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CHARACTERIZATION OF HILBERT SPACE OPERATORS WITH UNITARY CROSS SECTIONS

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1. INTRODUCTION

The purpose of this note is to characterize the operators on a separable Hilbert space which have local unitary cross sections in the following sense. Let $\mathcal{A}$ denote a C^*-algebra with identity and let $\mathcal{U}(\mathcal{A})$ denote the unitary group in $\mathcal{A}$. For an element X in $\mathcal{A}$, let $\mathcal{U}(X) = \{U^*XU : U \text{ in } \mathcal{U}(\mathcal{A})\}$ denote the unitary orbit of X and let π denote the norm continuous mapping of $\mathcal{U}(\mathcal{A})$ onto $\mathcal{U}(X)$ given by $\pi(U) = U^*XU$. A *local cross section* for π is a pair $(\varphi, \mathcal{B})$ such that $\mathcal{B}$ is a relatively open subset of $\mathcal{U}(X)$ that contains X and $\varphi : \mathcal{B} \to \mathcal{U}(\mathcal{A})$ is a norm continuous function such that $\varphi(X) = 1$ and $\pi\varphi = 1_{\mathcal{B}}$. If π admits a local cross section, we say that X has a *(local unitary) cross section*, and in this case X satisfies the following *sequential unitary lifting property:*

(P) If $\{U_n\} \subset \mathcal{U}(\mathcal{A})$ and $\lim \|U_n{}^*XU_n - X\| = 0$, then there exists a sequence $\{W_n\} \subset \mathcal{U}(\mathcal{A})$ such that $\lim \|W_n - 1\| = 0$ and such that $W_n{}^*XW_n = U_n{}^*XU_n$ for each n.

Let $\mathcal{H}$ denote a complex, separable, infinite dimensional Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. The fact that finite rank projections satisfy (P) (relative to the algebra $\mathcal{A}=\mathcal{L}(\mathcal{H})$) was proved in [5], Lemma 1, and in [3] it was shown that a normal or isometric operator satisfies (P) if and only if its spectrum is finite, in which case a cross section exists. In [4], Section 3 this characterization was extended to hyponormal operators and the analogue for hyponormal elements of the Calkin algebra was obtained. [3] and [4] contain examples of non-normal operators with cross sections, and in [4], Theorem 2.1 it was proved that a compact operator satisfies (P) if and only if it is of finite rank. In the present note we obtain the following characterization of operators with cross sections relative to the algebra $\mathcal{A}=\mathcal{L}(\mathcal{H})$, and our result affirms the conjecture of [4].

THEOREM 1.1. *For an operator T in $\mathcal{L}(\mathcal{H})$, the following are equivalent:*
i) *T satisfies the sequential unitary lifting property;*
ii) *$\mathcal{U}(T)$ is norm closed in $\mathcal{L}(\mathcal{H})$;*

iii) *The C^*-algebra generated by T and 1, $C^*(T)$, is finite dimensional;*

iv) *T is unitarily equivalent to an operator of the form*

$$A \oplus B \oplus B \oplus \ldots \oplus B \oplus \ldots,$$

where A and B are operators on finite dimensional spaces;

v) *T has a local unitary cross section.*

The implication i) $\Rightarrow$ ii) is contained in [4], Proposition 4.3 and the equivalence of ii) and iii) is a result of Dan Voiculescu [7], Proposition 2.4. The equivalence of iii) and iv) is a well known consequence of the reduction theory for C^*-algebras (see [1], Section 3), and the implication v) $\Rightarrow$ i) follows easily from the definitions. To complete the proof of Theorem 1.1 it thus suffices to establish that iii) $\Rightarrow$ v) and this we do in Theorem 2.1. As an application, we show that if an element of a C^*-algebra generates a finite dimensional C^*-subalgebra, then it has a cross section, and we discuss a possible converse of this result for the Calkin algebra. We conclude by showing that if the image of T in the Calkin algebra generates a finite dimensional C^*-subalgebra, then the projection of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra, when restricted to $\mathcal{U}(T)$, is an open mapping.

