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WEAK LIMITS OF ALMOST INVARIANT PROJECTIONS

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In this Note we give a new characterization (Corollary 4.6) of strongly reductive algebras of operators [1] based on a new characterization of the weak limits of orthogonal projections, almost invariant with respect to an algebra of operators (Theorem 2.1).

This paper originates in the following question (connected to the Note [2]) raised privately by C. Apostol: If T is an operator on a Hilbert space H such that any operator in the normclosure of the unitary orbit

$$\{UTU^{-1} | U : H \to H \text{ unitary}\}$$

of T, is reductive (i.e. $PT = PTP$ for $P = P^* = P^2$ implies $PT = TP$), does it follow that T is normal?

In the sequel we give an affirmative answer to this question (see Corollary 4.4., below) as well as a natural generalization of this answer to the case of operator algebras (see Theorem 4.1. and Corollary 4.6., below).

The proof of those results rely on a new characterization (given in Theorem 2.1.) of the weak limits Q of sequences $(P_n)_{n=1}^{\infty}$ formed by orthogonal projections such that:

$$\|(I - P_n)TP_n\| \to 0 \quad (n \to \infty)$$

for all T belonging to a (norm) separable algebra of operators on a Hilbert space.

We hope that this characterization connecting dilation theory to quasitriangularity, will lead in the future to applications of wider interest.

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§ 1

1.1. We begin by recalling the necessary terminology and notation. Let H be a complex Hilbert space with the scalar product $(.,.)$. The set of all bounded linear operators on H will be denoted by $\mathscr{L}(H)$. The identity operator on any Hilbert space will be loosely denoted by I; also by 0 we shall denote the null element in any Hilbert space, the null operator on any Hilbert space as well as the subspace $\{0\}$ of any Hilbert space. Each time the Hilbert space involved will be determined from the context. The ideal of compact operators in $\mathscr{L}(H)$ is denoted by $\mathscr{K}(H)$, the Calkin algebra by $\mathscr{L}(H)/\mathscr{K}(H)$ and the image of $T \in \mathscr{L}(H)$ in $\mathscr{L}(H)/\mathscr{K}(H)$ by $p(T)$. The restriction of $T \in \mathscr{L}(H)$ to a subspace $L \subset H$ will be denoted by $T_{|L}$. By id we shall note the identic representation of $\mathscr{L}(H)$ on H. If $M \subset \mathscr{L}(H)$, then $C^*(M)$ will denote the C^*-algebra generated in $\mathscr{L}(H)$ by M and $I(=I_H)$.

1.2. If $n \in \mathbf{N}$, $H^{(n)} := \underbrace{H \oplus H \oplus \ldots \oplus H}_{n\text{-times}}$ for every $B \subset \mathscr{L}(H)$ we define

$$M_n(B) = \{(X_{ij})_{i=1,\, j=1}^{n} \in \mathscr{L}(H^{(n)}) \mid X_{ij} \in B,\ 1 \leqslant i, j \leqslant n\}.$$

Let $\mathscr{C}$ be a unital C^*-algebra, $S \subset \mathscr{C}$ a subspace (not necessarily closed) and $\varphi : S \to \mathscr{L}(H)$ a linear norm-bounded mapping. For every $n \in \mathbf{N}$ we define $\varphi_n : M_n(S) \to \mathscr{L}(H^{(n)})$ by:

$$\varphi_n((a_{ij})_{i,j=1}^{n}) = (\varphi(a_{ij}))_{i,j=1}^{n}$$

for all $(a_{ij})_{i,j=1}^{n} \in M_n(S)$. We say that φ is a completely contractive mapping if for every $n \in \mathbf{N}$, φ_n is a contractive mapping.

We shall need in the sequel the following two theorems:

1.3. THEOREM. (W. B. Arveson, [4]). *Let $\mathscr{B}$ be a unital C^*-algebra $\mathscr{A} \subset \mathscr{B}$ a subspace (not necessarily closed), $1 \in \mathscr{A}$ and $\varphi : \mathscr{A} \to \mathscr{L}(H)$ a completely contractive mapping such that $\varphi(1) = I$. Then there exists a completely positive extension of φ to $\mathscr{B}$.*

1.4. THEOREM. (W. Stinespring, [9]). *Let $\mathscr{B}$ be a unital C^*-algebra and $\varphi : \mathscr{B} \to \mathscr{L}(H)$ a completely positive mapping. Then there exist a complex Hilbert space K, a bounded linear operator $V : H \to K$ and a representation $\pi : \mathscr{B} \to \mathscr{L}(K)$ such that:*

$$\varphi(x) = V^* \pi(x) V, \qquad x \in B$$

$$K = \bigvee_{x \in B} \pi(x)\, VH.$$

For the proofs see for instance [10] (p. 101—111 and p. 95—98).

1.5. An algebra $\mathscr{A} \subset \mathscr{L}(H)$ is called reductive if every subspace $L \subset H$ invariant for $\mathscr{A}$ is also invariant for $\mathscr{A}^*$ (i.e. L is reducing $\mathscr{A}$). An operator $T \in \mathscr{L}(H)$ is called reductive if the algebra $\mathscr{A}_T := \{q(T) \mid q = \text{polynomial}\}$ is reductive.

Let $\psi_1, \psi_2 : \mathscr{A} \to \mathscr{L}(H)$ be two representations of an algebra $\mathscr{A} \subset \mathscr{L}(H)$. We say that ψ_2 is in the norm-closed unitary orbit of ψ_1, if there exists a sequence $(U_n)_{n=1}^{\infty}$ of unitary operators such that:

$$\lim_{n \to \infty} \|\psi_2(T) - U_n \psi_1(T)\, U_n^{-1}\| = 0,$$

for all $T \in \mathscr{A}$.

