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COMPACT OPERATORS IN THE RADICAL OF A REFLEXIVE OPERATOR ALGEBRA

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In the appendix to [2], Fall, Arveson and Muhly characterize the compact operators which lie in the radical of a nest algebra. The purpose of this note is to show that this characterization remains valid for any reflexive operator algebra whose invariant subspace lattice is commutative. Furthermore, the proof we give is much shorter and simpler than the proof in [2]. It should be pointed out that essentially similar results are obtained in [1] for a nest algebra of a maximal nest.

Let $\mathscr{L}$ be a commutative subspace lattice acting on a separable Hilbert space and let $\mathscr{A}=\mathscr{A}\ell g\,\mathscr{L}$. Let X denote the carrier space of $\mathscr{A}$, the set of all lattice homomorphisms of $\mathscr{L}$ onto the trivial lattice $\{0, 1\}$. A projection E in $\mathscr{A}$ is said to be an *interval* from $\mathscr{L}$ if $E=P-Q$, where $P, Q \in \mathscr{L}$ and $Q \leqslant P$. If φ is an element of X such that $\varphi(P)=1$ and $\varphi(Q)=0$, we say that E is a *test interval* for φ. The family, $\mathscr{F}_{\varphi}$, of all test intervals for φ has the finite intersection property and is maximal with respect to possession of this property. An interval E is an *atom* if either $E \leqslant P$ or $E \perp P$, for all $P \in \mathscr{L}$. Each atom E gives rise to an element φ of the carrier space defined by $\varphi(P)=1$ if $E \leqslant P$ and $\varphi(P)=0$ if $E \perp P$. Any element of X which arises in this way is said to be *atomic*; the others are *non-atomic*. If φ is atomic, the family $\mathscr{F}_{\varphi}$ consists of all intervals which contain the associated atom E. If $\mathscr{F}_{\varphi}$ is non-atomic, $\mathscr{F}_{\varphi}$ is a directed set which converges strongly to 0. For each $\varphi \in X$, a semi-norm N_{φ} is defined on $\mathscr{A}$ by $\mathscr{N}_{\varphi}(T)=\inf\{\|ETE\| \,|E \in \mathscr{F}_{\varphi}\}$. The set $\mathscr{A}_{\varphi}=\{T|\mathscr{N}_{\varphi}(T)=0\}$ is a closed two sided ideal in $\mathscr{A}$. The intersection, $\mathscr{I}$, of all the ideals $\mathscr{A}_{\varphi}$ is always contained in the radical, $\mathscr{R}$, of $\mathscr{A}$. In a variety of cases, including all nest algebras, it is known that $\mathscr{I}=\mathscr{R}$. More information about these concepts and proofs of the various assertions above can be found in [3], [4].

Define a mapping δ from $\mathscr{B}(\mathscr{H})$ into the commutant of the set of atoms of $\mathscr{L}$ by the formula $\delta(T)=\sum ETE$. The sum is taken as E ranges over all atoms from $\mathscr{L}$; should the number of atoms be infinite (at most countable since $\mathscr{H}$ is separable) the sum converges in the strong operator topology. If $\mathscr{L}$ has no atoms, δ is taken to be the 0 map. Since every atom is semi-invariant for $\mathscr{A}\ell g\mathscr{L}$ (a difference of two nested invariant projections), δ is multiplicative when restricted to $\mathscr{A}\ell g\mathscr{L}$.

The following is the generalization of the result in [2] referred to above:

THEOREM. *Let T be a compact operator in $\mathcal{Alg}\,\mathcal{L}$. Then $T \in \mathcal{I}$ if, and only if $\delta(T) = 0$.*

Proof. First we show that δ annihilates the whole radical, $\mathcal{R}$, of $\mathcal{A}$. Since $\mathcal{I} \subseteq \mathcal{R}$, this immediately yields $\delta(T)=0$ for compact T in $\mathcal{I}$. If $\mathcal{L}$ has no atoms than $\delta = 0$ and we are done. Otherwise, let E be any atom from $\mathcal{L}$ and let T be any operator in $\mathcal{R}$. Since E is an atom, $EP = 0$ or $EP = E$, for any $P \in \mathcal{L}$. From this it follows that $ENE \in \mathcal{A}$, for any operator $N \in \mathcal{B}(\mathcal{H})$. (Just observe: $P^{\perp}ENEP = 0$, for all $P \in \mathcal{L}$.) In particular, $ET^*E \in \mathcal{A}$. Since $\mathcal{R}$ is a two sided ideal, we obtain $(ET^*E)(ETE) \in \mathcal{R}$. Thus ET^*ETE is a positive, quasi-nilpotent operator; the only possibility is 0. This, in turn, yields $ETE = 0$; as E is an arbitrary atom, $\delta(T) = 0$.

For the converse, suppose T is a compact operator in $\mathcal{A}$ for which $\delta(T) = 0$. To prove $T \in \mathcal{I}$, it suffices to show that $T \in \mathcal{A}_\varphi$, for all $\varphi \in X$. First suppose that φ is atomic, with associated atom E. Since $\delta(T) = 0$, $ETE = 0$ and hence, $\mathcal{N}_\varphi(T) = \|ETE\| = 0$. Thus $T \in \mathcal{A}_\varphi$. It remains to consider the case in which φ is non-atomic: we shall show that any compact operator in $\mathcal{A}$ lies in $\mathcal{A}_\varphi$. Suppose the contrary; let T be a compact operator in $\mathcal{A}$ for which $\mathcal{N}_\varphi(T) = \lambda > 0$. Then $\|ETE\| \geqslant \lambda$, for all $E \in \mathcal{F}_\varphi$. Fix an interval $E_1 \in \mathcal{F}_\varphi$. Since the directed set $\mathcal{F}_\varphi$ converges strongly to 0, we have $\lim_{E \in \mathcal{F}_\varphi} (E_1 - E)T(E_1 - E) = E_1TE_1$, with convergence in the strong operator topology. But the mapping $S \to \|S\|$ is lower semi-continuous in the strong operator topology, hence there is an interval $E_2 < E_1$ such that

$$\|(E_1 - E_2)T(E_1 - E_2)\| > \lambda/2.$$

Continuing in this fashion we obtain, by induction, a decreasing sequence $E_1 > E_2 > E_3 > \dots$ of test intervals for φ such that $\|(E_n - E_{n+1})T(E_n - E_{n+1})\| > \lambda/2$, for all n. The projections $E_n - E_{n+1}$ are all mutually orthogonal, yielding a contradiction since T is compact. This completes the proof.

An inspection of the proof yields an immediate corollary: the radical of $\mathcal{Alg}\,\mathcal{L}$ and the ideal $\mathcal{I}$ contain precisely the same compact operators.

REMARK. We can argue exactly as in [2] to show that if X is a trace class operator in $\mathcal{A}$ with $\delta(X) = 0$ then the ultraweakly continuous linear functional ρ given by $\rho(A) = tr(XA)$ vanishes on $\mathcal{A}$. [If $A \in \mathcal{A}$ then XA is trace class and $\delta(XA) = \delta(X)\delta(A) = 0$. Hence $XA \in \mathcal{I} \subseteq \mathcal{R}$, so XA is quasinilpotent and $tr(XA)=0$.] The converse of this result, which Fall, Arveson, and Muhly prove for nest algebras, is not true for the general $\mathcal{Alg}\,\mathcal{L}$. The corollary that every nest algebra is the ultraweakly closed linear span of its rank one operators is also not true in general. A continuous maximal abelian von Neumann algebra, e.g., the L^∞— multiplication algebra serves as an easy counterexample for both results.

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