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SPECTRA OF COMPRESSIONS OF AN OPERATOR WITH COMPACT IMAGINARY PART

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Let $\mathscr{H}$ stands for a complex separable Hilbert space with the norm $\|\cdot\|$ and the scalar product $\langle . , . \rangle$. $\mathscr{L}(\mathscr{H})(\mathscr{L}\mathscr{C}(\mathscr{H}))$ denotes the set of all bounded (compact) linear operators on $\mathscr{H}$. For an $A \in \mathscr{L}(\mathscr{H})$, $\sigma(A)$ denotes the spectrum of A, $\rho(A)$-the resolvent set, $W(A)$ $(W_e(A))$ - the (essential) numerical range (see e.g. [2]).

$\mathfrak{S}_\omega(\mathfrak{S}_2)$ denotes thes Macaev (Hilbert-Schmidt) ideal of all compact operators $A \in \mathscr{L}\mathscr{C}(\mathscr{H})$ such that

$$\|A\|_\omega = \sum_{n=1}^{\infty} s_n/(2n-1) < \infty, \qquad \left((\|A\|_2)^2 = \sum_{n=1}^{\infty} s_n^2 < \infty\right),$$

where $s_1, s_2, \ldots$ are the eigenvalues of $\sqrt{AA^*}$ arranged in decreasing order and repeated according to multiplicity.

For an operator A with the single valued extension property (see e.g. [1]) and $x \in \mathscr{H}$, $\rho_A(x)$ is the union of all open subsets G of the complex plane $\mathbf{C}$ such that there exists the analytic function $x(\cdot): G \to \mathscr{H}$ which satisfies the relation $(A-\lambda)x(\lambda) \equiv x$, and we set $\mathbf{C} \setminus \rho_A(x) = \sigma_A(x)$. For any set $F \subset \mathbf{C}$ we set $\{x \in \mathscr{H}; \sigma_A(x) \subset F\} = \mathscr{X}_A(F)$. A is called a decomposable operator if for any finite open covering $\{G_n\}_{n=1}^m$ of $\sigma(A)$, $\mathscr{X}_A(\bar{G}_n)$ are closed subspaces and for any $x \in \mathscr{H}$ there are $x_j \in \mathscr{X}_A(\bar{G}_j)$ such that $x = x_1 + x_2 + \ldots + x_m$. It is proved in [10] that an operator A whose imaginary part $\operatorname{Im} A = (A - A^*)/(2\mathrm{i})$ belongs to the Macaev ideal is decomposable.

For any $\lambda \in \mathbf{C}$, $F, G \subset \mathbf{C}$, we define

$$\mathrm{d}(\lambda, F) = \inf_{\mu \in F} |\lambda - \mu|,$$

$$\operatorname{dist}(F, G) = \max\{\sup_{\mu \in F} \mathrm{d}(\mu, G), \sup_{\mu \in G} \mathrm{d}(\mu, F)\}.$$

LEMMA 1. *If $\mathscr{H}$ is an n-dimensional space, $B \in \mathscr{L}(\mathscr{H})$,* $\operatorname{Im} B = K_\omega + K_2$, *where* $K_\omega = K_\omega^*$, $\|K_\omega\|_\omega \leqslant d/8$, *then for any* $\lambda \in \rho(B)$ *such that* $\mathrm{d}(\lambda, \sigma(B)) \geqslant d$,

$$\|(B-\lambda)^{-1}\| \leqslant (2/d)\exp(4(1+64\|K_2\|_2^2/d^2)).$$

Proof. By the theorem on the triangular matrix form there are orthogonal projections $0 = Q_0 < Q_1 < \ldots < Q_n = 1$ such that $BQ_j = Q_jBQ_j$. Setting $Q_j - Q_{j-1} = E_j$ we define the operators:

$$S = \sum_{j=1}^{n} E_jBE_j ,$$

$$N_\omega = 2\mathrm{i} \sum_{j=1}^{n} Q_{j-1}\Big(K_\omega - \sum_{l=1}^{n} E_lK_\omega E_l\Big)E_j ,$$

$$N_2 = 2\mathrm{i} \sum_{j=1}^{n} Q_{j-1}\Big(K_2 - \sum_{l=1}^{n} E_lK_2E_l\Big)E_j .$$

