J. OPERATOR THEORY
2(1979), 159–167

PATHOLOGY IN THE CALKIN ALGEBRA

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The Calkin algebra $\mathscr{Q}(\mathscr{H})$ is the quotient of $\mathscr{B}(\mathscr{H})$, the bounded linear operator acting on a separable Hilbert space $\mathscr{H}$, by the ideal $\mathscr{K}(\mathscr{H})$ of compact operators on $\mathscr{H}$. Since its appearance in [7], it has been known that the Calkin algebra has many pathological properties. For example, it does not contain an infinite increasing sequence with a least upper bound, but it does contain an uncountable set of mutually orthogonal projections.

However, while the known facts indicate that the structure of the Calkin algebra is complicated, few concrete results have been obtained. For example, a natural problem is to try to classify the masas (the maximal abelian selfadjoint subalgebras) in the Calkin algebra. The main result along these lines is due to Johnson and Parrott [11] who showed that if $\mathscr{A}$ is a masa in $\mathscr{B}(\mathscr{H})$, then $\nu(\mathscr{A})$ is a masa in the Calkin algebra. (Throughout this paper ν shall denote the quotient map of $\mathscr{B}(\mathscr{H})$ onto $\mathscr{Q}(\mathscr{H})$. Apparently, the only other information we have in this regard is that there is a masa in the Calkin algebra that is not generated by its projections [6], p. 126 and therefore since such a masa cannot lift to a masa in $\mathscr{B}(\mathscr{H})$, the Johnson-Parrott result does not give a complete classification. (A masa in the Calkin algebra with special properties is constructed in [5], but it is not known if it lifts to a masa in $\mathscr{B}(\mathscr{H})$.)

The results in this paper show that the Calkin algebra contains other masas with special properties and provide more evidence of the pathological nature of the Calkin algebra. In particular they seem to indicate that the problem of classifying masas is difficult. In section 1 it is shown that, assuming the continuum hypothesis, there is a masa in the Calkin algebra that is generated by its projections but does not lift to a masa in $\mathscr{B}(\mathscr{H})$. (This answers a question raised in [5].) Section 2 contains the surprising fact that if f is *any* state (not necessarily pure) on the Calkin algebra, then, assuming the continuum hypothesis, there is a masa $\mathscr{A}$ such that the restriction of f to $\mathscr{A}$ is a homomorphism.

Throughout this paper, ω shall denote the cardinality of the natural numbers $\mathbf{N}$, ω_1 shall denote the first uncountable ordinal, and c shall denote the cardinality of the continuum. We shall sometimes consider sets $\mathscr{S}$ such that all but a countable number of elements of $\mathscr{S}$ have a property P. In such a case we shall say that *almost all* of the elements of $\mathscr{S}$ have property P.

1. ALMOST CENTRAL SUBSETS

Let us say that a subset $\mathscr{S}$ is *almost central* in the Calkin algebra if $\mathscr{S}$ is uncountable, the elements of S commute with each other, and for each t in the Calkin algebra, almost all of the elements of $\mathscr{S}$ commute with t. In this section it will be shown that there is a masa in the Calkin algebra that contains an almost central subset of projections and is generated by its projections. Let us begin by showing that the presence of an almost central subset implies that such a masa cannot lift to a masa in $\mathscr{B}(\mathscr{H})$.

LEMMA 1. *If $R_1, R_2, \ldots$ is a sequence of mutually orthogonal nonzero projections in $\mathscr{B}(\mathscr{H})$ and $\mathscr{R}$ denotes the von Neumann algebra that they generate, then there is an element t in the Calkin algebra such that t does not commute with any nontrivial projection in $\nu(\mathscr{R})$.*

