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SOME RESULTS ON NORM-IDEAL PERTURBATIONS OF HILBERT SPACE OPERATORS

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The starting point for the present paper was a problem (attributed in [4] to P. R. Halmos) concerning Hilbert-Schmidt perturbations of normal operators, for which we provide an affirmative answer: *every normal operator on a separable Hilbert space is a Hilbert-Schmidt perturbation of a diagonal normal operator.*

In fact, we prove more, namely that n-tuples of commuting hermitian operators, for $n \geqslant 2$, are $\mathscr{C}_n$-perturbations of diagonal n-tuples of commuting hermitian operators.

Thus for n-tuples of commuting hermitian operators with $n \geqslant 2$ the norm-ideal $\mathscr{C}_n$ is not the right analogue of the trace-class for $n = 1$, where, by a corollary of the Kato-Rosenblum theorem trace-class perturbations conserve up to unitary equivalence the absolutely continuous part. We exhibit in the present paper for each $n \geqslant 2$ a norm-ideal $\mathscr{C}_n^-$ so that $\mathscr{C}_p \subset \mathscr{C}_n^-$ when $p < n$ and $\mathscr{C}_n^- \subset \mathscr{C}_n$, and which seems to be the right replacement of the trace-class for $n \geqslant 2$. We prove that for $n \geqslant 2$ a n-tuple of commuting hermitian operators can be diagonalized after a $\mathscr{C}_n^-$-perturbation if and only if its spectral measure is singular with respect to Lebesgue-measure. Moreover under the additional assumption that the multiplicity function of the absolutely continuous part is integrable we prove that up to unitary equivalence the ab solutely continuous part is invariant with respect to $\mathscr{C}_n^-$-perturbations. This improves a part of the results of J. Voigt [12] concerning $\mathscr{C}_p$-perturbations for $p < n$ and $n \geqslant 3$.

The method used to obtain these results grew out from the remark that the proof of the author's non-commutative Weyl-von Neumann type theorem ([14] see also [2]) can be adapted for norm-ideal perturbations other than compact, provided there are quasicentral approximate units for which the almost - commutation property is satisfied in the norm of the given ideal. This reduces the diagonalization problem modulo a given norm-ideal for a n-tuple of commuting hermitian operators to a simpler problem about the existence of quasicentral approximate units. Given a n-tuple τ of operators and a norm-ideal $\mathfrak{S}_\Phi^{(0)}$ we are thus led to consider an invariant $k_\Phi(\tau)$ (resembling Apostol's modulus of quasitriangularity [1], [13]), which measures the obstruction to the existence of a quasicentral approximate unit represented by τ and which turns out to be very useful in the study of the problems we are considering.

The paper has four sections.

In § 1 definitions are given and general properties of the invariant k_Φ are obtained.

In § 2 we give the adaption of the non-commutative Weyl-von Neumann type theorem.

In § 3 the invariant k_Φ is studied for certain special n-tuples of unitaries.

In § 4 combining the results of the preceding sections we obtain results on the diagonality modulo $\mathscr{C}_n$ for n-tuples of commuting hermitian operators and results about the invariance of the absolutely continuous part under $\mathscr{C}_n^-$perturbations.

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§ 1.

In this Section after preliminaries, general properties of the invariant k_Φ are presented.

We begin by recalling some facts from the theory of norm-ideals of operators (see [7] ch. III or [10]).

Consider $\hat{c}$ the space of sequences $(\xi_j)_{j\in\mathbf{N}}$, $\xi_j \in \mathbf{R}$ and $\xi_j \neq 0$ only for a finite number of indices. We denote by $\mathscr{F}$ the set of real-valued functions Φ, defined on $\hat{c}$, satisfying:

I) $\Phi(\xi) > 0$ if $\xi \neq 0$

II) $\Phi(\alpha\xi) = |\alpha| \Phi(\xi)$

III) $\Phi(\xi + \eta) \leqslant \Phi(\xi) + \Phi(\eta)$

IV) $\Phi((1, 0, 0, \ldots)) = 1$

V) $\Phi((\xi_j)_{j\in\mathbf{N}}) = \Phi((|\xi_{\pi(j)}|)_{j\in\mathbf{N}})$

where $\pi\colon \mathbf{N} \to \mathbf{N}$ is one-to-one and onto.

If $\mathscr{H}$ is a separable infinite-dimensional Hilbert space, $\mathscr{L}(\mathscr{H})$, $\mathscr{K}(\mathscr{H})$, $\mathscr{P}(\mathscr{H})$, $\mathscr{R}(\mathscr{H})$, $\mathscr{R}_1^+(\mathscr{H})$ (or simply $\mathscr{L}$, $\mathscr{K}$; $\mathscr{P}$, $\mathscr{R}$, $\mathscr{R}_1^+$ when $\mathscr{H}$ is fixed) will respectively denote the bounded operators on $\mathscr{H}$, the compact operators on $\mathscr{H}$, the finite rank orthogonal projections, the finite rank operators on $\mathscr{H}$ and the finite rank positive contractions on $\mathscr{H}$.

For $T \in \mathscr{R}(\mathscr{H})$ and $\Phi \in \mathscr{F}$ put

$$|T|_\Phi = \Phi((\lambda_j)_{j\in\mathbf{N}})$$

where λ_j are the eigenvalues of $(T^*T)^{1/2}$(multiple eigenvalues repeated according to multiplicity). Then

$$\sup_{P \in \mathscr{P}} |TP|_\Phi$$

extends $|\ \ |_\Phi$ to the set $\mathfrak{S}_\Phi$ of operators for which this supremum is finite. $\mathfrak{S}_\Phi$ is an ideal in $\mathscr{L}(\mathscr{H})$ and a Banach space with respect to the norm $|\ \ |_\Phi$. The closure $\mathfrak{S}_\Phi^{(0)}$ of $\mathscr{R}$ in $\mathfrak{S}_\Phi$ is again an ideal. For $T \in \mathfrak{S}_\Phi$ we have

$$|ATB|_\Phi \leqslant \|A\|\, |T|_\Phi\, \|B\|$$

and

$$|T|_\Phi = \|T\|$$

if T is of rank 1.

Consider

$$\Phi_p((\xi_j)_{j \in \mathbf{N}}) = (\sum |\xi_j|^p)^{1/p} \ (1 \leqslant p < \infty)$$

and

$$\Phi_\infty((\xi_j)_{j \in \mathbf{N}}) = \sup_{j \in \mathbf{N}} |\xi_j|.$$

We shall write $\mathscr{C}_p$ for $\mathfrak{S}_{\Phi_p}^{(0)}$ $(1 \leqslant p < \infty)$ and $|\ \ |_p$ for $|\ \ |_{\Phi_p}$. The ideals $\mathfrak{S}_{\Phi_p}$ and $\mathfrak{S}_{\Phi_p}^{(0)}$ are equal for $1 \leqslant p < \infty$. For $p = \infty$ we have $\mathfrak{S}_{\Phi_\infty}^{(0)} = \mathscr{K}$ and $\mathfrak{S}_{\Phi_\infty} = \mathscr{L}$.

For $1 \leqslant p < \infty$ we shall also consider

$$\Psi_p((\xi_j)_{j \in \mathbf{N}}) = \sum_{j=1}^{\infty} \xi_j^* j^{-1+\frac{1}{p}}$$

where $\xi_j^* = |\xi_{\pi(j)}|$ with $\pi : \mathbf{N} \to \mathbf{N}$ a bijection such that

$$|\xi_{\pi(1)}| \geqslant |\xi_{\pi(2)}| \geqslant |\xi_{\pi(3)}| \geqslant \ldots$$

In order not to complicate notations, we shall write

$$\mathscr{C}_p^-, \ |\ \ |_p^- \ \text{for} \ \mathfrak{S}_{\Psi_p}^{(0)}, \ |\ \ |_{\Psi_p}.$$

It is easy to see that $\mathscr{C}_1^- = \mathscr{C}_1$ and

$$\bigcup_{1 \leqslant s < p} \mathscr{C}_s \subset \mathscr{C}_p^- \subset \mathscr{C}_p$$

and $\mathscr{C}_p^- \neq \mathscr{C}_p$ for $p > 1$.

For $T \in \mathscr{R}(\mathscr{H})$ and arbitrary $\Phi \in \mathscr{F}$ we have:

$$\|T\| = |T|_{\Phi_\infty} \leqslant |T|_\Phi \leqslant |T|_1 \leqslant \|T\| \ (\text{rank } T).$$

If $T \notin \mathfrak{S}_\Phi$ we put $|T|_\Phi = \infty$.
Since we shall deal with n-tuples of operators we need some notations. Thus let

$$\tau^{(j)} = (T_1^{(j)}, \ldots, T_n^{(j)}) \in (\mathscr{L}(\mathscr{H}))^n,$$

$$\sigma = (S_1, \ldots, S_m) \in (\mathscr{L}(\mathscr{H}))^m$$

and $X, Y \in \mathscr{L}(\mathscr{H})$; we shall write

$$\tau^{(1)} \oplus \tau^{(2)} = (T_1^{(1)} \oplus T_1^{(2)}, \ldots, T_n^{(1)} \oplus T_n^{(2)})$$

$$(\tau, \sigma) = (T_1, \ldots, T_n, S_1, \ldots, S_m) \in (\mathscr{L}(\mathscr{H}))^{n+m}$$

$$\tau^{(1)} + \tau^{(2)} = (T_1^{(1)} + T_1^{(2)}, \ldots, T_n^{(1)} + T_n^{(2)})$$

$$X \tau Y = (X T_1 Y, \ldots, X T_n Y)$$

$$[X, \tau] = ([X, T_1], \ldots, [X, T_n])$$

$$\|\tau\| = \max_{1 \leqslant j \leqslant n} \|T_j\|$$

$$|\tau|_\Phi = \max_{1 \leqslant j \leqslant n} |T_j|_\Phi$$

$$\tau^* = (T_1^*, \ldots, T_n^*).$$

For $\tau \in (\mathscr{L}(\mathscr{H}))^n$ and $\Phi \in \mathscr{F}$ we consider the numbers

$$q_\Phi(\tau) = \liminf_{P \in \mathscr{P}} |(I - P)\tau P|_\Phi$$

$$qd_\Phi(\tau) = \liminf_{P \in \mathscr{P}} |[P, \tau]|_\Phi$$

$$k_\Phi(\tau) = \liminf_{A \in \mathscr{R}_1^+} |[A, \tau]|_\Phi$$

(the lim inf's are taken with respect to the natural orders on $\mathscr{P}$, $\mathscr{R}_1^+$).

Then q_Φ, qd_Φ are the moduli of Φ-quasitriangularity and respectively of Φ-quasidiagonality ([1], [13]). The following relations between q_Φ, qd_Φ, k_Φ are immediate

$$qd_\Phi(\tau) \geqslant q_\Phi(\tau), \quad qd_\Phi(\tau) \geqslant k_\Phi(\tau)$$

$$qd_\Phi(\tau) = qd_\Phi(\tau^*), \quad k_\Phi(\tau) = k_\Phi(\tau^*)$$

$$q_\Phi(\tau, \tau^*) \leqslant qd_\Phi(\tau) \leqslant 2q_\Phi(\tau, \tau^*).$$

In case $\Phi = \Phi_p$ $(1 \leqslant p < \infty)$ we shall write q_p, qd_p, k_p and in case $\Phi = \Psi_p$ we shall write q_p^-, qd_p^-, k_p^-. For $p = \infty$ we write simply q, qd, k. In fact $k(\tau)$ is always zero, this being equivalent to the existence of quasi-central approximate units for the ideal $\mathscr{K}(\mathscr{H})$ of $\mathscr{L}(\mathscr{H})$ (see [2]).

The following three propositions for k_Φ are quite similar as statements and proofs with known results on quasitriangularity and the simple proofs will be therefore omitted.

PROPOSITION 1.1. *If* $\operatorname{w-lim}_{\iota \in \mathscr{I}} A_\iota = I$, $A_\iota \in \mathscr{R}_1^+(\mathscr{H})$ *the limit being with respect to the weak operator topology, then*

$$k_\Phi(\tau) \leqslant \liminf_{\iota \in \mathscr{I}} \|[A_\iota, \tau]\|_\Phi$$

PROPOSITION 1.2. *Given* $\tau \in (\mathscr{L}(\mathscr{H}))^n$ *we can find an increasing sequence of* $A_n \in \mathscr{R}_1^+(\mathscr{H})$, $A_n \uparrow I$ *such that*

$$\lim_{n \to \infty} \|[A_n, \tau]\|_\Phi = k_\Phi(\tau).$$

PROPOSITION 1.3. *Let* $\tau \in (\mathscr{L}(\mathscr{H}))^n$ *and* $\sigma \in (\mathfrak{S}_\Phi^{(0)}(\mathscr{H}))^n$ *then*

$$k_\Phi(\tau + \sigma) = k_\Phi(\tau)$$

The next proposition gives a property of k_Φ which sharply distinguishes it from q_Φ and qd_Φ.

