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DIMENSION GROUPS AND FINITE DIFFERENCE EQUATIONS

EDWARD G. EFFROS and CHAO-LIANG SHEN

INTRODUCTION

A modified form of the classical Murray and von Neumann dimension theory for von Neumann algebras has begun to play an important role in C^*-algebra theory. Roughly speaking, one begins by identifying unitarily equivalent projections in a C^*-algebra A, and then in the matrix algebras $M_n(A)$. Letting $N_0(A)$ be the equivalence classes, the direct sum of projections provides an abelian semigroup operation on $N_0(A)$. One then produces an abelian group $K_0(A)$ from $N_0(A)$ in essentially the same way that one constructs the integers $\mathbf{Z}$ from the non-negative integers $\mathbf{N}_0$, i.e., $K_0(A)$ is the "Grothendieck completion" of $N_0(A)$. The *dimension* of a projection is defined to be its image in $K_0(A)$.

The set $\mathbf{N}_0$ determines the usual ordering of $\mathbf{Z}$, i.e., one has that $m \geqslant n$ if and only if $m - n \in \mathbf{N}_0$. One might therefore expect that the image of $N_0(A)$ should define something like a (partial) ordering of $K_0(A)$, and in fact this should generally be the case for Grothendieck completions. Intuitively, the ordering of $K_0(A)$ should enable one to "compare the sizes" of dimensions. This idea was used by Elliott [7], who showed that the approximately finite (AF) C^*-algebras A are classified to within stable isomorphism by the corresponding ordered groups $K_0(A)$. We recall that an AF algebra is just the closure of an ascending sequence of finite dimension C^*-algebras. The classification of such algebras had earlier been studied by Glimm, Dixmier, and Bratteli [1], [3], [10]. Elliott called the ordered groups that arise the countable *dimension groups*. The latter have recently been completely characterized by simple interpolation properties [4], and an extensive effort to classify these groups has begun (see [8], [18], [5], [6], [4]).

Given a countable dimension group G, it would be of some interest to have methods for finding specific AF C^*-algebras A with $G = K_0(A)$. In ordered group theoretic terms, this amounts to finding a direct system of positive homomorphisms

$$\mathbf{Z}^{r(0)} \xrightarrow{\varphi_1} \mathbf{Z}^{r(1)} \xrightarrow{\varphi_2} \dots \tag{0.1}$$

where $\mathbf{Z}^{r(n)}$ is given the usual ordering, such that $G = \varinjlim \mathbf{Z}^{r(n)}$. We call such an array a *Bratteli system* for G since one can immediately construct from it the "Bratteli diagram" for the desired AF algebra A (see [1]).

Bratteli systems of the form

$$\mathbf{Z}^r \xrightarrow{\varphi_1} \mathbf{Z}^r \xrightarrow{\varphi_2} \dots \tag{0.2}$$

where $\det \varphi_n \neq 0$ are of particular interest since they naturally arise in the theory of topological Markov shifts (see [20]). We say that such systems are *Elliott systems*, and if the φ_n are group isomorphisms, i.e., $\det \varphi_n = \pm 1$, we say they are *unimodular*. In §2 we show how one can identify the limit G of such a system by using the asymptotic theory of finite difference equations. In particular if the system (0.2) is *stationary*, i.e., all the φ_n are constant, and the φ_n have strictly positive entries, the Perron-Frobenius theory applies and we find that G has only one state (i.e., the corresponding AF algebra has only one, possibly unbounded, trace). This uniqueness result is due to Elliott.

If G is the limit of a unimodular system (0.2), then as a group $G \cong \mathbf{Z}^r$. Conversely it is natural to make the

CONJECTURE: Any finitely generated dimension group G is the limit of a unimodular system.

Elliott proved that this is the case if G is a dense subgroup of $\mathbf{R}$. We clarified this result for doubly generated dense subgroups of $\mathbf{R}$ by showing in [5] that the φ_n may be computed with continued fraction expansions. The latter tie in quite naturally with certain finite difference systems. Using this observation, we show in §3 that multidimensional continued fractions may be used to construct unimodular systems for arbitrary finitely generated dense subgroups of $\mathbf{R}$.

In § 4 we give an example of a dense subgroup of $\mathbf{R}^2$ that has a unimodular system. We also show that there exist groups with unimodular systems, which have the form $H \oplus_t \mathbf{Z}$, where H is a dense doubly generated subgroup of $\mathbf{R}$ and the "t" indicates the ordering $(a, k) \geqslant 0$ if $a > 0$ or $a = k = 0$. Since any simple (see § 2) finitely generated dimension group is isomorphic to a group of the form $H \oplus_t \mathbf{Z}^p$ where H is a dense subgroup of $\mathbf{R}^d$ for some $d \geqslant 1$, $p \geqslant 0$, this lends additional support to the Conjecture.

We are indebted to George Elliott for encouraging us to develop our theory for Elliott systems rather than the unimodular systems (see [7], § 6 for some interesting examples). We also wish to thank C. Lance for providing us with a simplified argument for the convergence of multi-dimensional continued fractions (Theorem 3.2).