Proof. Choose projections Q_n of rank one such that $Q_n \leqslant R_n$, $n = 1, 2, \ldots$ and select an isometry T on $\mathscr{H}$ such that

$$TQ_n = Q_{n+1}T, \ n = 1, 2, \ldots \ .$$

Suppose σ is a subset of $\mathbf{N}$ such that if

$$R_\sigma = \sum_{n \in \sigma} R_n ,$$

then $TR_\sigma - R_\sigma T \in \mathscr{K}(\mathscr{H})$. If $n \in \sigma$ and $n + 1 \notin \sigma$ then

$$Q_{n+1}(TR_\sigma - R_\sigma T)Q_n = Q_{n+1}TQ_n = TQ_n .$$

As $\|TQ_n\| \geqslant \|T^*TQ_n\| = 1$ and $TR_\sigma - R_\sigma T$ is compact,

$$\{n \text{ in } \mathbf{N} : n \in \sigma \text{ and } n + 1 \notin \sigma\}$$

is finite. It follows that $\nu(R_\sigma)$ is either 0 or 1. Thus, the element $t = \nu(T)$ has the desired property.

LEMMA 2. *If $\mathscr{P}$ is a set of commuting projections in the Calkin algebra such that for each t in the Calkin algebra there is a nontrivial projection p in $\mathscr{P}$ satisfying $tp = pt$, then $\mathscr{P}$ is uncountable. Hence, if, in addition, for each t almost all of the elements of $\mathscr{P}$ commute with t, then $\mathscr{P}$ is an almost central subset.*

Proof. If $\mathscr{P}$ were countable, then there would be mutually orthogonal projections $R_1, R_2, \ldots$ in $\mathscr{B}(\mathscr{H})$ such that $\mathscr{P} \subseteq \nu(\mathscr{R})$, where $\mathscr{R}$ denotes the von Neumann algebra generated by $R_1, R_2, \ldots$ ([6],(5.3); [5], proof of Theorem 3). By Lemma 1, there is a t in the Calkin algebra such that t does not commute with any nontrivial projection in $\nu(\mathscr{R})$, contradicting our hypothesis. Hence $\mathscr{P}$ is uncountable.

The problem of classifying masas in $\mathscr{B}(\mathscr{H})$ has been completely settled. There are two canonical masas in $\mathscr{B}(\mathscr{H})$ that are defined as follows. First, fix a sequence $\{P_n\}$ of mutually orthogonal projections of rank one on $\mathscr{H}$ such that $\sum P_n = 1$. The von Neumann algebra $\mathscr{D}$ generated by $\{P_n\}$ is a masa in $\mathscr{B}(\mathscr{H})$. A masa that is unitarily equivalent to $\mathscr{D}$ is called *atomic*. To define the second kind, regard $\mathscr{H}$ as $L^2(\Delta)$, the Lebesgue measurable, square-integrable functions on the unit circle Δ and let $\mathscr{M}$ denote the multiplication operators on $L^2(\Delta)$ defined by the essentially bounded functions on the unit circle. Thus, $\mathscr{M}=\{M_\varphi : \varphi \in L^\infty(\Delta)\}$, where $M_\varphi f = \varphi f$ for f in $L^2(\Delta)$. A masa that is unitarily equivalent to $\mathscr{M}$ is called *nonatomic*. If $\mathscr{A}$ is a masa in $\mathscr{B}(\mathscr{H})$, then A is either atomic, nonatomic, the direct sum of an atomic and a nonatomic masa, or else, the direct sum of a finite dimensional masa (acting on a finite dimensional Hilbert space) and a nonatomic masa. Note that a finite dimensional masa is atomic in the sense that it is generated by its minimal projections. (For further details and references see [4], Introduction.)

A masa $\mathscr{A}$ in the Calkin algebra is said to *lift* to a masa in $\mathscr{B}(\mathscr{H})$ in case there is a masa $\mathscr{B}$ in $\mathscr{B}(\mathscr{H})$ such that $\nu(\mathscr{B}) = \mathscr{A}$.

