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SINGLY GENERATED ALGEBRAS CONTAINING COMPACT OPERATORS

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Let T be an invertible operator on a complex Hilbert space H and let $\mathscr{A}(T)$ be the strongly closed algebra generated by T and the identity operator I. The problem of finding conditions which imply that $T^{-1} \in \mathscr{A}(T)$ has been considered in [1], [2], [3], [4]. In [3], Feintuch asks if $T^{-1} \in \mathscr{A}(T)$ whenever $\mathscr{A}(T)$ contains a compact injective operator. A partial answer is given below.

Standard terminology and notation will be used (see, for example [6]). For any operator X, the lattice of all invariant subspaces of X is denoted by Lat X. For any positive integer n, $H^{(n)}$ denotes the direct sum of n copies of H and $X^{(n)}$ denotes the operator on $H^{(n)}$ which is the direct sum of n copies of X. For any set $\mathscr{S}$ of operators on H, $\mathscr{S}^{(n)} = \{S^{(n)}: S \in \mathscr{S}\}$. A well known result ([6], Theorem 7.1) states that if $\mathscr{A}$ is any algebra of operators on H containing I then the strong closure of $\mathscr{A}$ is $\{B : \mathrm{Lat}\, \mathscr{A}^{(n)} \subseteq \mathrm{Lat}\, B^{(n)}$ for all $n\}$. We say that an invariant subspace of an operator X is *full* if $\overline{X(N)} = N$ and say that X is *full* if every invariant subspace of X is full. (In [5] full subspaces are called "regular" but this term is avoided since regular operator has a different meaning.) A subspace of the form $N \ominus M$ where N and M are invariant under X and $M \subseteq N$ is called a *semi-invariant* subspace of X, (cf.[7]).

An obvious necessary condition for T^{-1} to be in $\mathscr{A}(T)$ is that Lat $T \subseteq$ Lat T^{-1}. It is easy to see that a paraphrase of this condition is that T be a full operator. It is not known whether this condition is sufficient. From the result quoted above, $T^{-1} \in \mathscr{A}(T)$ if and only if Lat $T^{(n)} \subseteq$ Lat $(T^{-1})^{(n)}$ for all n or in other words, if and only if $T^{(n)}$ is full for all n. In Theorem 1 we give a condition on $\mathscr{A}(T)$ which implies that T is full. The subsequent results consist of finding cases when this condition is satisfied by $\mathscr{A}(T^{(n)})$ for every n.

THEOREM 1. *Let T be an invertible operator and suppose that the compression of the strongly closed algebra generated by T and I to any infinite-dimensional semi-invariant subspace contains a non-zero compact operator. Then every invariant subspace of T is invariant under T^{-1}.*

Proof. It is sufficient to prove that every cyclic invariant subspace of T is invariant under T^{-1}. Let x be any non-zero vector and, for any integer k, let

$$M_k = \bigvee_{n=k}^{\infty} T^n x.$$

Since T is invertible, $T(M_k) = M_{k+1}$ and the required conclusion is that $M_0 = M_1$.

Suppose that $M_0 \neq M_1$. Then clearly we have the proper inclusions $M_0 \supset M_1 \supset M_2 \supset \ldots$. Let e_n be a unit vector of the one-dimensional space $M_n \ominus \ominus M_{n-1}$. Then $\{e_n\}$ is an orthonormal sequence and, since $T(M_n) = M_{n+1}$,

$$Te_n = \alpha_n e_{n+1} + x_{n+2}$$

where α_n is a scalar and $x_{n+2} \in M_{n+2}$. Since

$$e_n = \alpha_n T^{-1} e_{n+1} + T^{-1} x_{n+2}$$

and $T^{-1} x_{n+2} \in M_{n+1}$, we have that

$$1 = \langle e_n, e_n \rangle = |\alpha_n| |\langle T^{-1} e_{n+1}, e_n \rangle| \leqslant |\alpha_n| \, \|T^{-1}\|.$$

That is,

$$|\alpha_n| \geqslant \|T^{-1}\|^{-1}.$$

Now let $M_* = \bigwedge_{n=1}^{\infty} M_n$. Clearly $M_0 \ominus M_*$ is an infinite-dimensional semi-invariant subspace of T and so $\mathscr{A}(T)$ contains and operator K such that the compression of K to $M_0 \ominus M_*$ is compact and $K(M_0) \nsubseteq M_*$. Let k be the largest integer such that $K(M_0) \subseteq M_k$. Then $K(M_n) = T^n K(M_0) \subseteq M_{n+k}$ and so

$$Ke_n = \beta_n e_{n+k} + y_{n+k+1}$$

where β_n is a scalar and $y_{n+k+1} \in M_{n+k+1}$. Also $\beta_n \neq 0$ since otherwise $K(M_n) \subseteq M_{n+k+1}$ which would imply that $K(M_0) \subseteq K(T^{-n} M_n) \subseteq T^{-n}(M_{n+k+1}) = M_{k+1}$, contradicting the choice of k. From the compactness of the compression of K we have that $(\beta_n) \to 0$. But

