J. OPERATOR THEORY
2(1979), 131–140

A VARIATION OF LOMONOSOV'S THEOREM

H. W. KIM, R. MOORE and C. M. PEARCY

Let $\mathcal{X}$ be an infinite-dimensional complex Banach space, and let $\mathcal{L}(\mathcal{X})$ denote the algebra of all (bounded, linear) operators on $\mathcal{X}$. One version of the pioneering theorem of Lomonosov [7] says that if T is a nonscalar operator in $\mathcal{L}(\mathcal{X})$ and T commutes with some nonzero compact operator, then T has a nontrivial hyperinvariant subspace. (Recall that a (closed) subspace $\mathcal{M}$ of $\mathcal{X}$ is a nontrivial hyperinvariant subspace for an operator T in $\mathcal{L}(\mathcal{X})$ if $(0)\neq\mathcal{M}\neq\mathcal{X}$ and $T'\mathcal{M}\subset\mathcal{M}$ for every operator T' in $\mathcal{L}(\mathcal{X})$ that commutes with T. For expository accounts of ramifications of the Lomonosov technique, the reader might consult [8, Chapter 7] or [9].) Additional results in this direction were obtained in [3], [5] and [6], and in 1976 the first and third authors, together with A. L. Shields, proved the following theorem, which was stated without proof as Theorem 7.17 of [8].

THEOREM A. *If T is a nonscalar operator in $\mathcal{L}(\mathcal{X})$ and there exists a nonzero compact operator K such that either*

(a) *$KT=\lambda TK$ for some scalar λ,*

(b) *$KT=Tp(K)$ for some polynomial p satisfying $p(0)=p'(0)=0$, or*

(c) *T is quasinilpotent and $KT=T^nK$ for some positive integer n,*

then T has a nontrivial hyperinvariant subspace.

Somewhat later the authors improved upon Theorem A and also discovered that parts of Theorem A had been proved independently by others. In [1], Scott Brown showed (among other things) that a nonscalar operator T in $\mathcal{L}(\mathcal{X})$ has a nontrivial hyperinvariant subspace under hypothesis (a), and in [4] it was shown that such a T has a nontrivial hyperinvariant subspace under a weaker hypothesis than (c). It is the main purpose of this note to prove that such an operator T has a nontrivial hyperinvariant subspace under a hypothesis weaker than (b), thereby making available a complete proof of a stronger theorem than Theorem A.

In what follows the spectrum of an operator T in $\mathcal{L}(\mathcal{X})$ will be denoted by $\sigma(T)$ and the point spectrum of T (i.e., the set of eigenvalues of T) by $\sigma_p(T)$. Furthermore, the commutant of T will be denoted by $\mathcal{A}'_T$; clearly T has no nontrivial hyperinvariant subspace if and only if $\mathcal{A}'_T$ is transitive, i.e., has no nontrivial invariant subspace. Our principal result is the following.

THEOREM 1. *Suppose K and T are operators in $\mathscr{L}(\mathscr{X})$ such that T is nonscalar and K is compact and nonzero. Suppose also that f is a function defined and analytic on an open neighborhood $\mathscr{U}$ of $\sigma(K)$ such that either $KT = Tf(K)$ or $TK = f(K)T$.*

Then T has a nontrivial hyperinvariant subspace provided any one of the following conditions is valid:

(*i*) $|f'(0)| < 1$,

(*ii*) $|f'(0)| > 1$ *and K is quasinilpotent, or*

(*iii*) $|f'(0)| > 1$ *and K has trival kernel.*

We remark that since the action in this paper is taking place on a (possibly nonreflexive) Banach space, and since, to our knowledge, the existence of a nontrivial hyperinvariant subspace for the adjoint T^* of an operator T does not imply the existence of one for T, there is a real lack of symmetry in the conditions $KT = Tf(K)$ and $TK = f(K)T$. We shall, however, treat only the first of these conditions, and leave it to the interested reader to supply the similar arguments in case the second condition is satisfied. (A similar dichotomy occurs in Lemma 1 below.)

The proof of Theorem 1 is based on a sequence of three lemmas. The first of these is related to Theorem 4 of [9].