1. SOME PRELIMINARIES

We use the terminology and notation of [4], § 1. In particular we let e_i and ε_i $(1 \leqslant i \leqslant r)$ be the canonical bases for $\mathbf{R}^r$ and $(\mathbf{R}^r)^*$, respectively. Although we distinguish the latter two spaces, we write $(\alpha_1, \dots, \alpha_r)$ for either $\sum \alpha_i e_i$ or $\sum \alpha_i \varepsilon_i$

(the correct interpretation will be clear from the context). We also write $\mathbf{Z}^r$ for either $\mathbf{Z}e_1 + \ldots + \mathbf{Z}e_r$ or $\mathbf{Z}\varepsilon_1 + \ldots + \mathbf{Z}\varepsilon_r$, and we refer to the elements as *integral vectors*. We define $(\mathbf{Z}^r)^+$, $(\mathbf{Q}^r)^+$, $(\mathbf{R}^r)^+$ and $(\mathbf{R}^r)^{*+}$ to be the vectors $\alpha = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_i \geqslant 0$, and we provide $(\mathbf{R}^r)^*$ with the norm

$$\|(\alpha_1, \ldots, \alpha_r)\| = |\alpha_1| + \ldots + |\alpha_r|. \tag{1.1}$$

We let $\Delta_0 \subseteq (\mathbf{R}^r)^*$ be the simplex spanned by $\varepsilon_1, \ldots, \varepsilon_r$. Given a linear map $\varphi: \mathbf{R}^r \to \mathbf{R}^r$, we let $\varphi^{tr}: (\mathbf{R}^r)^* \to (\mathbf{R}^r)^*$ be the adjoint map. Unless otherwise stated, all groups are assumed torsion free and abelian.

In contrast with [4], it will be necessary to consider finite rank groups which are not finitely generated. Given a linear spanning subgroup G of the rational vector space $\mathbf{Q}^r$, we may choose a rational vector basis v_i $(1 \leqslant i \leqslant r)$ in G. For any additive homomorphism $f: G \to \mathbf{R}$, there is a unique real linear function $g: \mathbf{R}^r \to \mathbf{R}$ satisfying $g(v_i) = f(v_i)$. Given $a \in G$ we have that $a = \sum \alpha_i v_i$ for unique $\alpha_i \in \mathbf{Q}$. Choosing $n \in \mathbf{Z}$ with $n\alpha_i \in \mathbf{Z}$ it follows that

$$\begin{aligned} ng(a) &= \textstyle\sum (n\,\alpha_i) g(v_i) = \\ &= \textstyle\sum (n\,\alpha_i) f(v_i) = \\ &= f(\textstyle\sum (n\,\alpha_i) v_i) = \\ &= f(na) = \\ &= nf(a), \end{aligned}$$

i.e., $g(a) = f(a)$ and g is an *extension* of f to an element of $(\mathbf{R}^r)^*$ (this argument does not work for subgroups G of $\mathbf{R}^r$ rather than $\mathbf{Q}^r$: consider the Galois automorphism of $\mathbf{Z}[\sqrt{2}]$). We conclude that if G was initially provided with an ordering (not necessarily the relative ordering), then we may identify the set $P^*(G)$ of positive homomorphisms $f: G \to \mathbf{R}$ with a subset of $(\mathbf{R}^r)^*$. Given an order unit $e \in G$, it follows that the corresponding state space

$$S_e(G) = \{f \in (\mathbf{R}^r)^* : f(e) = 1\}$$

is also a subset of $(\mathbf{R}^r)^*$.

From [6], [4], we have that if $G \subseteq \mathbf{Q}^r$ is a linearly spanning simple dimension group, then there exist linearly independent $\theta_1, \ldots, \theta_d \in (\mathbf{R}^r)^*$ with

$$P^*(G) = \mathbf{R}^+ \theta_1 + \ldots + \mathbf{R}^+ \theta_d$$

for which the map

$$\theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_d \end{bmatrix} : G \to \mathbf{R}^d$$

has a dense image, and that

$$G^+ = \{a \in G : \theta_i(a) > 0\} \cup \{0\} = P(\theta_1, \ldots, \theta_d).$$

Conversely, all such groups $(G, P(\theta_1, \ldots, \theta_d))$ with θ_i linearly independent and $\theta(G)$ dense in $\mathbf{R}^d$ are simple dimension groups. If $G = \mathbf{Z}^r$, the density of $\theta(G)$ is equivalent to

$$(1.2) \qquad (\mathbf{R}\,\theta_1 + \ldots + \mathbf{R}\,\theta_d) \cap \mathbf{Z}^r = \{0\}.$$

If G is any rank r (torsion free abelian) group, then we may identify G with a spanning subgroup of $\mathbf{Q}^r$ since we have

$$G \cong G \otimes \mathbf{Z} \subseteq G \otimes \mathbf{Q} \cong \mathbf{Q}^r$$

(note that any vector basis in $G \otimes \mathbf{Q}$ yields integrally independent elements in $G \otimes \mathbf{Z}$ when multiplied by a suitable integer). Thus any finite rank simple dimension group has the form $(G, P(\theta_1, \ldots, \theta_d))$, $G \subseteq \mathbf{Q}^r$. We may in fact assume that $\theta_1, \ldots, \theta_d$ lie in the simplex Δ_0. To see this, let

$$N = \{a \in \mathbf{R}^r : \theta_i(a) > 0\}.$$

Since N is an open cone, we may find an integral basis $e_1', \ldots, e_r' \in N$ (for example, one may fix a vector $v = (v_1, \ldots, v_r) \in N$ with integrally independent entries and then use the continued fraction method of § 3 to construct bases $X_i(n)$, $1 \leqslant i \leqslant r$, with $\mathbf{R}^+X_i(n)$ approximating $\mathbf{R}^+v$ as $n \to \infty$). Letting ε_i' be the dual integral basis, we have that

$$\mathbf{R}^+e_1' + \ldots + \mathbf{R}^+e_d' \subseteq N \cup \{0\}$$

implies that

$$\mathbf{R}^+\varepsilon_1' + \ldots + \mathbf{R}^+\varepsilon_d' \supseteq \{\theta_1, \ldots, \theta_d\}.$$