The unitary orbit of $T \in \mathscr{L}(H)$ is the set:

$$O_u(T) = \{UTU^{-1} | U \text{ is unitary in } \mathscr{L}(H)\}$$

and the norm closure of $O_u(T)$ will be denoted by $O_u(T)^-$.

In this paper we shall essentially use the following theorem.

1.6. THEOREM. (D. Voiculescu, [11]). *Let $\mathscr{A}$ be a separable C^*-algebra with unit and ρ a representation of $\mathscr{A}$ on the separable Hilbert space H. Let π be a representation of $p(\rho(\mathscr{A}))$ on a separable Hilbert space H_π. Then there exists a sequencet $(U_n)_{n=1}^{\infty}$ of unitary operators from $H \oplus H_\pi$ onto H such that:*

$$\rho(T) - U_n(\rho(T) \oplus \pi(p(\rho(T))))\, U_n^{-1} \in \mathscr{K}(H), \qquad n \in \mathbf{N}$$

$$\lim_{n\to\infty} \|\rho(T) - U_n(\rho(T) \oplus \pi(p(\rho(T))))\, U_n^{-1}\| = 0$$

for every $T \in \mathscr{A}$.

§ 2.

The main result of this paper is given by the following.

2.1. THEOREM. *Let H be a complex separable infinite-dimensional Hilbertt space. Let $\mathscr{A} \subset \mathscr{L}(H)$ be a norm-separable norm closed algebra, such that $I \in \mathscr{A}$, and $Q \in \mathscr{L}(H), 0 \leqslant Q \leqslant I$.*

Then the following conditions are equivalent:

(i) there exists $(P_n)_{n=1}^{\infty}, P_n = P_n^* = P_n^2 \in \mathscr{L}(H)$ for all $n \in \mathbf{N}$, such that $\lim\limits_{n\to\infty} \|(I - P_n)\, TP_n\| = 0$, for all $T \in \mathscr{A}$ and $w\text{-}\lim\limits_{n\to\infty} P_n = Q$.

(ii) there exists $(R_n)_{n=1}^{\infty}$, $R_n = R_n^* = R_n^2 \in \mathscr{L}(H)$ for all $n \in N$ such that $w\text{-}\lim\limits_{n\to\infty}(I - R_n)\, TR_n = 0$ for all $T \in \mathscr{A}$ and $w\text{-}\lim\limits_{n\to\infty} R_n = Q$.

(iii) there exists a representation ρ of $p(C^*(\mathscr{A}))$ on some separable Hilbert space H' and a subspace $L \subset H \oplus H'$ invariant for $(\mathrm{id} \oplus (\rho \circ p))(\mathscr{A})$, such that:

$$P_{H\oplus 0} P_{L|H\oplus 0} = Q. \tag{2.1.1.}$$

The proof of this theorem will involve also two other equivalent properties, namely:

(*iv*) there exist two linear multiplicative mappings $\varphi : \mathscr{A} \to \mathscr{L}(\overline{QH})$ and $\theta : p(\mathscr{A}) \to \mathscr{L}(\overline{Q(1 - Q)H})$ uniquely defined by the relations:

$$Q^{1/2}\varphi(T) = TQ^{1/2}_{|\overline{QH}}, \tag{2.1.2.}$$

$$T \in \mathscr{A}$$

$$(I - Q)^{1/2}\varphi(T) = \theta(p(T))\,(I - Q)^{1/2}_{|\overline{QH}} \tag{2.1.3.}$$

with θ completely contractive.

(v) there exist two linear multiplicative completely contractive mappings $\tilde{\varphi}: \mathscr{A} \to \mathscr{L}(\overline{QH})$ and $\tilde{\theta}: p(\mathscr{A}) \to \mathscr{L}(\overline{Q(I-Q)H})$ uniquely defined by the relations:

(2.1.4.) $$Q^{1/2}\tilde{\varphi}(T) = TQ^{1/2}_{|\overline{QH}},$$

$$T \in \mathscr{A}$$

2.1.5.) $$(I-Q)^{1/2}\tilde{\varphi}(T) = \tilde{\theta}(p(T))(I-Q)^{1/2}_{|\overline{QH}}.$$

Obviously, if the mappings from (iv) and (v) exist, then $\varphi = \tilde{\varphi}$ and $\theta = \tilde{\theta}$.

2.2. Proof of (iii) $\Rightarrow$ (i)

We consider the representation:

$$\psi = \mathrm{id} \oplus (\rho \circ p) \oplus (\rho \circ p) \oplus \dots$$

of $C^*(\mathscr{A}) + \mathscr{K}(H)$, on the separable Hilbert space:

$$H_1 = H \oplus H' \oplus H' \oplus \dots .$$

We also consider:

$$L_1 = L \oplus 0 \oplus 0 \oplus \dots \subset (H \oplus H') \oplus H' \oplus \dots = H_1$$

and the operators $S_n \in \mathscr{L}(H_1)$ such that:

$$S_n(h \oplus h_1 \oplus h_2 \oplus \dots) = h \oplus h_n \oplus h_2 \dots \oplus h_{n-1} \oplus h_1 \oplus h_{n+1} \oplus \dots$$

Then $S_n \in (\psi((C^*(\mathscr{A}) + \mathscr{K}(H)))'$ and:

$$w\text{-}\lim_{n\to\infty} S_n^{-1} P_{L_1} S_n = Q \oplus 0 \oplus 0 \oplus \dots \quad .$$

Using Theorem 1.6. there is a sequence of unitary operators $U_k : H_1 \to H$ such that:

$$\lim_{k\to\infty} \| U_k \psi(x) U_k^* - x \| = 0$$

for all $x \in C^*(\mathscr{A}) + \mathscr{K}(H)$.