LEMMA 1. *Let T be a nonscalar operator and $\{K_n\}_{n=0}^{\infty}$ be a sequence of operators in $\mathscr{L}(\mathscr{X})$ such that K_0 is nonzero and compact, $\|K_n\| \to 0$, and either $K_nT = TK_{n+1}$ or $K_{n+1}T = TK_n$ for every nonnegative integer n. Then T has a nontrivial hyperinvariant subspace.*

Proof. We treat the case in which $K_nT = TK_{n+1}$ for all n; the argument for the other case is similar. It follows by induction that $K_0T^n = T^nK_n$ for all positive integers n. If $\mathfrak{A}'_T$ were transitive, then by [9, Theorem 2], there would exist an operator T_1 in $\mathfrak{A}'_T$. such that $1 \in \sigma_p(T_1K_0)$, and hence, since $K_0^*T_1^*$ is also compact, there would be a unit vector x in $\mathscr{X}^*$ such that $K_0^*T_1^*x = x$. But then

$$\begin{aligned} K_n^*T_1^*T^{*n}x &= K_n^*T^{*n}T_1^*x = \\ &= T^{*n}K_0^*T_1^*x = \\ &= T^{*n}x, \qquad\qquad n = 1, 2, \ldots\ . \end{aligned}$$

If $T^{*n}x = 0$ for some n, then T^* has nontrivial kernel, which implies that the closure of the range of T is a nontrivial hyperinvariant subspace for T, contrary to the hypothesis that $\mathfrak{A}'_T$ is transitive. Thus $T^{*n}x \neq 0$ and $1 \in \sigma_p(K_n^*T_1^*)$ for all n, from which follows the contradiction

$$1 \leqslant \|K_n^*T_1^*\| \leqslant \|K_n^*\|\ \|T_1^*\| = \|K_n\|\ \|T_1\| \to 0.$$

Suppose now that $X \in \mathcal{L}(\mathcal{X})$ and f is a complex-valued function defined and analytic on an open neighborhood $\mathcal{U}$ of $\sigma(X)$. Then the operator $f(X)$ is defined by the Riesz functional calculus (cf. [2], p. 389), and one knows from the spectral mapping theorem (cf. [2], Proposition 17.27) that

$$\sigma(f(X)) = f(\sigma(X)).$$

Thus if f satisfies

$$f(\sigma(X)) = \sigma(X),$$

then f maps $\mathcal{U}$ onto an open neighborhood of $\sigma(X)$, and the operator

$$f^{(n)}(X) = f(f^{(n-1)}(X))$$

is defined by the Riesz functional calculus and induction for every positive integer n; clearly

$$\sigma(f^{(n)}(X)) = \sigma(X).$$

In this situation we define $f^{(0)}(X) = 1$.

LEMMA 2. *Suppose K and T are operators in $\mathcal{L}(\mathcal{X})$ with T nonscalar and K nonzero and compact, and suppose f is a function defined and analytic on an open neighborhood $\mathcal{U}$ of $\sigma(K)$ such that $KT = Tf(K)$. Suppose also that $f(0) = 0$ and that $\sigma_p(T) = \sigma_p(T^*) = \emptyset$. Then f maps $\sigma(K)$ onto itself in such a way that for every positive integer k, the function $f^{(k)}$ has no fixed point in $\sigma(K)$ except 0. Furthermore,*

$$f^{(n)}(K)\,T = Tf^{(n+1)}(K), \qquad n = 1,2, \dots , \tag{1}$$

and

$$KT^n = T^n f^{(n)}(K), \qquad n = 1, 2, \dots . \tag{2}$$

Proof. We show first that f maps $\sigma(K)$ onto itself. If K is quasinilpotent, this is trivial, so we suppose that $\lambda \neq 0$ belongs to $\sigma(K)$. Then, of course, $\lambda \in \sigma(K^*)$, and since K^* is compact on $\mathcal{X}^*$, there exists a nonzero vector y in $\mathcal{X}^*$ such that

$$K^*y = \lambda y.$$

Thus

$$\lambda T^*y = T^*K^*y = f(K)^*T^*y,$$

and since $T^*y \neq 0$ by hypothesis,

$$\lambda \in \sigma(f(K)^*) = \sigma(f(K)).$$

This shows that

$$\sigma(K) \subset \sigma(f(K)) = f(\sigma(K)).$$

To establish the reverse inclusion, observe that $f(\zeta)$ can be factored as $f(\zeta) = \zeta g(\zeta)$ for some analytic function g on $\mathscr{U}$, and hence

