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ULTRAWEAKLY CLOSED OPERATOR ALGEBRAS

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The "Invariant Subspace Problem" remains unsolved since more than 30 years. The question to be answered is the following: Does every bounded linear operator T acting in a separable complex Hilbert space have a proper invariant subspace? Scott Brown [3] answered "yes" this question in case T is subnormal (i.e. the restriction to an invariant subspace of a normal operator). The way he solved the problem is beautiful, but the most remarkable fact is that his techniques are strong enough for significant generalizations as well as for the study of the ultraweakly closed algebra generated by some operators. We already know three generalizations belonging to J. Agler [1], to S. Brown, B. Chevreau and C. Pearcy [4] and to J. Stampfli [14]. Given T, assume that we have

$$\|f(T)\| \leqslant a \sup \{|f(\lambda)| : \lambda \in \sigma(T)\}$$

where $a \geqslant 1$ is fixed and f is any rational function with poles outside $\sigma(T)$. Then "yes" holds again as shown by J. Agler for $a = 1$ and J. Stampfli for $a > 1$. The Brown-Chevreau-Pearcy's theorem solves the problem for contractions T for which $\sigma(T) \cap \{\lambda : |\lambda| < 1\}$ is dominating in the sense of Rubel and Shields (see the Remark after Theorem 2.3). Along the same lines is our Theorem 2.3. below, a generalization of Brown-Chevreau-Pearcy's result. The spectrum of our contraction T will include the unit circumference and the essential norm of the resolvent will grow quickly enough in the unit disk, near $\sigma(T)$.

In the sequel we shall use the following notation:

H : an infinite-dimensional complex Hilbert space,

B_H : the open unit ball of H,

$\mathscr{L}(H)$: the algebra of all bounded linear operators acting in H,

$\mathscr{T}(H)$: the set of all trace-class operators acting in H,

S : the adjoint of a unilateral shift acting in H,

$\mathscr{A}(T)$: the ultraweakly closed algebra generated by all polynomials in T, where T belongs to $\mathscr{L}(H)$,

D : the interior of the unit disk.

$\overline{\text{aco}}$: the closure of the absolutely convex hull.

Consider the bilinear functional on $\mathscr{L}(H)\times\mathscr{T}(H)$

$$(A, K) \to \operatorname{tr}(AK), \qquad A \in \mathscr{L}(H), \qquad K \in \mathscr{T}(H).$$

This functional allows an identification of $\mathscr{L}(H)$ with the conjugate space of $\mathscr{T}(H)$ and the corresponding w^*-topology of $\mathscr{L}(H)$ coincides with the ultraweak topology of $\mathscr{L}(H)$. Recall that the ultraweak topology in $\mathscr{L}(H)$ is the weakest topology which makes the map

$$A \to \sum_{n=1}^{\infty} \langle Ax_n , y_n\rangle, \; A \in \mathscr{L}(H),$$

continuous for any $\{x_n\}_{n=1}^{\infty} \subset H$, $\{y_n\}_{n=1}^{\infty} \subset H$, such that $\sum_{n=1}^{\infty} \|x_n\| \, \|y_n\| < \infty$, where $\langle .,.\rangle$ denotes the scalar product in H. The equivalence of the two topologies can be found in [6], pp. 35, 105.

Let $T \in \mathscr{L}(H)$ be given. The set

$$\{K \in \mathscr{T}(H) : \operatorname{tr}(AK) = 0, \quad (\forall) A \in \mathscr{A}(T)\}$$

is a subspace in $\mathscr{T}(H)$ and the corresponding quotient space $\mathscr{T}^T(H)$ is a predual of $\mathscr{A}(T)$, i.e. $\mathscr{A}(T)$ can be canonically identified with the dual space of $\mathscr{T}^T(H)$. The norm in $\mathscr{T}^T(H)$ will be denoted by $\|\cdot\|_*$. For any $x, y \in H$ define $x \otimes y \in \mathscr{T}(H)$ by the equation

$$(x \otimes y)h = \langle h, y\rangle x , \qquad h \in H$$

and denote by $x \overset{T}{\otimes} y$ the image of $x \otimes y$ in $\mathscr{T}^T(H)$. If σ and δ are subsets in H then the sets $\sigma \otimes \delta \subset \mathscr{T}(H)$ and $\sigma \overset{T}{\otimes} \delta \subset \mathscr{T}^T(H)$ will be pointwise defined.