2. OPERATORS WHICH GENERATE FINITE DIMENSIONAL C^*-ALGEBRAS

THEOREM 2.1. *Let $\mathcal{H}$ be a complex Hilbert space of arbitrary dimension. If T is in $\mathcal{L}(\mathcal{H})$ and $C^*(T)$ is finite dimensional, then T has a cross section $(\varphi, \mathcal{B})$ such that i V is in $\mathcal{U}(\mathcal{H})$ $(\equiv \mathcal{U}(\mathcal{L}(\mathcal{H})))$ and V^*TV is in $\mathcal{B}$, then $\varphi(V^*TV)$ is in $C^*(T, V)$, the C^* algebra generated by T, V, and 1.*

REMARK. Let $T = A \oplus B \oplus B \oplus \ldots \oplus B \oplus \ldots$, where A and B act on finite dimensional spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Analogues of Theorem 2.1 were given in [3] and [4] for certain special cases: for normal operators (that is, when A and B are normal) [3], Theorem 3.2, for finite rank operators [4], Theorem 2.1, and for the case when $\mathcal{H}_1$ is absent and $\dim(\mathcal{H}_2) \leqslant 3$ [4], Example 4.6. The proof of the last case uses essentially single-operator methods; these methods do not naturally generalize to the case $\dim(\mathcal{H}_2) > 3$ due to the difficulty, in this case, of obtaining a *usable* expression for the commutant of B solely in terms of the matrix entries of B. Thus, in the proof of Theorem 2.1, rather than considering T alone, we represent all of $C^*(T)$ relative to a Wedderburn-type decomposition. We further require that this representation be spatial since we must analyze *arbitrary* unitary operators U for which $\|UT - TU\|$ is small, not merely the unitary operators in $C^*(T)$ with this property.

Proof of Theorem 2.1. Since $C^*(T)$ is finite dimensional, there exist Hilbert spaces $\mathcal{K}, \mathcal{K}_1, \ldots, \mathcal{K}_p$ and positive integers $n_1, \ldots, n_p$ satisfying the following properties:

i) $\mathcal{H} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_p$, where $\mathcal{L}_i$ is an n_i-fold copy of $\mathcal{H}_i$, $1 \leqslant i \leqslant p$.

ii) There exists a unitary operator J mapping $\mathcal{H}$ onto $\mathcal{H}$ such that an operator B on $\mathcal{H}$ is in $C^*(T)$ if and only if $A = JBJ^*$ has the following properties:

a) The operator matrix of A, relative to the decomposition $\mathcal{H} = \mathcal{L}_1 \oplus \ldots \ldots \oplus \mathcal{L}_p$, is of the form $A = A_1 \oplus \ldots \oplus A_p$;

b) For each, i, $1 \leqslant i \leqslant p$, the operator matrix of A_i, relative to the decomposition $\mathcal{L}_i = \mathcal{H}_i \oplus \ldots \oplus \mathcal{H}_i$, is of the form $(\alpha_{j,k} 1_{\mathcal{H}_i})_{1 \leqslant j,k \leqslant n_i}$, where the $\alpha_{j,k}$ are arbitrary scalars. (Thus J induces a spatial C^*-isomorphism between $C^*(T)$ and the algebra $\mathcal{M}_{n_1}(\mathcal{H}_1) \oplus \ldots \oplus \mathcal{M}_{n_p}(\mathcal{H}_p)$.) A development of this representation is given in [6], Chapter 1, section 1.1, pages 1—14.

It now suffices to prove that $S = JTJ^*$ has a cross section; we first introduce ome notation. For A in $\mathcal{L}(\mathcal{H})$, let $(A_{ij})_{1 \leqslant i,j \leqslant p}$ denote the operator matrix of A relative to the decomposition $\mathcal{H} = \mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_p$. Let $A_i = A_{ii}$ and let $(A_{ijk})_{1 \leqslant j,k \leqslant n_i}$ denote she operator matrix of A_i with respect to the decomposition $\mathcal{L}_i = \mathcal{H}_i \oplus \ldots \oplus \mathcal{H}_i$.