These operators have the following properties: S is normal and $\sigma(S) = \sigma(B)$, $B = S + N_\omega + N_2$, the operators N_ω and N_2 are nilpotents. Using Theorems II.5.1.[4], III.4.1. [5], I.10.2.[5] from the books of Gohberg and Kreĭn we obtain the inequalities

$$\begin{aligned} &\|N_\omega\| \leqslant 2\Big\|K_\omega - \sum_{l=1}^{n} E_lK_\omega E_l\Big\|_\omega \leqslant 4\|K_\omega\|_\omega \leqslant d/2 , \\ &\|N_2\|_2 \leqslant 2\Big\|K_2 - \sum_{l=1}^{n} E_lK_2E_l\Big\|_2 \leqslant 4\|K_2\|_2. \end{aligned} \tag{1}$$

The operator S is normal, therefore

$$\|(S-\lambda)^{-1}\| = (\mathrm{d}(\lambda, \sigma(S)))^{-1} \leqslant d^{-1}, \tag{2}$$

and by (1), $\|(S-\lambda)^{-1}N_\omega\| \leqslant 1/2$. Hence there exists the operator $(1+(S-\lambda)^{-1}N_\omega)^{-1}$ and

$$\|(1+(S-\lambda)^{-1}N_\omega)^{-1}\| \leqslant (1-\|(S-\lambda)^{-1}N_\omega\|)^{-1} \leqslant 2. \tag{3}$$

Setting

$$(1+(S-\lambda)^{-1}N_\omega)^{-1}(S-\lambda)^{-1}N_2 = M ,$$

we have

$$B-\lambda = (S-\lambda)(1+(S-\lambda)^{-1}N_\omega)(1+M).$$

Note also that

$$\|M\|_2 \leqslant 2\|N_2\|_2/d \leqslant 8\|K_2\|_2/d$$

and M is a nilpotent.

To see this last assertion it suffices to observe that the subspaces $Q_j\mathscr{H}(1 \leqslant j \leqslant n)$ are invariant for the operators M, S, N_ω and N_2. Since $N_2Q_j = Q_{j-1}N_2Q_j$ it follows that $MQ_j = Q_{j-1}MQ_j$ and $M^n = Q_0M^nQ_n = 0$. Using the estimation of Sahnovič [11]

$$\|(1 + M)^{-1}\| \leqslant \exp(4(\|M\|_2^2+1)),$$

the inequalities (2), (3) and the identity

$$(B - \lambda)^{-1} = (1 + M)^{-1}(1 + (S - \lambda)^{-1}N_\omega)^{-1}(S - \lambda)^{-1},$$

we obtain the inequality that we need.

LEMMA 2. *If* $A \in \mathscr{L}(\mathscr{H})$, *($\mathscr{H}$ may be infinite-dimensional) and* $\operatorname{Im}A \in \mathfrak{S}_\omega$ *then there exists a function* $m\colon (0, \infty) \to (0, \infty)$ *such that for any finite-dimensional orthogonal projection* $Q \in \mathscr{L}(\mathscr{H})$

$$\|(Q(A - \lambda)|_{Q\mathscr{H}})^{-1}\| \leqslant m(\mathrm{d}(\lambda, \sigma(QA|_{Q\mathscr{H}}))).$$

Proof. Since the set of finite-dimensional operators is dense in $\mathfrak{S}_\omega$ with $\|\cdot\|_\omega$ norm, for any $d > 0$ there exists a finite-dimensional selfadjoint operator K_d such that

$$\|K_d - \operatorname{Im}A\|_\omega < d/8.$$

Since

$$\operatorname{Im}QA|_{Q\mathscr{H}} = QK_d|_{Q\mathscr{H}} + Q(\operatorname{Im}A - K_d)|_{Q\mathscr{H}}$$

and

$$\|QK_d|_{Q\mathscr{H}}\|_2 \leqslant \|K_d\|_2,$$

$$\|Q(\operatorname{Im}A - K_d)|_{Q\mathscr{H}}\|_\omega \leqslant \|\operatorname{Im}A - K_d\|_\omega < d/8,$$

the assertion of the lemma follows from Lemma 1.