$$\beta_n \alpha_{n+k} = \langle TKe_n, e_{n+k+1} \rangle =$$

$$= \langle KTe_n, e_{n+k+1} \rangle = \alpha_n \beta_{n+1}.$$

Put $\gamma_n = \alpha_n \cdot \alpha_{n+1} \cdot \ldots \cdot \alpha_{n+k-1}$ if $k > 0$ and $\gamma_n = 1$ if $k = 0$. It follows from above that $|\gamma_n| \geqslant \|T^{-1}\|^{-k}$ and that $\beta_n \gamma_{n+k} = \beta_{n+k} \gamma_n$. Thus β_{rk}/γ_{rk} is constant for all integers r. This contradicts the fact that $(\beta_n) \to 0$ and therefore $M_0 = M_1$.

A slight simplification of the above proof gives an alternative proof for the result of [3].

THEOREM 2. (Feintuch). *If the compact operators of $\mathscr{A}(T)$ are strongly dense in $\mathscr{A}(T)$ then $T^{-1} \in \mathscr{A}(T)$.*

Proof. With the notation of Theorem 1, if $M_0 \neq M_1$, since $\langle Te_0, e_1 \rangle \neq 0$ there is a compact operator K in $\mathscr{A}(T)$ such that $\langle Ke_0, e_1 \rangle \neq 0$. This leads to a contradiction just as in the proof of Theorem 1. Hence every cyclic invariant subspace of T, and so every invariant subspace of T, is full. The proof is completed by observing that if $\mathscr{A}(T)$ satisfies the hypothesis of the theorem, so does $\mathscr{A}(T^{(n)})$.

We now find conditions on T which imply that $T^{(n)}$ satisfies the hypothesis of Theorem 1 for all n. It is not clear whether the hypothesis itself is such a condition.

THEOREM 3. *If the strongly closed algebra $\mathscr{A}(T)$ generated by T and I contains an operator of the form $K = D + R$, where D is a compact injective dissipative operator and R has finite rank, then $T^{-1} \in \mathscr{A}(T)$.*

Proof. It is clear that if $\mathscr{A}(T)$ contains an operator of the given form, so does $\mathscr{A}(T^{(n)})$. In view of Theorem 1, it only remains to show that the compression of such an operator to any infinite-dimensional semi-invariant subspace is non-zero.

Let M and N be invariant subspaces of T such that $M \subset N$ and suppose that the compression of K to $N \ominus M$ is zero. Then $K(N) \subseteq M$. We prove that $N \ominus M$ has finite dimension. Let X and Y be the real and imaginary parts of K. If $L = \ker \left[\frac{1}{2i} (R - R^*) \right]$ then, since D is dissipative, Y is non-negative on L and L has finite co-dimension. Suppose $x \in L \cap (N \ominus M)$. Then $\langle Kx, x \rangle = \langle Xx, x \rangle + i\langle Yx, x \rangle = 0$ and so $\langle Yx, x \rangle = 0$. Since $x \in L$, this implies that $Yx = 0$, showing that $Kx = K^*x$. But $K(N) \subseteq M$ and M is invariant under K so

$$0 = \langle K^2x, x \rangle = \|Kx\|^2.$$

Therefore $L \cap (N \ominus M) \subseteq \ker K$. If L has co-dimension r, it is easy to prove that

$$\dim [L \cap (N \ominus M)] \geqslant \dim(N \ominus M) - r.$$

However, since D is injective, $\ker K$ has finite dimension. Thus $N \ominus M$ has finite dimension. Therefore, by Theorem 1, $T^{-1} \in \mathscr{A}(T)$ and the proof is complete.

COROLLARY 4. *If $\mathscr{A}(T)$ contains a compact injective dissipative operator then $T^{-1} \in \mathscr{A}(T)$.*

Corollary 4 is immediate from Theorem 3. It also follows directly from Lemma 1 of [1] applied to Theorem 1.

We conclude with some open questions.

1. It was observed above that the question whether Lat $T \subseteq$ Lat T^{-1} implies that $T^{-1} \in \mathscr{A}(T)$ is equivalent to the question whether T full implies that $T^{(n)}$ is

full for every n. This latter formulation makes sense for all operators (not just invertible ones). In particular one may ask the same question for compact operators; that is, if K is full and compact, is $K^{(n)}$ full for every n?

2. Let us call an operator X *nearly full* if, for every invariant subspace N of X, the dimension of $N \ominus \overline{X(N)}$ is finite. It follows from the proof of Theorem 1 that if for every n, $\mathscr{A}(T^{(n)})$ contains a nearly full compact operator then $T^{-1} \in \mathscr{A}(T)$. Thus the following question is of interest: if X is nearly full, is $X^{(n)}$ nearly full for every n; in particular does this implication hold if X is compact. Note that the proof of Theorem 3 shows that a finite-dimensional perturbation of an injective dissipative operator is nearly full. Are there any other easily identifiable classes of operators that are nearly full?

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