PROPOSITION. 3. *If a masa in the Calkin algebra contains an almost central subset of projections, then it does not lift to a masa in $\mathscr{B}(\mathscr{H})$.*

Proof. Suppose $\mathscr{P}$ is an almost central subset of projections contained in a masa $\mathscr{A}$ in the Calkin algebra. If there were a masa $\mathscr{B}$ in $\mathscr{B}(\mathscr{H})$ such that $\nu(\mathscr{B}) = \mathscr{A}$, then either $\mathscr{B}$ would be atomic or nonatomic or else there would be a projection P in $\mathscr{B}$ such that $P\mathscr{B}$ is an atomic masa in $\mathscr{B}(P\mathscr{H})$ and $P^\perp \mathscr{B}$ is a nonatomic masa in $\mathscr{B}(P^\perp \mathscr{H})$. Furthermore, either $\nu(P)\mathscr{P}$ or $\nu(P^\perp)\mathscr{P}$ would be uncountable so that either $\nu(P)\mathscr{P}$ would be an almost central subset for $\mathscr{Q}(P\mathscr{H})$ or else $\nu(P^\perp)\mathscr{P}$ would be an almost central subset for $\mathscr{Q}(P^\perp \mathscr{H})$. It follows from these observations that to prove the theorem, it suffices to show that neither $\nu(\mathscr{D})$ nor $\nu(\mathscr{M})$ contain an almost central subset of projections.

Lemma 1 implies that $\nu(\mathscr{D})$ does not contain an almost central subset of projections.

Now consider $\mathscr{M}$ acting on $L^2(\Delta)$. Recall that the functions $\{z^n : -\infty < n < \infty\}$ form an orthonormal basis for $L^2(\Delta)$. If P denotes the projection onto the closed linear span of $\{z^n : n \geqslant 0\}$ then $\nu(P)$ does not commute with any nontrivial projection in $\nu(\mathscr{M})$. Indeed, if δ denotes the characteristic function of a measurable subset of the unit circle and if

$$PM_\delta - M_\delta P \in \mathscr{K}(\mathscr{H}),$$

then the image of the Toeplitz operator

$$T_\delta = PM_{\delta|P\mathscr{H}}$$

in $\mathcal{Q}(P\mathcal{H})$ is a projection. It follows that the spectrum of T_δ is countable and so the selfadjoint operator T_δ has an eigenvalue. Therefore ([10], problem 198) δ is a constant almost everywhere and M_δ is trivial. Thus, $\nu(\mathcal{M})$ does not contain an almost central subset of projections and the proposition is proved.

THEOREM 4. *If the continuum hypothesis is true, then there is a masa in the Calkin algebra that contains an almost central subset of projections and is generated by its projections.*

Proof. Fix a well-ordering of $\{a_\alpha\}_{\alpha<\omega_1}$ of the self-adjoint elements in the Calkin algebra.

Assertion: There is a well-ordered collection $\{\mathcal{S}_\alpha\}_{\alpha<\omega_1}$ of subsets of the Calkin algebra such that

(1) For each ordinal α, $\mathcal{S}_\alpha$ is countable and consists of commuting projections.

(2) If $\beta < \alpha$, then $\mathcal{S}_\beta \subseteq \mathcal{S}_\alpha$.

(3) If for some ordinal α, a_α commutes with the elements of $\mathcal{S}_\alpha$, then $a_\alpha \in C^*(\mathcal{S}_\alpha)$ the C^*-algebra generated by $\mathcal{S}_\alpha$ and the identity.

(4) For each ordinal α, there is a nontrivial projection p_α in $\mathcal{S}_\alpha$ such that $p_\alpha a_\beta = a_\beta p_\alpha$ for all ordinals $\beta \leqslant \alpha$.