Let $q_k \in \mathscr{K}(H)$ be selfadjoint finite-rank projections such that $q_k \nearrow I$. Replacing the sequence $\{U_k\}_{k=1}^{\infty}$ by a subsequence we can assume that:

(2.2.1.) $$\left\| U_k \psi(q_k Q q_k) U_k^* - q_k Q q_k \right\| < \frac{1}{k}$$

(2.2.2.) $$\left\| U_k \psi(q_k) U_k^* - q_k \right\| < \frac{1}{k}.$$

Because $\psi(q_k) = q_k \oplus 0 \oplus 0 \oplus \dots$, we have:

$$\lim_{n\to\infty} \|\psi(q_k)(S_n^{-1}P_{L_1}S_n - (Q \oplus 0 \oplus 0 \dots))\,\psi(q_k)\| = 0.$$

Then we can choose $n_1 < n_2 < \dots$ such that:

$$\left\|\psi(q_k)\,(S_{n_k}^{-1}P_{L_1}\,S_{n_k} - (Q \oplus 0 \oplus 0 \oplus \dots))\psi(q_k)\right\| < \frac{1}{k}. \tag{2 2.3.}$$

Denoting $V_k = U_k S_{n_k}^{-1}$ we have:

$$\lim_{k\to\infty} \|V_k\psi(x)\,V_k^* - x\| = \lim_{k\to\infty}\|U_k\psi(x)\,U_k^* - x\| = 0$$

for all $x \in C^*(\mathscr{A}) + \mathscr{K}(H)$.

We have:

$$\|q_k(V_kP_{L_1}V_k^* - Q)q_k\| \leqslant \|V_k\psi(q_k)\,P_{L_1}\,\psi(q_k)\,V_k^* - q_kQq_k\| + 2\|V_k\psi(q_k) - q_kV_k\|. \tag{2.2.4.}$$

Using (2.2.2.) we have:

$$\|V_k\psi(q_k) - q_kV_k\| = \|V_k\psi(q_k)\,V_k^* - q_k\| =$$

$$= \left\|\,U_kS_{n_k}^{-1}\,\psi(q_k)\,S_{n_k}U_k^* - q_k\right\| = \|U_k\psi(q_k)\,U_k^* - q_k\| < \frac{1}{k} \tag{2.2.5.}$$

and using (2.2.3.), (2.2.1.):

$$\|V_k\psi(q_k)\,P_{L_1}\psi(q_k)\,V_k^* - q_kQq_k\| =$$

$$= \|U_k\psi(q_k)\,S_{n_k}^{-1}P_{L_1}S_{n_k}\psi(q_k)\,U_k^* - q_kQq_k\| \leqslant \tag{2.2.6.}$$

$$\leqslant \|U_k\psi(q_k)(S_{n_k}^{-1}P_{L_1}S_{n_k} - Q \oplus 0 \oplus 0 \oplus \dots)\,\psi(q_k)U_k^*\| +$$

$$+ \|U_k\psi(q_kQq_k)U_k^* - q_kQq_k\| < \frac{2}{k}.$$

By virtue of (2.2.4.), (2.2.5.) and (2.2.6.) we have:

$$\|q_k(V_kP_{L_1}V_k^* - Q)q_k\| < \frac{3}{k}.$$

Because $q_k \nearrow I$, it follows that:

$$w\text{-}\lim_{k\to\infty} V_kP_{L_1}V_k^* = Q.$$

Thus, taking $P_k = V_k P_{L_1} V_k^*$, we have:

$$w\text{-}\lim_{k\to\infty} P_k = Q.$$

For all $x \in \mathscr{A}$, we can write:

$$\|(I - P_k)xP_k\| = \|V_k(I - P_{L_1}) V_k^* x V_k P_{L_1} V_k^*\| \leqslant$$
$$\leqslant \|V_k\psi(x) V_k^* - x\| + \|V_k(I - P_{L_1})\psi(x) P_{L_1} V_k^*\| =$$
$$= \|V_k\psi(x)V_k^* - x\|.$$

It follows that:

$$\lim_{k\to\infty}\|(I - P_k)xP_k\| = 0, \quad \text{for all } x \in \mathscr{A},$$

this finishes the proof of *(iii)* $\Rightarrow$ *(i)*.

Proof of *(iv)* $\Rightarrow$ *(iii)*.

We observe that $\varphi(I) = I$ and $\theta(p(I)) = I$.

Using Theorem 1.3. of W. B. Arveson it follows that θ is the restriction of a completely positive mapping $\bar{\theta} : p(C^*(\mathscr{A})) \to \mathscr{L}(\overline{Q(I - Q)H})$.

Theorem 1.4 of W. Stinespring used for $\bar{\theta}$ provides a Hilbert space $H' \supset \supset \overline{Q(I - Q)H}$ and a representation ρ of $p(C^*(\mathscr{A}))$ on H', such that:

$$P_{H_1}\rho(p(T))_{|H_1} = \bar{\theta}(p(T)), \qquad T \in \mathscr{A} \tag{2.2.7.}$$
$$H' = \bigvee_{T\in C^*(\mathscr{A})} \rho(p(T))\, H_1$$

where $H_1 := \overline{Q(I - Q)H}$.

We define $F := \bigvee_{T\in\mathscr{A}} \rho(p(T))\, H_1$. Using (2.2.7.) and the fact that $\bar{\theta}$ is multiplicative on $p(\mathscr{A})$, it follows that:

$$P_{H_1}\rho(p(T))_{|F} = \theta(p(T))\, P_{H_1\,|F}, \quad \text{for all } T \in \mathscr{A}. \tag{2.2.8.}$$

We define the subspaces $N = \{Q^{1/2}h \oplus (I - Q)^{1/2}h \mid h \in \overline{QH}\} \subset H \oplus H'$ and $L = \bigvee_{T\in\mathscr{A}} (T \oplus \rho(p(T))N \subset H \oplus H'$.