THEOREM 1. *Suppose that* $A \in \mathscr{L}(\mathscr{H})$, $\operatorname{Im}A \in \mathfrak{S}_\omega$, $\{P_n\}_1^\infty$ *is a sequence of finite-dimensional orthogonal projections in* $\mathscr{H}$, $P_n \to 1$ *(strongly). Set* $P_nA|_{P_n\mathscr{H}} = A_n$. *Let* F *be a closed subset of* $\mathbf{C}$. *Then*

i) *if* $x_n \in \mathscr{X}_{A_n}(F)$, *and* $x_n \to x$ *(weakly) then* $x \in \mathscr{X}_A(F)$,

ii) *if* $x \in \mathscr{X}_A(F)$, G *is an open subset of* $\mathbf{C}$ *containing* F *and* Q_n *stands for the orthogonal projection on* $\mathscr{X}_{A_n}(\bar{G})$ *then* $\|x - Q_nx\| \to 0$,

iii) *if* $\lambda \in \sigma(A)$ *then* $\mathrm{d}(\lambda, \sigma(A_n)) \to 0$.

Proof. i) Let Ω be an open neighbourhood of F. Then

$$d = \inf\{|\lambda - \mu|;\ \lambda \in F, \mu \notin \Omega\} > 0.$$

$\mathscr{X}_{A_n}(F)$ is an invariant subspace of A_n and setting $A_n|\mathscr{X}_{A_n(F)}=B_n$, we see that $\sigma(B_n) \subset F$ ([1], p. 23). Hence by Lemma 2 there is a constant m such that

$$\|(B_n - \lambda)^{-1}\| \leqslant m \quad \text{for } \lambda \notin \Omega,\ n = 1, 2, \ldots . \tag{4}$$

Let $x_n(\lambda)$ be the analytic extension of

$$(A_n - \lambda)^{-1}x_n = (B_n - \lambda)^{-1}x_n.$$

By (4) $\|x_n(\lambda)\| \leqslant m\|x_n\|$ for $\lambda \notin \Omega$. Since the sequence $\{x_n\}$ is bounded, the sequence $\{x_n(\lambda)\}$ for $\lambda \notin \Omega$ is bounded too. Thus for a fixed $y \notin \Omega$ there is a subsequence $\{x_{n_k}(\lambda)\}$ which converges weakly to some $y \in \mathscr{H}$. The equality $(A_n - \lambda)x_n(\lambda) = x_n$ implies that for any $z \in \mathscr{H}$

$$\langle x_n, z\rangle = \langle (A - \lambda)x_n(\lambda), z\rangle - \langle (A - \lambda)x_n(\lambda), z - P_n z\rangle.$$

This and (4) give the inequality

$$|\langle x_n, z\rangle - \langle (A - \lambda)x_n(\lambda), z\rangle| \leqslant \|A - \lambda\| m \cdot \|x_n\| \cdot \|z - P_n z\|.$$

Passing to the subsequence and tending to the limit in this inequality we see that $\langle x, z\rangle = \langle (A - \lambda)y, z\rangle$. Thus $x = (A - \lambda)y$, since z was arbitrary. This implies that

$$x_n(\lambda) \xrightarrow{w} x(\lambda) = (A - \lambda)^{-1}x, \quad \text{for any } \lambda \in \rho(A) \setminus \Omega \tag{5}$$

because any subsequence of $\{x_n(\lambda)\}$ contains a subsequence which converges weakly to $x(\lambda)$. We shall show that (5) holds also for $\lambda \in \sigma(A) \setminus \Omega$. If $\lambda, u \notin \Omega$ then

$$x_n(\lambda) - x_n(\mu) = (\lambda - \mu)(B_n - \lambda)^{-1}(B_n - \mu)^{-1}x_n.$$

Thus it follows from (4) that

$$\|x_n(\lambda) - x_n(\mu)\| \leqslant |\lambda - \mu| m^2 \|x_n\| \quad \text{for } \lambda, \mu \notin \Omega. \tag{6}$$