PROPOSITION 1.4. *Let* $\tau^{(j)} \in (\mathscr{L}(\mathscr{H}))^n$, $(j \in \mathbf{N})$; *we have*

$$\max_{j=1,2} (k_\Phi(\tau^{(j)})) \leqslant k_\Phi(\tau^{(1)} \oplus \tau^{(2)}) \leqslant k_\Phi(\tau^{(1)}) + k_\Phi(\tau^{(2)}),$$

$$k_\Phi\left(\bigoplus_{j=1}^{\infty} \tau^{(j)}\right) = \lim_{m \to \infty} k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right).$$

Proof. In view of Proposition 1.2 we can find $A_m^{(j)} \uparrow I$, $A_m^{(j)} \in \mathscr{R}_1^+(\mathscr{H})$, $(j = 1, 2$ and $m \in \mathbf{N})$ such that

$$\lim_{m \to \infty} \|[A_m^{(j)}, \tau^{(j)}]\|_\Phi = k_\Phi(\tau^{(j)}).$$

Then

$$\limsup_{m \to \infty} \|[A_m^{(1)} \oplus A_m^{(2)}, \tau^{(1)} \oplus \tau^{(2)}]\|_\Phi \leqslant k_\Phi(\tau^{(1)}) + k_\Phi(\tau^{(2)})$$

and since $A_m^{(1)} \oplus A_m^{(2)} \uparrow I_{\mathscr{H} \oplus \mathscr{H}}$ it follows from Proposition 1.1 that

$$k_\Phi(\tau^{(1)} \oplus \tau^{(2)}) \leqslant k_\Phi(\tau^{(1)}) + k_\Phi(\tau^{(2)}).$$

For the other inequality let $B_m \uparrow I$, $B_m \in \mathscr{R}_1^+(\mathscr{H} \oplus \mathscr{H})$ be such that

$$\lim_{m\to\infty} |[B_m, \tau^{(1)} \oplus \tau^{(2)}]|_\Phi = k_\Phi(\tau^{(1)} \oplus \tau^{(2)}).$$

Consider $A_m = P_{\mathscr{H}\oplus 0} B_m|_{\mathscr{H}\oplus 0}$. Then we have $A_m \uparrow I$, $A_m \in \mathscr{R}_1^+(\mathscr{H})$ and

$$|[A_m, \tau^{(1)}]|_\Phi \leqslant |[B_m, \tau^{(1)} \oplus \tau^{(2)}]|_\Phi,$$

so that using Proposition 1.1 we have

$$k_\Phi(\tau^{(1)} \oplus \tau^{(2)}) \geqslant k_\Phi(\tau^{(1)}).$$

Similarly we prove

$$k_\Phi(\tau^{(1)} \oplus \tau^{(2)}) \geqslant k_\Phi(\tau^{(2)}).$$

By the inequalities just proved we have that

$$k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right) \leqslant k_\Phi\left(\bigoplus_{j=1}^{m+1} \tau^{(j)}\right) \leqslant k_\Phi\left(\bigoplus_{j=1}^{\infty} \tau^{(j)}\right).$$

Thus

$$\lim_{m\to\infty} k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right) \leqslant k_\Phi\left(\bigoplus_{j=1}^{\infty} \tau^{(j)}\right).$$

The reverse inequality must only be proved when $\lim_{m\to\infty} k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right) < \infty$. To this end fix $(e_k)_{k=1}^{\infty}$ an orthonormal basis of $\mathscr{H}$ and using the definition of k_Φ choose recurrently $A_m \in \mathscr{R}_1^+(\underbrace{\mathscr{H} \oplus \ldots \oplus \mathscr{H}}_{m\text{-times}})$ such that

$$A_{m+1} \geqslant A_m \oplus 0,\ A_m e_k^{(i)} = e_k^{(i)}$$

$(1 \leqslant i, k \leqslant m)$ where $e_k^{(i)} = 0 \oplus \ldots \oplus 0 \oplus e_k \oplus 0 \oplus \ldots \oplus 0$, e_k being placed in the i-th position, and

$$\left|\left|\left[A_m, \bigoplus_{j=1}^{m} \tau^{(j)}\right]\right|_\Phi - k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right)\right| < \frac{1}{j}.$$

Defining $B_m \in \mathscr{R}_1^+(\mathscr{H} \oplus \mathscr{H} \oplus \ldots)$ as $B_m = A_m \oplus 0 \oplus 0 \oplus \ldots$ we have $B_m \uparrow I$ and

$$\lim_{m\to\infty} \left|\left[B_m, \bigoplus_{j=1}^{\infty} \tau^{(j)}\right]\right|_\Phi = \lim_{m\to\infty} k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right).$$

Hence

$$\lim_{m\to\infty} k_\Phi\left(\bigoplus_{j=1}^{m} \tau^{(j)}\right) \geqslant k_\Phi\left(\bigoplus_{j=1}^{\infty} \tau^{(j)}\right). \qquad \text{Q.E.D.}$$

In case $\Phi = \Phi_p$ $(1 \leqslant p < \infty)$ the preceding proposition can be given a more precise form.

PROPOSITION 1.5. *Let $\tau^{(j)} \in (\mathscr{L}(\mathscr{H}))^n$ $(j \in \mathbf{N})$; then we have*

$$k_p\left(\bigoplus_{j=1}^{\infty} \tau^{(j)}\right) = \left(\sum_{j=1}^{\infty} (k_p(\tau^{(j)}))^p\right)^{1/p},$$

where $1 \leqslant p < \infty$.

Proof. In view of the second part of Proposition 1.4 it will be sufficient to prove that

$$(k_p(\tau^{(1)} \oplus \tau^{(2)}))^p = (k_p(\tau^{(1)}))^p + (k_p(\tau^{(2)}))^p.$$

Thus, let $A_m^{(j)} \uparrow I$, $A_m^{(j)} \in \mathscr{R}_1^+(\mathscr{H})$ be such that

$$k_p(\tau^{(j)}) = \lim_{m\to\infty} |[A_m^{(j)}, \tau^{(j)}]|_p.$$

Then

$$\lim_{m\to\infty} |[A_m^{(1)} \oplus A_m^{(2)}, \tau^{(1)} \oplus \tau^{(2)}]|_p =$$

$$= \lim_{m\to\infty}((|[A_m^{(1)}, \tau^{(1)}]|_p)^p + (|[A_m^{(2)}, \tau^{(2)}]|_p)^p)^{1/p}$$

so that

$$k_p(\tau^{(1)} \oplus \tau^{(2)}) \leqslant ((k_p(\tau^{(1)}))^p + (k_p(\tau^{(2)}))^p)^{1/p}.$$

For the reverse inequality let $B_m \uparrow I$, $B_m \in \mathscr{R}_1^+(\mathscr{H} \oplus \mathscr{H})$ be such that

$$\lim_{m\to\infty} |[B_m, \tau^{(1)} \oplus \tau^{(2)}]|_p = k_p(\tau^{(1)} \oplus \tau^{(2)}).$$

Denoting $U = (-I) \oplus I$ there are $A_m^{(j)} \in \mathscr{R}_1^+(\mathscr{H})$ such that

$$1/2(B_m + U^* B_m U) = A_m^{(1)} \oplus A_m^{(2)}.$$

Clearly $A_m^{(j)} \uparrow I$ $(j = 1, 2)$. We have

$$|[B_m, \tau^{(1)} \oplus \tau^{(2)}]|_p =$$

$$= 1/2(|[B_m, \tau^{(1)} \oplus \tau^{(2)}]|_p + |[U^* B_m U, \tau^{(1)} \oplus \tau^{(2)}]|_p) \geqslant$$

$$\geqslant 1/2 |[B_m + U^* B_m U, \tau^{(1)} \oplus \tau^{(2)}]|_p =$$

$$= (|[A_m^{(1)}, \tau^{(1)}]|_p^p + |[A_m^{(2)}, \tau^{(2)}]|_p^p)^{1/p}.$$

This gives

$$k_p(\tau^{(1)}\oplus\tau^{(2)}) \geqslant ((k_p(\tau^{(1)}))^p + (k_p(\tau^{(2)}))^p)^{1/p}. \qquad \text{Q.E.D.}$$

PROPOSITION 1.6. *Consider* $\tau^{(j)} \in (\mathscr{L}(\mathscr{H}))^n, 1 \leqslant j \leqslant m$ *and consider* $\lambda^{(j)} = (\lambda_1^{(j)}I, \ldots \ldots, \lambda_n^{(j)}I) \in (\mathscr{L}(\mathscr{H}))^n$ $(1 \leqslant j \leqslant m$ *and* $\lambda_i^{(j)} \in \mathbf{C})$. *Then we have:*

$$k_\Phi(\tau^{(1)} \oplus \ldots \oplus \tau^{(m)}) = k_\Phi((\tau^{(1)} - \lambda^{(1)}) \oplus \ldots \oplus (\tau^{(m)} - \lambda^{(m)})).$$

Proof. It will be clearly sufficient to prove that

$$k_\Phi(\tau^{(1)} \oplus \ldots \oplus \tau^{(m)}) = k_\Phi((\tau^{(1)} - \lambda^{(1)}) \oplus \ldots \oplus (\tau^{(m)} - \lambda^{(m)})).$$

Consider $A_s \in \mathscr{R}_1^+(\mathscr{H} \oplus \ldots \oplus \mathscr{H})$ such that $A_s \uparrow I$ and

$$\lim_{s\to\infty} |[A_s, \tau^{(1)} \oplus \ldots \oplus \tau^{(m)}]|_\Phi = k_\Phi(\tau^{(1)} \oplus \ldots \oplus \tau^{(m)}).$$

Let further for every $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_m) \in (\{-1, 1\})^m$, U_ε be the unitary operator $\varepsilon_1 I \oplus \varepsilon_2 I \oplus \ldots \oplus \varepsilon_m I$. Then we have

$$|[A_s, \tau^{(1)} \oplus \ldots \oplus \tau^{(m)}]|_\Phi \geqslant$$

$$\geqslant 2^{-m}|\sum_{\varepsilon\in\{-1,1\}^m} U_\varepsilon[A_s, \tau^{(1)} \oplus \ldots \oplus \tau^{(m)}]\, U_\varepsilon^*|_\Phi =$$

$$= |[2^{-m} \sum_{\varepsilon\in\{-1,1\}^m} U_\varepsilon A_s U_\varepsilon^*, \tau^{(1)} \oplus \ldots \oplus \tau^{(m)}]|_\Phi.$$

Now, there are $A_s^{(j)} \in \mathscr{R}_1^+(\mathscr{H})$ such that

$$2^{-m} \sum_{\varepsilon\in\{-1,1\}^m} U_\varepsilon A_s U_\varepsilon^* = A_s^{(1)} \oplus \ldots \oplus A_s^{(m)}.$$

Clearly $A_s^{(j)} \uparrow I$ for $s \to \infty$ and for every $1 \leqslant j \leqslant m$. We have

$$k_\Phi(\tau^{(1)} \oplus \ldots \oplus \tau^{(m)}) =$$

$$= \lim_{s\to\infty} |[A_s^{(1)} \oplus \ldots \oplus A_s^{(m)}, \tau^{(1)} \oplus \ldots \oplus \tau^{(m)}]|_\Phi =$$

$$= \lim_{s\to\infty} |[A_s^{(1)} \oplus \ldots \oplus A_s^{(m)}, (\tau^{(1)} - \lambda^{(1)}) \oplus \ldots \oplus (\tau^{(m)} - \lambda^{(m)})]|_\Phi \geqslant$$

$$\geqslant k_\Phi((\tau^{(1)} - \lambda^{(1)}) \oplus \ldots \oplus (\tau^{(m)} - \lambda^{(m)})). \qquad \text{Q.E.D.}$$

For the next proposition we shall consider a variant of k_Φ, denoted $\tilde{k}_\Phi$ and which, as can be easily checked, has similar properties with k_Φ. Though we shall not work with the invariant $\tilde{k}_\Phi$, we introduce it here because it shows how by a slight modification we get an invariant with very good symmetry properties.

Let $\tau = (T_1, \ldots, T_n) \in (\mathscr{L}(\mathscr{H}))^n$ and consider $d_\tau\colon \mathscr{H} \to \mathscr{H} \otimes \mathbf{C}^n$ the operator defined by

$$d_\tau \xi = T_1\xi \otimes e_1 + \ldots + T_n\xi \otimes e_n$$

where $\xi \in \mathscr{H}$ and $e_1, \ldots, e_n$ is the canonical orthonormal basis of the n-dimensional Hilbert space $\mathbf{C}^n$.

