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INVARIANT OPERATOR RANGES OF NEST ALGEBRAS

SING-CHEONG ONG

Since the publication of the poineering paper [3] of Foiaş on invariant operator ranges of operator algebras, much progress on this subject has been made by many others. Most of the existing literature on this subject is of the form assuming some conditions on the lattice of invariant operator ranges of operator algebras, then something can be said about the algebras. In [8], Voiculescu calculates, explicitly, the lattices of invariant operator ranges of von Neumann algebras with Schwartz property P. Still very little is known about the lattice of invariant operator ranges of a given operator algebra. In this paper we determine the lattices of invariant operator ranges of some special nest algebras.

1. For a complex Hilbert space $\mathscr{H}$, $\mathscr{B}(\mathscr{H})$ denotes the algebra of (bounded linear) operators on $\mathscr{H}$. A submanifold $\mathscr{R} \subseteq \mathscr{H}$ is an operator range if there exists $T(\geqslant 0$, cf. [2]) in $\mathscr{B}(\mathscr{H})$ such that $T\mathscr{H} = \mathscr{R}$. For an algebra $\mathscr{A} \subseteq \mathscr{B}(\mathscr{H})$, we denote the lattice (cf. [2]) of invariant operator ranges [closed subspaces] of (every element of) $\mathscr{A}$ by $\mathrm{Lat}_{1/2}\mathscr{A}$ [$\mathrm{Lat}\,\mathscr{A}$]. We adapt standard notation in [7].

The following theorem of Foiaş [3] is of great importance.

THEOREM A. *If $\mathscr{A}$ is a uniformly closed algebra, and if $T\mathscr{H} \in \mathrm{Lat}_{1/2}\mathscr{A}$, then there exists a unique bounded algebra homomorphism π of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ such that $AT = T\pi(A)$ and $\pi(A)\mathscr{H} \subseteq \mathrm{N}(T)^{\perp}$ for all $A \in \mathscr{A}$. If $\mathscr{A}$ contains a commutative self-adjoint subalgebra $\mathscr{A}_0$, then T can be chosen positive and in $\mathscr{A}_0'$, the commutant of $\mathscr{A}_0$.*

Combining the structure theorem of [5] and a moment's consideration of the null space, we obtain the following useful theorem.

THEOREM B. *Let P be a positive operator with $P \leqslant I$, and E the spectral measure of P. If an algebra $\mathscr{A}$ leaves invariant the spectral subspaces $E((2^{-k}, 1])\mathscr{H}$, $k = 1, 2, \ldots$, then $\mathscr{A}$ leaves the range $P\mathscr{H}$ invariant.*

2. Let $\mathscr{H} = \sum_{j=1}^{\infty} \oplus \mathscr{H}_j$, $\mathscr{H}_j$ a Hilbert space of dimension $\geqslant 1$, be an orthogonal decomposition of $\mathscr{H}$. Let $\mathscr{A}$ be the algebra of operators on $\mathscr{H}$ leaving invariant $H_1, H_1 \oplus H_2, \ldots$, i.e., $\mathscr{A}$ is the algebra of operators with upper triangular matrices relative to this decomposition. Let I_j denote the identity operator on $\mathscr{H}_j$.

THEOREM 1. $\mathrm{Lat}_{1/2}\,\mathscr{A} = \left\{ D\mathscr{H} : D = \sum_{j=1}^{\infty} \oplus \lambda_j I_j\,,\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0 \right\}.$

Proof. Let $D = \sum_{j=1}^{\infty} \oplus \lambda_j I_j$. We may assume, without loss of generality, that $\lambda_j \leqslant 1$ for all $j = 1, 2, \ldots$. If E is the spectral measure of D, then $E((2^{-k}, 1])\mathscr{H}$ is of the form $\mathscr{H}_1 \oplus \ldots \oplus \mathscr{H}_l$, where l is such that $2^{-k} < \lambda_l$ and $\lambda_{l+1} \leqslant 2^{-k}$. So, by definition of $\mathscr{A}$, $E((2^{-k},1])\mathscr{H}$ is invariant under $\mathscr{A}$. Hence, by Theorem B, $D\mathscr{H}$ is invariant under $\mathscr{A}$.

Conversely, if $T\mathscr{H}$ is invariant under $\mathscr{A}$, then, since $\mathscr{A}$ contains the commutative von Neumann algebra consisting of all operators of the form $\sum_{j=1}^{\infty} \oplus \alpha_j I_j$, where $\{\alpha_j\}$ are bounded sequences of complex numbers, T can be chosen in the commutant of this von Neumann algebra by Theorem A. We may assume T is in the commutant and is positive to begin with; hence, in particular, the projections E_j of $\mathscr{H}$ onto $\mathscr{H}_j$ commute with T. We write T_j for the restriction of T to $\mathscr{H}_j$. Let $\lambda_j = \sup\{\|T_k\| : k = j, j+1, \ldots\}$, and $D = \sum_{j=1}^{\infty} \oplus \lambda_j I_j$. Then, since the restriction D_j of D to $\mathscr{H}$ is $\lambda_j I_j$ which dominates T_j, we have $T \leqslant D$.