Suppose now that $\lambda \notin \Omega$, and $\{x_{k_n}(\lambda)\}$, $\{x_{l_n}(\lambda)\}$ are weakly convergent subsequences to y and y' respectively. Since $\overline{\rho(A)} = \mathbf{C}$, there is a sequence $\{\mu_j\} \subset \rho(A)$ such that $\mu_j \to \lambda$. Writting the relations

$$\begin{aligned} |\langle y - y', z\rangle| = |\langle [y - x_{k_n}(\lambda)] + [x_{k_n}(\lambda) - x_{k_n}(\mu_j)] + \\ + [x_{k_n}(\mu_j) - x_{l_n}(\mu_j)] + [x_{l_n}(\mu_j) - x_{l_n}(\lambda)] + [x_{l_n}(\lambda) - y'], z\rangle| \leqslant \\ \leqslant |\langle y - x_{k_n}(\lambda), z\rangle| + |\langle x_{k_n}(\mu_j) - x_{l_n}(\mu_j), z\rangle| + \\ + |\langle x_{l_n}(\lambda) - y', z\rangle| + m^2|\lambda - \mu_j|\, \|z\| \, (\|x_{k_n}\| + \|x_{l_n}\|), \end{aligned}$$

and letting firstly $n \to \infty$ and secondly $j \to \infty$, we obtain $\langle y, z\rangle = \langle y', z\rangle$. This shows that there exists $w\text{-lim } x_n(\lambda) \overset{\text{def}}{=} x(\lambda)$ for any $\lambda \notin \Omega$. Moreover we know that $(A - \lambda)\, x(\lambda) \equiv x$. It follows from (6) that the functions $\langle x_n(\lambda), z\rangle$ are uniformly continuous functions in $\mathbf{C} \setminus \Omega$. Thus their limit is also an analytic function. Hence the function $x(\lambda)$ is the analytic extension of $(A - \lambda)^{-1}x$, and $x = w\text{-lim } x_n \in \mathscr{X}_A(\Omega)$. But Ω was an arbitrary open set containing F. Thus $x \in \mathscr{X}_A(F)$.

ii) Since the sequence $y_n = Q_n x$ is bounded, it contains a weakly convergent subsequence. Thus ii) will be proved if we show that if $y_n \to y$ weakly then $y = x$ and $\|y_n - x\| \to 0$.

Therefore let us assume that $y_n \to y$ weakly. Note that

$$P_n x - y_n \in (\mathscr{X}_{A_n}(\overline{G}))^{\perp} = (\text{ran } E(\overline{G}, A_n))^{\perp} =$$

$$= \text{ran } E(\mathbf{C} \setminus \overline{G}^*, A_n^*) \subset \text{ran } E(\mathbf{C} \setminus G^*, A_n^*) = \mathscr{X}_{A_n^*}(\mathbf{C} \setminus G^*),$$

where $G^* = \{\lambda \in \mathbf{C};\, \bar{\lambda} \in G\}$, and $E(.,.)$ stands for the suitable spectral projection. Since $P_n x - y_n \to x - y$ weakly, i) implies that $x - y \in \mathscr{X}_{A^*}(\mathbf{C} \setminus G^*)$. But $\mathscr{X}_{A^*}(\mathbf{C} \setminus G^*)$ is orthogonal to $\mathscr{X}_A(F)$ (see e.g. [3]), hence x is orthogonal to $x - y$ and $\|y\|^2 = \|x\|^2 + \|x - y\|^2 \geqslant \|x\|^2$. On the other hand $\|y\| \leqslant \sup \|y_n\| \leqslant \|x\|$. The above inequalities imply that $x = y$ and ([6], Lemma V.1.2.) $\|y_n - x\| \to 0$.

iii) Suppose the contrary, namely that $\lambda \in \sigma(A)$ and $\text{d}(\lambda, \sigma(A_n)) > d > 0$. Then by Lemma 2

$$\|P_{n_k} y\| \leqslant m \|P_{n_k}(A - \lambda)\, P_{n_k} y\|,$$

for some positive m and every $y \in \mathscr{H}$. Hence $\|y\| \leqslant m\|(A - \lambda) y\|$ for all $y \in \mathscr{H}$. But this is impossible, since $\lambda \in \sigma(A)$ is a boundary point of $\rho(A)$, and there is a sequence $\{x_n\}$ of unit vectors such that $\|(A - \lambda) x_n\| \to 0$.

REMARK. In the above theorem, as well as in Lemma 2, it is not essential that the projections P_n are finite-dimensional. One may prove this using a suitable limit process. The assumption $\text{Im} A \in \mathfrak{S}_\omega$ cannot be relaxed. This will be shown in Theorem 3.