$$f(K) = Kg(K).$$

In particular, $f(K)$ is compact. Thus if $0 \neq \mu \in \sigma(f(K))$, then there exists a nonzero vector v in $\mathscr{X}$ such that

$$f(K)\, v = \mu v$$

and therefore such that

$$KTv = Tf(K)v = \mu Tv.$$

Since $Tv \neq 0$ by hypothesis, $\mu \in \sigma(K)$ and hence

$$f(\sigma(K)) = \sigma(K).$$

It follows from what was said above that $f^{(n)}(K)$ is defined by the Riesz functional calculus for every positive integer n and satisfies $\sigma(f^{(n)}(K)) = \sigma(K)$. We next show that (1) and (2) are satisfied. To this end, let $\mathscr{X} \oplus \mathscr{X}$ denote the Banach space that is the l_1 direct sum of $\mathscr{X}$ with itself, and note that the operator $K \oplus f(K)$ belongs to $\mathscr{L}(\mathscr{X} \oplus \mathscr{X})$ and satisfies $\sigma(K \oplus f(K)) = \sigma(K)$. Thus, for every positive integer n, the operator

$$f^{(n)}(K \oplus f(K)) = f^{(n)}(K) \oplus f^{(n+1)}(K)$$

is well-defined and commutes with the operator

$$\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$$

on $\mathscr{X} \oplus \mathscr{X}$ since $K \oplus f(K)$ does. Equation (1) results, and equation (2) follows easily from (1), via an induction argument.

To complete the proof, let k be a positive integer and suppose there exists some $\eta \neq 0$ in $\sigma(K)$ such that

$$f^{(k)}(\eta) = \eta.$$

Let w be a nonzero vector satisfying $Kw = \eta w$, and observe that it follows easily from the Riesz functional calculus that

$$f^{(k)}(K)w = f^{(k)}(\eta)w = \eta w.$$

Thus, using (2), we have

$$KT^k w = T^k f^{(k)}(K)w = \eta T^k w.$$

Thus the finite-dimensional space kernel$(K - \eta I)$ is invariant under T^k. This forces T^k (and thus T) to have an eigenvalue, contradicting the hypothesis that the point spectrum of T is empty. Thus $f^{(k)}$ has no fixed points in $\sigma(K)$ other than zero, and the lemma is proved.

LEMMA 3. *Suppose K, T, f, and $\mathcal{U}$ are as in Lemma 2, including the hypotheses that $f(0) = 0$ and $\sigma_p(T) = \sigma_p(T^*) = \emptyset$. Then the compact operator K is quasinilpotent provided either $|f'(0)| < 1$ or $|f'(0)| > 1$ and K has a trivial kernel.*

Proof. We argue first the case in which $|f'(0)| < 1$. Suppose that there exists a nonzero λ in $\sigma(K)$. Since f maps $\sigma(K)$ onto itself by Lemma 2, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ of nonzero numbers in $\sigma(K)$ such that $f^{(n)}(\lambda_n)=\lambda$ for every positive integer n. Furthermore, the numbers λ_n are pairwise distinct, for if $\lambda_m = \lambda_{m+k}$ for some positive integers k and m, then

$$f^{(k)}(\lambda) = f^{(k)}(f^{(m)}(\lambda_m)) = f^{(m+k)}(\lambda_{m+k}) = \lambda,$$

which contradicts Lemma 2. Thus, since K is compact, $\lambda_n \to 0$. On the other hand, since $f(0) = 0$ and $|f'(0)| < 1$, there exists a positive number δ such that for all ζ satisfying $0 < |\zeta| < \delta$, $|f(\zeta)| < |\zeta|$. Let n_0 be any positive integer such that $|\lambda_{n_0}| < \min\{\delta, |\lambda|\}$. Then $|f(\lambda_{n_0})| < |\lambda_{n_0}| < \min\{\delta, |\lambda|\}$, and iterating n_0 times, we obtain the contradiction that

$$|\lambda| = |f^{(n_0)}(\lambda_{n_0})| < \ldots < |f(\lambda_{n_0})| < |\lambda_{n_0}| < |\lambda|.$$