We shall prove that:

$$P_{H\oplus 0}P_{L\,|H\oplus 0} = P_{H\oplus 0}\,P_{N\,|H\oplus 0}. \tag{2.2.9.}$$

To this end we consider the operators $S := I \oplus P_{H_1} \in L(H \oplus H')$. By virtue of (2.1.2.), (2.1.3.), (2.2.8.) and because $F \supset \overline{Q(I - Q)H} = H_1$, we have for all $T \in \mathscr{A}$ and $h \in \overline{QH}$:

$$S(TQ^{1/2}h \oplus \rho(p(T))(I - Q)^{1/2}h) = TQ^{1/2}h \oplus P_{H_1}\rho(p(T))(I - Q)^{1/2}h =$$
$$= TQ^{1/2}h \oplus \theta(p(T))(I - Q)^{1/2}h = Q^{1/2}\varphi(T)h \oplus (I - Q)^{1/2}\varphi(T)h.$$

Because $\varphi(I) = I$ it follows that $S(L) = N$. But $N \subset L$ and therefore

(2.2.10.) $$S_{|L} = P_N^L$$

where P_N^L denotes the orthogonal projection of L onto N.

We consider $h \oplus 0 \in H \oplus 0 \subset H \oplus H'$. Writing

$$h \oplus 0 = u_1 \oplus u_2 + v_1 \oplus v_2$$

with $u_1 \oplus u_2 \in L$ and $v_1 \oplus v_2 \in (H \oplus H') \ominus L$, and using (2.2.10.) it follows that:

$$P_{H\oplus 0} P_N (h \oplus 0) = P_{H\oplus 0} P_N^L (u_1 \oplus u_2) = P_{H\oplus 0} (u_1 \oplus P_{H_1} u_2) =$$

$$= u_1 \oplus 0 = P_{H\oplus 0} (u_1 \oplus u_2) = P_{H\oplus 0} P_L (h \oplus 0).$$

Therefore (2.2.9.) is proved, thus it remains to prove the relation:

(2.2.11.) $$P_{H\oplus 0} P_{N\,|H\oplus 0} = Q.$$

Therefore we consider $h \oplus 0 \in H \oplus 0$. Then

$$h \oplus 0 = Qh \oplus (I - Q)^{1/2} Q^{1/2} h + (I - Q)h \oplus (-(I - Q)^{1/2} Q^{1/2} h)$$

where evidently $Qh \oplus (I - Q)^{1/2} Q^{1/2} h \in N$ and

$$(I - Q)\, h \oplus (-(I - Q)^{1/2} Q^{1/2} h) \in (H \oplus H') \ominus N.$$

It follows:

$$P_{H\oplus 0} P_N (h \oplus 0) = P_{H\oplus 0} (Qh \oplus (I - Q)^{1/2} Q^{1/2} h) = Qh.$$

Thus the relation (2.2.11.) is established and the proof of $(iv) \Rightarrow (iii)$ concluded.

Proof of $(ii) \Rightarrow (v)$.

For every $n \in \mathbf{N}$, $T = (T_{ij})_{i,j=1}^n \in M_n(\mathscr{A})$ and $\bigoplus_{j=1}^n h_j$, $\bigoplus_{i=1}^n k_i \in H^{(n)}$ we can write:

$$\left| \sum_{i,j=1}^n (T_{ij} Q h_j, k_i) \right| = \lim_{m\to\infty} \left| \sum_{i,j=1}^n (T_{ij} R_m h_j, k_i) \right| =$$

$$= \lim_{m\to\infty} \left| \sum_{i,j=1}^n (R_m T_{ij} R_m h_j, k_i) \right| =$$

$$= \lim_{m\to\infty} \left| \sum_{i,j=1}^n (T_{ij} R_m h_j, R_m k_i) \right| \leqslant$$

$$\leqslant \lim_{m\to\infty} \|T\| \cdot \left\| \bigoplus_{j=1}^n R_m h_j \right\| \cdot \left\| \bigoplus_{i=1}^n R_m k_i \right\| =$$

$$= \|T\| \left\| \bigoplus_{j=1}^n Q^{1/2} h_j \right\| \left\| \bigoplus_{i=1}^n Q^{1/2} k_i \right\|.$$

It follows that we can define a completely contractive mapping $\tilde{\varphi}:\mathscr{A}\to\mathscr{L}(\overline{QH})$ such that $\tilde{\varphi}(I)=I$, by:

$$(\tilde{\varphi}(T)Q^{1/2}h,\ Q^{1/2}k)=(TQh,k),$$

where $T\in\mathscr{A}$ and $h,k\in H$.

Evidently:

$$Q^{1/2}\tilde{\varphi}(T)=TQ^{1/2}_{|\overline{QH}}$$

for every $T\in\mathscr{A}$.

On the other hand, for every $T_1, T_2\in\mathscr{A}$ and $h,k\in H$, we have:

$$(\tilde{\varphi}(T_1)\,\tilde{\varphi}(T_2)Q^{1/2}h,\ Q^{1/2}k)=(Q^{1/2}\tilde{\varphi}(T_1)\,\tilde{\varphi}(T_2)\,Q^{1/2}h,k)=$$

$$=(T_1Q^{1/2}\,\tilde{\varphi}(T_2)Q^{1/2}h,k)=(T_1T_2Qh,k)=$$
$$=(\tilde{\varphi}(T_1T_2)\,Q^{1/2}h,\ Q^{1/2}k).$$

Therefore $\tilde{\varphi}$ is multiplicative.