We define

$$\tilde{k}_\Phi(\tau) = \liminf_{A \in \mathscr{R}_1^+(\mathscr{H})} |(A \otimes I_n)d_\tau - d_\tau A)|_\Phi.$$

Clearly

$$k_\Phi(\tau) \leqslant \tilde{k}_\Phi(\tau) \leqslant n\, k_\Phi(\tau).$$

Let now G be a compact topological group, $\rho : G \to \mathscr{L}(\mathscr{H})$ a continuous unitary representation and

$$P_G(T) = \int_G \rho(g)T\rho(g^{-1})\, \mathrm{d}\mu,$$

where $\mathrm{d}\mu$ is Haar-measure on G, the projection of norm one onto $(\rho(G))'$. Moreover, assume there is a projective unitary representation $g \to V_g$ of G on $\mathbf{C}^n$ such that

$$(\rho(g) \otimes I_n)d_\tau\, \rho(g^{-1}) = (I_{\mathscr{H}} \otimes V_g)d_\tau.$$

PROPOSITION 1.7. *Consider τ, G, ρ, V_g, P_G as above. Then we have*

$$\tilde{k}_\Phi(\tau) = \liminf_{A \in \mathscr{R}_1^+(\mathscr{H}) \cap (\rho(G))'} |(A \otimes I_n)d_\tau - d_\tau A|_\Phi.$$

Proof. It is clear that the first term of the equality to be proved is $\leqslant$ than the second term, so it will be sufficient to prove the reverse inequality. For $A \in \mathscr{R}_1^+(\mathscr{H})$ we have:

$$|(A \otimes I_n)d_\tau - d_\tau A|_\Phi =$$

$$= \int_G |(I_{\mathscr{H}} \otimes V_g)((A \otimes I_n)d_\tau - d_\tau A)|_\Phi\, \mathrm{d}\mu(g) =$$

$$= \int_G |(A \otimes I_n)(\rho(g) \otimes I_n)d_\tau\, \rho(g^{-1}) - (\rho(g) \otimes I_n)d_\tau \rho(g^{-1})A|_\Phi \mathrm{d}\mu(g) =$$

$$= \int_G |(\rho(g^{-1})A\rho(g) \otimes I_n)d_\tau - d_\tau \rho(g^{-1})A\rho(g)|_\Phi \mathrm{d}\mu(g) \geqslant$$

$$\geqslant \left| \int_G ((\rho(g^{-1})A\rho(g) \otimes I_n)d_\tau - d_\tau\, \rho(g^{-1})A\, \rho(g))\mathrm{d}\mu(g) \right|_\Phi =$$

$$= |(P_G(A) \otimes I_n)d_\tau - d_\tau P_G(A)|_\Phi.$$

It is easily seen that $\|P_G(A)\| \leqslant 1$ and, $P_G(A) \in \mathfrak{S}_\Phi^{(0)}$. Now if $A_m \uparrow I$ and $|(A_m \otimes I_n)d_\tau - d_\tau A_m|_\Phi \to \tilde{k}_\Phi(\tau)$ for $m \to \infty$, we then have

$$\tilde{k}_\Phi(\tau) \geqslant \liminf_{m\to\infty} |(P_G(A_m) \otimes I_n)d_\tau - d_\tau A_m|_\Phi.$$

Since $P_G(A_m) \uparrow I$ it is easily seen that this gives the desired inequality. Q.E.D.

For the next proposition let $P_k (k \in \mathbf{N})$, be orthogonal projections such that $\sum_{k\in\mathbf{N}} P_k = I$. Consider the von Neumann algebra $\mathscr{B} = ((P_k)_{k\in\mathbf{N}})'$ and the normal projection of norm one onto $\mathscr{B}$

$$\Delta(T) = \sum_{k\in\mathbf{N}} P_k T P_k.$$

It is easy to see that

$$\Delta(\mathfrak{S}_\Phi^{(0)}) \subset \mathfrak{S}_\Phi^{(0)}$$

and

$$|\Delta(T)|_\Phi \leqslant |T|_\Phi$$

for $T \in \mathfrak{S}_\Phi^{(0)}$.

Let further $\sigma = (S_1, \ldots, S_n) \in (\mathscr{L}(\mathscr{H}))^n$ be a n-tuple of isometries $(S_j^* S_j = I)$ and assume

$$S_j \Delta(T) S_j^* = \Delta(S_j T S_j^*)$$

for $1 \leqslant j \leqslant n$ and all $T \in \mathscr{L}(\mathscr{H})$.

Proposition 1.8. *Let σ, $\mathscr{B}$, Δ be as above. Then we have*

$$k_\Phi(\sigma) = \liminf_{A\in\mathscr{R}_1^+(\mathscr{H})\cap\mathscr{B}} |[\sigma, A]|_\Phi.$$

Proof. It is clear that the first term of the inequality to be proved is $\leqslant$ than the second term and so it is sufficient to prove the reverse inequality. We have

$$\begin{aligned}|AS_j - S_jA|_\Phi &= |(AS_j - S_jA)S_j^*|_\Phi = \\ &= |(A - S_jAS_j^*)\, S_jS_j^*|_\Phi \geqslant \\ &\geqslant |\Delta((A - S_jAS_j^*)\, S_jS_j^*)|_\Phi = \\ &= |(\Delta(A) - \Delta(S_jAS_j^*))S_jS_j^*|_\Phi = \\ &= |(\Delta(A) - S_j\Delta(A)S_j^*)S_jS_j^*|_\Phi = \\ &= |\Delta(A)S_j - S_j\Delta(A)|_\Phi.\end{aligned}$$

Now if $A_m \uparrow I$, $A_m \in \mathscr{R}_1^+(\mathscr{H})$ and

$$|[A_m, \sigma]|_\Phi \to k_\Phi(\sigma)$$

for $m \to \infty$, then we have

$$k_\Phi(\sigma) \geqslant \liminf_{m\to\infty} |[\Delta(A_m), \sigma]|_\Phi.$$

It is easily seen that we can find $k_1 \leqslant k_2 \leqslant \ldots$ so that

$$\left| \Delta(A_m) - \sum_{s=1}^{k_m} P_s A_m P_s \right|_\Phi \leqslant \frac{1}{m}.$$

Then for $A'_m = \sum_{s=1}^{k_m} P_s A_m P_s$ we have $A'_m \in \mathscr{R}_1^+(\mathscr{H}) \cap \mathscr{B}$, $A'_m \uparrow I$, and

$$k_\Phi(\sigma) \geqslant \liminf_{m\to\infty} |[A'_m, \sigma]|_\Phi,$$

which is easily seen to give the desired inequality. Q.E.D.

§ 2.

In this section we shall give an adaption of the main result of [14] to the case when $\mathscr{K}(\mathscr{H})$ is replaced by some norm-ideal $\mathfrak{S}_\Phi^{(0)}$, of course under the hypothesis of the existence of appropriate quasicentral approximate units.

DEFINITION 2.1. *Let $\mathscr{X} \subset \mathscr{L}(\mathscr{H})$ be a vector subspace with an at most countable algebraic basis. We shall say that $\mathscr{X}$ is Φ-well-behaved if for every $n \in \mathbf{N}$ and $T_1, \ldots, T_n \in \mathscr{X}$ we have*

$$k_\Phi(T_1, \ldots, T_n) = 0.$$

LEMMA 2.2. *Let $\mathscr{X} \subset \mathscr{L}(\mathscr{H})$ be a vector subspace with an at most countable algebraic basis. Assume moreover $\mathscr{X}$ is Φ-well-behaved. Then, given $\varepsilon > 0$ and $T_1, \ldots, T_n \in \mathscr{X}$ there are $B_m \in \mathscr{R}_1^+(\mathscr{H})$ such that*

$$(i) \sum_{m=1}^{\infty} B_m^2 = I,$$

$$(ii) \sum_{m=1}^{\infty} |[B_m, T_j]|_\Phi \leqslant \varepsilon, \quad \text{for } 1 \leqslant j \leqslant n,$$

$$(iii) \sum_{m=1}^{\infty} |[B_m, T]|_\Phi < \infty, \quad \text{for all } T \in \mathscr{X}.$$

Proof. We complete the given set $T_1, \ldots, T_n \in \mathscr{X}$ to a sequence $(T_j)_{j=1}^\infty$ which spans the vector space $\mathscr{X}$.

Taking $\varepsilon_m = \varepsilon .\ 2^{-11m}$ we can find $A_m \in \mathscr{R}_1^+$, $A_1 = 0$ such that

$$A_m \uparrow I,\ A_{m+1}A_m = A_m \text{ and } \|[A_m, T_j]\|_\Phi < \varepsilon_m \quad \text{for } 1 \leqslant j \leqslant m+n.$$

We shall prove that taking

$$B_m = \left(\left(1 - \frac{1}{m+1}\right) A_{m+1}^2 - \left(1 - \frac{1}{m}\right) A_m^2\right)^{1/2}$$

conditions (*i*), (*ii*), (*iii*) are satisfied.

It is immediate that (*i*) holds, so we shall concentrate on (*ii*) and (*iii*). First we remark that since $A_{m+1}A_m = A_m$ we have

$$B_m = \left(\left(1 - \frac{1}{m+1}\right) I - \left(1 - \frac{1}{m}\right) A_m^2\right)^{1/2} - \left(1 - \frac{1}{m+1}\right)^{1/2} (I - A_{m+1}).$$

It follows that for $1 \leqslant j \leqslant m+n$ we have

$$\|[B_m, T_j]\|_\Phi \leqslant \varepsilon_{m+1} + \left\| \left[\left(\left(1 - \frac{1}{m+1}\right) I - \left(1 - \frac{1}{m}\right) A_m^2\right)^{1/2},\ T_j \right] \right\|_\Phi .$$

To evaluate the last commutator let us write

$$X_m = \left(1 - \frac{1}{m+1}\right) I - \left(1 - \frac{1}{m}\right) A_m^2$$

and remark that

$$\frac{1}{2} m^{-2} I \leqslant X_m \leqslant I$$

and

$$\|[X_m, T_j]\|_\Phi \leqslant 2\varepsilon_m .$$

We have

$$\|[X_m^{1/2}, T_j]\|_\Phi = \frac{1}{2\pi} \left| \int_\Gamma z^{1/2} [(X_m - zI)^{-1}, T_j] \mathrm{d}z \right|_\Phi ,$$

where $z^{1/2}$ is defined on $\mathbf{C} \setminus (-\infty, 0]$ and positive on $(0, \infty)$, and where Γ is the boundary of the rectangle with vertices $\frac{1}{4} m^{-2} - \mathrm{i}$, $\frac{1}{4} m^{-2} + \mathrm{i}$, $2 + \mathrm{i}$, $2 - \mathrm{i}$.

It follows that

$$\|[X_m^{1/2}, T_j]\|_\Phi \leqslant 2 \sup_{z \in \Gamma} \|[(X_m - zI)^{-1}, T_j]\|_\Phi \leqslant$$

$$\leqslant 2 \|[X_m, T_j]\|_\Phi \sup_{z \in \Gamma} \|(X_m - zI)^{-1}\|^2 \leqslant$$

$$\leqslant 32\, m^4\, \varepsilon_m \leqslant 2^{9m} \varepsilon_m .$$

Thus we have

$$\|[B_m, T_j]\|_\Phi \leqslant \varepsilon_{m+1} + 2^{9m}\varepsilon_m \leqslant \varepsilon.2^{-m}.$$

It follows that for $1 \leqslant j \leqslant n$ we have

$$\sum_{m=1}^{\infty} \|[B_m, T_j]\|_\Phi \leqslant \sum_{m=1}^{\infty} \varepsilon.2^{-m} = \varepsilon,$$

which proves (ii).

For arbitrary $j \in \mathbf{N}$ we have

$$\sum_{m=1}^{\infty} \|[B_m, T_j]\|_\Phi \leqslant$$

$$\leqslant \sum_{m=1}^{j} \|[B_m, T_j]\|_\Phi + \sum_{m=j+1}^{\infty} \varepsilon.2^{-m} \leqslant$$

$$\leqslant \sum_{m=1}^{j} \|[B_m, T_j]\|_\Phi + \varepsilon < \infty.$$

Since the $(T_j)_{j=1}^{\infty}$ span the vector space $\mathscr{X}$, this proves *(iii)*. Q.E.D.

For the next lemma we shall denote by $\mathscr{L}/\mathscr{K}(\mathscr{H})$ the Calkin algebra and by

$$p : \mathscr{L}(\mathscr{H}) \to \mathscr{L}/\mathscr{K}(\mathscr{H})$$

the canonical map.