The assertion is proved by transfinite induction. Suppose that for some ordinal $\alpha < \omega_1$ and all ordinals $\beta<\alpha$ subsets $\mathcal{S}_\beta$ satisfying (1), (2), (3) and (4) have been selected. (Note that if α is the first ordinal, nothing has been selected and the conditions are vaccuously satisfied.) To choose $\mathcal{S}_\alpha$ write

$$\mathcal{B}_\alpha = C^*(\{\mathcal{S}_\beta\}_{\beta<\alpha}) \quad \text{and} \quad \mathcal{C}_\alpha = C^*(\mathcal{B}_\alpha, \{a_\beta\}_{\beta\leqslant\alpha}).$$

Note that $\mathcal{B}_\alpha$ and $\mathcal{C}_\alpha$ are separable because α is a countable ordinal. Now consider two cases. First, suppose $a_\alpha \in \mathcal{B}_\alpha$. By Voiculescu's Weyl-von Neumann theorem ([13], Theorem 1.3), there is a nontrivial projection p_α in the Calkin algebra that commutes with the elements of $\mathcal{C}_\alpha$. In this case the subset

$$\mathcal{S}_\alpha = (\bigcup_{\beta<\alpha} \mathcal{S}_\beta) \cup \{p_\alpha\}$$

has the required properties. If, on the other hand, $a_\alpha \notin \mathcal{B}_\alpha$, then we may apply Voiculescu's double commutant theorem ([13], Corollary 1.9) to obtain a projection q_α that commutes with the elements of $\mathcal{B}_\alpha$ but does not commute with a_α, and then choose a nontrivial projection p_α in the commutant of $\mathcal{C}_\alpha \cup \{q_\alpha\}$. If in this case we write

$$\mathcal{S}_\alpha = (\bigcup_{\beta<\alpha} \mathcal{S}_\beta) \cup \{p_\alpha\} \cup \{q_\alpha\},$$

then $\mathcal{S}_\alpha$ has the desired properties. The proof of the assertion is complete.

To complete the proof of the theorem, let us show that the C^*-algebra $\mathscr{A}$ generated by the subsets $\{\mathscr{S}_\alpha\}_{\alpha<\omega_1}$ and the identity is a masa in the Calkin algebra and that the set

$$\mathscr{P} = \{p_\alpha\}_{\alpha<\omega_1}$$

of the projections satisfying property (4) is an almost central subset of the Calkin algebra.

Suppose a is a self-adjoint element of the Calkin algebra that commutes with the elements of $\mathscr{A}$. Then $a=a_\alpha$ for some ordinal α and since $\mathscr{B}_\alpha \subseteq \mathscr{A}$, a_α commutes with $\mathscr{B}_\alpha$. Hence by property (3) $a_\alpha \in \mathscr{B}_\alpha \subseteq \mathscr{A}$ and $\mathscr{A}$ is a masa. Fix t in the Calkin algebra. Then $t = a_\beta + ia_\gamma$ for some ordinals β and γ. If α is an ordinal greater than both β and γ, then $p_\alpha t = tp_\alpha$. This fact and Lemma 2 imply that $\mathscr{P}$ is an almost central subset in the Calkin algebra.

REMARK. If $\mathscr{P}$ is an almost central subset of projections in the Calkin algebra, then $\mathscr{P}$ contains at most a countable number of mutually orthogonal projections. To see this, fix p in $\mathscr{P}$ and choose an isometry v in the Calkin algebra such that $v^*pv = 1$. If q is a projection that commutes with v and is orthogonal to p, then

$$q = v^*pvq = v^*pqv = 0.$$

As almost all of the elements of $\mathscr{P}$ commute with v, there are at most countably many projections in $\mathscr{P}$ that are orthogonal to p.

2. DETERMINING SETS

A C^*-algebra is said to have the *restriction property* if each pure state on it restricts to a homomorphism on a masa. Aarnes and Kadison showed in [1] that separable C^*-algebras have the restriction property. The main result of this section is that, assuming the continuum hypothesis, *every* state on the Calkin algebra restricts to a homomorphism on a masa. In fact for every countable set of states a masa can be found so that each state in the set restricts to a homomorphism on the masa. It is convenient to begin the proof by presenting some preliminary propositions. The first result to be given is probably fairly widely known.