Defining $\eta \in GL(r, \mathbf{Z})$ by $\eta(e_i) = e_i'$, we have that $\eta(G) \subseteq \mathbf{Q}^r$, $\eta^{tr}(\varepsilon_i') = \varepsilon_i$, and thus $\eta^{tr}(\theta_i) \in (\mathbf{R}^r)^{*+}$. Letting

$$\zeta_i = \eta^{tr}(\theta_i)/\|\eta^{tr}(\theta_i)\|,$$

$\zeta_i \in \Delta_0$ and

$$(G, P(\theta_1, \ldots, \theta_d)) \cong (\eta^{-1}(G), P(\zeta_1, \ldots, \zeta_d)).$$

We will say that a simple dimension group G is *non-degenerate* if $\theta : G \to \mathbf{R}^d$ is one-to-one, i.e., we may identify G with a dense subgroup of $\mathbf{R}^d$. In particular, we have that a simple dimension group is totally ordered if and only if it is non-degenerate and has only one state, i.e., $d = 1$. If $G = \mathbf{Z}^r$ and $\theta_1 = \alpha = (\alpha_1, \ldots \alpha_r)$, this will be the case if and only if $\alpha_1, \ldots, \alpha_r$ are integrally independent. More generally, the finitely generated simple groups with one state have the form $(\mathbf{Z}^r, P(\alpha))$, where $\mathbf{R}\alpha \cap \mathbf{Z} = \{0\}$.

2. ELLIOTT SYSTEMS AND FINITE DIFFERENCE EQUATIONS

Let us suppose that we are given an Elliott system (0.2). We shall identify φ_n with its matrix $[a_{ij}(n)]$, where $a_{ij}(n) = \varepsilon_i(\varphi_n(e_j)) \in \mathbf{Z}^+$. Regarding φ_n as a linear map $\mathbf{Q}^r \to \mathbf{Q}^r$, we must have that φ_n is one-to-one, i.e., $\varphi_n \in GL(r, \mathbf{Q})$. This is an immediate consequence of the fact that for suitable $\sigma, \tau \in GL(r, \mathbf{Z})$,

$$\sigma \circ \varphi_n \circ \tau = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & \end{bmatrix} \qquad (d_k \in \mathbf{Z})$$

see [17], p. 29). The system will be unimodular if and only if we have $\varphi_n \in GL(r, \mathbf{Z})$. We will often restrict our attention to *strictly positive systems*, i.e., those for which $a_{ij}(n) > 0$. This implies that the limit group is simple (see [1], Cor. 3.5). Conversely if the limit group of an Elliott system (0.2) is simple, then for each n there will be a p with $\varphi_{n+p} \circ \ldots \circ \varphi_n$ strictly positive. Thus by composing consecutive sequences of maps, we obtain a strictly positive system for G.

Following Elliott [7], § 6.3, we may use the linear isomorphism

$$\rho_n = \varphi_n \circ \ldots \circ \varphi_1 : \mathbf{Q}^r \to \mathbf{Q}^r \quad (\rho_0 = I)$$

to transform (0.2) into a system of inclusions. Specifically, if we define $G_n = \rho_n^{-1}(\mathbf{Z}^r) \subseteq \mathbf{Q}^r$ and $G_n^+ = \rho_n^{-1}((\mathbf{Z}^r)^+)$, we may replace (0.2) by

(2.1) $$G_0 \hookrightarrow G_1 \hookrightarrow \ldots$$

Letting $G_\infty = \cup\, G_n$ and $G_\infty^+ = \cup\, G_n^+$, the maps $G_n \hookrightarrow G_\infty$ determine an isomorphism

$$G_\infty \cong \varinjlim G_n.$$

If the system is unimodular, we have that as groups $G_\infty = G_n = \mathbf{Z}^r$. Since the ρ_n are strictly positive, $e = e_1 + \ldots + e_n$ is an order unit in each of the groups G_n. Letting $\Delta_n = S_e(G_n)$ $(n = 0, 1, \ldots, \infty)$, it is evident that

(2.2) $$\Delta_0 \supseteq \Delta_1 \supseteq \ldots \supseteq \Delta_\infty.$$

We recall that Δ_0 is the simplex spanned by $\varepsilon_1, \ldots, \varepsilon_r$. In general Δ_n is the simplex spanned by

$$J_i(n) = X_i(n) \,/\, \|X_i(n)\|,$$

where

$$X_i(n) = \rho_n^{tr}(\varepsilon_i) \quad (n \geqslant 0) \tag{2.3}$$

and (see (1.1)),

$$\|X_i(n)\| = \sum_j |X_i(n)(e_j)| = X_i(n)(e).$$

To see this, note that if $f \in (\mathbf{R}^r)^*$, f is positive on G_n if and only if

$$f \circ \rho_n^{-1} = (\rho_n^{tr})^{-1}(f)$$

is positive on $(\mathbf{Z}^r)^+$, hence ρ_n^{tr} is an affine isomorphism of $P^*(\mathbf{Z}^r)$ onto $P^*(G_n)$. From the latter it also follows that

$$\Delta_n = \{\alpha \in \Delta_0 : \alpha = \rho_n^{tr}(\beta) \text{ for some } \beta \in (\mathbf{R}^r)^{*+}\}.$$

We have

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \Delta_n$$

since $f \in P^*(G)$ if and only if $f(G_n^+) \geqslant 0$ for all n (see [11], Prop. 3.2 for a generalization to general Bratteli systems using projective limits of state spaces). We conclude that