This proves the lemma in case $|f'(0)| < 1$, so we now assume that $|f'(0)| > 1$ and kernel $K = (0)$. Once again, suppose $\lambda \neq 0$ belongs to $\sigma(K)$ and let x be a nonzero vector in $\mathcal{X}$ satisfying $Kx = \lambda x$. Then, by Lemma 2, the sequence $\{f^{(n)}(\lambda)\}_{n=1}^{\infty}$ lies in $\sigma(K)$, and we show that the numbers in this sequence are pairwise distinct. Suppose, to the contrary, that

$$f^{(n)}(\lambda) = f^{(n+k)}(\lambda)$$

for some positive integers k and n. Then

$$f^{(n)}(\lambda) = f^{(n+k)}(\lambda) = f^{(k)}(f^{(n)}(\lambda)),$$

so $f^{(n)}(\lambda)$ is a fixed point for $f^{(k)}$. By Lemma 2, $f^{(n)}(\lambda) = 0$, and hence

$$KT^n x = T^n f^{(n)}(K)x = f^{(n)}(\lambda)T^n x = 0.$$

Since K has trivial kernel, $T^n x = 0$, contradicting the hypothesis that $\sigma_p(T) = \emptyset$. Thus the numbers $f^{(n)}(\lambda)$ are pairwise distinct, and since K is compact, $f^{(n)}(\lambda) \to 0$. On the other hand, since $f(0) = 0$ and $|f'(0)| > 1$, there exists a positive number δ such that for $0 < |\zeta| < \delta$, $|f(\zeta)| > |\zeta|$. This is clearly incompatible with $f^{(n)}(\lambda) \to 0$, so the lemma is proved.

Proof of Theorem 1. If $f(0) \neq 0$, the equation

$$KT = Tf(K)$$

implies that T is a compact operator, so we may assume that $f(0) = 0$. Furthermore, if either $\sigma_p(T)$ or $\sigma_p(T^*)$ is nonempty, then T has a nontrivial hyperinvariant subspace for elementary reasons, so we may suppose that

$$\sigma_p(T) = \sigma_p(T^*) = \emptyset.$$

Thus whether we are treating case *i*), *ii*) or *iii*), we may assume, via Lemma 3, that K is quasinilpotent. We shall first treat the case $|f'(0)| < 1$. Then, as above, there exist positive numbers α and δ such that $\alpha < 1$ and such that for $0 < |\zeta| < \delta$, $|f(\zeta)| < \alpha|\zeta|$. We define $K_0 = K$ and, in general, by induction,

$$K_n = f^{(n)}(K_0).$$

Then (1) becomes

$$K_n T = TK_{n+1},$$

and by virtue of Lemma 1, it suffices to show that $\|K_n\| \to 0$.
Since

$$|f(\zeta)| \leqslant \alpha|\zeta|$$

for all ζ satisfying $|\zeta| < \delta$, we have

$$|f^{(n)}(\zeta)| \leqslant \alpha^n|\zeta|$$

for all such ζ and for all $n = 1, 2, \ldots$.

Let $0 < \varepsilon < \delta$. Then by the Riesz functional calculus, for all positive integers n,

$$K_n = f^{(n)}(K) = \frac{1}{2\pi i} \int_{|\zeta|=\varepsilon} f^{(n)}(\zeta)(\zeta - K)^{-1}\, d\zeta,$$

and we have

$$\|K_n\| \leqslant \varepsilon \sup_{|\zeta|=\varepsilon} |f^{(n)}(\zeta)| \sup_{|\zeta|=\varepsilon} \|(\zeta - K)\|^{-1} \leqslant$$

$$\leqslant \varepsilon\, \alpha^n M$$

for some positive number M. Hence $\|K_n\| \to 0$, and the theorem is proved in the case $|f'(0)| < 1$.