PROPOSITION 5. *If $\mathscr{F}$ is a countable set of states on the Calkin algebra and p is a nonzero projection, then there is a nonzero projection $q \leqslant p$ such that $f(q) = 0$ for each f in $\mathscr{F}$.*

Proof. Fix an uncountable set $\mathscr{P}$ of mutually orthogonal projections in the Calkin algebra such that each element of $\mathscr{P}$ is dominated by p. For each f in $\mathscr{F}$ and each n in $\mathbf{N}$

$$\{r \text{ in } \mathscr{P} : f(r) \geqslant 1/n\}$$

has cardinality at most n. Therefore for each f in $\mathscr{F}$

$$\{r \text{ in } \mathscr{P} : f(r) > 0\}$$

is countable and so

$$\{r \text{ in } \mathscr{P} : f(r) > 0 \text{ for some } f \text{ in } \mathscr{F}\}$$

is countable. Hence there are an uncountable number of projections in $\mathscr{P}$ that are annihilated by all f in $\mathscr{F}$.

PROPOSITION 6. *If $\{f_n\}$ is a sequence of states on a separable C^*-subalgebra $\mathscr{A}$ of the Calkin algebra, then there is a sequence $\{p_n\}$ of nonzero mutually orthogonal projections in the Calkin algebra such that*

$$p_n a p_n = f_n(a) p_n$$

for a in $\mathscr{A}$ and $n = 1, 2, \ldots$.

Proof. By the G.N.S. construction each f_n gives rise to a representation $\pi_n : \mathscr{A} \to \mathscr{B}(\mathscr{H}_n)$ and a unit vector ξ_n in $\mathscr{H}_n$ such that

$$f_n(a) = (\pi_n(a)\xi_n, \xi_n)$$

for all a in $\mathscr{A}$. For each n, let $\mathscr{M}_n$ denote the direct sum of ω copies of $\mathscr{H}_n$ and let ρ_n denote the direct sum of ω copies of π_n. Fix n and write $\xi_{n1} = \xi_n \oplus 0 \oplus 0 \oplus \ldots$, $\xi_{n2} = 0 \oplus \xi_n \oplus 0 \oplus \ldots$, $\xi_{n3} = 0 \oplus 0 \oplus \xi_n \oplus \ldots$, etc. Note that

$$(\rho_n(a)\xi_{nj}, \xi_{nk}) = \delta_{jk} f_n(a)$$

for all natural numbers j and k and all a in $\mathscr{A}$. It follows that if Q_n denotes the projection of $\mathscr{M}_n$ onto the closed linear span of $\{\xi_{nk} : k \in \mathbf{N}\}$, then

$$Q_n \rho_n(a) Q_n = f_n(a) Q_n$$

for all a in $\mathscr{A}$. If we write $\mathscr{M} = \mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \ldots$ and $\rho = \rho_1 \oplus \rho_2 \oplus \ldots$, then the projections Q_n extend to projections on $\mathscr{M}$ in a natural way so that

$$Q_n \rho(a) Q_n = f_n(a) Q_n$$

for all a in $\mathscr{A}$ and all n in $\mathbf{N}$. By Voiculescu's theorem [13], Theorem 1.3 there is an isometry V mapping $\mathscr{M}$ onto a subspace of $\mathscr{H}$ such that if P denotes the projection of $\mathscr{H}$ onto the range of V, then

$$\nu(P) a \nu(P) = \nu(V \rho(a) V^*)$$

for a in $\mathscr{A}$. Write $p_n = \nu(VQ_nV^*)$, $n=1, 2, \ldots$. Note that the p_n's are nonzero mutually orthogonal projections and for each n, $p_n \leqslant \nu(P)$. If $a \in \mathscr{A}$ then

$$\begin{aligned} p_nap_n &= p_n\nu(P)a\nu(P)p_n = \\ &= \nu(VQ_nV^*)\nu(V\rho(a)V^*)\nu(VQ_nV^*) = \\ &= \nu(VQ_n\rho(a)Q_nV^*) = \nu(f_n(a)VQ_nV^*) = \\ &= f_n(a)p_n. \end{aligned}$$