Let H^∞ denote the disk algebra of all bounded holomorphic functions defined in D and endowed with the norm

$$\|f\|_\infty = \sup \{|f(\lambda)| : \lambda \in D\}, \qquad f \in H^\infty.$$

The w^*-topology in H^∞ will be determined by the predual of H^∞ as defined in [9], Ch.9. If T is a completely nonunitary contraction (i.e. no subspace reduces T to a unitary operator) then by [10], Ch. III, Theorem 2.1 we may consider the operator $f(T)$ for any $f \in H^\infty$. More precisely, we have

$$f(T) = s\text{-}\lim_{r \to 1-0} f_r(T),$$

where $f_r(\lambda) = f(r\lambda)$. Now if T is a completely non-unitary contraction we may consider the maps

$$\Phi^T : H^\infty \to \mathscr{A}(T), \qquad \Phi^T_* : (H^\infty, w^*) \to (\mathscr{A}(T), w^*)$$

defined by the equations

$$\Phi^T(f) = f(T), \qquad \Phi^T_*(f) = f(T), \qquad f \in H^\infty.$$

The properties of Φ^T and Φ^T_* are listed in [10], Ch. III, Theorem 2.1 and [4], Theorem 3.2. We mention only the following consequence of the proof of [4], Theorem 3.2, (g) : *If Φ^T is bounded from below then Φ^T_* maps homeomorphically (H^∞, w^*) onto $(\mathscr{A}(T), w^*)$*. In case Φ^T_* is a homeomorphism we shall denote by $\mathscr{E}^T_\lambda$ the w^*-continuous multiplicative functional in $\mathscr{A}(T)$ determined by the equation

$$\mathscr{E}^T_\lambda \circ \Phi^T_* = \mathscr{E}_\lambda,$$

where $\lambda \in D$ and $\mathscr{E}_\lambda$ is the evaluation at λ in H^∞. As seen in the proof of [4], Lemma 4.2, to any w^*-continuous linear functional in $\mathscr{A}(T)$ we can associate an element in $\mathscr{T}^T(H)$.

In the present paper we use the techniques of Scott Brown to put in evidence some properties of the predual of $\mathscr{A}(S)$ (§ 1), whence we shall derive a generalization of the result of S. Brown, B. Chevreau and C. Pearcy (see Theorem 2.3). Moreover we reduce the invariant subspace problem for a contraction T with $\sigma(T) \supset \partial D$, to the case when Φ^T_* is a homeomorphism (see Theorem 2.2).

1. PROPERTIES OF $\mathscr{T}^S(H)$.

Because by [2], Theorem 7, $\mathscr{A}(S)$ is isometrically isomorphic with H^∞, we may consider the functional $\mathscr{E}^S_\lambda$, $\lambda \in D$. If $e_\lambda \in \operatorname{Ker}(S - \lambda)$, $\|e_\lambda\| = 1$, we have

$$(e_\lambda \overset{S}{\otimes} e_\lambda) f(S) = \langle f(S) e_\lambda, e_\lambda \rangle = f(\lambda), \qquad f \in H^\infty,$$

consequently $\mathscr{E}^S_\lambda = e_\lambda \overset{S}{\otimes} e_\lambda$.

Let X be an invariant subspace of S and let P, P_λ denote the orthogonal projections of H onto X, resp. onto $\operatorname{Ker}(S-\lambda)$. If $A \in \mathscr{L}(H)$ is given we denote by $\tilde{A}$ the image of A in the Calkin algebra. Recall that a set $\Gamma \subset D$ is called dominating in D (see [5], [11]) if

$$\sup\{|f(\lambda)| : \lambda \in \Gamma\} = \|f\|_\infty, \qquad f \in H^\infty.$$

For any $0 < a < 1$ we put

$$\Gamma_a(X) = \{\lambda \in D : \|PP_\lambda\| > (1 - a^2)^{1/2}\},$$

$$\tilde{\Gamma}_a(X) = \{\lambda \in D : \|\tilde{P}\tilde{P}_\lambda\| > (1 - a^2)^{1/2}\}.$$

It is plain that we have $\Gamma_a(X) \supset \tilde{\Gamma}_a(X)$.

PROPOSITION 1.1. *Let $0 < a < 1$ be such that $\Gamma_a(X)$ is a dominating set. Then we have*

$$\{\varphi \in \mathscr{T}^S(H) : \|\varphi\|_* \leqslant 1 - a\} \subset \overline{\operatorname{aco}}\,(B_X \overset{S}{\otimes} B_X).$$

Proof. Let $\varphi \in \mathcal{T}^S(H)$ be given. For any $\{\lambda_k\}_{k=1}^m \subset \Gamma_a(X)$, $\{\alpha_k\}_{k=1}^m$ complex numbers, $e_k \in \mathrm{Ker}(S-\lambda_k)$ such that

$$\sum_{k=1}^m |\alpha_k| \leqslant \|\varphi\|_*, \quad \|e_k\| = 1, \quad \|Pe_k\| > (1-a^2)^{1/2},$$

we have

$$\|\varphi - \sum_{k=1}^m \alpha_k Pe_k \overset{S}{\otimes} Pe_k\|_* = \|\varphi - \sum_{k=1}^m \alpha_k Pe_k \overset{S}{\otimes} e_k\|_* \leqslant$$

$$\leqslant \|\varphi - \sum_{k=1}^m \alpha_k \mathscr{E}^S_{\lambda_k}\|_* + \sum_{k=1}^m |\alpha_k| \, \|\mathscr{E}^S_{\lambda_k} - Pe_k \overset{S}{\otimes} e_k\|_* =$$

$$= \|\varphi - \sum_{k=1}^m \alpha_k \mathscr{E}^S_{\lambda_k}\|_* + \sum_{k=1}^m |\alpha_k| \, \|(I-P)e_k\| \leqslant$$

$$\leqslant \|\varphi - \sum_{k=1}^m \alpha_k \mathscr{E}^S_{\lambda_k}\|_* + a\|\varphi\|_*.$$