We remark that for every $T\in\mathscr{A}$ and $K\in\mathscr{K}(H)$, we have:

$$(I-Q)TQ=TQ-QTQ=w\text{-}\lim_{n\to\infty}TR_n-QTQ=$$

$$=w\text{-}\lim_{n\to\infty}R_nTR_n-w\text{-}\lim_{n\to\infty}R_nTQ=$$

$$=w\text{-}\lim_{n\to\infty}R_nT(R_n-Q)=w\text{-}\lim_{n\to\infty}R_n(T+K)(R_n-Q)=$$

$$=w\text{-}\lim_{n\to\infty}(R_n-Q)(T+K)(R_n-Q).$$

Using this fact it follows that for every $n\in\mathbf{N}$, $T=(T_{ij})_{i,j=1}^n\in M_n(\mathscr{A})$, $K=(K_{ij})_{i,j=1}^n\in M_n(\mathscr{K}(H))$ and $\bigoplus_{j=1}^n h_j$, $\bigoplus_{i=1}^n k_i\in H^{(n)}$ we can write:

$$\left|\sum_{i,j=1}^n((I-Q)T_{ij}Qh_j,k_i)\right|=\left|\sum_{i,j=1}^n\lim_{m\to\infty}((R_m-Q)T_{ij}(R_m-Q)h_j,k_i)\right|=$$

$$=\left|\sum_{i,j=1}^n\lim_{m\to\infty}((T_{ij}+K_{ij})\,(R_m-Q)\,h_j,(R_m-Q)k_i)\right|\leqslant$$

$$\leqslant\lim_{m\to\infty}\|T+K\|\left\|\bigoplus_{j=1}^n(R_m-Q)h_j\right\|\left\|\bigoplus_{i=1}^n(R_m-Q)k_i\right\|=$$

$$=\|T+K\|\left\|\bigoplus_{j=1}^nQ^{1/2}(I-Q)^{1/2}h_j\right\|\left\|\bigoplus_{i=1}^nQ^{1/2}(I-Q)^{1/2}k_i\right\|.$$

It follows that we can define a completely contractive mapping $\tilde{\theta} : p(\mathscr{A}) \to \to \mathscr{L}(\overline{Q(I-Q)H})$ such that $\tilde{\theta}(p(I)) = I$, by:

$$(\tilde{\theta}(p(T))Q^{1/2}(I-Q)^{1/2}h, Q^{1/2}(I-Q)^{1/2}k) = ((I-Q)TQh, k),$$

where $T \in \mathscr{A}$ and $h, k \in H$.

Then:

$$((I-Q)^{1/2}\tilde{\varphi}(T)Q^{1/2}h, Q^{1/2}(I-Q)^{1/2}k) =$$

$$= ((I-Q)^{1/2}Q^{1/2}\tilde{\varphi}(T)Q^{1/2}h, \ (I-Q)^{1/2}k) =$$

$$= ((I-Q)^{1/2}TQh, \ (I-Q)^{1/2}k) = ((I-Q)TQh, k) =$$

$$= (\tilde{\theta}(p(T))(I-Q)^{1/2} Q^{1/2}h, Q^{1/2}(I-Q)^{1/2}k),$$

for every $T \in \mathscr{A}$ and $h, k \in H$.

Therefore:

$$(I-Q)^{1/2}\, \tilde{\varphi}(T) = \tilde{\theta}(p(T))(I-Q)^{1/2}_{|\overline{QH}}.$$

for every $T \in \mathscr{A}$.

Using the last relation, for every $T_1, T_2 \in \mathscr{A}$ and $h, k \in H$ we can write:

$$(\tilde{\theta}(p(T_1))\tilde{\theta}(p(T_2))Q^{1/2}(I - Q)^{1/2}h, \ Q^{1/2}(I - Q)^{1/2}k) =$$

$$=(\tilde{\theta}(p(T_1))(I-Q)^{1/2}\, \tilde{\varphi}(T_2)Q^{1/2}h, Q^{1/2}(I-Q)^{1/2}k) =$$

$$= ((I-Q)^{1/2}\tilde{\varphi}(T_1)\, \tilde{\varphi}(T_2)\, Q^{1/2}h, Q^{1/2}(I-Q)^{1/2}k) =$$

$$= ((I-Q)^{1/2}T_1Q^{1/2}\, \tilde{\varphi}(T_2)Q^{1/2}h, (I-Q)^{1/2}k) =$$

$$= ((I-Q)T_1T_2Qh, k) = (\tilde{\theta}(p(T_1T_2))Q^{1/2}(I-Q)^{1/2}h, Q^{1/2}(I-Q)^{1/2}k).$$

Therefore $\tilde{\theta}$ is multiplicative and the proof of *(ii)* $\Rightarrow$ *(v)* is concluded.

Since the implications *(i)* $\Rightarrow$ *(ii)* and *(v)*$\Rightarrow$*(iv)* are obvious, the proof of Theorem 2.1. is completed.

§ 3.

The core of the theorem proved in §2, namely the equivalence of the properties *(i)* and *(iii)* was already obtained several years ago [6] with a different proof for the non obvious implication *(i)* $\Rightarrow$ *(iii)*.

Since this proof might have an intrinsic interest, we shall give it here:

Proof. Let ω be a free ultrafilter on **N** and the W^*-algebra:

$$\ell^\infty \mathbin{\overline{\otimes}} \mathscr{L}(H) = \{(x_n)_{n=1}^\infty \mid x_n \in \mathscr{L}(H), \ \sup_n \|x_n\| < +\infty\}$$

and the ideal:

$$I_\omega = \{(x_n)_{n=1}^\infty \in \ell^\infty \overline{\otimes} \mathscr{L}(H) \mid \lim_{n\to\omega} \|x_n\| = 0\}.$$

By $\mathscr{L}_\omega(H)$ we denote the quotient C^*-algebra:

$$\mathscr{L}_\omega(H) = (\ell^\infty \overline{\otimes} \mathscr{L}(H))/I_\omega.$$

The mapping which associates to $(x_n)_{n=1}^\infty \in \ell^\infty \overline{\otimes} \mathscr{L}(H)$ the element $w\text{-}\lim_{n\to\omega} x_n \in \mathscr{L}(H)$ is zero on I_ω and hence induces a mapping:

$$\psi : \mathscr{L}_\omega(H) \to \mathscr{L}(H).$$

We observe that ψ is a unital completely positive mapping.