PROPOSITION 2.1. *A vector $\alpha \in \Delta_0$ lies in the state space of G_∞ if and only if for all n, $\alpha = \rho_n^{tr}(\alpha_n)$ for some $\alpha_n \in (\mathbf{R}^r)^{*+}$.*

Since we have

$$X_i(n) = \varphi_1^{tr} \circ \ldots \circ \varphi_{n-1}^{tr}(\varphi_n^{tr}\, \varepsilon_i) = $$

$$= \sum_j a_{ij}(n)\, \varphi_1^{tr} \circ \ldots \circ \varphi_{n-1}^{tr}(\varepsilon_j),$$

these vectors are completely determined by the initial conditions $X_i(0) = \varepsilon_i$ and the simultaneous difference equations

$$X_i(n) = \sum_{j=1}^{r} a_{ij}(n) X_j(n-1) \tag{2.4}$$

$(a_{ij}(n) \in \mathbf{Z}^+,\ i = 1, \ldots, r; n \geqslant 1)$.

In general we say that a set of difference equations of the form (2.4) is an *Elliott system* (resp., a *unimodular* system) if for each n, $\det[a_{ij}(n)] \neq 0$ (resp., $\det[a_{ij}(n)] = \pm 1$), and *strictly positive* if in addition we have $a_{ij}(n) > 0$. If $a_{ij}(n) =$

$= a_{ij}(m)$ we say the system is *stationary*. If $X_i(n)$ is the solution of an Elliott (resp., unimodular) system (2.4) satisfying the initial conditions $X_i(0) = \varepsilon_i$, then it is evident that $(X_1(n), \ldots, X_r(n))$ is a vector (resp., integral) basis for $(\mathbf{R}^r)^*$. Thus for each n, the vectors $J_i(n) = X_i(n)/\|X_i(n)\|$ span an r-simplex Δ_n, and we have $\Delta_0 \supseteq \Delta_1 \supseteq \supseteq \ldots$. The system will be strictly positive if and only if

$$\Delta_n \subseteq \operatorname{int} \Delta_{n+1} \quad (n \geqslant 0). \tag{2.5}$$

We call the set $\Delta_\infty = \bigcap_{n=1}^{\infty} \Delta_n$ the *asymptotic simplex* of the system (2.4).

Given an Elliott system of difference equations (2.4) we may let $G_\infty = = \varinjlim (\mathbf{Z}^r, \varphi_n)$ where $\varphi_n = [a_{ij}(n)]$. Identifying G with a subgroup of $\mathbf{Q}^r$ as above, it is immediate that $\Delta_\infty = S_e(G_\infty)$ (and thus, in particular, Δ_∞ is a d-simplex with $d \leqslant r$). We conclude that *the explicit determination of the state spaces of the limits of Elliott systems of groups is equivalent to the calculation of the asymptotic simplexes for Elliott systems of difference equations*. In particular, we may reformulate the conjecture stated in the introduction as follows: Given linearly independent $\theta_1, \ldots, \theta_d \in (\mathbf{R}^r)^*$ satisfying (1.2), there exists a unimodular strictly positive system (0.2) for which Δ_∞ is the simplex spanned by the θ_j. That the conditions are necessary (and in particular $d \leqslant r - 1$) follows from § 1.

In the stationary positive coefficient case, the calculation of the asymptotic simplex Δ_∞ is provided by the classical Perron-Frobenius theory (see [9], Ch. III, § 2; [19], p. 143). Letting $\varphi = \varphi_n = [a_{ij}(n)]$, φ is a matrix with strictly positive entries. It follows that φ has a unique eigenvalue λ of maximum absolute value, λ is positive, it is a simple eigenvalue, and for each i

$$X_i(n) \,/\, \lambda^n = (\varphi^n)^{tr} (\varepsilon_i)/\lambda^n \tag{2.6}$$

converges to a non-zero λ-eigenvector. Since $X_i(n)/\lambda^n \in (\mathbf{R}^n)^{*+}$, we may let α be the unique λ-eigenvector in Δ_0. It follows that $J_i(n)$ converges to α for each i, hence $\Delta_\infty = \{\alpha\}$. We conclude:

PROPOSITION 2.2. (Elliott). *The limit of a stationary strictly positive Elliott system has only one state.*

In § 4 we give an example of such a system with a degenerate, i.e., non-totally ordered limit group.

Given $a = (a_1, \ldots, a_r) \in (\mathbf{Z}^r)^+$, we define the *cyclic* homomorphism $\varphi(a) : \mathbf{Z}^r \to \mathbf{Z}^r$ by the matrix

$$\varphi(a) = \begin{bmatrix} a_1 & a_2 \ldots a_{r-1} & a_r \\ 1 & 0 \ldots 0 & 0 \\ & \ldots & \\ 0 & 0 \ldots 1 & 0 \end{bmatrix} \tag{2.7}$$

we have that $\det \varphi(a) = (-1)^{r+1}a_r$, and thus φ is an injection (resp., $\varphi \in GL(r, \mathbf{Z})$) if and only if $a_r > 0$ (resp., $a_r = 1$). We say that an Elliott system (0.2) is *cyclic* if each φ_n has that property. The corresponding difference equations (2.4) are given by

$$X_1(n) = a_1(n)X_1(n-1) + \ldots + a_r(n)X_r(n-1)$$
$$X_i(n) = X_{i-1}(n-1) \quad (2 \leqslant i \leqslant r).$$