LEMMA 2.3. *Let $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ be a C^*-subalgebra, $I \in \mathscr{A}$, and let $\mathscr{B} \subset \mathscr{A}$ be a $*$-subalgebra which has an at most countable basis as a vector space over* $\mathbf{C}$. *Let further $\rho : p(\mathscr{A}) \to \mathscr{L}(\mathscr{H})$ be a unital $*$-homomorphism such that $\rho(p(\mathscr{B}))$ is Φ-well-behaved. Then there are isometries $L_j \in \mathscr{L}(\mathscr{H})$ $(j \in \mathbf{N})$ such that*

$$L_i^* L_j = \delta_{ij} I$$

and

$$L_j \rho(p(b)) - bL_j \in \mathfrak{S}_\Phi^{(0)}$$

for all $b \in \mathscr{B}$, and

$$\lim_{j\to\infty} | L_j\, \rho(p(b)) - b\, L_j|_\Phi = 0$$

for all $b \in \mathscr{B}$.

Proof. Let $(b_k)_{k=1}^{\infty} \subset \mathscr{B}$ be a sequence which spans the vector space $\mathscr{X}$. In view of Lemma 2.2 there are $B_{ij} \in \mathscr{R}_1^+(\mathscr{H})$ $(i, j \in \mathbf{N})$ such that

$$\sum_{j=1}^{\infty} B_{ij}^2 = I,$$

$$\sum_{j=1}^{\infty} |[\rho(p(b_k)), B_{ij}]|_{\Phi} \leqslant 2^{-i} \qquad \text{for } 1 \leqslant k \leqslant i,$$

$$\sum_{j=1}^{\infty} |[\rho(p(b)), B_{ij}]|_{\Phi} < \infty \qquad \text{for all } b \in \mathscr{B}.$$

Using the results of [14] we can find isometries L_{ij} such that $L_{ij}^* L_{ij} = I$ and $L_{ij}^* L_{rs} = 0$ when $(r, s) \neq (i, j)$ and

$$\|L_{ij}\rho(p(b_k)) - b_k L_{ij}\| \cdot |B_{ij}|_{\Phi} < 2^{-i-j}$$

for $1 \leqslant k \leqslant i + j$.

We define $L_i = \sum_{j=1}^{\infty} L_{ij} B_{ij}$. It is immediate that $L_i^* L_j = \delta_{ij} I$.

We have

$$|L_i \rho(p(b_k)) - b_k L_i|_{\Phi} \leqslant$$

$$\leqslant \sum_{j=1}^{k} |L_{ij} B_{ij} \rho(p(b_k)) - b_k L_{ij} B_{ij}|_{\Phi} +$$

$$+ \sum_{j=k+1}^{\infty} (\|L_{ij}\rho(p(b_k)) - b_k L_{ij}\| \cdot |B_{ij}|_{\Phi} +$$

$$+ |[\rho(p(b_k)), B_{ij}]|_{\Phi}) \leqslant$$

$$\leqslant \sum_{j=1}^{k} |L_{ij} B_{ij} \rho(p(b_k)) - b_k L_{ij} B_{ij}|_{\Phi} +$$

$$+ \sum_{j=k+1}^{\infty} 2^{-i-j} + \sum_{j=k+1}^{\infty} |[\rho(p(b_k)), B_{ij}]|_{\Phi} < \infty.$$

Since $(b_k)_{k=1}^{\infty}$ spans $\mathscr{B}$ it follows that

$$L_i \rho(p(b)) - b\, L_i \in \mathfrak{S}_{\Phi}^{(0)}$$

for all $b \in \mathscr{B}$.

If $1 \leqslant k \leqslant i$ then we have

$$|L_i \rho(p(b_k)) - b_k L_i|_{\Phi} \leqslant$$

$$\leqslant \sum_{j=1}^{\infty} \|L_{ij}\rho(p(b_k)) - b_k L_{ij}\| \cdot |B_{ij}|_{\Phi} +$$

$$+ \sum_{j=1}^{\infty} |[\rho(p(b_k)), B_{ij}]|_{\Phi} \leqslant \sum_{j=1}^{\infty} 2^{-i-j} + 2^{-i} = 2^{-i+1}.$$

We infer

$$\lim_{i\to\infty} |L_i\rho(p(b_k)) - b_kL_i|_\Phi = 0,$$

and since $(b_k)_{k=1}^{\infty}$ spans $\mathscr{B}$ it follows that

$$\lim_{i\to\infty} |L_i\rho(p(b)) - b\, L_i|_\Phi = 0$$

for all $b \in \mathscr{B}$. Q.E.D.

THEOREM 2.4. *Let $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$ be a C^*-subalgebra, $I \in \mathscr{A}$ and let $\mathscr{B} \subset \mathscr{A}$ be a $*$-subalgebra which has an at most countable basis as a vector space over* **C**. *Let further $\rho: p(\mathscr{A}) \to \mathscr{L}(\mathscr{H})$ be a unital $*$-homomorphism such that $\rho(p(\mathscr{B}))$ is Φ-well-behaved. Then there are unitaries $U_n \in \mathscr{L}(\mathscr{H}, \mathscr{H} \oplus \mathscr{H})$ such that*

$$U_n^*(b \oplus \rho(p(b)))U_n - b \in \mathfrak{S}_\Phi^{(0)},$$

$$\lim_{n\to\infty} |U_n^*(b \oplus \rho(p(b)))U_n - b|_\Phi = 0$$

for all $b \in \mathscr{B}$.

Proof. Let $(b_k)_{k=1}^{\infty} \subset \mathscr{B}$, $b_k = b_k^*$ be a sequence of selfadjoint elements which spans the vector space $\mathscr{B}$. Using Lemma 2.3 and passing to a subsequence of the sequence of isometries given by this lemma, it follows that we can find $L_i \in \mathscr{L}(\mathscr{H})$ $(i \in \mathbf{N})$, with $L_i^*L_j = \delta_{ij}I$ such that

$$bL_i - L_i\rho(p(b)) \in \mathfrak{S}_\Phi^{(0)}$$

for $b \in \mathscr{B}$, $i \in \mathbf{N}$ and

$$|b_kL_i - L_i\rho(p(b_k))|_\Phi < 2^{-i} \quad \text{for } 1 \leqslant k \leqslant i.$$

We define

$$S_j = I - \sum_{i=j}^{\infty} L_iL_i^* + \sum_{i=j}^{\infty} L_{i+1}L_i^*.$$

We have

$$\begin{aligned}
&|[S_j, b_k]|_\Phi \leqslant \\
&\leqslant \sum_{i=j}^{\infty}(|[L_i\, L_i^*, b_k]|_\Phi + |[L_{i+1}\, L_i^*, b_k]|_\Phi) \leqslant \\
&\leqslant \sum_{i=j}^{\infty} (2|\rho(p(b_k))L_i^* - L_i^*b_k|_\Phi + |L_i\rho(p(b_k)) - b_kL_i|_\Phi + \\
&\quad + |L_{i+1}\rho(p(b_k)) - b_kL_{i+1}|_\Phi) \leqslant \\
&\leqslant 4 \sum_{i=j}^{\infty} |L_i\rho(p(b_k)) - b_kL_i|_\Phi.
\end{aligned}$$

This easily gives now

$$[S_j, b_k] \in \mathfrak{S}_\Phi^{(0)}$$

and in case $1 \leqslant k \leqslant j$ we have

$$\|[S_j, b_k]\|_\Phi \leqslant 2^{-j+3}.$$

This again implies

$$[S_j, b] \in \mathfrak{S}_\Phi^{(0)}$$

and

$$\lim_{j\to\infty} \|[S_j, b]\|_\Phi = 0$$

for all $b \in \mathscr{B}$.
Defining

$$U_n^* : \mathscr{H} \oplus \mathscr{H} \to \mathscr{H}$$

by

$$U_n^*(h_1 \oplus h_2) = S_n h_1 + L_n h_2$$

it is now easily seen that U_n is unitary and the desired properties hold. Q.E.D.

COROLLARY 2.5. *Let $\mathscr{A}$ be a C^*-algebra with unit, and let $\mathscr{B} \subset \mathscr{A}$ be a $*$-subalgebra which has an at most countable basis as a vector space over* $\mathbf{C}$. *Let $\rho_j : \mathscr{A} \to \mathscr{L}(\mathscr{H})$ $(j = 1, 2)$ be unital $*$-monomorphisms such that $\rho_j(\mathscr{A}) \cap \mathscr{K}(\mathscr{H}) = 0$ $(j = 1, 2)$ and moreover $\rho_j(\mathscr{B})$ $(j = 1, 2)$ are Φ-well-behaved. Then there are unitaries $U_n \in \mathscr{L}(\mathscr{H})$ such that*

$$U_n \rho_1(b) - \rho_2(b) U_n \in \mathfrak{S}_\Phi^{(0)}$$

and

$$\lim_{n\to\infty} |U_n \rho_1(b) - \rho_2(b) U_n|_\Phi = 0$$

for all $b \in \mathscr{B}$.

Proof. In view of Theorem 2.4 there are $W_n^{(j)} \in \mathscr{L}(\mathscr{H}, \mathscr{H} \oplus \mathscr{H})$ $(j = 1, 2)$ unitaries such that

$$W_n^{(j)} \rho_j(b) - (\rho_1(b) \oplus \rho_2(b)) W_n^{(j)} \in \mathfrak{S}_\Phi^{(0)}$$

and

$$\lim_{n\to\infty} |W_n^{(j)} \rho_j(b) - (\rho_1(b) \oplus \rho_2(b)) W_n^{(j)}|_\Phi = 0.$$

We can take $U_n = W_n^{(2)*} W_n^{(1)}$. Q.E.D.

COROLLARY 2.6. *Let δ be a n-tuple of commuting hermitian operators. Then the following assertions are equivalent;*

(i) $k_{\Phi}(\delta) = 0$

(ii) $qd_{\Phi}(\delta) = 0$

(iii) there is a diagonal n-tuple δ' of commuting hermitian operators such that $\delta - \delta' \in \mathfrak{S}_{\Phi}^{(0)}$

(iv) given $\varepsilon > 0$ there is a diagonal n-tuple δ' of commuting hermitian operators such that $\delta - \delta' \in \mathfrak{S}_{\Phi}^{(0)}$ and $|\delta - \delta'|_{\Phi} < \varepsilon$.

Proof. It is easily seen that *(iv)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)* so that all we have to prove is that *(i)* $\Rightarrow$ *(iv)*.

To prove that *(i)* $\Rightarrow$ *(iv)* it will be sufficient to consider the case when the spectrum and the essential spectrum of δ are equal. Indeed $\delta = \delta_1 \oplus \delta_2$, where δ_1 is the diagonal part of δ and δ_2 has no eigenvectors.

By Proposition 1.4 the hypothesis $k_{\Phi}(\delta) = 0$ implies $k_{\Phi}(\delta_2) = 0$. Clearly the spectrum and essential spectrum of δ_2 are equal and it is also clearly sufficient to prove (iv) for δ replaced by δ_2.

Thus we can assume the spectrum and essential spectrum of δ are equal.

Let $\delta = (D_1, \ldots, D_n)$ and consider $\mathscr{A}$ the C^*-algebra generated by $D_1, \ldots, D_n$ and I. Let further $\mathscr{B} \subset \mathscr{A}$ be the $*$-subalgebra consisting of polynomials in $D_1, \ldots, D_n$. Consider

$$\rho_1 \colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$$

the identical representation of $\mathscr{A}$ and consider also

$$\rho_2 \colon \mathscr{A} \to \mathscr{L}(\mathscr{H})$$

a faithful representation of $\mathscr{A}$ of infinite multiplicity which is an infinite direct sum of one-dimensional representations.

It is easily seen that $k_{\Phi}(\delta) = 0$ implies that $\rho_1(\mathscr{B})$ is Φ-well-behaved and it is obvious that $\rho_2(\mathscr{B})$ is also Φ-well-behaved.

After these remarks assertion *(iv)* follows by applying Corollary 2.5 to the representations ρ_1 and ρ_2. Q.E.D.

§ 3.

Let $J = \{j_1, j_2, j_3, \ldots\}$ be a countably infinite set and let $\mathrm{Aut}(J)$ be the set of bijections of J onto J. Let $\hat{c}(J)$ be the vector space of systems of numbers $(\xi_j)_{j \in J}$, $\xi_j \in \mathbf{C}$ such that $\xi_j \neq 0$ only for a finite number of indices $j \in J$. By $l_{\Phi}^{(0)}(J)$ we shall denote the Banach space obtained by completing $\hat{c}(J)$ with respect to the norm

$$|(\xi_j)_{j \in J}|_{\Phi} = \Phi(|\xi_{j_1}|, \ |\xi_{j_2}|, \ldots).$$

We shall write l^p for $l_{\Phi_p}^{(0)}$ and $l_{\bar{p}}^{-}$ for $l_{\Phi_{\bar{p}}}^{(0)}$ and $|\cdot|_p, |\cdot|_{\bar{p}}^{-}$ for the corresponding norms.