Since by [4], Proposition 2.8, $\|\varphi - \sum_{k=1}^m \alpha_k \mathscr{E}^S_{\lambda_k}\|_*$ can be made arbitrarily small we derive

$$\mathrm{dist}\,(\varphi, \|\varphi\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X)) \leqslant a\|\varphi\|_*.$$

We shall prove by induction the relation

$$(*) \qquad \inf\{\|\varphi - \sum_{k=1}^n \varphi_k\|_* : \varphi_k \in a^{k-1}\|\varphi\|_* \overline{\mathrm{aco}}(B_X \overset{S}{\otimes} B_X)\} \leqslant a^n \|\varphi\|_*.$$

Because we already proved $(*)$ for $n = 1$, assume $(*)$ holds true for $n \leqslant m$ and pick $\varphi_k \in a^{k-1}\|\varphi\|_* \overline{\mathrm{aco}}(B_X \overset{S}{\otimes} B_X)$, $1 \leqslant k \leqslant m$, $\varepsilon > 0$. But we have

$$\mathrm{dist}\,(\varphi - \sum_{k=1}^m \varphi_k, \|\varphi - \sum_{k=1}^m \varphi_k\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X)) \leqslant a\|\varphi - \sum_{k=1}^m \varphi_k\|_*;$$

consequently we can find $\varphi_{m+1} \in \|\varphi - \sum_{k=1}^m \varphi_k\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X)$ such that

$$\|\varphi - \sum_{k=1}^{m+1} \varphi_k\|_* < (a+\varepsilon)\|\varphi - \sum_{k=1}^m \varphi_k\|_*.$$

Since $\varepsilon > 0$ is arbitrarily small by the induction hypothesis, $(*)$ follows for $n = m+1$. Now we observe that we have

$$\sum_{k=1}^n \|\varphi_k\|_* \leqslant \|\varphi\| \sum_{k=1}^n a^{k-1} \leqslant \|\varphi\|_*(1-a)^{-1}$$

whence $\sum_{k=1}^{n} \varphi_k \leqslant (1-a)^{-1}\|\varphi\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X)$ and this implies

$$\mathrm{dist}\,(\varphi, (1-a)^{-1}\|\varphi\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X)) = 0.$$

Thus if $\|\varphi\|_* \leqslant 1-a$ we have

$$\varphi \in (1-a)^{-1}\|\varphi\|_* \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X) \subset \overline{\mathrm{aco}}\,(B_X \overset{S}{\otimes} B_X).$$

LEMMA 1.2. *For any $h \in H$ the function*

$$x \to h \otimes x \quad , \quad x \in H,$$

is weakly sequentially continuous.

Proof. Since $H = \mathrm{span}\,\{\mathrm{Ker}\,(S-\lambda)\}_{\lambda \in D}$ it suffices to assume $h \in \mathrm{Ker}\,(S-\lambda)$ for some $\lambda \in D$. But in this case we have

$$\|h \overset{S}{\otimes} x\|_* = \sup\,\{|\langle f(S)h, \; x\rangle| : f \in B_{H^\infty}\} =$$

$$= |\langle h, x\rangle| \sup\,\{|f(\lambda)| : f \in B_{H^\infty}\} \leqslant$$

$$\leqslant |\langle h, x\rangle|$$

and the continuity becomes obvious.

LEMMA 1.3. *Let $0 < a < 1$, $\varphi \in \mathcal{T}^S(H), x, y \in H$ be such that $\tilde{\Gamma}_a(X)$ is a dominating set and put $\|\varphi - x \overset{S}{\otimes} y\|_* = b$. Then we have*

$$\mathrm{dist}\,(\varphi, (x + \sqrt{b}\, B_X) \overset{S}{\otimes} (y + \sqrt{b}\, B_X)) \leqslant a\sqrt{b}\,(\|y\|^2 + b)^{1/2}.$$

If, moreover, the function

$$x \to x \overset{S}{\otimes} h \quad , \quad x \in X$$

is weakly sequentially continuous for any $h \in X$ then

$$\mathrm{dist}\,(\varphi, (x + \sqrt{b}\, B_X) \overset{S}{\otimes} (y + \sqrt{b}\, B_X)) \leqslant ab.$$