Defining $X(n) \in (\mathbf{Z}^r)^+$ by

$$X(k-r) = \varepsilon_k \qquad (1 \leqslant k \leqslant r) \tag{2.8}$$
$$X(n) = X_1(n) \qquad (n \geqslant 1),$$

$X(n)$ is determined by the r-th order equation

$$X(n) = a_1(n)X(n-1) + \ldots + a_r(n)X(n-r) \quad (n \geqslant 1). \tag{2.9}$$

The solutions $X_1(n), \ldots, X_r(n)$ to (2.4) are then obtained by extracting consecutive sequences of length r from the sequence $\{X(n) : n \geqslant -r+1\}$. Since the matrices (2.7) do not have strictly positive entries, to obtain a simple limit group it is necessary to verify that for each n, $\varphi_{n+p} \circ \ldots \circ \varphi_n$ is strictly positive for sufficiently large p. In particular, this will be the case if $a_1(n) \geqslant 1$ for each n.

Let us suppose that we are given $\alpha \in \Delta_0$ and that we wish to find a cyclic system (2.4) with $\Delta_\infty = \{\alpha\}$. From Proposition 2.1, we must have a sequence $\varphi_n = \varphi(a(n))$ such that for each n, $\alpha = \rho_n^{tr}(\alpha_n)$ for some $\alpha_n \in (\mathbf{R}^r)^{*+}$. This however is not enough, since all this will imply is that $\alpha \in \Delta_\infty$. The remaining problem is to prove that the sequence of vertices

$$J(n) = X(n)/\|X(n)\| \tag{2.10}$$

converges to α. This is solved for α with integrally independent components in § 3.

3. CONTINUED FRACTIONS AND TOTALLY ORDERED GROUPS

In this section we shall assume that $\alpha = (\alpha_1, \ldots, \alpha_r) \in (\mathbf{R}^r)^{*+}$ is a fixed vector. To clarify the general situation, we shall not initially suppose that $\alpha_1, \ldots, \alpha_r$ are integrally independent. We shall, however, assume that the first non-zero entry α_k is strictly maximal, i.e., $\alpha_k > \alpha_j$ for all $j \neq k$. If α is integrally independent, this may be arranged by first applying a permutation matrix. Motivated by the terminology for fractions (we are thinking of α_1/α_2) we say that such a vector is *improper*.

Given an improper vector $\alpha = (\alpha_1, \ldots, \alpha_r)$ with $\alpha_r \neq 0$, let us divide $\alpha_1, \ldots, \alpha_{r-1}$ by α_r. Letting $\alpha_1', \ldots, \alpha_{r-1}'$ be the remainders, $(\alpha_1', \ldots, \alpha_{r-1}', \alpha_r)$ is no longer improper, but the cyclic permutation $\beta = (\alpha_r, \alpha_1', \ldots, \alpha_{r-1}')$ is once again improper. If $\alpha_r = 0$, we will just let β be the improper vector $(\alpha_r, \alpha_1, \ldots, \alpha_{r-1})$. Formalizing this procedure, which we shall iterate, we have

LEMMA 3.1. *If $\alpha \in (\mathbf{R}^r)^{*+}$ is improper, then there exists a vector $a = (a_1, \ldots, a_{r-1}, 1) \in (\mathbf{Z}^r)^+$ and a unique improper vector $\beta \in (\mathbf{R}^r)^{*+}$ with*

$$\alpha = \begin{bmatrix} a_1 & 1 & 0 \ldots 0 \\ a_2 & 0 & 0 \ldots 0 \\ \ldots & & \\ a_{r-1} & & 0 \ldots 1 \\ 1 & 0 & 0 \ldots 0 \end{bmatrix} \beta = \varphi(a)^{tr}\beta. \tag{3.1}$$

If $\alpha_r \neq 0$, then the a_k are also unique, and the first non-zero a_k is maximal.

Proof. If $\alpha_r \neq 0$ we may use the Euclidean algorithm to find unique $a_k \in \mathbf{Z}^+$ with

$$\alpha_k = a_k\alpha_r + \alpha_k', \quad 0 \leqslant \alpha_k' < \alpha_r \quad (1 \leqslant k \leqslant r-1). \tag{3.2}$$

Since α is improper, the first non-zero a_k is maximal. If $\alpha_r = 0$, we obtain the equality in (3.2) by letting $a_k \in \mathbf{Z}^+$ be arbitrary and $\alpha_k' = \alpha_k$. In either case $\beta = (\alpha_r, \alpha_1', \ldots, \alpha_{r-1}')$ is improper and satisfies (3.1). Conversely, given (3.1) for improper vectors α and β, we let $\beta = (\alpha_r, \alpha_1', \ldots, \alpha_{r-1}')$. If $\alpha_r \neq 0$, (3.2) is an immediate consequence, and if $\alpha_r = 0$, we still have the equality. Whether or not $\alpha_r = 0$, the entries α_k' in (3.2) are unique. Q.E.D.