We turn now to the case in which $|f'(0)| > 1$, recalling that we may assume that K is quasinilpotent. Since f is a schlicht map of sufficiently small neighborhoods of the origin, there exists a function h defined and analytic on a small open neighborhood $\mathscr{U}_1$ of the origin such that for $\zeta \in \mathscr{U}_1$, $h(f(\zeta)) = \zeta$. Moreover, since

$$|h'(0)| = |1/f'(0)| < 1,$$

the open set $\mathscr{U}_1$ may be chosen so that

$$h(\mathscr{U}_1) \subset \mathscr{U}_1.$$

Since

$$h(f(K)) = K$$

(cf. [2, Proposition 17.28]) and

$$h(K)T = Th(f(K))$$

by an argument like that used to establish (1), we obtain

$$TK = h(K)T.$$

Another argument like that used to establish (1) yields

$$Th^{(n)}(K) = h^{(n+1)}(K)\,T$$

for all positive integers n, and since one may show that

$$\|h^{(n)}(K)\| \to 0$$

by the same argument used in the first case, the result now follows from Lemma 1.

We conclude this note with some remarks concerning this circle of ideas that seem to be of interest.

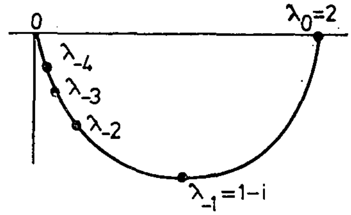
REMARK 1. It would be nice to be able to prove Theorem 1 without assuming that one of *i*), *ii*) or *iii*) is valid. This would be quite a strong theorem, however. For example, if one could simply prove Theorem 1 under either of the hypotheses $|f'(0)| > 1$ or $|f'(0)| = 1$, this would have as an immediate corollary that every weighted bilateral shift on a separable, infinite-dimensional, complex Hilbert space $\mathscr{H}$ has a nontrivial hyperinvariant subspace. We sketch the argument in the case $|f'(0)| > 1$. Suppose $T \in \mathscr{L}(\mathscr{H})$, $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $\mathscr{H}$, and $\{\alpha_n\}_{n=-\infty}^{\infty}$ is a bounded sequence of complex numbers such that

$$Te_n = \alpha_n e_{n+1}, \qquad n = 0, \pm 1, \ldots .$$

Let $\{\lambda_n\}_{n=-\infty}^{\infty}$ be the sequence of numbers defined by $\lambda_n = 0$ for $n > 0$ and

$$\lambda_{-n} = 1 - \exp(i\pi/2^n), \qquad n = 0, 1, 2, \ldots ,$$

which pictorially looks like this:



If we take $f(\zeta)$ to be the polynomial

$$f(\zeta) = 2\zeta - \zeta^2,$$

then a routine computation shows that

$$f(\lambda_n) = \lambda_{n+1}, \qquad -\infty < n < +\infty. \tag{3}$$

(Although it may not be obvious from the form taken by the above proofs, the existence or nonexistence of such two-way infinite sequences $\{\lambda_n\}$ satisfying (3) and the conditions $\lambda_n \to 0$ as $n \to \pm\infty$ is in some sense what this paper is all about.) Let K be the normal compact operator in $\mathscr{L}(\mathscr{H})$ satisfying

$$Ke_n = \lambda_n e_n$$

for all integers n.
Then

$$KTe_n = K(\alpha_n e_{n+1}) = \alpha_n \lambda_{n+1} e_{n+1}$$

and

$$Tf(K)e_n = T(f(\lambda_n)e_n) = T(\lambda_{n+1}e_n) = \alpha_n \lambda_{n+1} e_{n+1}$$

for all integers n, so

$$KT = Tf(K).$$

Since $f'(0) = 2$, if Theorem 1 were true under the hypothesis that $|f'(0)| > 1$, then T would have a nontrivial hyperinvariant subspace.

REMARK 2. Observe that the example given in Remark 1 shows that one can not hope to prove Lemma 3 under the sole assumption that $|f'(0)| > 1$, since the compact operator K constructed there satisfies

$$KT = Tf(K)$$

where T is the (unweighted) bilateral shift (and thus satisfies $\sigma_p(T) = \sigma_p(T^*) = \emptyset$), and yet K is far from being quasinilpotent.

