Lemma 4.3, $X_T(J)$ is a closed hyperinvariant subspace of T. Denoting by T^J the linear operator induced by T on $X^J = X/X_T(J)$, we have $\sigma(T^J) \subset \mathbf{C} \setminus$ the interior of J. Indeed, assuming that the interior of J intersects $\sigma(T^J)$, since

$$\|(\lambda - T^J)^{-1}\| \leqslant f\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

by Lemma 4.6 we should have $(X^J)_{T^J}(J) \neq \{0\}$, in contradiction with $(X^J)_{T^J}(J) = X_T(J)/X_T(J) = \{0\}$ (see [19], Th. 2.10(d)).

Now, the family of all open subarcs of Γ is a base for the topology of Γ, so by the above part of the proof, by [19], Lemma 3.2 and by [19], Th. 3.3 T is decomposable.

q.e.d.

Using Corollary 3.2 and Theorem 4.7, we get immediately:

4.8. COROLLARY. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space, $T \in \mathscr{L}(X)$ and K in the norm-closure of the set of finite-rank operators on X, such that*

$$\sigma(T) \subset \Gamma, \ \sigma(T + K) \subset \Gamma.$$

If there exists an increasing function $f : (0, +\infty) \to (0, +\infty)$ such that

$$\|(\lambda - T)^{-1}\| \leqslant f\left(\frac{1}{\mathrm{d}(\lambda, \Gamma)}\right), \quad \lambda \in \mathbf{C} \setminus \Gamma,$$

$$\int_1^{+\infty} \frac{\ln^+ \ln^+ f(t)}{t^2} \, \mathrm{d}t < +\infty,$$

$$\int_1^{+\infty} \frac{\ln^+ n_K(af(t) + b)}{t^2} \, \mathrm{d}t < +\infty, \quad a, b > 0,$$

then $T + K$ is decomposable.

Similarly to the case of Hilbert spaces (see [15]), for each complex Banach space X one can define the Macaev ideal

$$\mathscr{C}_{\omega}(X) = \left\{K \in \mathscr{L}(X); \sum_{j=1}^{\infty} \frac{\alpha_j(K)}{j} < +\infty\right\}.$$

If $K \in \mathscr{C}_{\omega}(X)$ then

$$\lim_{j\to\infty} \alpha_j(K) \ln j = 0$$

(see [3], Lemma 1.5), so K belongs to the norm-closure of the set of finite-rank operators on X. Moreover,

$$\int_1^{+\infty} \frac{\ln^+ n_K(t)}{t^2}\, \mathrm{d}t < +\infty, \quad K \in \mathscr{C}_{\omega}(X)$$

(see [3], Th. 1.10). Applying Corollary 4.8 with $f(t) = ct$, it follows:

4.9. COROLLARY. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space and $T \in \mathscr{L}(X)$ such that $\sigma(T) \subset \Gamma$ and for some $c > 0$*

$$\|(\lambda - T)^{-1}\| \leqslant \frac{c}{\mathrm{d}(\lambda, \Gamma)}, \quad \lambda \in \mathbf{C} \setminus \Gamma.$$

If $K \in \mathscr{C}_{\omega}(X)$ and $\sigma(T + K) \subset \Gamma$ then $T + K$ is decomposable.

We remark that, if $T \in \mathscr{L}(X)$ is such that $\sigma(T) \subset \mathbf{R}$ and for some $c > 0$

$$\|(\lambda - T)^{-1}\| \leqslant \frac{c}{|\mathrm{Im}\lambda|}, \quad \lambda \in \mathbf{C} \setminus \mathbf{R},$$

then for every $K \in \mathscr{C}_{\omega}(X)$ with $\sigma(T + K) \subset \mathbf{R}$, defining the entire function ω by

$$\omega(z) = \prod_{k=1}^{\infty} \left(1 + \mathrm{i}\,\frac{\alpha_k(K)}{k}\, z\right)$$

and using Theorem 3.1 and [3; Th. 6.8], one can easily deduce that $T + K$ is ω-self-adjoint. A similar result may be proved in the case $\sigma(T) \subset \Gamma$, $\sigma(T + K) \subset \Gamma$, where Γ is an analytic curve.

Finally we prove the invariant subspace result in Banach spaces, corresponding to those from the papers quoted at the beginning of the section:

4.10. THEOREM. *Let Γ be a closed C^2 Jordan curve, X a complex Banach space and $T \in \mathscr{L}(X)$ such that $\sigma(T) \subset \Gamma$ and for some $c > 0$*

$$\|(\lambda - T)^{-1}\| \leqslant \frac{c}{\mathrm{d}(\lambda, \Gamma)}, \quad \lambda \in \mathbf{C} \setminus \Gamma.$$

If $K \in \mathscr{C}_\omega(X)$ and $T + K$ is not a scalar multiple of the identity then $T + K$ has a non-trivial closed hyperinvariant subspace.