We next define matrix units for $C^*(S)$ which allow us to give a particularly simle description of the unitary operators on $\mathcal{H}$ which commute or "nearly commute" with S. Fix integers i, r and s with $1 \leqslant i \leqslant p$ and $1 \leqslant r, s \leqslant n_i$. The operator A n $C^*(S)$ such that $A_{jk} = 0$ if $j \neq i$ or $k \neq i$, and such that $A_{ijk} = \delta_{jr}\delta_{ks} 1_{\mathcal{H}_i}$, will be denoted by E_{irs}. Let $f_{irs}(x, y)$ be a (non-commutative) polynomial such that $E_{irs} = f_{irs}(S, S^*)$; choose $\delta > 0$ such that if R is in $\mathcal{U}(S)$ and $\|R - S\| < \delta$, then $\|f_{irs}(R, R^*) - f_{irs}(S, S^*)\| < 1$ for $1 \leqslant r, s \leqslant n_i$, $1 \leqslant i \leqslant p$.

We shall define a cross section for S with domain

$$\mathcal{B} = \{R \in \mathcal{U}(S) \colon \|R - S\| < \delta\}.$$

Let $V \in \mathcal{U}(\mathcal{H})$ be such that $R = V^*SV$ is in $\mathcal{B}$. Then

$$\|VE_{i11}V^* - E_{i11}\| = \|V^*E_{i11}V - E_{i11}\| = \|f_{i11}(R, R^*) - f_{i11}(S, S^*)\| < 1,$$

and a matrix calculation shows that $\|1_{\mathcal{H}_i} - (V_{i11})^* V_{i11}\| < 1$ and $\|1_{\mathcal{H}_i} - V_{i11}(V_{i11})^*\| < 1$, so that V_{i11} is invertible, $1 \leqslant i \leqslant p$. Let $V_{i11} = X_i P_i$ denote the polar decomposition of V_{i11}. Let U_V denote the block diagonal operator whose matrix, relative to the decomposition

$$(*) \qquad \mathcal{H} = (\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_1) \oplus \ldots \oplus (\mathcal{H}_p \oplus \ldots \oplus \mathcal{H}_p),$$

is of the form

$$U_V = (X_1^* \oplus \ldots \oplus X_1^*) \oplus \ldots \oplus (X_p^* \oplus \ldots \oplus X_p^*),$$

where each X_i^* is repeated as a direct summand n_i times. A cross section $(\varphi, \mathcal{B})$ for S is defined by $\varphi(V^*SV) = U_V V$.

To verify that φ is well defined, suppose $V^*SV = (V')^*SV'$ for V' in $\mathcal{U}(\mathcal{H})$. Since $W = V'V^*$ is a unitary operator that commutes with S, W commutes with

S^* and therefore with each E_{irs}. Thus, relative to the decomposition (*),

$$W = (Y_1 \oplus \ldots \oplus Y_1) \oplus \ldots \oplus (Y_p \oplus \ldots \oplus Y_p),$$

where Y_i is a unitary operator on $\mathcal{H}_i$. Since $V' = WV$, the polar decomposition of $(V')_{i11}$ is of the form $(V')_{i11} = Y_i V_{i11} = (Y_i X_i) P_i$.

Now

$$U_{V'} = (X_1^* Y_1^* \oplus \ldots \oplus X_1^* Y_1^*) \oplus \ldots \oplus (X_p^* Y_p^* \oplus \ldots \oplus X_p^* Y_p^*),$$

so clearly $U_{V'} V' = (U_{V'} W)V = U_V V$, and thus φ is well defined. From the above definition $\varphi(S) = 1$, and since U_V commutes with S,

$$\varphi(V^* S V)^* S \varphi(V^* S V) = V^* S V.$$

We next verify that $\varphi(V^*SV)$ is a member of $C^*(S, V)$. For each i, $C^*(S, V)$ contains $E_{i11} V E_{i11}$, which is of the form $(V_{i11} \oplus 0 \oplus \ldots \oplus 0) \oplus 0_{\mathcal{L}_i^\perp}$ relative to the decomposition $\mathcal{H} = (\mathcal{H}_i \oplus \ldots \oplus \mathcal{H}_i) \oplus (\mathcal{L}_i)^\perp$. Thus $C^*(S, V)$ contains $(V_{i11} \oplus 1_{\mathcal{H}_i} \oplus \oplus \ldots \oplus 1_{\mathcal{H}_i}) \oplus 1_{\mathcal{L}_i^\perp}$. Also, since $X_i^* = ((V_{i11})^* V_{i11})^{-1/2} (V_{i11})^*$ and $C^*(S, V)$ is closed under inverses, it follows that $C^*(S, V)$ contains $(X_i^* \oplus 1_{\mathcal{H}_i} \oplus \ldots \oplus 1_{\mathcal{H}_i}) \oplus 1_{\mathcal{L}_i^\perp}$, and thus also the operator $F_i = (X_i^* \oplus 0 \ldots \oplus 0) \oplus 0_{\mathcal{L}_i^\perp}$. Now $U_V = \sum_{i=1}^{p} \sum_{s=1}^{n_i} E_{is1} F_i E_{i1s}$, and it follows that $\varphi(V^*SV) = U_V V$ is in $C^*(S, V)$.