The dilation of ψ, provides a Hilbert space $H_\omega \supset H$ and a representation π_ω of $\mathscr{L}_\omega(H)$ on H_ω, such that:

$$P_H \pi_\omega(x)|_H = \psi(x), \qquad x \in \mathscr{L}_\omega(H)$$

$$\overline{\pi_\omega(\mathscr{L}_\omega(H))\, H} = H_\omega.$$

We can also define a mapping

$$j : \mathscr{L}(H) \to \mathscr{L}_\omega(H)$$

by:

$$j(T) = (T_n)_{n=1}^\infty + I_\omega$$

where $T_n = T$ for all $n \in \mathbf{N}$.

Since $(\psi \circ j)(T) = T$, for every $T \in \mathscr{L}(H)$, it follows that H is reducing $(\pi_\omega \circ j)(\mathscr{L}(H))$ and that for $T \in \mathscr{L}(H)$ and $h \in H$, we have:

$$\pi_\omega(j(T))h = Th, \tag{3.1.}$$

whence:

$$\pi_\omega(j(\mathscr{K}(H)))H_\omega \supset \pi_\omega(j(\mathscr{K}(H)))H = H.$$

Actually, we have even more, namely:

$$\pi_\omega(j(\mathscr{K}(H)))\, H_\omega = H. \tag{3.2.}$$

For $y \in \mathscr{K}(H)$, $(x_n)_{n=1}^\infty \in \ell^\infty \overline{\otimes} \mathscr{L}(H)$, $(z_n)_{n=1}^\infty \in \ell^\infty \overline{\otimes} \mathscr{L}(H)$ we have:

$$\psi(((x_n)_{n=1}^\infty + I_\omega)\, j(y)\, ((z_n)_{n=1}^\infty + I_\omega)) = w\text{-}\lim_{n\to\infty} x_n y z_n.$$

But because y is compact, we have:

$$w\text{-}\lim_{n\to\omega} x_n y z_n = (w\text{-}\lim_{n\to\omega} x_n)\, y(w\text{-}\lim_{n\to\omega} z_n) =$$

$$= \psi((x_n)_{n=1}^{\infty} + I_\omega)\, \psi(j(y))\, \psi((z_n)_{n=1}^{\infty} + I_\omega).$$

Thus, for $y \in \mathscr{K}(H)$ and $x, z \in \mathscr{L}_\omega(H)$ we can write:

$$P_H \pi_\omega(x)\, \pi_\omega(j(y))\, \pi_\omega(z)\, P_H = P_H\, \pi_\omega(x)\, P_H\, \pi_\omega(j(y)) P_H \pi_\omega(z)\, P_H.$$

Because $\overline{\pi_\omega(\mathscr{L}_\omega(H))H} = H_\omega$ it follows that:

$$P_H\, \pi_\omega(j(y))\, P_H = \pi_\omega(j(y))$$

and hence:

$$\pi_\omega(j(\mathscr{K}(H)))\, H_\omega \subset H.$$

It follows that (3.2.) is proved.

We are now in state to complete the proof of the implication.

First we remark that it is enough to prove *(iii)* without the separability condition on H' (because it is always possible to replace L by a separable subspace and consequently to replace ρ by its restriction to a separable reducing subspace, such that *(iii)* be valid).

We define $q := (P_n)_{n=1}^{\infty} + I_\omega \in \mathscr{L}_\omega(H)$. By virtue of *(i)*, $\pi_\omega(q)$ is an invariant projection for $\pi_\omega(j(\mathscr{A}))$ and $\psi(q) = Q$. We have:

$$\text{(3.3.)} \qquad P_H \pi_\omega(q)_{|H} = Q.$$

By (3.1.) and (3.2.) there is a representation ρ of $\mathscr{L}(H)/\mathscr{K}(H)$ on $H_\omega \ominus H$ such that $\rho \circ p = (\pi_\omega \circ j)_{H_\omega \ominus H}$. Thus the restriction of $\pi_\omega \circ j$ to $\mathscr{A}$ agrees with the restriction of id $\oplus$ $(\rho \circ p)$ to $\mathscr{A}$ and consequently setting $L = \pi_\omega(q)H_\omega$, we see, by virtue of (3.3.), that *(iii)* is valid.

§4.

As an application of Theorem 2.1. we shall give the following:

4.1. THEOREM. *Let H be a complex Hilbert space and let $\mathscr{A} \subset L(H)$ be a norm-separable commutative algebra containing $I(=I_H)$ and such that for every representation π of $\mathscr{A}$ in the norm-closed unitary orbit of the identic representation of $\mathscr{A}$ on H, the algebra $\pi(\mathscr{A})$ is reductive. Then the norm-closure $\overline{\mathscr{A}}$ of $\mathscr{A}$ coincides with the C^*-algebra generated by $\mathscr{A}$.*

Evidently it is enough to study the case when H is a complex separable Hilbert space. Therefore, in the proof of Theorem 4.1., we shall assume that H is a complex separable Hilbert space.

We will divide the proof of this theorem into several steps.