Proof. Let $\{\lambda_k\}_{k=1}^m \subset \tilde{\Gamma}_a(X)$, $\{\alpha_k\}_{k=1}^m$ be given. Because we have $\|\tilde{P}\tilde{P}_{\lambda_k}\| > (1-a^2)^{1/2}$ we can find $z_k \in \mathrm{Ker}\,(S-\lambda_k)$, $\|z_k\| = 1$ such that (see [8])

$$\|Pz_k\| > (1-a^2)^{1/2}, \; z_k \perp z_j, (I-P)z_k \perp (I-P)z_j, \; k \neq j.$$

If we put

$$u = \sum_{k=1}^{m} \sqrt{\alpha_k}\, z_k \,, \; v = \sum_{k=1}^{m} \sqrt{\overline{\alpha_k}}\, z_k \,, \; x' = Pu \quad , \quad y' = Pv,$$

then

$$x' \overset{S}{\otimes} y' = x' \overset{S}{\otimes} v = u \overset{S}{\otimes} v - (I - P)u \overset{S}{\otimes} v,$$

$$x' \overset{S}{\otimes} y = u \overset{S}{\otimes} y - (I - P)u \overset{S}{\otimes} y,$$

$$u \overset{S}{\otimes} v = \sum_{k=1}^{m} \alpha_k \mathscr{E}^S_{\lambda_k},$$

$$u \overset{S}{\otimes} y = 0,$$

Hence we derive

$$\varphi - (x + x') \overset{S}{\otimes} (y + y') = \varphi - x \overset{S}{\otimes} y - x' \overset{S}{\otimes} y' - x \overset{S}{\otimes} y' - x' \overset{S}{\otimes} y =$$

$$= \varphi - x \overset{S}{\otimes} y - \sum_{k=1}^{m} \alpha_k \mathscr{E}^S_{,k} + (I - P)u \overset{S}{\otimes} (v + y) -$$

$$- x \overset{S}{\otimes} y'.$$

Let $\eta > 0$ be fixed. Applying [4], Proposition 2.8 we may suppose $\sum_{k=1}^{m} |\alpha_k| \leqslant b$ and $\|\varphi - x \overset{S}{\otimes} y - \sum_{k=1}^{m} \alpha_k \mathscr{E}^S_{\lambda_k}\|_* < \eta$. But it is easy to see that we may choose y' orthogonal to any given finite-dimensional subspace, thus by Lemma 1.2, we may also suppose $\|x \otimes y'\|_* < \eta$. Consequently we may determine $x', y' \in X$ such that

$$\|\varphi - (x + x') \overset{S}{\otimes} (y + y')\|_* < 2\eta + \|(I - P)u \overset{S}{\otimes} (v + y)\|_*,$$

$$\|x'\| \leqslant \sqrt{b}, \quad \|y'\| \leqslant \sqrt{b}, \quad \|v + y\|^2 \leqslant b + \|y\|^2,$$

$$\|(I - P)u\|^2 = \sum_{k=1}^{m} |\alpha_k| \, \|(I - P)z_k\|^2 \leqslant a^2 b$$

and this implies

$$\operatorname{dist}(\varphi, (x + \sqrt{b}\, B_X) \overset{S}{\otimes} (y + \sqrt{b}\, B_X)) \leqslant a\sqrt{b}\,(\|y\|^2 + b)^{1/2}.$$

If the function

$$x \to x \overset{S}{\otimes} h, \qquad x \in X$$

is weakly sequentially continuous for any $h \in X$, then as before we may suppose $\|Pu \overset{S}{\otimes} y\|_* < \eta$, thus

$$\|\varphi - (x + x') \overset{S}{\otimes} (y + y')\|_* < 3\eta + \|(I - P)\, u \overset{S}{\otimes} v\|_* \leqslant 3\eta + ab$$

and the proof is concluded.

THEOREM 1.4. *If $\tilde{\Gamma}_a(X)$ is dominating for any $0 < a < 1$ then we have*

$$\{\varphi \in \mathcal{T}^S(H) : \|\varphi\|_* < 1\} = B_X \overset{S}{\otimes} B_X.$$

Proof. Let $\varphi \in \mathcal{T}^S(H)$ be given. Applying Lemma 1.3 we can find by induction two sequences $\{x_k\}_{k=0}^\infty$, $\{y_k\}_{k=0}^\infty \subset X$ such that

$$x_0 = y_0 = 0,\ b_{k+1} \leqslant a_{k+1}\sqrt{b_k}(\|y_k\|^2 + b_k)^{1/2},\ \|x_{k+1} - x_k\| \leqslant \sqrt{b_k}, \|y_{k+1} - y_k\| \leqslant \sqrt{b_k},$$

where $b_k = \|\varphi - x_k \overset{S}{\otimes} y_k\|_*$, $0 < a_k < 1$. If $\{a_k\}_{k=1}^\infty$ converges quickly enough to zero then $x_k \xrightarrow{s} x$, $y_k \xrightarrow{s} y$, $\varphi = x \overset{S}{\otimes} y$ and $|\,\|x\| - \|\varphi\|_*^{1/2}|, |\,\|y\| - \|\varphi\|_*^{1/2}|$ can be made arbitrarily small. This implies the inclusion "$\subset$" and because the opposite inclusion is trivial, the proof is concluded.