To complete the proof if suffices to prove the continuity of φ, and to this end we first show that T satisfies property (P). Suppose $\{V_n\} \subset \mathcal{U}(\mathcal{H})$ and $\lim \|V_n^* S V_n - S\| = 0$. Since

$$\lim \|V_n S - S V_n\| = \lim \|V_n^* S - S V_n^*\| = 0,$$

then

$$\lim \|V_n E_{irs} - E_{irs} V_n\| = \lim \|V_n^* E_{irs} - E_{irs} V_n^*\| = 0$$

for $1 \leqslant r, s \leqslant n_i$, $1 \leqslant i \leqslant p$, and the following limits result:

i) $\lim \|(V_n)_{ij}\| = 0$ for $1 \leqslant i \neq j \leqslant p$;

ii) $\lim \|(V_n)_{ijk}\| = 0$ for $1 \leqslant j \neq k \leqslant n_i$, $1 \leqslant i \leqslant p$;

iii) $\lim \|(V_n)_{iss} - (V_n)_{irr}\| = 0$ for $1 \leqslant r, s \leqslant n_i$, $1 \leqslant i \leqslant p$;

iv) $\lim \|1_{\mathcal{H}_i} - ((V_n)_{iss})^* (V_n)_{iss}\| = \lim \|1_{\mathcal{H}_i} - (V_n)_{iss} ((V_n)_{iss})^*\| = 0$

for $1 \leqslant s \leqslant n_i$, $1 \leqslant i \leqslant p$.

Now for n large, $(V_n)_{i11}$ is invertible, $1 \leqslant i \leqslant p$, and we may define $W_n = \varphi(V_n^* S V_n)$ by the preceding method. Thus $W_n^* S W_n = V_n^* S V_n$, and a straightforward but tedious matrix calculation using i) — iv) readily implies that $\lim \|W_n - 1\| = 0$. (For details of a somewhat similar calculation, see [3], Lemma 2.5.)

Turning to the continuity of φ, suppose $\{V_n\}_{n=0}^{\infty} \subset \mathcal{U}(\mathcal{H})$ and $V_n^* S V_n \to V_0^* S V_0$ in $\mathcal{B}$. Since $V_0 V_n^* S V_n V_0^* \to S$, property (P) implies that there exists a sequence $\{W_n\} \subset \mathcal{U}(\mathcal{H})$ such that $W_n \to 1$ and $W_n^* S W_n = V_0 V_n^* S V_n V_0^*$ for each n.

Thus $V_n^* S V_n = V_0^* W_n^* S W_n V_0$, and since φ is well defined, we may calculate $\varphi(V_n^* S V_n)$ using the sequence $\{W_n V_0\}$; moreover, since $W_n V_0 \to V_0$, it is clear from the definition of φ that

$$\varphi(V_0^* W_n^* S W_n V_0) \to \varphi(V_0^* S V_0).$$

The proof is now complete.

COROLLARY 2.2. *Let $\mathcal{A}$ be a C^*-algebra with identity. If X is an element of $\mathcal{A}$ and $C^*(X)$ is finite dimensional, then X has a local unitary cross section.*

Proof. Let $\rho : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ denote an isometric C^*-isomorphism of $\mathcal{A}$ onto a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ (where $\mathcal{H}$ is a Hilbert space of suitable dimension). If $T = \rho(X)$, then $C^*(T)$ is finite dimensional, and the preceding result implies the existence of a cross section $(\varphi, \mathcal{B})$ for T such that if V is in $\mathcal{U}(\mathcal{H})$ and V^*TV is in $\mathcal{B}$, then $\varphi(V^*TV)$ is in $C^*(T, V)$. If, additionally, $V = \rho(U)$ (U in $\mathcal{U}(\mathcal{A})$), then $\varphi(V^*TV)$ is in $\rho(\mathcal{A})$, and $\rho^{-1}(\varphi(V^*TV))$ is a unitary lifting for U^*XU. This procedure defines a local cross section for X with domain $\rho^{-1}(\rho(\mathcal{U}(X)) \cap \mathcal{B})$.