Given an improper vector $\alpha \in (\mathbf{R}^r)^{*+}$, we define sequences of improper *remainder* vectors

$$\alpha(n) = (\alpha_1(n), \ldots, \alpha_r(n)) \in (\mathbf{R}^r)^{*+} \quad (n \geqslant 1)$$

and *integral quotient* vectors

$$a(n) = (a_1(n), \ldots, a_{r-1}(n), 1) \in (\mathbf{Z}^r)^+ \quad (n \geqslant 1)$$

as follows. We let $\alpha(1) = \alpha$. Having defined $\alpha(n)$, we let $\alpha(n+1)$ be the unique improper vector with $\alpha(n) = \varphi(a)^{tr}\alpha(n+1)$ for some $a = (a_1, \ldots, a_{r-1}, 1) \in (\mathbf{Z}^r)^+$. If $\alpha_r(n) \neq 0$, a is uniquely determined, and we let $a(n) = a$. If $\alpha_r(n) = 0$, we let $a(n) = (0, \ldots, 0, 1)$. Deleting the last components of the integral quotients, we call the resulting array of non-negative integers

$$\begin{bmatrix} a_1(1) & a_1(2) & \\ a_2(1) & a_2(2) & \\ \vdots & \vdots & \ldots \\ a_{r-1}(1) & a_{r-1}(2) & \end{bmatrix} \tag{3.3}$$

the *cyclic continued fraction* determined by α. If $r = 2$ and $\beta = \alpha_2/\alpha_1$ is irrational, $[a_1(1), a_1(2), \ldots]$ is just the usual continued fraction for β.

Given a continued fraction as in (3.3), the maps

$$\varphi_n = \varphi(a(n)) = \begin{bmatrix} a_1(n) & \ldots & a_{r-1}(n) & 1 \\ 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \ldots & 1 & 0 \end{bmatrix} \tag{3.4}$$

determine a cyclic unimodular Bratteli system (0.2), and equivalently, a cyclic unimodular system of difference equations (2.9). We let Δ_∞ be the asymptotic set for (2.9). Since we have

$$\alpha = \varphi_1^{tr} \circ \ldots \circ \varphi_{r-1}^{tr}\, \alpha(n) = \rho_{r-1}^{tr}\, \alpha(n) \tag{3.5}$$

where $\alpha(n) \in (\mathbf{R}^r)^{*+}$, it follows from Proposition 2.1 that $\alpha \in \Delta_\infty$. The following is just a reformulation of the classical Jacobi-Perron convergence theorem for multidimensional continued fractions [14], [16].

THEOREM 3.2. *Suppose that $\alpha \in \Delta_0$ has integrally independent components, and the continued fraction expansion* (3.3). *Then letting $\varphi_n = \varphi(a(n))$, $\{\alpha\} = \Delta_\infty$ for the cyclic unimodular system* (2.9).

Proof. We claim that for each n, the components of $\alpha(n) = (\alpha_1(n), \ldots, \alpha_r(n))$ are strictly positive. To see this, note that from (3.5), (2.3), and (2.8)

$$\alpha = [X(n-1) \ldots X(n-r)] \begin{bmatrix} \alpha_1(n) \\ \vdots \\ \alpha_r(n) \end{bmatrix} =$$

$$= \alpha_1(n)X(n-1) + \ldots + \alpha_r(n)X(n-r).$$

If $\alpha_k(n) = 0$, α must lie in the linear hyperplane spanned by the $X(n-j)$, $1 \leqslant j \leqslant r$, $j \neq k$. Since the $X(n-j)$ are integral vectors, the latter must have the form

$$\{\beta \in (\mathbf{R}^d)^* : m_1\beta_1 + \ldots + m_r\beta_r = 0\} \quad (m_j \in \mathbf{Z}).$$

This contradicts the fact that the components of α are integrally independent.

Since $\alpha(n) = (\alpha_1(n), \ldots, \alpha_r(n))$ is an improper vector with non-zero entries, we have that $\alpha_1(n) > \alpha_j(n)$ for all $j \neq 1$, and thus dividing by $\alpha_r(n)$ we conclude that $a_1(n) \geqslant a_j(n)$ $(1 \geqslant j \geqslant r)$.

From (2.9) and (2.10), we have

$$J(n) = b_1(n)J(n-1) + \dots + b_r(n)J(n-r), \tag{3.6}$$

where the coefficients

$$b_j(n) = a_j(n)\,\|X(n-j)\|/\|X(n)\|$$

satisfy $b_j(n) \geqslant 0$, $\sum b_j(n) = 1$. It is evident from (2.9) that $\|X(n)\|$ is an increasing sequence, hence

$$b_1(n) = a_1(n)\|X(n-1)\|/\|X(n)\| \geqslant a_j(n)\|X(n-j)\|/\|X(n)\| = b_j(n).$$

It follows that $b_1(n) \geqslant 1/r$. As we shall see, the convergence of the $J(n)$ follows from this inequality. We are indebted to C. Lance for the following elegant argument.

As above we let Δ_n be the simplex spanned by $J(n), \dots, J(n-r+1)$, and we have $\Delta_0 \supseteq \Delta_1 \supseteq \dots$. We use (3.6) to recursively determine the barycentric coordinates of $J(n+k)$ in Δ_n:

$$J(n+1) = c_1^1 J(n) + \dots + c_r^1 J(n-r+1)$$

$$J(n+2) = c_1^2 J(n) + \dots + c_r^2 J(n-r+1)$$

$$\dots$$

We have that

$$c_1^1 = b_1(n+1) \geqslant 1/r$$

$$c_1^2 \geqslant b_1(n+2)b_1(n+1) \geqslant 1/r^2$$

$$\dots$$

Letting Δ_n' be the sub-simplex of Δ_n spanned by $J(n)$ and

$$J(n-j)' = 1/r^r\, J(n) + (r^r - 1)/r^r\, J(n-j) \qquad (1 \leqslant j \leqslant r-1),$$

it follows that the simplex Δ_{n+r} spanned by $J(n+1), \dots, J(n+r)$ lies in Δ_n'. Thus

$$\operatorname{diam} \Delta_{n+r} \leqslant \operatorname{diam} \Delta_n' = (r^r - 1)/r^r \operatorname{diam} \Delta_n,$$

and $\operatorname{diam} \Delta_{n+kr} \to 0$ as $k \to \infty$. It follows that $J(n)$ is Cauchy and thus converges. Since $\alpha \in \Delta_\infty = \cap \Delta_n$, $J(n)$ converges to α. Q.E.D.