In $l^2(J)$ we consider the orthonormal basis $(e_j)_{j\in J}$ where $e_j = (\xi_i^{(j)})_{i\in J}$ with $\xi_i^{(j)} = \delta_{ij}$.

There is an isometric linear map

$$\mathrm{diag}\colon l_\Phi^{(0)}(J) \to \mathfrak{S}_\Phi^{(0)}(l^2(J))$$

which associates to an element of $l_\Phi^{(0)}(J)$ the corresponding diagonal matrix, i.e.

$$\mathrm{diag}\,((\xi_j)_{j\in J})\,e_i = \xi_i e_i \quad \text{for all } i \in J.$$

Let also P_j denote the rank one orthogonal projection of $l^2(J)$ onto $\mathbf{C}e_j$ and consider $\mathscr{B}$ the von Neumann algebra $((P_j)_{j\in J})' = \mathrm{diag}(l^\infty(J))$.

By Δ we shall denote the projection of norm one

$$\Delta(T) = \sum_{j\in J} P_j T P_j$$

for $T \in \mathscr{L}(l^2(J))$.

Consider now $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathrm{Aut}\,(J))^n$ a n-tuple of automorphisms of J. Let $T(\alpha_1), \ldots, T(\alpha_n)$ denote the operators on $l_\Phi^{(0)}(J)$ defined by

$$T(\alpha_i)(\xi_j)_{j\in J} = (\eta_j)_{j\in J}$$

where $\eta_j = \xi_{\alpha_i^{-1}(j)}$. For $\Phi = \Phi_2$ we get a n-tuple $u(\alpha) = (U(\alpha_1), \ldots, U(\alpha_n))$ of unitaries given by

$$U(\alpha_i)e_j = e_{\alpha_i(j)}.$$

Proposition 3.1. *We have*

$$k_\Phi((U(\alpha_1), \ldots, U(\alpha_n))) =$$

$$= \sup_{m\in\mathbf{N}} (\inf \{ \max_{1\leqslant i\leqslant n} |T(\alpha_i)\eta - \eta|_\Phi \mid \eta = (\eta_j)_{j\in J} \in l_\Phi^{(0)}(J), \eta_{j_1} = \ldots = \eta_{j_m} = 1\}) =$$

$$= \sup_{m\in\mathbf{N}} (\inf \{ \max_{1\leqslant i\leqslant n} |T(\alpha_i)\eta - \eta|_\Phi \mid \eta = (\eta_j)_{j\in J} \in l_\Phi^{(0)}(J), \quad 0 \leqslant \eta_j \leqslant 1$$

for all $j \in J$ *and* $\eta_{j_1} = \ldots = \eta_{j_m} = 1\})$.

Proof. Let us denote by E_1, E_2 the two quantities we want to prove equal with $k_\Phi(u(\alpha))$.

It is clear that $E_1 \leqslant E_2$.

For the reverse inequality it is sufficient to remark that for $\eta = (\eta_j)_{j\in J} \in l_\Phi^{(0)}(J)$ and $\eta' = (\eta'_j)_{j\in J}$ where $\eta'_j = \min(1, |\eta_j|)$ we have

$$|T(\alpha_i)\eta - \eta|_\Phi \geqslant |T(\alpha_i)\eta' - \eta'|_\Phi.$$

Thus it will be sufficient to prove that $k_\Phi(u(\alpha)) = E_2$. It is easy to see that $u(\alpha)$, $\mathscr{B}$, Δ satisfy the conditions of Proposition 1.8 so that

$$k_\Phi(u(\alpha)) = \liminf_{A\in\mathscr{R}_1^+\cap\mathscr{B}} |[A, u(\alpha)]|_\Phi.$$

But now

$$\mathscr{R}_1^+ \cap \mathscr{B} = \operatorname{diag}(\{\eta \in \hat{c}(J) \mid 0 \leqslant \eta_j \leqslant 1 \text{ for all } j \in J\})$$

and for $A = \operatorname{diag}(\eta)$, $\eta \in \hat{c}(J)$ we have

$$\begin{aligned}|[A, U(\alpha_i)]|_\Phi &= |U(\alpha_i) A\, U(\alpha_i)^* - A|_\Phi = \\ &= |\operatorname{diag}(T(\alpha_i)\eta) - \operatorname{diag}(\eta)|_\Phi = \\ &= |\eta - T(\alpha_i)\eta|_\Phi .\end{aligned}$$

This now easily gives

$$k_\Phi(u(\alpha)) =$$

$$= \sup_{m \in \mathbf{N}} (\inf \{ \max_{1 \leqslant i \leqslant n} |T(\alpha_i)\eta - \eta|_\Phi \mid \eta = (\eta_j)_{j \in J} \in \hat{c}(J),\ 0 \leqslant \eta_j \leqslant 1 \text{ for all}$$

$$j \in J \text{ and } \eta_{j_1} = \ldots = \eta_{j_m} = 1\}).$$

This now easily gives $k_\Phi(u(\alpha)) = E_2$. Q.E.D.

Consider now $f_1, \ldots, f_m \in l^\infty(J)$. Then the proof of the preceding proposition immediately gives.

COROLLARY 3.2. *We have*

$$k_\Phi((U(\alpha_1), \ldots, U(\alpha_n))) =$$

$$= k_\Phi((U(\alpha_1), \ldots, U(\alpha_n), \operatorname{diag}(f_1), \ldots, \operatorname{diag}(f_m)).$$

We shall now apply Proposition 3.1 to certain concrete examples.

For the rest of this section we shall take $J = \mathbf{Z}^n$ and $\alpha_i \in \operatorname{Aut}(\mathbf{Z}^n)$ will be given by

$$\alpha_i((m_1, \ldots, m_n)) = (m_1', \ldots, m_n')$$

where $m_j' = m_j + \delta_{ij}$. Also in order to simplify notations we shall write U_i for $U(\alpha_i)$, u for $u(\alpha)$ and T_i for $T(\alpha_i)$.

PROPOSITION 3.3. *We have for* $n = 1$

$$k_1(U_1) = 2.$$

Proof. Let $\xi = (\xi_k)_{k \in \mathbf{Z}} \in l^1(\mathbf{Z})$ with $\xi_0 = 1$. Then

$$|T_1\xi - \xi|_1 = \sum_{k=-\infty}^{\infty} |\xi_{k-1} - \xi_k| =$$

$$= \sum_{k=-\infty}^{0} |\xi_{k-1} - \xi_k| + \sum_{k=1}^{\infty} |\xi_{k-1} - \xi_k| \geqslant 2|\xi_0| = 2.$$

Because of Proposition 3.1 this gives $k_1(U_1) \geqslant 2$.

To get the reverse inequality consider $\xi^{(m)} \in l^1(\mathbf{Z})$ given by

$$\xi_j^{(m)} = \begin{cases} 1 & \text{if } |j| \leqslant m \\ 0 & \text{if } |j| > m. \end{cases}$$

Then $|T(\alpha_1)\,\xi^{(m)} - \xi^{(m)}|_1 = 2$ so that using again Proposition 3.1 we have $k_1(U_1) = 2$.

Q.E.D.

Since in what follows we shall be interested to determine whether $k_\Phi(U_1, \ldots, U_n)$ is zero or not, we shall use the following fact

LEMMA 3.4. *The following conditions are equivalent:*

(i) $k_\Phi(U_1, \ldots, U_n) = 0$

(ii) $\inf\{\max_{1\leqslant i\leqslant n} |T_i\eta - \eta|_\Phi \mid \eta \in l_\Phi^{(0)}(\mathbf{Z}^n), \eta_{(0,\ldots,0)} = 1\} = 0$

(iii) $\inf\{\max_{1\leqslant i\leqslant n} |T_i\eta - \eta|_\Phi \mid \eta \in \hat{c}(\mathbf{Z}^n), \eta_{(0,\ldots,0)} = 1\} = 0.$

Proof. It is immediate that *(ii)* $\Leftrightarrow$ *(iii)*. Also *(i)* $\Rightarrow$ *(ii)* is an immediate consequence of Proposition 3.1. Conversely, assume *(ii)* holds; we shall prove that $k_\Phi(U_1, \ldots, U_n) = 0$. Indeed, then there are $\eta^{(s)} \in l_\Phi^{(0)}(\mathbf{Z}^n)$ such that $\eta^{(s)}_{(0,\ldots,0)} = 1$ and

$$\max_{1\leqslant i\leqslant n} |T_i\eta^{(s)} - \eta^{(s)}|_\Phi < 2^{-s}.$$

Then defining $\eta'^{(s)}$ by

$$\eta'^{(s)}_{(m_1,\ldots,m_n)} = \begin{cases} 1 & \text{if } \max_{1\leqslant i\leqslant n} |m_i| \leqslant s \\ \eta^{(s)}_{(m_1,\ldots,m_n)} & \text{if } \max_{1\leqslant i\leqslant n} |m_i| > s \end{cases}$$

it is easily seen that

$$|\eta'^{(s)} - \eta^{(s)}|_\Phi \leqslant (2s+1)^n ns2^{-s}.$$

Hence it follows that

$$\eta'^{(s)}_{(m_1,\ldots,m_n)} = 1 \qquad \text{for } \max_{1\leqslant i\leqslant n} |m_i| \leqslant s$$

and

$$\lim_{s\to\infty} \max_{1\leqslant i\leqslant n} |T_i\eta'^{(s)} - \eta'^{(s)}|_\Phi = 0$$

so that $k_\Phi(U_1, \ldots, U_n) = 0$ follows from Proposition 3.1. Q.E.D.

PROPOSITION 3.5. *For* $n \geqslant 2$ *we have*

$$k_n(U_1, \ldots, U_n) = 0.$$

Proof. In view of Lemma 3.5 this reduces to finding $\eta^{(s)} \in l^n(Z^n)$ with $\eta^{(s)}_{(0,\ldots,0)} = 1$ such that

$$\lim_{s\to\infty} (\max_{1\leqslant i\leqslant n} |T_i\,\eta^{(s)} - \eta^{(s)}|_n) = 0.$$

Let $\mathbf{T}^n = (\{z \in \mathbf{C} \mid |z|=1\})^n$ be the n-dimensional torus and let $d\mu$ be normalized Haar measure on $\mathbf{T}^n$. Then for $n \geqslant 2$, by the Hausdorff-Young inequality, the Fourier-transform gives a bounded linear map

$$\mathscr{F} : L^q(\mathbf{T}^n) \to l^n(\mathbf{Z}^n)$$

where $q = \dfrac{n}{n-1}$. Thus, if we can find $\chi^{(s)} \in L^\infty(\mathbf{T}^n)$ such that

$$(*)\qquad \begin{cases} 0 \leqslant \chi_s, \quad \int_{\mathbf{T}^n} \chi_s \, d\mu = 1 \\ \text{and} \\ \lim\limits_{s\to\infty} \left(\max\limits_{1 \leqslant i \leqslant n} \int_{\mathbf{T}^n} (d_i \chi_s)^q \, d\mu \right) = 0 \end{cases}$$

where $d_i(z_1, \ldots, z_n) = |z_i - 1|$, then taking

$$\eta^{(s)} = \mathscr{F}(\chi_s)$$

we will have

$$\eta^{(s)}_{(0, \ldots, 0)} = 1$$

and

$$\lim_{s\to\infty} \left(\max_{1 \leqslant i \leqslant n} |T_i \eta^{(s)} - \eta^{(s)}|_n \right) = 0$$

so that $k_n(U_1, \ldots, U_n) = 0$.

Now we will show that in order to find $\chi_s \in L^\infty(\mathbf{T}^n)$ satisfying (*) it will be sufficient to find $\psi_s \in L^\infty([-1, 1]^n)$ such that

$$(**)\qquad \begin{cases} \psi_s \geqslant 0, \ 2^{-n} \int_{[-1,1]^n} \psi_s \, d\lambda = 1 \\ \text{and} \\ \lim\limits_{s\to\infty} \int_{[-1,1]^n} (r\psi_s)^q \, d\lambda = 0 \end{cases}$$

where $d\lambda$ is Lebesgue-measure and $r(t_1, \ldots, t_n) = (t_1^2 + \ldots + t_n^2)^{1/2}$. Indeed, then we may define

$$\chi_s(e^{i\pi t_1}, \ldots, e^{i\pi t_n}) = \psi_s(t_1, \ldots, t_n)$$

for $-1 < t_j \leqslant 1$ $(1 \leqslant j \leqslant n)$ and it is easily seen that the functions χ_s will satisfy (*).