Proof. Let us assume firstly that there exists $\lambda_0 \in \sigma(T + K) \setminus \Gamma$. Then, by the equality

$$\lambda_0 - T - K = (\lambda_0 - T)(I - (\lambda_0 - T)^{-1}K),$$

1 belongs to the spectrum of the compact operator $(\lambda_0 - T)^{-1}K$, hence it is an eigenvalue. Therefore there exists $x_0 \in X$, $\|x_0\| = 1$, such that

$$(\lambda_0 - T)^{-1}K(x_0) = x_0,$$

$$(\lambda_0 - T - K)x_0 = 0,$$

so $\mathrm{Ker}\,(\lambda_0 - T - K) \neq \{0\}$. Since $T + K$ is not a scalar multiple of the identity, $\mathrm{Ker}(\lambda_0 - T - K) \neq X$. Consequently, $\mathrm{Ker}(\lambda_0 - T - K)$ is a non-trivial closed hyperinvariant subspace of $T + K$.

Thus we may restrict ourselves to the case $\sigma(T + K) \subset \Gamma$.

Let us assume now that $\sigma(T + K)$ contains at least two different points λ_1, λ_2. We choose a closed subarc J of Γ such that λ_1 belongs to the interior of J and $\lambda_2 \notin J$. By Corollary 3.2 and Lemma 4.3 $X_{T+K}(J)$ is a closed linear subspace of X. It is easy to see that the spectrum of $(T + K)|X_{T+K}(J)$ is contained in J and $\lambda_2 \notin J$, so $X_{T+K}(J) \neq X$. On the other hand, by Corollary 3.2 and Lemma 4.6, $X_{T+K}(J) \neq \{0\}$. Hence $X_{T+K}(J)$ is a non-trivial closed hyperinvariant subspace of $T + K$.

It remains to treat only the case $\sigma(T + K) = \{\mu_0\}$ with $\mu_0 \in \Gamma$. We may assume, without restricting the generality, that $\mu_0 = 0$. We shall prove that in this case $T + K$ is compact and then the existence of a non-trivial closed hyperinvariant subspace of $T + K$ is a consequence of the Lomonosov theorem (see [20; Corollary 8.24]).

Let $\mathscr{C}(X)$ be the ideal of all compact operators from $\mathscr{L}(X)$, $\mathscr{A}(X) = \mathscr{L}(X)/\mathscr{C}(X)$ and $\tilde{S}$ the canonical image of $S \in \mathscr{L}(X)$ in $\mathscr{A}(X)$. We must prove that $\tilde{T} = \widetilde{T + K} = 0$. Taking in account that $\sigma(\tilde{T}) = \{0\}$ and

$$\|(\lambda - \tilde{T})^{-1}\| \leqslant \frac{c}{\mathrm{d}(\lambda, \Gamma)}, \quad \lambda \in \mathbf{C} \setminus \Gamma, \tag{4.16}$$

this follows by an argument from the proof of [24], Lemma 4: for $\varepsilon > 0$ sufficiently small, denoting by $\mu_{1,\varepsilon}, \mu_{2,\varepsilon}$ the points on Γ with $|\mu_{1,\varepsilon}| = |\mu_{2,\varepsilon}| = \varepsilon$, by $\Gamma_{1,\varepsilon}, \Gamma_{2,\varepsilon}$ the normals to Γ at $\mu_{1,\varepsilon}$ respectively $\mu_{2,\varepsilon}$ and by Γ_3 the counter clockwise oriented boundary of the intersection of $\{\lambda \in \mathbf{C};\, \mathrm{d}(\lambda, \Gamma) < \varepsilon\}$ with the component of $\mathbf{C} \setminus (\Gamma_{1,\varepsilon} \cup \Gamma_{2,\varepsilon})$ containing 0, we have

$$(\tilde{T} - \mu_{1,\varepsilon})(\tilde{T} - \mu_{2,\varepsilon}) = \frac{1}{2\pi i} \int_{\Gamma_3} (\lambda - \mu_{1,\varepsilon})(\lambda - \mu_{2,\varepsilon})(\lambda - \tilde{T})^{-1}\mathrm{d}\lambda;$$

letting $\varepsilon \to 0$ and using (4.16), we get $\tilde{T}^2 = 0$, so

$$(\lambda - \tilde{T})^{-1} = \lambda^{-1} + \lambda^{-2}\tilde{T}, \ \lambda \neq 0.$$

Using again (4.16), it follows that $\tilde{T} = 0$.

q.e.d.

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