COROLLARY 3.3. *Suppose that the components of $\alpha \in \Delta_0$ are integrally independent, and that α has the continued fraction expansion* (3.3). *Then* $(\mathbf{Z}^r, P(\alpha))$ *is the direct limit of the system*

$$\mathbf{Z}^r \xrightarrow{\varphi_1} \mathbf{Z}^r \xrightarrow{\varphi_2} \dots,$$

where $\varphi_n = \varphi(a(n))$.

4. SOME EXAMPLES AND PROBLEMS

We first consider the system

$$\mathbf{Z}^3 \overset{\varphi_1}{\to} \mathbf{Z}^3 \overset{\varphi_2}{\to} \dots, \tag{4.1}$$

where φ_n has the cyclic form

$$\varphi_n = \begin{bmatrix} 0 & a_n & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (a_n \in \mathbf{N}).$$

PROPOSITION 4.1. *Suppose that* $\sum_{n=1}^{\infty} a_n^{-1} < 1$. *Then* $G = \varinjlim (\mathbf{Z}^3, \varphi_n)$ *has two pure states.*

Proof. Using the notation of § 2, we have that

$$X(3-k) = \varepsilon_k \quad (1 \leqslant k \leqslant 3),$$

$$X(n) = a_n X(n-2) + X(n-3) \quad (n \geqslant 1),$$

and in particular, $\|X(n)\|$ $(n \geqslant 1)$ is an increasing sequence in $\mathbf{N}$. Thus $J(n) = X(n)/\|X(n)\|$ satisfies

$$J(3-k) = \varepsilon_k \quad (1 \leqslant k \leqslant 3),$$

$$J(n) = b_n J(n-2) + c_n J(n-3) \quad (n \geqslant 1),$$

where b_n, $c_n \geqslant 0$ and $b_n + c_n = 1$. We have

$$1 \geqslant b_n = a_n \|X(n-2)\|/\|X(n)\| \geqslant$$

$$\geqslant a_n \|X(n-3)\|/\|X(n)\| = a_n c_n,$$

and thus $c_n \leqslant a_n^{-1}$. Since $\|Y - Z\| \leqslant 2$ for all $Y, Z \in \Delta_0$ (the edges have maximal length),

$$\|J(n) - J(n-2)\| = \|(b_n - 1)J(n-2) + c_n J(n-3)\| =$$

$$= c_n \|J(n-3) - J(n-2)\| \leqslant$$

$$\leqslant 2a_n^{-1}.$$

Thus for $0 \leqslant m \leqslant n$ we have

$$\|J(2n) - J(2m)\| \leqslant 2 \sum_{k=m}^{\infty} a_{2k}^{-1},$$

and $J(2n)$ is Cauchy. Letting $m = 0$,

$$\|J(2n) - \varepsilon_3\| \leqslant 2 \sum_{k=1}^{\infty} a_{2k}^{-1},$$

hence the limit point β of $J(2n)$ satisfies

$$\|\beta - \varepsilon_3\| \leqslant 2 \sum_{k=1}^{\infty} a_{2k}^{-1}.$$

Similarly, $J(2n + 1)$ is Cauchy, and its limit point α satisfies

$$\|\alpha - \varepsilon_2\| \leqslant 2 \sum_{k=0}^{\infty} a_{2k+1}^{-1} ,$$

and thus

$$\|\alpha - \beta\| = \|\varepsilon_3 - \varepsilon_2 - (\beta - \varepsilon_2) + (\alpha - \varepsilon_3)\| \geqslant$$

$$\geqslant 2 - 2 \sum_{k=1}^{\infty} a_k^{-1} > 0.$$

Since $\alpha, \beta \in \Delta_\infty$, Δ_∞ must be a 2- simplex. Q.E.D.

For our second example, we consider the stationary sequence (4.1) with

$$\varphi_n = \varphi = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Although the entries are not all strictly positive, the Perron-Frobenius theory as stated in § 2 still applies. For nonnegative matrices it is known that φ must have an eigenvalue λ with $\lambda \geqslant |\alpha|$ for all other eigenvalues α (see [9], Ch. III, § 3). Since φ^3 has strictly positive entries, it has an eigenvalue μ with $\mu > |\beta|$ for all other eigenvalues β. Using the fact that sp $\varphi^3 = (\text{sp } \varphi)^3$ (see [12], § 3.3, Th. 1) we conclude that $\mu = \lambda^3$. Letting

$$\varphi^{3n}(\varepsilon_i)/\lambda^{3n} \to \alpha ,$$

it follows that

$$\varphi^{3n+1}(\varepsilon_i)/\lambda^{3n+1} \to \varphi(\alpha)/\lambda,$$

$$\varphi^{3n+2}(\varepsilon_i)/\lambda^{3n+2} \to \varphi^2(\alpha)/\lambda^2.$$

If the three limit points did not coincide, the sequence $\varphi^{4n}(\varepsilon_i)/\lambda^{4n}$ would oscillate, and thus not converge. This would contradict the fact that φ^4 has strictly positive entries, and λ^4 is its Perron eigenvalue. We conclude that $\Delta_\infty = \{\alpha/\|\alpha\|\}$ and $G_\infty \cong \cong (\mathbf{Z}^3, P(\alpha))$. Letting $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we claim that $\alpha_1, \alpha_2, \alpha_3$ are integrally dependent, and thus G is degenerate (see §1).