If f is a state on the Calkin algebra let us say that a subset $\mathscr{P}$ of the Calkin algebra is *determining for f* if

(1) $\mathscr{P}$ is uncountable and consists of mutually orthogonal projections,
(2) for each p in $\mathscr{P}$, $f(p) = 0$, and
(3) if $t \in \mathscr{Q}(\mathscr{H})$, then $ptp = f(t)p$ for almost all p in $\mathscr{P}$.

THEOREM 7. *If $f_1, f_2, \ldots$ are states on the Calkin algebra, then, assuming the continuum hypothesis, there are subsets $\mathscr{P}_1, \mathscr{P}_2, \ldots$ in the Calkin algebra such that $\mathscr{P}_n$ is a determining set for f_n, $n = 1, 2, \ldots$ and such that if $m \neq n$, then the elements of $\mathscr{P}_m$ are orthogonal to the elements of $\mathscr{P}_n$.*

Proof. The proof is by transfinite induction. Fix a well-ordering $\{t_\alpha\}_{\alpha<\omega_1}$ of the Calkin algebra and suppose that for some ordinal $\alpha < \omega_1$ and all ordinals $\beta < \alpha$ sets $\{p_{\beta n} : n \in \mathbf{N}\}$ have been selected such that

(1) $\{p_{\beta n} : \beta < \alpha,\ n \in \mathbf{N}\}$ consists of nonzero mutually orthogonal projections,
(2) $f_n(p_{\beta m}) = 0$ for all $\beta < \alpha$ and all m and n in $\mathbf{N}$, and
(3) If $\gamma \leqslant \beta$, then $p_{\beta n}t_\gamma p_{\beta n} = f_n(t_\gamma)p_{\beta n}$ for $n = 1, 2, \ldots$.

To select $\{p_{\alpha n} : n \in \mathbf{N}\}$, write

$$\mathscr{A}_\alpha = C^*(\{t_\beta : \beta \leqslant \alpha\},\ \{p_{\beta n} : \beta < \alpha,\ n \in \mathbf{N}\}).$$

As α is a countable ordinal, $\mathscr{A}_\alpha$ is a separable C^*-algebra and so by Proposition 6, there are mutually orthogonal projections p_n such that

$$(*) \qquad p_nap_n = f_n(a)p_n$$

for a in $\mathscr{A}_\alpha$ and $n = 1, 2, \ldots$. Then, by property (2)

$$p_np_{\beta m}p_n = f_n(p_{\beta m})p_n = 0$$

if $\beta < \alpha$ and m and n are in $\mathbf{N}$. Since $\|p_{\beta m}p_n\|^2 = \|p_np_{\beta m}p_n\|$, each p_n is orthogonal to all the $p_{\beta m}$'s. By Proposition 5, there are nonzero projections $p_{\alpha n} \leqslant p_n$ such that $f_m(p_{\alpha n}) = 0$ for all m and n in $\mathbf{N}$. Clearly $\{p_{\beta n} : \beta \leqslant \alpha, n \in \mathbf{N}\}$ satisfies (1) and (2) and by (*) (3) is also satisfied. The induction is complete. To finish the proof, write

$$\mathscr{P}_n = \{p_{\alpha n}\}_{\alpha < \omega_1}$$

$n = 1, 2, \ldots$.