The following definition will be usefull in order to characterize the asymptotical behaviour of spectra in the method of orthogonal projections.

DEFINITION. For $A \in \mathscr{L}(\mathscr{H})$ let $S(A)$ be the family of all compact nonvoid subsets of the complex plane $\mathbf{C}$ such that for every $\Omega \in S(A)$ there is a sequence $\{P_n\}_1^\infty$ of orthogonal projections in $\mathscr{H}$ such that $P_n \to 1$ (strongly) and dist $(\Omega, \sigma(P_n A|_{P_n \mathscr{H}})) \to 0$.

It may be proved ([8]) that if in the above definition we require the projections P_n to be finite-dimensional and ordered, then still $S(A)$ is unchanged.

With the help of this definition, the main result of this paper may be written as follows.

THEOREM 2. *Assume that* $\mathrm{Im}A \in \mathfrak{S}_\omega$. *Then* $\Omega \in S(A)$ *if and only if* $\sigma(A) \subset \subset \Omega = \overline{\Omega} \subset \sigma(A) \cup W_e(A)$.

Proof. It follows from the Weyl-von Neuman theorem ([6]) that there exists an operator $K = K^* \in \mathscr{LC}(\mathscr{H})$ and a sequence $\{P_n\}$ of finite-dimensional orthogonal projections in $\mathscr{H}$ such that $P_n \to 1$ strongly and

$$P_n(\mathrm{Re}A - K) = (\mathrm{Re}\,A - K)P_n.$$

Then

$$\|P_nA - AP_n\| = \|P_n(K + \mathrm{i}\mathrm{Im}A) - (K + \mathrm{i}\mathrm{Im}A)P_n\| \to 0, \tag{7}$$

since the operator $K + \mathrm{i}\mathrm{Im}A$ is compact. For any $\lambda_0 \in \rho(A)$ we have

$$\|P_n(A - \lambda_0)\, P_nx\| = \|(A - \lambda_0)P_nx + (P_nA - AP_n)P_nx\| \geqslant$$
$$\geqslant (\|(A - \lambda_0)^{-1}\|^{-1} - \|P_nA - AP_n\|)\|P_nx\|.$$

It follows now from the continuity of the resolvent and from (7) that

$$\|P_n(A - \lambda)P_nx\| \geqslant (2\|(A - \lambda_0)^{-1}\|)^{-1}\|P_nx\|$$

for all λ in a sufficiently small neighbourhood of λ_0 and for sufficiently large n. Consequently

$$\liminf \mathrm{d}(\lambda, \sigma(P_nA|_{P_n\mathscr{H}})) > 0 \quad \text{for all } \lambda \in \rho(A).$$

This and assertion iii) of Theorem 1 imply that $\mathrm{dist}(\sigma(A), \sigma(P_nA|_{P_n\mathscr{H}})) \to 0$. The "if" part follows now from [7], Lemma 7.

The "only if" part follows from Theorem 1 and [7], Theorem 1.

An example of a nondecomposable operator with a compact imaginary part is given in [9]. The following theorem says something more.

THEOREM 3. *Let K be a selfadjoint compact operator. The following conditions are equivalent:*

i) $K \in \mathfrak{S}_\omega$,

ii) *for every selfadjoint operator A, the operator $A + \mathrm{i}K$ is decomposable,*

iii) *for every selfadjoint operator A, if $\Omega \in S(A + \mathrm{i}K)$ then $\sigma(A + \mathrm{i}K) \subset \Omega$.*

Proof. The implication i) $\Rightarrow$ ii) is proved in [10]. i) $\Rightarrow$ iii) is contained in Theorem 1. To complete the proof it suffices to show that if $K \notin \mathfrak{S}_\omega$ then the conditions ii) and iii) are not satisfied.