Now taking

$$\psi_s(t_1, \ldots, t_n) = \begin{cases} \dfrac{2^n}{c_n \ln s} (t_1^2 + \ldots + t_n^2)^{-n/2} & \text{when } 1/s \leqslant (t_1^2 + \ldots + t_n^2)^{1/2} \leqslant 1 \\ 0 & \text{when } 1/s > (t_1^2 + \ldots + t_n^2)^{1/2} \text{ or} \\ & \text{when } t_1^2 + \ldots + t_n^2 > 1, \end{cases}$$

where c_n is the $(n-1)$-dimensional volume of the unit sphere in $\mathbf{R}^n$, we have

$$2^{-n}\int_{[-1,1]^n} \psi_s \mathrm{d}\lambda = 2^{-n} c_n \int_{1/s}^{1} \frac{2^n}{c_n \ln s} r^{-n} \cdot r^{n-1} \mathrm{d}r = \frac{1}{\ln s}\int_{1/s}^{1} r^{-1}\,\mathrm{d}r = 1,$$

and

$$\int_{[-1,1]^n} (\psi_s r)^q \,\mathrm{d}\lambda = c_n \left(\frac{2^n}{c_n \ln s}\right)^q \int_{1/s}^{1} (r^{-n+1})^q\, r^{n-1}\,\mathrm{d}r =$$

$$= c_n \left(\frac{2^n}{c_n \ln s}\right)^q \int_{1/s}^{1} r^{-1}\mathrm{d}r = 2^{\frac{n^2}{n-1}} \cdot c_n^{-\frac{1}{n-1}} \cdot (\ln s)^{-\frac{1}{n-1}}$$

and clearly

$$\lim_{s\to\infty} (\ln s)^{-\frac{1}{n-1}} = 0.$$

Thus the functions ψ_s satisfy (**), which ends the proof. Q.E.D.

Our next aim is to show that for $n \geqslant 2$ we have $k_n^-(U_1, \ldots, U_n) > 0$. To this end we must first estimate the Fourier-coefficients of the functions

$$F_j(z) = (z_j - 1) \Big/ \left(\sum_{k=1}^{n} |z_k - 1|^2\right)$$

where $z \in \mathbf{T}^n = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n \mid |z_1| = \ldots = |z_n| = 1\}$.

Before doing this let us recall a few facts we shall use.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set $\Omega \ni 0 = (0, \ldots, 0)$ and let $G\colon \Omega \to \mathbf{C}$ be a function which is C^∞ on $\Omega \setminus \{0\}$ and assume $\partial G/\partial x_j$ is L^1 with respect to Lebesgue-measure on $\Omega\setminus\{0\}$. Then, if $n \geqslant 2$ the function $\partial G/\partial x_j$ which is defined on $\Omega\setminus\{0\}$, hence almost everywhere defined on Ω, coincides with the corresponding distributional partial derivative of G on Ω.

On the n-torus $\mathbf{T}^n$ we shall consider the differential operators D^α for $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geqslant 0$ integers so that

$$(D^\alpha f)(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n}) = \frac{\partial^{|\alpha|}}{\partial\theta_1^{\alpha_1}\ldots\partial\theta_n^{\alpha_n}} f(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n}).$$

Also we shall consider the neighbourhood $\Omega \subset \mathbf{T}^n$ of $1 = (1, \ldots, 1)$ defined by

$$\Omega = \{z \in \mathbf{T}^n \mid z_j \neq -1 \quad \text{for } 1 \leqslant j \leqslant n\}.$$

On Ω there are local coordinates $\theta_1, \ldots, \theta_n$ such that $\theta_j \in (-\pi, \pi)$ and $e^{i\theta_j} = z_j$. On Ω we shall also consider the function $r(\theta_1, \ldots, \theta_n) = (\theta_1^2 + \ldots + \theta_n^2)^{1/2}$.

We shall need the following fact which is a consequence of the results of [15] (see also [11], Theorem 4.1 ch. IV and the proof of Theorem 2.17 ch. VII):

For $n \geqslant 2$ there is an integrable function H_j on $\mathbf{T}^n$, such that H_j is C^∞ on $\mathbf{T}^n \setminus \{1\}$, $H_j - i\,\theta_j r^{-2}$ is C^∞ on Ω and for the Fourier-transform $\mathscr{F}(H_j) = (c_m)_{m \in \mathbf{Z}^n}$ we have

$$c_0 = 0 \text{ and } c_m = \gamma\, m_j \|m\|^{-n}\, (m \neq 0),$$

where

$$\|m\| = (m_1^2 + \ldots + m_n^2)^{1/2}$$

and γ is a constant.

LEMMA 3.6. *For $n \geqslant 2$ let F_j, H_j be the functions defined above and let $E_j = H_j - F_j$. Then for the Fourier-transform $\mathscr{F} E_j = (d_m)_{m \in \mathbf{Z}^n}$ we have*

$$\lim_{\|m\| \to \infty} d_m \|m\|^{n-1} = 0.$$

Proof. It is easily seen that it will be sufficient to prove that the partial derivatives of E_j up to order $n-1$ are L^1, or equivalently in view of a remark made before, to prove that $D^\alpha E_j$ are in L^1 on $\Omega \setminus \{1\}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $0 \leqslant \alpha_i$; $\sum \alpha_i \leqslant n-1$. Working on $\Omega \setminus \{1\}$ it will be sufficient to prove that for $r \to 0$ we have $D^\alpha E_j = O(r^{-|\alpha|})$ where $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Passing to the local coordinates θ we have

$$E_j = (i\,\theta_j - e^{i\theta_j})\, r^{-2} + (e^{i\theta_j} - 1)\left(r^{-2} - \left(4 \sum_{k=1}^{n} \sin^2 \theta_k/2\right)^{-1}\right) + b$$

where b is a C^∞-function of $\theta_1, \ldots, \theta_n$ in some neighborhood of $[-\pi, \pi]^n$. We have

$$E_j = L\theta_j^2 r^{-2} + G\theta_j \left(\sum_{k=1}^{n} \theta_k^4 Q_k\right)\left(4 \sum_{k=1}^{n} \theta_k^2\right)^{-1} \left(\sum_{k=1}^{n} \sin^2 \theta_k/2\right)^{-1} + b$$

where L, G, Q_k are C^∞-functions in some neighbourhood of $[-\pi, \pi]^n$. It is easily seen that

$$\frac{\partial^{|\alpha|}}{\partial \theta^\alpha} (\theta_j^2 r^{-2}) = O(r^{-|\alpha|})$$

$$\frac{\partial^{|\alpha|}}{\partial \theta^\alpha} \left(\theta_j . \theta_k^4 \left(\sum_{p=1}^{n} \theta_p^2\right)^{-1}\right) = O(r^{3-|\alpha|}).$$

Also since $y_p = \sin\theta_p$ can be taken as local coordinates in some neighbourhood of $(0, \ldots, 0)$ we easily obtain

$$\frac{\partial^{|\alpha|}}{\partial\theta^\alpha}\left(\sum_{p=1}^{n}\sin^2\theta_p\right)^{-1} = O(r^{-2-|\alpha|}).$$

Now because $(\partial^{|\alpha|}/\partial\theta^\alpha)(K_1K_2)$ is a sum of products of the form

$$\left(\frac{\partial^{|\beta|}}{\partial\theta^\beta}K_1\right)\left(\frac{\partial^{|\gamma|}}{\partial\theta^\gamma}K_2\right)$$

with $|\beta| + |\gamma| = |\alpha|$ it follows that

$$\frac{\partial^{|\alpha|}}{\partial\theta^\alpha}(L\,\theta_j^2\,r^{-2}) = O(r^{-|\alpha|})$$

$$\frac{\partial^{|\alpha|}}{\partial\theta^\alpha}\left(G\,\theta_j\left(\sum_{k=1}^{n}\theta_k^4 Q_k\right)\left(4\sum_{k=1}^{n}\theta_k^2\right)^{-1}\left(\sum_{k=1}^{n}\sin^2\theta_k/2\right)^{-1}\right) = O(r^{1-|\alpha|})$$

and hence

$$\frac{\partial^{|\alpha|}}{\partial\theta^\alpha}E_j = O(r^{-|\alpha|})$$

which is the desired result. Q.E.D.

Proposition 3.7. *For $n \geqslant 2$ we have*

$$k_n^-(U_1, \ldots, U_n) > 0.$$

Proof. Let

$$\mathscr{F}F_j = (a_m^{(j)})_{m\in\mathbf{Z}^n}.$$

Then because of Lemma 3.6 there is a constant $C > 0$ such that

$$|a_m^{(j)}| \leqslant C(1 + |m|)^{-n+1}$$

where $|m| = \max_{1\leqslant j\leqslant n}|m_j|$. Indeed it is obvious that the Fourier-coefficients of H_j satisfy such an estimate and the Fourier-coefficients of the difference $H_j - F_j$ have been estimated in Lemma 3.6.

Consider now $(\xi_k^{(j)})$ the decreasing rearrangement of the numbers $(|a_m^{(j)}|)_{m\in\mathbf{Z}^n}$, i.e. $\xi_k^{(j)} = |a_{\sigma(k)}^{(j)}|$ where $\sigma: \mathbf{N} \to \mathbf{N}$ is a bijection such that

$$|a_{\sigma(1)}^{(j)}| \geqslant |a_{\sigma(2)}^{(j)}| \geqslant |a_{\sigma(3)}^{(j)}| \geqslant \ldots\ .$$

Then the preceding estimate easily gives

$$\xi_k^{(j)} \leqslant C k^{-1+1/n}$$

for some $C > 0$.

Assume now that $k_n^-(U_1, \ldots, U_n) = 0$. Then it follows from Lemma 3.4 that we can find $g^{(s)} = (g_m^{(s)})_{m \in \mathbf{Z}^n} \in \hat{c}(\mathbf{Z}^n)$ such that

$$g^{(s)}_{(0,\ldots,0)} = 1$$

$$\lim_{s \to \infty} |T_j g^{(s)} - g^{(s)}|_n^- = 0$$

for $1 \leqslant j \leqslant n$. Now for $G^{(s)}$, $G_j^{(s)}$ the Fourier-transforms of $g^{(s)}$ and of $T_j g^{(s)} - g^{(s)}$ we have

$$G_j^{(s)} = (z_j - 1)\, G^{(s)}.$$

Thus we shall have

$$\sum_{j=1}^{n} \int_{\mathbf{T}^n} G_j^{(s)} \overline{F}_j \, \mathrm{d}\lambda = \int_{\mathbf{T}^n} G^{(s)} \, \mathrm{d}\lambda = g^{(s)}_{(0,\ldots,0)} = 1.$$

On the other hand

$$\sum_{j=1}^{n} \int_{\mathbf{T}^n} G_j^{(s)} \overline{F}_j \, \mathrm{d}\lambda = \sum_{j=1}^{n} \Big(\sum_{m \in \mathbf{Z}^n} (T_j g^{(s)} - g^{(s)})_m \overline{a}_m^{(j)} \Big).$$

Denoting $(\eta_k^{(j)})_{k=1}^{\infty}$ the decreasing rearrangement of the $|(T_j g^{(s)} - g^{(s)})_m|$ ($m \in \mathbf{Z}^n$), we have

$$1 = \left| \sum_{j=1}^{n} \int_{\mathbf{T}^n} G_j^{(s)} \overline{F}_j \mathrm{d}\lambda \right| \leqslant$$

$$\leqslant \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} \xi_k^{(j)} \eta_k^{(j)} \right) \leqslant C \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} \eta_k^{(j)} k^{-1+1/n} \right) =$$

$$= C \sum_{j=1}^{n} |T_j g^{(s)} - g^{(s)}|_n^-$$

which contradicts the assumption

$$\lim_{s \to \infty} |T_j g^{(s)} - g^{(s)}|_n^- = 0.$$

Thus we must have $k_n^-(U_1, \ldots, U_n) > 0$. Q.E.D.

§ 4.

In this section we study $\mathscr{C}_n$ and $\mathscr{C}_n^-$-perturbations of n-tuples of commuting hermitian operators. The reader of this section should keep in mind that for such n-tuples of hermitian operators diagonalizability modulo $\mathfrak{S}_\Phi^0$ is equivalent to the vanishing of k_Φ by Corollary 2.6.

PROPOSITION 4.1. *Let $\delta = (D_1, \ldots, D_n) \in (\mathscr{L}(\mathscr{H}))^n$ be a n-tuple of commuting hermitian operators the spectral measure of which is singular with respect to Lebesgue measure on* $\mathbf{R}^n$. *Then we have*

$$k_n^-(\delta) = 0.$$

Proof. Because of Proposition 1.4 it will be sufficient to prove the proposition in case δ has a cyclic vector ξ. Since the spectral measure of δ is singular with respect to Lebesgue measure we can find for every $j \in \mathbf{N}$ a sequence of disjoint Borel sets $(\omega_k^{(j)})_{k=1}^\infty$ in $\mathbf{R}^n$ such that

$$\sum_{k=1}^\infty (\operatorname{diam}(\omega_k^{(j)}))^n < 1/j$$

and

$$\sum_{k=1}^\infty E(\omega_k^{(j)}) = I,$$

where for a Borel set ω, $\operatorname{diam}(\omega)$ denotes the diameter of ω and $E(\omega)$ denotes the spectral projection of δ corresponding to ω.