PROPOSITION 8. *If f is a state on the Calkin algebra and $\mathscr{P}$ is a determining set for f, then f restricts to a homomorphism on the commutant $\mathscr{P}'$ of $\mathscr{P}$ in the Calkin algebra.*

Proof. For each t in the Calkin algebra write $\mathscr{S}_t = \{p \in \mathscr{P} : ptp = f(t)p\}$. Since $\mathscr{P}$ is a determining set $\mathscr{S}_t$ has a countable complement in $\mathscr{P}$. Therefore for any a and b in the Calkin algebra $\mathscr{S}_a \cap \mathscr{S}_b \cap \mathscr{S}_{ab}$ is not empty. Fix a and b in $\mathscr{P}'$ and p in $\mathscr{S}_a \cap \mathscr{S}_b \cap \mathscr{S}_{ab}$. As a and b are in $\mathscr{P}'$,

$$pabp = (pap)\,(pbp)$$

and therefore

$$f(ab)p = pabp = (pap)(pbp) = f(a)f(b)p.$$

Thus, f is a homomorphism on $\mathscr{P}'$.

THEOREM 9. *If $\mathscr{E}$ is a countable set of states on the Calkin algebra, then, assuming the continuum hypothesis, there is a masa $\mathscr{A}$ in the Calkin algebra such that the restriction of each state in $\mathscr{E}$ to $\mathscr{A}$ is a homomorphism.*

Proof. By Theorem 7 there is a collection $\{\mathscr{P}_f\}_{f \in \mathscr{E}}$ of mutually orthogonal subsets of projections in the Calkin algebra such that for each f in $\mathscr{E}$, $\mathscr{P}_f$ is a determining set for f. If $\mathscr{A}$ is a masa that contains each $\mathscr{P}_f$, then $\mathscr{A} \subset \mathscr{P}_f'$ for each f in $\mathscr{E}$. Hence by Proposition 8 if $f \in \mathscr{E}$, then f is a homomorphism on $\mathscr{A}$.

REMARKS.

1. Theorem 9 cannot be extended to the case where $\mathscr{E}$ has cardinality c. For example, if f is a fixed state and $\mathscr{E}$ consists of all unitary translates of f, then $\mathscr{E}$ is weak*-dense in the state space of Calkin algebra ([8], (3.4.1); [9], Theorem 4′). If every element of $\mathscr{E}$ restricted to a homomorphism on a fixed masa, then *every* state on the Calkin algebra would also restrict to a homomorphism on this masa and this is clearly impossible. It seems possible, however, that a family of c *orthogonal* states may restrict to homomorphisms on a fixed masa. (The states f and g are orthogonal if $\|f - g\| = 2$.)

2. Suppose f and g are states on the Calkin algebra whose restrictions h and k to a masa are homomorphisms. If f and g are not orthogonal, then $h = k$. For, if $h \neq k$, then $\|f - g\| \geqslant \|h - k\| = 2$.

3. Note that if f is a state on the Calkin algebra that is not pure and f restricts to a homomorphism h on a masa, then h does not have a unique state extension

to the Calkin algebra. For, by the Kreĭn-Milman theorem, there is a pure state on the Calkin algebra that extends h and this state cannot equal f.

4. Akemann showed in [2] that if f is a pure state on a separable C^*-algebra, then f restricts to a homomorphism h on a masa and, furthermore, the masa can be chosen so that f is the unique state extension of h. It seems likely that this result remains true for states on the Calkin algebra.

5. If $\mathscr{M}$ is a nonatomic masa in $\mathscr{B}(\mathscr{H})$, then ν is injective on $\mathscr{M}$ so that $\nu(\mathscr{M})$ does not contain an uncountable subset of mutually orthogonal projections. Therefore $\nu(\mathscr{M})$ contains no determining set. On the other hand if $\mathscr{D}$ is an atomic masa in $\mathscr{B}(\mathscr{H})$, then $\nu(\mathscr{D})$ does contain an uncountable family of mutually orthogonal projections and, furthermore, it is not too difficult to show (assuming the continuum hypothesis) that $\nu(\mathscr{D})$ contains determining sets.

Partially supported by a grant from the National Science Foundation.

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Received April 20, 1978; revised April 18, 1979.