From above, $\varphi(\alpha) = \lambda\alpha$. The characteristic polynomial for φ is

$$x^3 - 4x^2 + 4x - 1 = (x^2 - 3x + 1)(x - 1).$$

The maximal root of $x^2 - 3x + 1$ is $(3 + \sqrt{5})/2$, which is larger than 1, hence $\lambda = (3 + \sqrt{5})/2$. Letting $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, if $\alpha_1 = 0$, we are done. We may thus assume that $\alpha_1 = 1$. From the second two rows of

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},$$

we have $\alpha_2 = \lambda^{-1}$ and $\alpha_3 = \alpha_2(\lambda - 2)^{-1} = \lambda^{-1}(\lambda - 2)^{-1}$. It follows that

$$1 - \alpha_2 - \alpha_3 = (\lambda^2 - 3\lambda + 1)/\lambda(\lambda - 2) = 0.$$

Further calculations show that in fact

$$G_\infty = G \oplus_t \mathbf{Z}$$

where $G = \mathbf{Z}\left[\dfrac{-1 + \sqrt{5}}{2}\right]$.

Similar arguments may be used to show that the stationary *cyclic* system (0.2) with

$$\varphi_n = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

also gives rise to a degenerate group.

In addition to the above (admittedly anecdotal) evidence, we feel that a general result of Davenport (see [2], §I, 2.4) lends support to our conjecture. He proved that given any vectors $\alpha_1, \ldots, \alpha_{r-1}$ in Δ_0, one can find a sequence of bases $(X_1(n), \ldots$

$\ldots, X_r(n))$ for which $J_i(n) = X_i(n)/\|X_i(n)\|$ converges to α_i $(1 \leqslant i \leqslant r-1)$. Although suggestive, this result does not appear to prove the conjecture, even when $r = 2$.

Groups with more than one state are usually not the limits of cyclic unimodular systems. For example, consider the group $(\mathbf{Z}^3, P(\alpha, \beta))$, where $\alpha, \beta \in \Delta_0$ are such that $\alpha_2 < \alpha_3 < \alpha_1$, and $\beta_1 < \beta_3 < \beta_2$. If there is a cyclic system (0.2) with this group as its limit, we must in particular have that $\alpha, \beta \in \Delta_\infty$. From Lemma 2.1 it will follow that $\alpha = \varphi_1^{tr}\gamma$ and $\beta = \varphi_1^{tr}\,\delta$ for $\gamma, \delta \in (\mathbf{R}^r)^{*+}$. Letting $\varphi_1 = \varphi(a)$, $a = (a_1, a_2, 1) \in \mathbf{Z}^3$, it follows that

$$\alpha_1 = \alpha_3 a_1 + \gamma_2, \quad \beta_1 = \beta_3 a_1 + \delta_2,$$

$$\alpha_2 = \alpha_3 a_2 + \gamma_3, \quad \beta_2 = \beta_3 a_2 + \delta_3,$$

where $\gamma_i, \delta_i \geqslant 0$. Since $\alpha_3 > \alpha_2$, $a_2 = 0$, and since $\beta_3 > \beta_1$, $a_1 = 0$. It follows that $\varphi(a) = \varphi((0, 0, 1))$ is just the cyclic permutation matrix and γ and δ are just permutations of α and β, respectively. Letting $\varphi_2 = \varphi(b)$ we have that $\gamma = \varphi(b)\varepsilon$, $\delta = \varphi(b)\eta$, and the same considerations (now we are dividing by α_2 and β_2) shows that $b = (0, 0, 1)$. Continuing once more we find that $\alpha(3) = \alpha$ and we are back where we started. Thus all the $\varphi(a(n))$ are cyclic permutations and the limit group is $(\mathbf{Z}^3, (\mathbf{Z}^3)^+)$, a contradiction.

It is thus evident that to prove the conjecture it will be necessary to find an analogue of Lemma 3.1 which uses non-cyclic maps. A related problem arises if one attempts to factor maps $\varphi \in GL(r, \mathbf{Z})$ into cyclic maps with non-negative components. This can be done if $r = 2$. In general any element $\varphi \in GL(r, \mathbf{Z})$ may be factored into a product of permutation matrices and matrices of the form

$$M(a) = \begin{bmatrix} 1 & a_1 & \ldots & a_{r-1} \\ 0 & 1 & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & 1 \end{bmatrix} \qquad (a = (a_1, \ldots, a_{r-1}, 1))$$

see [15], Ch. 4). For example, we have

$$\varphi(a) = M(a) \begin{bmatrix} 0 & \ldots & 0 & 1 \\ 1 & \ldots & 0 & 0 \\ & \ldots & & \\ 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Unfortunately, the entries of the factors are generally not positive. For example, letting

$$\varphi = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \\ 0 & 2 & 5 \end{bmatrix} \in GL(3, \mathbf{Z}),$$

the equation $\varphi = M(a)\psi$, where $\psi \in GL(3, \mathbf{Z})$ has non-negative components implies that $a = (0, 0, 1)$ (the matrix does not permit an elementary operation in which one row is subtracted from another).

We conclude by noting that even the stationary case has not been fully clarified. In particular we must ask what vectors $\alpha \in \Delta_0$ can arise as the Perron eigenvector of a unimodular matrix φ with positive integral entries? A solution of this problem would generalize the classical result of Lagrange characterizing the numbers with periodic continued fraction (see [13], § 10.12).

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EDWARD G. EFFROS and CHAO-LIANG SHEN
Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania 19104
U.S.A.

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