Then we can choose for every $j \in \mathbf{N}$ an integer n_j such that

$$\left\| \sum_{k=1}^{n_j} E(\omega_k^{(j)})\, \xi - \xi \right\| < 1/j.$$

For some sufficiently great $m_j \in \mathbf{N}$ we can then find $q_k^{(j)} \in \mathbf{N}$ such that

$$m_j^{-1} q_k^{(j)} \geqslant \operatorname{diam}(\omega_k^{(j)}) \qquad \text{for } 1 \leqslant k \leqslant n_j$$

and

$$\sum_{k=1}^{n_j} (m_j^{-1} q_k^{(j)})^n < 1/j.$$

Then each $\omega_k^{(j)}$ is the disjoint union of at most $(q_k^{(j)})^n$ Borel sets of diameters $\leqslant m_j^{-1}$. Replacing each $\omega_k^{(j)} (1 \leqslant k \leqslant n_j)$ by these at most $(q_k^{(j)})^n$ Borel sets we get a new finite sequence $(\omega_k^{(j)})_{k=1}^{n_j}$ (of course the new n_j is bigger) of disjoint Borel sets such that

$$\operatorname{diam}(\omega_k^{(j)}) \leqslant m_j^{-1} \qquad \text{for } 1 \leqslant k \leqslant n_j$$

$$m_j^{-n} n_j < 1/j$$

$$\left\| \sum_{k=1}^{n_j} E(\omega_k^{(j)})\, \xi - \xi \right\| < 1/j.$$

We then define $P_j \in \mathscr{P}(\mathscr{H})$ as the orthogonal projection onto the finite-dimensional subspace

$$\mathbf{C}E(\omega_1^{(j)})\xi + \ldots + \mathbf{C}E(\omega_{n_j}^{(j)})\,\xi.$$

Then we have

$$[P_j, D_i] = \sum_{k=1}^{n_j} E(\omega_k^{(j)})\,[P_j, D_i]\,E(\omega_k^{(j)}).$$

Now $E(\omega_k^{(j)})\,[P_j, D_i]\,E(\omega_k^{(j)})$ has rank $\leqslant 2$ and it is easily seen that

$$\|E(\omega_k^{(j)})\,[P_j, D_i]\,E(\omega_k^{(j)})\| \leqslant 2\,\mathrm{diam}(\omega_k^{(j)}) \leqslant 2m_j^{-1}.$$

Hence we have

$$|[P_j, D_i]|_n^- \leqslant 2\ \Psi_n(\underbrace{m_j^{-1}, \ldots, m_j^{-1}}_{2n_j-\text{times}}, 0, 0, \ldots) =$$

$$= 2m_j^{-1} \sum_{k=1}^{2n_j} k^{-1+1/n} \leqslant 2m_j^{-1} \int_1^{2n_j+1} x^{-1+1/n}\,dx \leqslant$$

$$\leqslant 8nm_j^{-1}n_j^{1/n} < 8nj^{-1/n}.$$

Hence we have

$$\lim_{j\to\infty} |[P_j, \delta]|_n^- = 0.$$

Passing to some subsequence we may suppose the P_j's have a weak limit Q. Then we have $Q\xi = \xi$ and $[Q, \delta] = 0$. Because of the cyclicity of ξ this implies $Q = I$. This shows that the P_j's converge weakly to I and the proposition follows from Proposition 1.1. Q.E.D.

THEOREM 4.2. *Let $\delta = (D_1, \ldots, D_n) \in (\mathscr{L}(\mathscr{H}))^n$ be a n-tuple of commuting hermitian operators with $n \geqslant 2$. Then we have*

$$k_n(\delta) = 0.$$

Proof. Using Proposition 4.1 and the fact that $\mathscr{C}_n^- \subset \mathscr{C}_n$ it follows that it will be sufficient to prove the theorem under the additional assumption that the spectral measure of δ is absolutely continuous with respect to Lebesgue-measure.

Now in Proposition 3.5 it was shown that $k_n(U_1, \ldots, U_n) = 0$ for $n \geqslant 2$. Then defining $A_j = 1/2(U_j + U_j^*)$ we obtain a n-tuple $(A_1, \ldots, A_n)$ of hermitian operators with spectrum $[-1, 1]^n \subset \mathbf{R}^n$ and with spectral measure absolutely continuous with respect to Lebesgue measure and such that $k_n(A_1, \ldots, A_n) = 0$.

Consider now $\beta = (B_1, \ldots, B_n) = (A_1 \otimes I_{\mathscr{H}}, \ldots, A_n \otimes I_{\mathscr{H}})$. We have $k_n(\beta) = 0$ and β has infinite multiplicity. Now if δ has spectral measure absolutely continuous with respect to Lebesgue measure and $\|\delta\| < 1$ (which is an inessential assumption here) then we can write $\beta = \beta_1 \oplus \beta_2$ with β_1 unitarily equivalent with δ. It follows $k_n(\delta) = 0$. Q.E.D.

LEMMA 4.3. *Let $D \in \mathscr{L}(\mathscr{H})$ be a hermitian operator with spectrum* [0, 1]. *Assume the spectral measure of D is absolutely continuous with respect to Lebesgue measure and the multiplicity function is* 1 *on* [0, 1]. *Then we have*

$$k_1(D) = \frac{1}{\pi}\cdot$$

Proof. Consider $V_n = \exp\left(\frac{2\pi i}{n} D\right)$. Then $\bigoplus_{k=1}^{n} e^{\frac{2k\pi i}{n}} V_n$ is unitarily equivalent to the bilateral shift of multiplicity one, so that by Proposition 3.3 we have

$$k_1\left(\bigoplus_{k=1}^{n} e^{\frac{2k\pi i}{n}} V_n\right) = 2$$

and hence by Proposition 1.5 we infer $k_1(V_n) = 2/n$. Then, for

$$C_n = \frac{1}{2i}(V_n - V_n^*),\ A \in \mathscr{R}_1^+$$

we have

$$\|[V_n, A]\|_1 \geqslant \frac{1}{2}\|[V_n - V_n^*, A]\|_1 = \|[C_n, A]\|_1$$

so that $k_1(C_n) \leqslant 2/n$.

Now, C_n and $\sin\frac{2\pi}{n} D$ are unitarily equivalent so that

$$k_1(D) \leqslant \lim_{n\to\infty} \frac{2}{n}\cdot\frac{1}{\sin\frac{2\pi}{n}} = \frac{1}{\pi}\cdot$$

On the other hand for $A \in \mathscr{R}_1^+$ we have

$$\|[V_n, A]\|_1 = \left\|\left[\exp\left(\frac{2\pi i}{n} D\right), A\right]\right\|_1 \leqslant$$

$$\leqslant \frac{2\pi}{n}\,|[D, A]\|_1 \exp\left(\frac{2\pi}{n}\,\|D\|\right)$$

and hence

$$k_1(V_n) \leqslant \frac{2\pi}{n}\exp\left(\frac{2\pi}{n}\right)k_1(D).$$

This gives

$$1/\pi \leqslant \exp\left(\frac{2\pi}{n}\right)k_1(D)$$

and taking the limit for $n \to \infty$ we obtain

$$k_1(D) \geqslant 1/\pi. \qquad \text{Q.E.D.}$$

LEMMA 4.4. *Let $\delta = (D_1, \ldots, D_n)$ be a n-tuple of commuting hermitian operators with spectrum $[0, 1]^n$, spectral measure absolutely continuous with respect to Lebesgue measure and multiplicity function equal 1 on $[0, 1]^n$. Then for $n \geqslant 2$ we have*

$$0 < k_n^-(\delta) < \infty.$$

Proof. a) We shall prove first that $k_n^-(\delta) > 0$. Indeed, if we would have $k_n^-(\delta) = 0$, then considering

$$(U_1, \ldots, U_n) = (\exp(2\pi i D_1), \ldots, \exp(2\pi i D_n))$$

it would follow that $k_n^-(U_1, \ldots, U_n) = 0$, because if $A_m \uparrow I$, $A_m \in \mathscr{R}_1^+$ are such that

$$\|[A_m, \delta]\|_n^- \to 0 \quad \text{for} \quad m \to \infty,$$

then, since

$$\|[A_m, \exp(2\pi i D_j)]\|_n^- \leqslant 2\pi\|[A_m, D_j]\|_n^- \exp(2\pi\|D_j\|),$$

we also have

$$\|[A_m, (U_1, \ldots, U_n)]\|_n^- \to 0 \quad \text{for } m \to \infty.$$

But the n-tuple $(U_1, \ldots, U_n)$ considered here is unitarily equivalent to the n-tuple $(U_1, \ldots, U_n)$ appearing in Proposition 3.7 and so $k_n^-(U_1, \ldots, U_n)$ cannot be zero.

b) Let ξ be a cyclic vector for δ and consider for each $m \in \mathbf{N}$ and $(i_1, \ldots, i_n) \in \{0, 1, \ldots, m-1\}^n$ the set

$$\omega_{i_1, \ldots, i_n} = [i_1/m, (i_1 + 1)/m) \times \ldots \times [i_n/m, (i_n + 1)/m).$$

Then denoting by $E(i_1, \ldots, i_n)$ the corresponding spectral projection of δ we have

$$\sum_{(i_1, \ldots, i_n) \in \{0, \ldots, m-1\}^n} E(i_1, \ldots, i_n) = I.$$

Let further P_m denote the projection onto the finite-dimensional subspace spanned by the vectors $E(i_1, \ldots, i_n)\xi$, for $(i_1, \ldots, i_n)$ running over the set $\{0, 1, \ldots, m-1\}^n$. Then it is easily seen that the projections P_m are weakly convergent to I. Moreover we have

$$[P_m, D_j] = \sum_{(i_1, \ldots, i_n) \in \{0, 1, \ldots, m-1\}^n} E(i_1 \ldots i_n)[P_m, D_j]\, E(i_1, \ldots, i_n)$$

and $E(i_1, \ldots, i_n)\,[P_m, D_j]\,E(i_1, \ldots, i_n)$ has rank at most 2 and norm $\leqslant 2/m$. This gives

$$\|[P_m, D_j]\|_n^- \leqslant \Psi_n(\underbrace{2/m, \ldots, 2/m}_{2m^n\text{-times}}, 0, 0, \ldots) \leqslant 2(mn)^{-1}(2m^n + 1)^{1/n} \leqslant 8.$$

Thus we have $k_n^-(\delta) < \infty$. Q.E.D.

THEOREM 4.5. *Let* $\delta = (D_1, \ldots, D_n)$ *be a n-tuple of commuting hermitian operators. Then we have*

$$(k_n^-(\delta))^n = \gamma_n \int_{\mathbf{R}^n} m(s) \, \mathrm{d}\lambda\,(s)$$

where $0 < \gamma_n < \infty$ *is a constant independent of* δ, m *is the multiplicity function of the Lebesgue absolutely continuous part of* δ *and* $\mathrm{d}\lambda$ *is Lebesgue measure. For* $n = 1$ *we have* $\gamma_1 = \dfrac{1}{\pi}$.

Proof. First, using Proposition 4.1 it follows that it will be sufficient to prove the theorem only for δ with spectral measure absolutely continuous with respect to Lebesgue measure. Second, using Proposition 1.4 it follows that it will be sufficient to prove the theorem in case δ has finite multiplicity. Third, using Proposition 1.6 it follows the proof of the theorem is reduced from the case of finite multiplicity, to the case when δ has multiplicity one.

Thus, let $\Omega \subset \mathbf{R}^n$ be a bounded measurable set and let δ_Ω be the n-tuple of commuting hermitian operators given by the multiplication operators by the coordinate functions in $L^2(\Omega, \mathrm{d}\lambda)$. Then let $\nu(\Omega) = (k_n^-(\delta_\Omega))^n$ in case $\lambda(\Omega) > 0$ and $\nu(\Omega) = 0$ in case $\lambda(\Omega) = 0$. We must prove that there is a constant γ_n independent of Ω such that $\gamma(\Omega) = \gamma_n \lambda(\Omega)$. It is immediate that $\nu(\Omega)$ is translation invariant, $\nu(r\Omega) = r^n \nu(\Omega)$ for $r \in \mathbf{R}$, $r \geqslant 0$ and in case $\Omega_p \uparrow \Omega$ we have $\nu(\Omega_p) \uparrow \nu(\Omega)$ because of Proposition 1.4. Also clearly $\nu(\Omega) = \nu(\Omega')$ in case $\lambda(\Omega' \setminus \Omega) = \lambda(\Omega \setminus \Omega') = 0$.