REMARK. The converse of Corollary 2.2 is false, since if $\mathcal{A}$ is commutative every element in $\mathcal{A}$ has a cross section. The analogue of Theorem 1.1 for arbitrary C^*-algebras is also false; an example showing this is given in [4]. In particular, for the Calkin algebra, the analogues of the implications ii) $\Rightarrow$ i) and ii) $\Rightarrow$ iii) are both false [4]. On the other hand, [4], Corollary 3.7 and our Corollary 2.2 imply that a hyponormal element of the Calkin algebra has a cross section if and only if it generates a finite dimensional C^*-algebra; this result suggests the following question.

QUESTION 2.3. Does an element of the Calkin algebra have a local unitary cross section if and only if it generates a finite dimensional C^*-algebra?

Let $\mathcal{C}$ denote the ideal of all compact operators in $\mathcal{L}(\mathcal{H})$ and let $\gamma : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{C}$ denote the canonical (norm continuous and open) mapping of $\mathcal{L}(\mathcal{H})$ onto the Calkin algebra. For $T \in \mathcal{L}(\mathcal{H})$, let $\tilde{T} = \gamma(T)$ and let $\mathcal{U}(\tilde{T})$ denote the unitary orbit of $\tilde{T}$ relative to the Calkin algebra. A question of [2], page 121 asks whether the restriction $\gamma|\mathcal{U}(T) : \mathcal{U}(T) \to \mathcal{U}(\tilde{T})$ is an open mapping if $\tilde{T}$ is normal. In [4], Proposition 3.8 this question was answered affirmatively for the case when $\tilde{T}$ is normal and has finite spectrum. This hypothesis on $\tilde{T}$ is used only to insure that $\tilde{T}$ has a cross section; thus, using Corollary 2.2 and the same method of proof as in [4], we obtain the following extension of this result.

COROLLARY 2.4. *If T is in $\mathcal{L}(\mathcal{H})$ and $C^*(\tilde{T})$ is finite dimensional, then $\gamma|\mathcal{U}(T)$: $\mathcal{U}(T) \to \mathcal{U}(\tilde{T})$ is an open mapping; if $\{U_n\} \subset \mathcal{L}(\mathcal{H})$, $\tilde{U}_n$ unitary for each n, and $\gamma(U_n^* T U_n) \to \gamma(T)$, then there exists $\{V_n\} \subset \mathcal{U}(\mathcal{H})$ such that $V_n \to 1$ and*

$$\gamma(V_n^* T V_n) = \gamma(U_n^* T U_n).$$

QUESTION 2.5. For which operators T is $\gamma|\mathfrak{U}(T)$ an open mapping?

We plan to study the preceding questions in future research; in a forthcoming paper, we study analogues of the preceding results for similarity orbits and "similarity cross sections".

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REFERENCES

1. APOSTOL, C., Inner derivations with closed range, *Rev. Roumaine Math. Pures et Appl.*, **21** (1976), 249–265.
2. BROWN, L. G.; DOUGLAS, R. G.; FILLMORE, P. A., *Unitary equivalence modulo the compact operators and extensions of C*-algebras*, Lecture Notes in Mathematics, volume 345, Springer-Verlag, 1973.
3. FIALKOW, L. A., A note on limits of unitarily equivalent operators, *Trans. Amer. Math. Soc.*, **232** (1977), 205–220.
4. FIALKOW, L. A., A note on unitary cross sections for operators, *Canadian J. Math.*, **30** (1978), 1215–1227.
5. HALMOS, P. R., Quasitriangular operators, *Acta Sci. Math. (Szeged)*, **29** (1968), 283–293.
6. SCHWARTZ, J. T., *W*-algebras*, Gordon-Breach, New York, 1967.
7. VOICULESCU, D., A non-commutative Weyl-von Neumann theorem, *Rev. Roumaine Math. Pures et Appl.*, **21** (1976), 97–113.

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