4.2. LEMMA. *The norm-closure $\mathscr{B}$ of $p(\mathscr{A})$ in $\mathscr{L}(H)/\mathscr{K}(H)$ is a C^*-algebra (obviously commutative).*

Proof. We want to prove that $p(\mathscr{A})^* \subset \mathscr{B}$. If $p(\mathscr{A})^* \not\subset \mathscr{B}$ then by the reflexivity theorem 1.8. of [11], there exist $A \in \mathscr{A}$ and $P = P^* = P^2 \in \mathscr{L}(H)$ such that:

$$(4.2.1.) \qquad \begin{aligned} &(I - p(P))p(T)p(P) = 0 \quad \text{for all } T \in \mathscr{A}, \text{ and:} \\ &(I - p(P))\, p(\mathscr{A})^* p(P) \neq 0. \end{aligned}$$

We define $\mathscr{C} := p(C^*(\mathscr{A} \cup \{P\}))$. Since $\mathscr{C}$ is a separable C^*-algebra, there exists a faithful C^*-representation ρ of $\mathscr{C}$ on some separable Hilbert space H_ρ(i.e. ρ maps $\mathscr{C}$ into $\mathscr{L}(H_\rho)$). By virtue of Theorem 1.6. there exists a sequence $(U_j)_{j=1}^{\infty}$ of unitary operators from $H \oplus H_\rho$ onto H such that:

$$(4.2.2.) \qquad \|X - U_j(X \oplus \rho(p(X)))\, U_j^{-1}\| \to 0 \quad (j \to \infty)$$

for all $X \in C^*(\mathscr{A} \cup \{P\})$.

On the other hand, we observe that the algebra $\rho(p(\mathscr{A})) \subset L(H_\rho)$ is not reductive, because:

$$(I - \rho(p(P)))\, \rho(p(T))\, \rho(p(P)) = 0 \quad \text{for all } T \in \mathscr{A} \text{ and}$$
$$(I - \rho(p(P)))\, \rho(p(\mathscr{A}))^*\, \rho(p(P)) \neq 0,$$

on account of (4.2.1) and the faithfulness of the C^*-representation ρ.

We define now the representation $\pi : \mathscr{A} \to \mathscr{L}(H \oplus H_\rho)$ by the relation:

$$\pi(T) := T \oplus \rho(p(T)), \text{ for all } T \in \mathscr{A}.$$

But the algebra $\pi(\mathscr{A})$ is not reductive. This fact and (4.2.2.) contradict our assumption on $\mathscr{A}$.

It follows that $p(\mathscr{A})^* \subset \mathscr{B}$ and therefore $\mathscr{B}$ is a C^*-algebra.

4.3. LEMMA. *If* $\dim H > 1$, *then there exists a non-trivial subspace of H reducing all operators of $\mathscr{A}$.*

Proof. If $\dim H < \infty$, the lemma is obviously true because of the commutativity of $\mathscr{A}$. Thus we shall assume that H is of infinite dimension. Also we can (and shall) assume that $\mathscr{A}$ is norm-closed.

Let χ be a Gelfand character of $\mathscr{A}$. If there exists $T \in \mathscr{A}$ such that:

$$\chi(T) = 1 \text{ and } \|p(T)\| < 1,$$

then, with the arguments of [1] we can obtain that there exists a nontrivial subspace invariant for all operators of $\mathscr{A}$ and therefore reducing all operators of $\mathscr{A}$.

So, it remains to elucidate the case when, for every character χ of $\mathscr{A}$,

(4.3.1.) $\qquad \chi(T) = 1$ implies $\|p(T)\| \geqslant 1$, for any $T \in \mathscr{A}$.

From (4.3.1.) and Lemma 4.2. it follows that $\tilde{\chi} : p(T) \mapsto \chi(T)$ defines a unique Gelfand character of $\mathscr{B}$. Let M be the Gelfand spectrum of the C^*-algebra $\mathscr{B}$ and let $(\mathscr{B} \ni) Y \mapsto \hat{Y}(\cdot) (\in C(M))$ denote the Gelfand representation of $\mathscr{B}$ (where $C(M)$ is the algebra of all complex-valued, continuous functions on M). Since $\mathscr{B}$ is a C^*-algebra, this representation is an isomorphism. We can identify the Gelfand spectrum of $\mathscr{A}$ with M (identifying every Gelfand character χ of $\mathscr{A}$ with $\tilde{\chi}$). So, the Gelfand representation $X \mapsto \hat{X}(\cdot)$ of $\mathscr{A}$ in $C(M)$ is given by:

$$\hat{X}(\chi) = \widehat{p(X)}(\chi)$$

for all $X \in \mathscr{A}$, $\chi \in M$.

Using the technique from [1] we can obtain a sequence $(P_n)_{n=1}^{\infty}$, $P_n = P_n^* = P$ for all $n \in \mathbf{N}$ and an operator Q, $0 \leqslant Q \leqslant I$, $0 \neq Q \neq I$, such that $\|(I - P_n)TP_n\| \to 0$ $(n \to \infty)$ and $w\text{-}\lim_{n\to\infty} P_n = Q$. By virtue of Theorem 2.1. there exists a representation ρ of $p(C^*(\mathscr{A}))$ on some separable Hilbert space H' and a subspace $L \subset H \oplus H$ invariant for $(\mathrm{id} \oplus (\rho \circ p))(\mathscr{A})$ such that $P_{H\oplus 0}P_{L|H\oplus 0} = Q$. Using Theorem 1.6. and the hypothesis, it follows that $P_L(T \oplus \rho(p(T))) = (T \oplus \rho(p(T)))\, P_L$ for all $T \in \mathscr{A}$, and therefore $TQ = QT$, for all $T \in \mathscr{A}$ (since $P_{H\oplus 0}P_{L|H\oplus 0} = Q$). If Q is not of the form λI, for some scalar λ, then any non-trivial spectral subspace of Q reduces $\mathscr{A}$. It remains to study the case when $Q = \lambda I$, $0 < \lambda < 1$. Using again Theorem 2.1. we infer that there exists a multiplicative completely contractive mapping $\theta : p(\mathscr{A}) \to \mathscr{L}(H)$ such that the subspace $\{\lambda^{1/2}h \oplus (1 - \lambda)^{1/2}h \mid h \in H\}$ is invariant for $T + \theta(p(T))$ for all $T \in \mathscr{A}$. It follows that $\theta(p(T)) = T$, for every $T \in \mathscr{A}$. We have

$$\|T\| = \|\theta(p(T))\| \leqslant \|p(T)\| \leqslant \|T\|$$

for all $T \in \mathscr{A}$, and therefore:

$$\|p(T)\| = \|T\|,$$

for all $T \in \mathscr{A}$.