Let $\mathscr{H}_0 = \ker K$. We shall define recursively infinite-dimensional invariant subspaces $\mathscr{H}_j$ of the operator K such that for $j \geqslant 1$

$$(8) \qquad K_j = K|_{\mathscr{H}_j} \in \mathfrak{S}_\omega \quad \text{and} \quad 1 \leqslant \|K_j\|_\omega \leqslant 2 + \|K\|.$$

If these subspaces are defined for $j \leqslant n$, then let $\tilde{\mathscr{H}}_n = \left(\bigoplus_{j=0}^{n} \mathscr{H}_j\right)^{\perp}$, and $\{e_{n,s}\}_{s=1}^{\infty}$, $\{\mu_{n,s}\}_{s=1}^{\infty}$ be the orthonormal eigenvectors and eigenvalues of the operator $K|_{\tilde{\mathscr{H}}_n} = \tilde{K}_n$ ($|\mu_{n,1}| \geqslant |\mu_{n,2}| \geqslant \dots$). Since $\tilde{K}_n \notin \mathfrak{S}_\omega$ there is a number $m = (m_n)$ such that

$$1 \leqslant \sum_{s=1}^{m} \frac{|\mu_{n,s}|}{2s-1} \leqslant 1 + |\mu_{n,1}| \leqslant 1 + \|K\|.$$

Since $\mu_{n,s} \to 0$ with $s \to \infty$, a subsequence $\{\mu_{n,k_s}\}_{s=m+1}^{\infty}$ of the sequence $\{\mu_{n,s}\}_{s=m+1}^{\infty}$ may be chosen in such a way that $\sum_{s=m+1} \frac{|\mu_{n,k_s}|}{2s-1} < 1$. Now we see that defining $\mathscr{H}_{n+1}$ as the closed linear subspace of H spaned by $e_{n,1}, \dots, e_{n,m}, e_{n,k_{m+1}}, e_{n,k_{m+2}}, \dots,$ we obtain (8) for $j = n+1$. It follows from the above construction that $\mathscr{H} = \bigoplus_{j=0}^{\infty} \mathscr{H}_j$.

It follows from Theorem III.4.2. of [5] that there exist compact selfadjoint operators $A_n \in \mathscr{L}(\mathscr{H}_n)$ such that

$$\sigma(A_n + iK_n) = \{0\} \quad \text{and} \quad \|K_n\|_\omega/\pi \leqslant \|A_n\| \leqslant 4\|K_n\|_\omega/\pi, \qquad n = 1, 2, \dots .$$

Now set $\bigoplus_{1}^{\infty} A_n = A$, and let P_n be the orthogonal projection on the space $\bigoplus_{0}^{n} \mathscr{H}_j$. It is obvious that

$$(9) \qquad P_n \to 1 \text{ strongly} \quad \text{and} \quad \sigma(P_n(A + iK)|_{P_n\mathscr{H}}) = \{0\}.$$

Let D be the closed disc with center 0 and radius 1/4. Since $\sigma(A + iK|_{\mathscr{H}_n}) = \{0\}$, $H_n \subset X_{A+iK}(D)$. Thus $X_{A+iK}(D)$ is dense in $\mathscr{H}$.

Since $A_n = A_n^*$ is compact, there is a unit vector $x_n \in \mathscr{H}_n$ such that $A_n x_n = \lambda_n x_n$, where $|\lambda_n| = \|A_n\| \geqslant \pi^{-1}$. Hence for some subsequence we have

$$A x_{n_k} = \lambda_{n_k} x_{n_k}, \; \lambda_{n_k} \to \lambda \quad \text{where } |\lambda| \geqslant \pi^{-1}.$$

This shows that λ belongs to the essential spectrum of A, thus also $\lambda \in \sigma(A + iK)$. This and (9) show that iii) is not satisfied. But also $\mathscr{X}_{A+iK}(D)$ cannot be closed, because $\overline{\mathscr{X}_{A+iD}(D)} = \mathscr{H}$ and $\lambda \notin D$. Therefore the operator $A + iK$ is not decomposable.

OPEN PROBLEM

What is the characterization of $S(A)$ in the case when the imaginary part of A is compact but does not belong to the Macaev ideal? The author conjectures that then the condition

$$\sigma(A)\setminus W_e(A) \subset \Omega = \bar{\Omega} \subset \sigma(A) \cup W_e(A) \text{ and } W_e(A) \cap \Omega \neq \emptyset$$

is necessary and sufficient in order that $\Omega \in S(A)$. The above condition is equivalent to $\Omega \in S(A)$, for operators A whose essential numerical range has nonvoid interior [7].

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