THEOREM 1.5. *Let $0 < a < 1$ be such that $\tilde{\Gamma}_a(X)$ is a dominating set. If the function*

$$x \to x \overset{S}{\otimes} h,\ x \in X$$

is weakly sequentially continuous for any $h \in X$, then we have

$$\{\varphi \in \mathcal{T}^S(H) : \|\varphi\|_* < 1 - a\} \subset B_X \overset{S}{\otimes} B_X.$$

Proof. Let $\varphi \in \mathcal{T}^S(H)$ be given. Applying Lemma 1.3 we can find two sequences $\{x_k\}_{k=0}^\infty$, $\{y_k\}_{k=0}^\infty \subset X$ such that

$$x_0 = y_0 = 0,\quad \|\varphi - x_k \overset{S}{\otimes} y_k\|_* < a^k\|\varphi\|_*,\quad \|x_{k+1} - x_k\| \leqslant (a\|\varphi\|_*)^{n/2},$$

$$\|y_{k+1} - y_k\| \leqslant (a\|\varphi\|_*)^{n/2}.$$

It is easy to see that we may also suppose $x_{k+1} - x_k \perp x_{j+1} - x_j$, $y_{k+1} - y_k \perp$ $\perp y_{j+1} - y_j$, $k \neq j$, thus we deduce

$$x_k \xrightarrow{s} x,\ y_k \xrightarrow{s} y,\ \varphi = x \overset{S}{\otimes} y,$$

$$\|x\|^2 = \sum_{k=0}^{\infty} \|x_{k+1} - x_k\|^2 \leqslant (1 - a)^{-1}\|\varphi\|_*,$$

$$\|y\|^2 = \sum_{k=0}^{\infty} \|y_{k+1} - y_k\|^2 \leqslant (1 - a)^{-1}\|\varphi\|_*.$$

Now the inclusion in our Theorem becomes obvious.

In the remainder of this section we shall denote by A the restriction of S to X. Since A is a completely nonunitary operator we may consider the maps

$$\Phi^A : H^\infty \to \mathcal{A}(A),\ \Phi_*^A : (H^\infty, w^*) \to (\mathcal{A}(A), w^*).$$

We already mentioned that in case Φ^A is bounded from below, Φ^A_* becomes a homeomorphism.

THEOREM 1.6. *(i) If $\tilde{\Gamma}_a(X)$ is dominating for some $0 < a < 1$ then we have*

$$\|\Phi^A(f)\| \geqslant (1-a)\|f\|_\infty \quad , \quad f \in H^\infty,$$

consequently Φ^A_* is a homeomorphism.

(ii) If $\tilde{\Gamma}_a(X)$ is dominating for any $0 < a < 1$ then for any $\lambda \in D$ there exists $x_\lambda \in X$, $\|x_\lambda\| = 1$ such that

$$\mathscr{E}^A_\lambda = x_\lambda \overset{A}{\otimes} x_\lambda.$$

(iii) If $\tilde{\Gamma}_a(X)$ is dominating for some $0 < a < 1$ and the function

$$x \to x \overset{S}{\otimes} h, \quad x \in X$$

is continuous for any $h \in X$, then for any $\lambda \in D$ there exists $x_\lambda \in X$, $\|x_\lambda\| = 1$ such that

$$\mathscr{E}^A_\lambda = x_\lambda \overset{A}{\otimes} x_\lambda.$$

Proof. (*i*) Let $f \in H^\infty$ be given and let $\varphi \in \mathscr{T}^S(H)$, $\varepsilon > 0$ be such that $\|\varphi\|_* = 1$, $|\varphi(f(S))| + \varepsilon > \|f\|_\infty$. Now using Proposition 1.1 we can find $K \in \mathscr{T}(X)$, $\|K\|_{\mathrm{tr}} \leqslant (1-a)^{-1}$ such that $|\mathrm{tr}\,(f(S)KP)| + \varepsilon > \|f\|_\infty$. Using the relation

$$\mathrm{tr}\,(f(S)KP) = \mathrm{tr}\,(f(A)K)$$

we derive

$$\|f(A)\| \geqslant (1-a)|\mathrm{tr}\,(f(A)K)| > (1-a)\,(\|f\|_\infty - \varepsilon)$$

whence it follows

$$\|\Phi^A(f)\| \geqslant (1-a)\,\|f\|_\infty.$$

(*ii*) By Theorem 1.4 we know that we have $\mathscr{E}^S_\lambda = x \overset{S}{\otimes} y$ for some $x, y \in X$. Let B denote the restriction of A to the invariant subspace $\mathrm{span}\{A^k x\}_{k=0}^\infty$. Since $\mathscr{E}^S_\lambda((S-\lambda)^k) = 0 = \langle (B-\lambda)^k x, y\rangle$, $k \geqslant 1$ and $\langle x, y\rangle = 1$ it follows that the range of $B - \lambda$ is not dense. If we take $x_\lambda \in \mathrm{Ker}\,(B-\lambda)^*$, $\|x_\lambda\| = 1$ then for any polynomial p we have

$$(x_\lambda \overset{A}{\otimes} x_\lambda)\,(p(A)) = \langle p(A)x_\lambda, x_\lambda\rangle = \langle p(B)x_\lambda, x_\lambda\rangle = p(\lambda).$$

Since by (*i*) $\mathscr{E}^A_\lambda$ is well defined and coincides with $x_\lambda \overset{A}{\otimes} x_\lambda$ on the set of all polynomials we infer $\mathscr{E}^A_\lambda = x_\lambda \overset{A}{\otimes} x_\lambda$.