It follows that $\mathscr{B} = p(\mathscr{A})$ and that the Gelfand map $(\mathscr{A} \ni) T \mapsto \hat{T}(\cdot)\ (\in C(M))$ is a bicontinuous isomorphism between $\mathscr{A}$ and $C(M)$. By virtue of the Sz.-Nagy-Dixmier-Mackey Theorem [5] $\mathscr{A}$ is similar to a commutative C^*-subalgebra of $\mathscr{L}(H)$, and the conclusion of the lemma easily follows.

The remaining part of the proof of Theorem 4.1 is similar to that of [1].

4.4. COROLLARY. *Let $T \in \mathscr{L}(H)$ be a bounded linear operator such that every $T' \in O_u(T)^-$ is reductive. Then T is a normal operator the spectrum of which neither divides the (complex) plane nor has interior (in the plane).*

Proof. By virtue of Theorem 4.1., T is a normal operator and $\lim_{n\to\infty}\|T^*-q_n(T)\|= 0$, where q_n is polynomial for all $n \in \mathbf{N}$. By Corollary 3.9. from [8] it follows that the spectrum of T neither divides the (complex) plane nor has interior (in the plane).

4.5. Let H be a complex Hilbert space. Let us recall the following definitions ([1]): for a set $\mathscr{A} \subset \mathscr{L}(H)$ we denote by Appr. Lat. $(\mathscr{A})$ the family of all sequences $(P_n)_{n=1}^{\infty}$ of orthogonal projections on H, such that:

$$\lim_{n\to\infty} \|(I - P_n)\, TP_n\| = 0 \tag{4.5.1.}$$

for every $T \in \mathscr{A}$. For an arbitrary family F of sequences $(P_n)_{n=1}^{\infty}$ of orthogonal projections of H, we denote by Appr. Alg.(F) the set of all $T \in \mathscr{L}(H)$ such that (4.5.1.) holds for all $(P_n)_{n=1}^{\infty} \in F$.

In [1] an algebra $\mathscr{A} \subset \mathscr{L}(H)$ is called strongly reductive if:

$$\mathscr{A}^* \subset \text{Appr. Alg. [Appr. Lat.}(\mathscr{A})].$$

4.6. Corollary. *Let H be a complex Hilbert space and let $\mathscr{A} \subset \mathscr{L}(H)$ be a norm-separable commutative algebra containing $I(=I_H)$. The following three properties are equivalent:*

(*i*) *$\mathscr{A}$ is strongly reductive,*

(*ii*) *the norm-closure of $\mathscr{A}$ is a C^*-algebra,*

(*iii*) *for every representation π of $\mathscr{A}$ in the norm-closed unitary orbit of the identical representation of $\mathscr{A}$ on H, the algebra $\pi(\mathscr{A})$ is reductive.*

Proof. It is obvious that $(ii) \Rightarrow (i)$.The implication $(iii) \Rightarrow (ii)$ follows from Theorem 4.1. It remains to verify the implication $(i) \Rightarrow (iii)$. To this end, let $\pi : \mathscr{A} \to \mathscr{L}(H)$ be a representation of the algebra $\mathscr{A}$ such that there exists a sequence $(U_n)_{n=1}^{\infty}$ of unitary operators satisfying:

$$\lim_{n\to\infty} \|T - U_n^{-1}\pi(T)\, U_n\| = 0, \tag{4.6.1.}$$

for all $T \in \mathscr{A}$.

Let $P \in \mathscr{L}(H)$ be an orthogonal projection such that:

$$(I - P)\, \pi(T)\, P = 0 \tag{4.6.2.}$$

for all $T \in \mathscr{A}$.

We set:

$$P_n := U_n^{-1} P U_n,$$

for every $n \in \mathbf{N}$. Then, for any $T \in \mathscr{A}$, we have:

$$\begin{aligned}\|(I - P_n)TP_n\| &\leqslant \|(I - P_n)\, U_n^{-1}\, \pi(T)\, U_n P_n\| + \|(I - P_n)(T - U_n^{-1}\, \pi(T) U_n) P_n\| \leqslant \\ &\leqslant \|(I - P)\, \pi(T) P\| + \|T - U_n^{-1}\, \pi(T) U_n\| = \\ &= \|T - U_n^{-1}\, \pi(T) U_n\| \to 0 \quad (\text{for } n \to \infty)\end{aligned}$$

by virtue of (4.6.1.) and (4.6.2.). It follows that $(P_n)_{n=1}^{\infty} \in$ Appr. Lat.$(\mathscr{A})$. Since $\mathscr{A}$ is strongly reductive, for every $T \in \mathscr{A}$, we have:

$$\lim_{n\to\infty} \|(I - P_n)\, T^* P_n\| = 0$$

and consequently:

$$\|(I - P)\, \pi(T)^* P\| = \|(I - P_n)\, U_n^{-1}\, \pi(T)^*\, U_n P_n\| \leqslant$$
$$\leqslant \|(I - P_n)\, T^* P_n\| + \|(I - P_n)\, (U_n^{-1}\, \pi(T)^*\, U_n - T^*) P_n\| \leqslant$$
$$\leqslant \|(I - P_n)\, T^* P_n\| + \|U_n^{-1}\, \pi(T)\, U_n - T\| \to 0 \qquad (\text{for } n\to\infty).$$

It follows that $(I - P)\, \pi(T)^* P = 0$, for all $T \in \mathscr{A}$. The proof is completed.

Let us finish with the following question: find a direct proof of the implication *(iii)* ⇒ *(i)* (i.e. avoiding the use of *(ii)*), as simple as that given above for the implication *(i)* ⇒ *(iii)*.

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