We shall take $\gamma_n = \nu([0, 1]^n)$.

Then $0 < \gamma_n < \infty$ because of Lemma 4.3 and of Lemma 4.4 and for $n = 1$ we have $\gamma_1 = \dfrac{1}{\pi}$ because of Lemma 4.3.

We shall first prove that $\nu(\Omega) = \gamma_n \lambda(\Omega)$ in case Ω is a disjoint union

$$\Omega = \bigcup_{j=1}^{s} \Omega^{(j)}, \quad \text{where}$$

$$\Omega^{(j)} = [t_1^{(j)}, t_1^{(j)} + 1/m) \times \ldots \times [t_n^{(j)}, t_n^{(j)} + 1/m)$$

with $t_i^{(j)} \in \mathbf{R}$ $(1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)$ and $m \in \mathbf{N}$. Thus Ω consists of s disjoint subsets of the form

$$[t_1, t_1 + 1/m) \times \ldots \times [t_n, t_n + 1/m)$$

which because of Proposition 1.6 may be moved into different positions via translations leaving $\nu(\Omega)$ invariant provided they remain disjoint. This gives

$$([s^{1/n}])^n m^{-n} \gamma_n \leqslant \nu(\Omega) \leqslant ([s^{1/n}] + 1)^n m^{-n} \gamma_n$$

where $[x]$ denotes the integer part of x.

Now, dividing each $\Omega^{(j)}$ into p^n disjoint subsets of the form

$$[t_1, t_1 + (pm)^{-1}) \times \ldots \times [t_n, t_n + (pm)^{-1})$$

the same argument gives

$$([ps^{1/n}])^n (pm)^{-n}\gamma_n \leqslant \nu(\Omega) \leqslant ([ps^{1/n}] + 1)^n (pm)^{-n}\gamma_n.$$

For $p \to \infty$ we obtain

$$\nu(\Omega) = s\, m^{-n}\gamma_n = \lambda(\Omega)\gamma_n.$$

The next step will be to prove that $\nu(\Omega) = \gamma_n\lambda(\Omega)$ when Ω is a bounded open set. To this end, let Ω_m denote the union of all sets of the form

$$[k_1 2^{-m}, (k_1 + 1)2^{-m}) \times \ldots \times [k_n 2^{-m}, (k_n + 1)2^{-m})$$

with $k_i \in \mathbf{Z}$ $(1 \leqslant i \leqslant n)$ and $m \in \mathbf{N}$, which are contained in Ω. Then it is easily seen that $\Omega_1 \subset \Omega_2 \subset \ldots$ and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$. By what has been already proved, we have

$$\nu(\Omega_m) = \gamma_n\lambda(\Omega_m)$$

and thus

$$\nu(\Omega) = \gamma_n\lambda(\Omega)$$

for Ω a bounded open set, follows from Proposition 1.4.

Let now $\Omega \subset \mathbf{R}^n$ be a bounded measurable set and let $\varepsilon > 0$. Then we can find a compact set $K \subset \Omega$ and a bounded open set G such that $\lambda(G \setminus K) < \varepsilon$. Because of Proposition 1.4 we have

$$\nu(K) \leqslant \nu(\Omega) \leqslant \nu(G)$$

and

$$(\nu(G))^{1/n} \leqslant (\nu(K))^{1/n} + (\nu(G \setminus K))^{1/n}.$$

It follows that

$$(\nu(G))^{1/n} - (\nu(G \setminus K))^{1/n} \leqslant (\nu(\Omega))^{1/n} \leqslant (\nu(G))^{1/n}.$$

Now since G and $G \setminus K$ are open sets, we have

$$\nu(G) = \gamma_n\lambda(G)$$

$$\nu(G \setminus K) = \gamma_n\lambda(G \setminus K) < \varepsilon\gamma_n.$$

This gives

$$(\gamma_n\lambda(G))^{1/n} - (\varepsilon\gamma_n)^{1/n} \leqslant$$
$$\leqslant (\nu(\Omega))^{1/n} \leqslant (\gamma_n(\lambda(\Omega) + \varepsilon))^{1/n}.$$

Since $\varepsilon > 0$ is arbitrary, we must have $\nu(\Omega) = \gamma_n\lambda(\Omega)$. Q.E.D.

The next two corollaries are immediate consequences of the preceding theorem.

COROLLARY 4.6. *Let δ be a n-tuple of commuting hermitian operators. Then $k_n^-(\delta) = 0$ if and only if the spectral measure of δ is singular with respect to Lebesgue measure on $\mathbf{R}^n$.*

COROLLARY 4.7. *Let $\delta^{(j)}$ $(j \in \mathbf{N})$ be n-tuples of commuting hermitian operators with $\sup_{j\in\mathbf{N}} \|\delta^{(j)}\| < \infty$. Then we have*

$$k_n^-(\bigoplus_{j\in\mathbf{N}} \delta^{(j)}) = (\sum_{j\in\mathbf{N}} (k_n^-(\delta^{(j)}))^n)^{1/n}.$$

COROLLARY 4.8. *Let $F = (f_1, \ldots, f_n) : \mathbf{R}^n \to \mathbf{R}^n$ be a C^1-map. Let further $\delta = (D_1, \ldots, D_n) \in (\mathscr{L}(\mathscr{H}))^n$ be a n-tuple of commuting hermitian operators and m the multiplicity function of the absolutely continuous part of δ. Then defining*

$$F(\delta) = (f_1(D_1, \ldots, D_n), \ldots, f_n(D_1, \ldots, D_n))$$

we have

$$(k_n^-(F(\delta)))^n = \gamma_n \int_{\mathbf{R}^n} \left| \det \left(\frac{\partial f_i}{\partial x_j} \right)_{1\leqslant i,\, j\leqslant n} \right| m(s)\, \mathrm{d}\lambda(s)$$

where γ_n is the same constant as in the statement of Theorem 4.5.

Proof. Let $K \subset \mathbf{R}^n$ be the set where $\det(\partial f_i/\partial x_j)_{1\leqslant i,\, j\leqslant n}$ vanishes. Then by Sard's theorem $F(K)$ has Lebesgue-measure zero and hence the restriction of $F(\delta)$ to $E(K)\mathscr{H}$ ($E(\omega)$ denotes the spectral projection of δ for a Borel set $\omega \subset \mathbf{R}^n$) has spectral measure singular with respect to Lebesgue measure. Now $\mathbf{R}^n \setminus K$ is a disjoint union of bounded Borel sets $\Omega_j, j \in \mathbf{N}$ such that F is a C^1-diffeomorphism of some neighbourhood of $\overline{\Omega_j}$ onto some neighbourhood of $F(\overline{\Omega_j})$. It is now immediate that the multiplicity function of the absolutely continuous part of $F(\delta)|E(\Omega_j)$ is zero outside $F(\Omega_j)$ and equal $m\circ(F|\Omega_j)^{-1}$ on $F(\Omega_j)$. Thus by Theorem 4.5 we have

$$(k_n^-(F(\delta)) \mid E(\Omega_j)\mathscr{H})^n = \gamma_n \int_{F(\Omega_j)} m\circ(F \mid \Omega_j)^{-1}\, \mathrm{d}\lambda =$$

$$= \gamma_n \int_{\Omega_j} |\det(\partial f_i/\partial x_j)_{1\leqslant i,\, j\leqslant n}| m\, \mathrm{d}\lambda.$$

Now since

$$F(\delta) = (F(\delta) \mid E(K)\mathscr{H}) \oplus \bigoplus_{j=1}^{\infty} (F(\delta) \mid E(\Omega_j)\mathscr{H})$$

we get the desired result by using Corollary 4.7. Q.E.D.

THEOREM 4.9. *Let δ, δ' be two n-tuples of commuting hermitian operators. Assume $k_n^-(\delta) < \infty$ and $\delta - \delta' \in \mathscr{C}_n^-$. Then the absolutely continuous parts of δ and δ' are unitarily equivalent.*

Proof. Let m, m' denote the multiplicity functions of the absolutely continuous parts of δ and δ'. Clearly $k_n^-(\delta') = k_n^-(\delta) < \infty$ and this means in view of Theorem 4.5 that m and m' are integrable. Consider now $F = (f_1, \ldots, f_n)$: $\mathbf{R}^n \to \mathbf{R}^n$ a n-tuple of polynomial functions and $F(\delta), F(\delta')$ as in the statement of Corollary 4.8. Then $F(\delta) - F(\delta') \in \mathscr{C}_n^-$ and hence $k_n^-(F(\delta)) = k_n^-(F(\delta'))$. Taking for $2 \leqslant j \leqslant n$, $f_j(x_1, \ldots, x_n) = x_j$ and using Corollary 4.8 we have

$$\int |\partial f_1/\partial x_1| m \, \mathrm{d}\lambda = \int |\partial f_1/\partial x_1| \, m' \, \mathrm{d}\lambda$$

for all polynomials f_1. Now $\partial f_1/\partial x_1$ may be any polynomial and since m and m' are integrable with bounded support it follows that m and m' must be equal almost everywhere with respect to Lebesgue measure and this is just the desired result. Q.E.D.

The method used to prove the results of this section is in fact quite flexible and by using a C^∞-functional calculus instead of the polynomial functional calculus one can obtain extensions of the above results to manifolds. We briefly sketch below such extensions, leaving the details to the reader.

The basic fact concerning the C^∞-functional calculus we shall need, is: given δ, δ' n-tuples of commuting hermitian operators with $\delta - \delta' \in \mathfrak{S}_\Phi^{(0)}$ and $f : \mathbf{R}^n \to \mathbf{R}$ a C^∞-function, we have $f(\delta) - f(\delta') \in \mathfrak{S}_\Phi^{(0)}$. Moreover for fixed f assuming $\|\delta\|, \|\delta'\|$ are bounded by some fixed constant, then given $\varepsilon > 0$ there is $\beta > 0$ such that

$$|\delta - \delta'|_\Phi < \beta \Rightarrow |f(\delta) - f(\delta')|_\Phi < \varepsilon.$$

This follows by using the Fourier transform method from [9] and remarking that the replacement of the trace-class by $\mathfrak{S}_\Phi^{(0)}$ is inessential.

Let $\mathscr{X}$ be a n-dimensional C^∞-manifold, countable at infinity and consider $C.(\mathscr{X})$ the C^*-algebra of continuous function on $\mathscr{X}$ which converge to zero at infinity. Consider further $C_0^\infty(\mathscr{X}) \subset C.(\mathscr{X})$ the $*$-subalgebra of infinitely differentiable functions with compact support. Given a non-degenerate $*$-representation ρ: $C.(\mathscr{X}) \to \mathscr{L}(\mathscr{H})$ there is a direct sum decomposition $\rho = \rho_a \oplus \rho_s$ into an absolutely continuous and a singular part ($\xi \in \mathscr{H}$ is in the space of ρ_a if the associated measure μ_ξ on $\mathscr{X}$ is absolutely continuous with respect to Lebesgue measure when restricted to any coordinate neighborhood).

Concerning diagonability results modulo $\mathfrak{S}_\Phi^{(0)}$ it is easily seen by splitting ρ into a direct sum of representations corresponding to the spectral projections associated with certain Borel sets on $\mathscr{X}$, that: if for every point $p \in \mathscr{X}$ we can find real functions $f_1, \ldots, f_n \in \mathscr{C}_0^\infty$ which give local coordinates in some neighbourhood of p such that $k_\Phi(\rho(f_1), \ldots, \rho(f_n)) = 0$, then there is a representation ρ' of $C.(\mathscr{X})$ which is a direct sum of one-dimensional representations and such that

$$\rho(f) - \rho'(f) \in \mathfrak{S}_\Phi^{(0)}$$

for all $f \in \mathscr{C}_0^\infty(\mathscr{X})$. Using this remark it follows immediately from the results about n-tuples that for $\mathfrak{S}_\Phi^{(0)} = \mathscr{C}_n$ there is always such a ρ' and that for $\mathfrak{S}_\Phi^{(0)} = \mathscr{C}_n^-$ there is such a ρ' provided ρ is reduced to its singular part.

Concerning the invariance up to unitary equivalence of absolutely continuous parts, we have the following results.

Let ρ_1 and ρ_2 be non-degenerate $*$-representations of $C.(\mathscr{X})$ such that

$$\rho_1(f) - \rho_2(f) \in \mathscr{C}_n^-$$

for all $f \in \mathscr{C}_0^\infty(\mathscr{X})$. Assume moreover that the multiplicity function of $\rho_{1,a}$ is locally integrable (or equivalently

$$k_n^-(\rho_1(f_1), \ldots, \rho_1(f_n)) < \infty$$

whenever $f_1, \ldots, f_n \in \mathscr{C}_0^\infty$, $f_j = f_j^*$). Then $\rho_{1,a}$ and $\rho_{2,a}$ are unitarily equivalent.

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