(*iii*) We proceed as in (*ii*).

COROLLARY 1.7. *If either the condition* (*ii*) *or the condition* (*iii*) *of Theorem* 1.6 *is fulfilled then A has a proper invariant subspace.*

Proof. Let $\lambda \in D$ be given. By Theorem 1.6 we can find $x_\lambda \in X$, $\|x_\lambda\| = 1$ such that $\overset{A}{\mathscr{E}_\lambda^A} = x_\lambda \otimes x_\lambda$. If we put $X_\lambda = \text{span}\,\{(A-\lambda)^k x_\lambda\}_{k=1}^\infty$ then $X_\lambda \perp x_\lambda$, thus X_λ is an invariant subspace and $X_\lambda \neq X$. In case $X_\lambda \neq \{0\}$ the proper invariant subspace of A, looked for, is X_λ; if $X_\lambda = \{0\}$, x_λ is an eigenvector and A has a one-dimensional invariant subspace.

2. INVARIANT SUBSPACES

Throughout this section we shall denote by T a fixed contraction acting in H. For any $0 < a < 1$ put

$$\zeta_a(T) = D \cap (\sigma(T) \cup \{\lambda \in \rho(T) : a\,\|(\lambda - T)^{-1}\| > (1 - |\lambda|)^{-1}\}),$$

$$\zeta_a(\tilde{T}) = D \cap (\sigma(\tilde{T}) \cup \{\lambda \in \rho(\tilde{T}) : a\|(\lambda - \tilde{T})^{-1}\| > (1 - |\lambda|)^{-1}\}).$$

Recall that T is by definition a contraction of class $C_{0\cdot}$ if $T^n \xrightarrow{s} 0$ (see [10], Ch. II, §4).

LEMMA 2.1. *If $\sigma(T) \supset \partial D$ and $\zeta_a(T)$ is not dominating for some $0 < a < 1$ then T has a proper hyperinvariant subspace (i.e. there exists a proper subspace in H invariant for all operators which commute with T).*

Proof. It is easy to see that the set

$$\zeta_a'(T) = D \cap (\sigma(T) \cup \{\lambda \in \rho(T) : a\|(\lambda - T)^{-1}\| \geqslant (1 - |\lambda|)^{-1}\})$$

is not dominating. Arguing as in [3], Lemma 3.1 (see also [5]) we can find a measurable set $\sigma \subset \partial D$ of positive Lebesgue measure such that for any $\mu \in \sigma$ we have

$$\Gamma_\mu' = \{t\mu : \varepsilon_\mu < t < 1\} \subset D \diagdown \zeta_a'(T)$$

for some $0 < \varepsilon_\mu < 1$. Further we observe that because σ is not countable and $D \diagdown \zeta_a'(T)$ being open has at most countable many connected components, there exists a connected component G of $D \diagdown \zeta_a'(T)$ such that

$$\Gamma_{\mu_k}' \subset G, \quad 1 \leqslant k \leqslant 4, \quad \mu_k \in \sigma, \quad \mu_k \neq \mu_j, \quad k \neq j.$$

If $\mu_k = e^{i\theta_k}$, $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$ and if we put

$$\Gamma_{1,2}' = \{e^{i\theta} : \theta_1 < \theta < \theta_2\}, \quad \Gamma_{3,4}' = \{e^{i\theta} : \theta_3 < \theta < \theta_4\},$$

$$\Gamma_{\mu_k} = \{t\mu_k : \varepsilon_{\mu_k} < t < 1 + \varepsilon_{\mu_k}\},$$

then we can find two simple rectifiable closed curves $\Gamma_{1,2}, \Gamma_{3,4}$ enjoying the properties:

$$\Gamma_{1,2} \cap \partial D = \{\mu_1\} \cup \{\mu_2\}, \quad \Gamma_{3,4} \cap \partial D = \{\mu_3\} \cup \{\mu_4\}, \quad \Gamma_{1,2} \cap \Gamma_{3,4} = \emptyset \quad ,$$

$$\Gamma_{1,2} \cap D \subset G, \quad \Gamma_{3,4} \cap D \subset G, \quad \Gamma_{1,2} \supset \Gamma_{\mu_1} \cup \Gamma_{\mu_2}, \quad \Gamma_{3,4} \supset \Gamma_{\mu_3} \cap \Gamma_{\mu_4},$$

$\Gamma'_{1,2}$ is surrounded by $\Gamma_{1,2}$, $\Gamma'_{3,4}$ is surrounded by $\Gamma_{3,4}$.