REMARK 3. Observe that it follows immediately from Theorem 1 that if T is a nonscalar operator in $\mathscr{L}(\mathscr{X})$, K is a nonzero compact operator, and $KT = \lambda TK$ where $|\lambda| < 1$, then T has a nontrivial hyperinvariant subspace. Suppose now that $|\lambda| \geqslant 1$. Equation (2) becomes

$$KT^n = \lambda^n T^n K,$$

and if we assume that the commutant $\mathcal{A}'_T$ is transitive, then by [9, Theorem 2] it follows that there exists a unit vector x and an operator T' in $\mathcal{A}'_T$ such that $KT'x = x$. Thus

$$(KT' - \lambda^n)T^n x = KT'T^n x - \lambda^n T^n x =$$

$$= KT^n T'x - \lambda^n T^n x = \lambda^n T^n KT'x - \lambda^n T^n x = 0.$$

Since $\mathcal{A}'_T$ is transitive, $T^n x$ cannot vanish, and hence $\lambda^n \in \sigma(KT')$ for all positive integers n. For $|\lambda| > 1$ this is impossible, and if $|\lambda| = 1$, then λ must be a root of unity, since KT' is compact and thus $\sigma(KT')$ can only contain finitely many points on the unit circle. But then $\lambda^k = 1$ for some positive integer k, which implies that

$$KT^k = T^k K,$$

and thus by Lomonosov's theorem that T^k (and hence T) has a nontrivial hyperinvariant subspace. This proves part (a) of Theorem A.

REMARK 4. It is worthwhile to note that there exist operators that satisfy the hypotheses of Theorem 1 and yet do not commute with any nonzero compact operator, so Theorem 1 does apply to operators that are not covered by Lomonosov's original theorem. Perhaps the simplest example of such an operator is a unilateral shift of multiplicity one acting on the Hilbert space $\mathcal{H}$ of Remark 1. In particular, let $\{d_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, and let T be the unilateral shift defined by $Td_n = d_{n+1}$ for all $n \geqslant 0$. If λ is any complex number satisfying $|\lambda| < 1$ and K is the normal compact operator in $\mathcal{L}(\mathcal{H})$ satisfying $Kd_n = \lambda^n d_n$ for all $n \geqslant 0$, then one sees easily that $KT = \lambda TK$, so Theorem 1 applies to T, but as is well-known, T commutes with no nonzero compact operator.

REFERENCES

1. BROWN, S. W., Connections between an operator and a compact operator that yield hyperinvariant subspaces, *J. Operator Theory*, **1** (1979), 117–121.
2. BROWN, A.; PEARCY, C., *Introduction to Operator Theory I. Elements of Functional Analysis*, Springer-Verlag, Berlin–New York, 1978.
3. DAUGHTRY, J., An invariant subspace theorem, *Proc. Amer. Math. Soc.*, **49** (1975), 267–268.
4. FONG, C. K.; NORDGREN, E. A.; RADJABALIPOUR, M.; RADJAVI, H.; ROSENTHAL, P., Extensions of Lomonosov's invariant subspace theorem, to appear.
5. KIM, H. W.; PEARCY, C. M.; SHIELDS, A. L., Rank-one commutators and hyperinvariant subspaces, *Michigan Math. J.*, **22** (1975), 193–194.
6. KIM, H. W.; PEARCY, C. M.; SHIELDS, A. L., Sufficient conditions for rank-one commutators and hyperinvariant subspaces, *Michigan Math. J.*, **23** (1976), 235–243.
7. LOMONOSOV, V., Invariant subspaces for operators commuting with compact operators (Russian), *Funkcional Anal. i Priložen*, **7** (1973), 55–56.

8. PEARCY, C. M., *Some recent developments in operator theory*, C.B.M.S. Regional Conference Series in Mathematics, no. 36, Amer. Math. Soc., Providence, 1978.
9. PEARCY, C. M.; SHIELDS, A. L., A survey of the Lomonosov technique in the theory of invariant subspaces, *Topics in operator theory*, 220—229, A.M.S. Survey, no. 13, Providence, 1974.
10. ROTA, G., On models for linear operators, *Comm. Pure Appl. Math.*, **13** (1960), 469—472.

H. W. KIM
Bucknell University
Lewisburg, PA 17837
U.S.A.

R. MOORE
University of Alabama
Tuscaloosa, Alabama 35401
U.S.A.

C. M. PEARCY
University of Michigan
Ann Arbor, MI 48109
U.S.A.

Received March 15, 1979.