Since $G \subset \rho(T)$ and $a\|(\lambda - T)^{-1}\| < (1 - |a|)^{-1}$, $\lambda \in G$, the operators

$$B_{1,2} = \frac{1}{2\pi i}\int_{\Gamma_{1,2}} (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - T)^{-1}\,d\lambda,$$

$$B_{3,4} = \frac{1}{2\pi i}\int_{\Gamma_{3,4}} (\lambda - \mu_3)(\lambda - \mu_4)(\lambda - T)^{-1}\,d\lambda$$

are well defined and the inclusion $\partial D \supset \Gamma'_{1,2} \cup \Gamma'_{3,4}$ implies (via Gelfand Representation Theorem) $B_{1,2} \neq 0$, $B_{3,4} \neq 0$.

Let Ω denote the unbounded domain whose boundary is $\Gamma_{1,2}$ and let H' denote the linear manifold of all vectors $h \in H$ such that the function

$$\lambda \to (\lambda - T)^{-1}h, \qquad |\lambda| > 1$$

has an analytic extension in Ω. It is plain that H' is left invariant by every operator which commutes with T and its norm-closure $\overline{H'}$ enjoys the same property. Let $x \in H'$ be given and let g_x be the analytic extension in Ω of the function

$$\lambda \to (\lambda - T)^{-1}x, \ |\lambda| > 1.$$

Because $\Gamma_{3,4} \subset \Omega$ we have

$$B_{3,4}x = \frac{1}{2\pi i}\int_{\Gamma_{3,4}} (\lambda - \mu_3)(\lambda - \mu_4)g_x(\lambda)\,d\lambda = 0$$

thus $\overline{H'} \subset \operatorname{Ker} B_{3,4} \neq H$. On the other hand if $y \notin \ker B_{1,2}$ and we put $z = B_{1,2}y$ then the function

$$g_z(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_{1,2}} (\mu - \mu_1)(\mu - \mu_2)(\mu - \lambda)^{-1}(\mu - T)^{-1}y\,d\mu$$

is analytic in Ω and

$$(\lambda - T)g_z(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_{1,2}} (\mu - \mu_1)(\mu - \mu_2)(\mu - \lambda)\,d\mu + z = z.$$

This shows that g_z is an analytic extension in Ω of the function

$$\lambda \to (\lambda - T)^{-1}z, \ \ |\lambda| > 1$$

and $z \in H' \neq \{0\}$. It follows that $\overline{H'}$ is a proper subspace and the proof is concluded.

REMARK. The idea of the proof used in the above Lemma involves a functional calculus similar with the techniques of [13].

THEOREM 2.2. *If* $\sigma(T) \supset \partial D$ *and* T *has no proper hyperinvariant subspace then* Φ^T *is well defined and* $\|\Phi^T(f)\| = \|f\|_\infty$, $f \in H^\infty$, *consequently* Φ^T_* *is a homeomorphism.*

Proof. Using [8], Ch. II, Theorem 5.4 we may suppose either $T \in C_{0\cdot}$ or $T^* \in C_{0\cdot}$, thus anyway T is a completely non-unitary contraction and Φ^T will be well defined. Because Φ^T and Φ^{T^*} could be isometries only simultaneously we may suppose $T \in C_{0\cdot}$ and by [10] Ch. II, Theorem 2.1, T will be unitarily equivalent with $A = S|X$, as defined at the end of § 1. To avoid the existence of proper hyperinvariant subspaces for T we have to assume $\sigma(T) = \sigma_l(T) = \sigma_r(T) = \sigma_{le}(\tilde{T}) = \sigma_{re}(\tilde{T})$. Now we have to prove that Φ^A is an isometry. To this aim put

$$S(\lambda) = (\lambda - S)^*[(\lambda - S)(\lambda - S)^*]^{-1}, \qquad \lambda \in D$$

and observe that $S(\lambda)$ is a right inverse of $\lambda - S$, $S(\lambda)H$ is orthogonal to $\mathrm{Ker}(S - \lambda)$ and $S(\lambda)(\lambda - S) = I - P_\lambda$. This implies

$$\|y\| = \|(\lambda - S)S(\lambda)y\| \geqslant \|S(\lambda)y\| \inf\{\|(\lambda - S)h\| : h \perp \mathrm{Ker}(S - \lambda),\ \|h\| = 1\} =$$
$$= \|S(\lambda)y\| \inf\{\|(\lambda - S)^*h\| : \|h\| = 1\} =$$
$$= \|S(\lambda)y\|(1 - |\lambda|), \qquad (\forall)\ y \in H,$$

thus $\|S(\lambda)\| = 1 - |\lambda|$. Since for any $\lambda \in \rho(A)$ we obviously have

$$\inf\{\|(\lambda - A)x\| : x \in X,\ \|x\| = 1\} = \|(\lambda - A)^{-1}\|^{-1}$$

we infer that for any $\lambda \in \zeta_a(T)$ the inequality

$$\inf\{\|(\lambda - A)x\| : x \in X,\ \|x\| = 1\} < a(1 - |\lambda|)$$

holds true. It follows that

$$\|PP_\lambda\|^2 = \|P_\lambda P\|^2 = \sup\{\|P_\lambda x\|^2 : x \in X,\ \|x\| = 1\} =$$
$$= 1 - \inf\{\|(I - P_\lambda)x\|^2 : x \in X,\ \|x\| = 1\} =$$
$$= 1 - \inf\{\|S(\lambda)(\lambda - S)x\|^2 : x \in X,\ \|x\| = 1\} \geqslant$$
$$\geqslant 1 - \|S(\lambda)\|^2 \inf\{\|(\lambda - A)x\|^2 : x \in X,\ \|x\| = 1\} >$$
$$> 1 - a^2$$

which means $\lambda \in \Gamma_a(X)$ and $\zeta_a(T) \subset \Gamma_a(X)$. But $\zeta_a(T)$ is dominating for any $0 < a < 1$, by Lemma 2.1, thus by Theorem 1.6 (i)

$$\|\Phi^A(f)\| \geqslant (1 - a)\|f\|_\infty, \quad f \in H^\infty, \qquad 0 < a < 1$$

and the proof is concluded.

THEOREM 2.3. *If $\zeta_a(\tilde{T})$ is dominating for any $0 < a < 1$, then T has a proper invariant subspace.*

Proof. Making the same reductions as in the proof of Theorem 2.2 we know in particular that T is unitarily equivalent to $A = S|X$. Let $\lambda \in \zeta_a(\tilde{T})$ be given. Then it is easy to see that there exists an orthonormal sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that

$$\overline{\lim_{n\to\infty}} \|(\lambda - A)\, x_n\| < a(1 - |\lambda|)$$

and hence

$$\|\tilde{P}\, \tilde{P}_\lambda\|^2 = \|\tilde{P}_\lambda \tilde{P}\|^2 \geqslant \underline{\lim_{n\to\infty}}\, \|P_\lambda x_n\|^2 = 1 - \overline{\lim_{n\to\infty}} \|(I - P_\lambda)x_n\|^2 =$$

$$= 1 - \overline{\lim_{n\to\infty}} \|S(\lambda)\,(\lambda - A)x_n\|^2 \geqslant 1 - \|S(\lambda)\|^2\, \overline{\lim_{n\to\infty}} \|(\lambda - A)x_n\|^2 >$$

$$> 1 - a^2.$$

This shows that we have $\zeta_a(\tilde{T}) \subset \tilde{\Gamma}_a(X)$ consequently $\tilde{\Gamma}_a(X)$ is dominating for any $0 < a < 1$. To conclude the proof we apply Corollary 1.7.

REMARK. If $D \cap \sigma(T)$ is dominating then T has a proper invariant subspace by the Theorem of Brown-Chevreau-Pearcy [4].

COROLLARY 2.4. *If $\sigma(T) \supset \partial D$ and $\|(\lambda - T)^{-1}\| = \|(\lambda - \tilde{T})^{-1}\|$, $\lambda \in \rho(T)$ then T has a proper invariant subspace.*

Proof. Because we may suppose $\sigma(T) = \sigma_l(\tilde{T}) = \sigma_r(\tilde{T})$ we derive $\zeta_a(T) = \zeta_a(\tilde{T}), 0 < a < 1$. If $\zeta_a(T)$ is not dominating for some $0 < a < 1$ we apply Lemma 2.1 and in the contrary case we apply Theorem 2.3.

THEOREM 2.5. *If $T \in C_{0\cdot}$, $T^* \in C_{0\cdot}$ and $\zeta_a(\tilde{T})$ is dominating for some $0<a<1$ then T has a proper invariant subspace.*

Proof. As seen in the proof of Theorem 2.3 we may suppose that T is unitarily equivalent with $A = S|X$ and $\Gamma_a(X)$ is a dominating set. Because by [10], Ch. II, Theorem 2.1, the isometric dilation W of A is a unilateral shift, if we put

$$\tilde{f}(\lambda) = \sum_{n=0}^{\infty} \bar{a}_n \lambda^n \quad \text{if } f \in H^\infty, f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

we have for any $x, y \in X$

$$|(x \overset{A}{\otimes} h) f(A)| = |\langle f(A)x, h\rangle| = |\langle x, \tilde{f}(A^*)h\rangle| =$$

$$= |\langle x, \tilde{f}(W^*)h\rangle| = |(h \overset{W^*}{\otimes} x) f(W^*)|$$

Applying Lemma 1.2 we infer that the function

$$x \to x \otimes h, \quad x \in X$$

is weakly sequentially continuous and the proof is concluded if we use Corollary 1.7

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