Claim: each T_j is either 0 or invertible. Indeed, if $T_j \neq 0$, then $T_j x_j \neq 0$ for some $x_j \in \mathscr{H}_j$, and

$$0 \oplus \ldots \oplus 0 \oplus T_j x_j \oplus 0 \oplus \ldots = T(0 \oplus \ldots \oplus 0 \oplus x_j \oplus 0 \oplus \ldots)$$

is in $T\mathscr{H}$. For any $S_j \in \mathscr{B}(\mathscr{H}_j)$, $S = (0 \oplus \ldots \oplus 0 \oplus S_j \oplus 0 \oplus \ldots)$ is in $\mathscr{A}$, and

$$S(0 \oplus \ldots \oplus 0 \oplus T_j x_j \oplus 0 \oplus \ldots) = (0 \oplus \ldots \oplus 0 \oplus S_j T_j x_j \oplus 0 \oplus \ldots)$$

is in $T\mathscr{H}$. If we let S_j vary we will get

$$(0 \oplus \ldots \oplus 0 \oplus \mathscr{H}_j \oplus 0 \oplus \ldots) \subseteq T\mathscr{H}.$$

This implies $\mathscr{H}_j \subseteq E_j T\mathscr{H} = T_j \mathscr{H}_j$, and thus T_j has full range. Since T_j is positive, it is invertible as asserted.

Let j be such that $T_j \neq 0$. Let

$$\delta_j = \sup\{\delta > 0 : \|T_j x_j\| \geqslant \delta \|x_j\| \text{ for all } x_j \in \mathscr{H}_j\} = \inf(\sigma(T_j)).$$

Also let π be the bounded algebra homomorphism of $\mathscr{A}$ into $\mathscr{B}(\mathscr{H})$ induced by T. Given $\varepsilon > 0$, there exist unit vectors x in $\sum_{k=j}^{\infty} \oplus \mathscr{H}_k$ and y_j in $\mathscr{H}_j$ such that

$\lambda_j - \varepsilon \leqslant \|Tx\|$ and $\delta_j + \varepsilon \leqslant \|Ty_j\|$. The rank-one partial isometry V, defined by $Vz = (z, x)y_j$ for $z \in \mathscr{H}$, is easily seen to be in $\mathscr{A}$. Now, we have

$$\lambda_j - \varepsilon \leqslant \|Tx\| = \|TV^*y_j\| = \|\pi(V)^*Ty_j\| \leqslant \|\pi\|\ \|Ty_j\| \leqslant \|\pi\|\ (\delta_j + \varepsilon).$$

Since ε is arbitrary, we have $\lambda_j I_j \leqslant \|\pi\| T_j$ for $T_j \neq 0$.

Next, we show that if $T_j = 0$ then $T_{j+k} = 0$ for all $k = 0, 1, 2, \ldots$. Suppose this is not so. Then some $T_{j+k} \neq 0$. Let $x_{j+k} \in \mathscr{H}_{j+k}$ be such that $T_{j+k}x_{j+k}$ is a unit vector in $\mathscr{H}_{j+k}$. Let y_j be any unit vector in $\mathscr{H}_j$. The partial isometry V defined by $Vz = (z, Tx_{j+k})y_j$ is, as a simple argument shows, in $\mathscr{A}$; and $VTx_{j+k} = x_j$ is in $T\mathscr{H}$. So $y_j = VTx_{j+k} = Tz$ for some z in $\mathscr{H}$. It is now clear that this implies $T_j \neq 0$ which contradicts our assumption.

These all together show that $\lambda_j I_j \leqslant \|\pi\| T_j$ for all $j = 1, 2, \ldots$; hence

$$D = \sum \oplus \lambda_j I_j \leqslant \|\pi\| \left(\sum_{j=1}^{\infty} \oplus T_j \right) = \|\pi\| T \leqslant \|\pi\|\ D.$$

Since T and D commute, a routine application of a theorem of Douglas ([11]) shows that $T\mathscr{H} = D\mathscr{H}$. The proof is thus complete.

NOTE. This proof can be simplified by observing that the maximal abelian self-adjoint algebra consisting of all diagonal operators is contained in $\mathscr{A}$. But this proof proves the following:

COROLLARY 2. *If $\mathscr{A}$ is the uniformly closed algebra generated by all rank-one operators that leave invariant $\mathscr{H}_1, \mathscr{H}_1 \oplus \mathscr{H}_2, \ldots,$ and all operators of the form $\sum_{j=1}^{\infty} \oplus \alpha_j I_j$, $\alpha_j \in \mathbf{C}$, then*

$$\mathrm{Lat}_{1/2}\mathscr{A} = \left\{ D\mathscr{H} : D = \sum_{j=1}^{\infty} \oplus \lambda_j I_j,\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0 \right\}.$$

COROLLARY 3. *Suppose $\mathscr{H}$ is separable and $\{e_n : n = 1, 2, \ldots\}$ is an orthonormal basis of $\mathscr{H}$. Let $\mathscr{A}$ be the algebra of all operators on $\mathscr{H}$ with upper triangular matrices. Then*

$$\mathrm{Lat}_{1/2}\mathscr{A} = \{D\mathscr{H} : D = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots\},\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0\}.$$

It is proved in [6] that any algebra whose lattice of invariant operator ranges consists only of $\mathscr{H}$ and ranges of compact operators either is strongly dense in $\mathscr{B}(\mathscr{H})$ or has a finite dimensional (non-zero) invariant subspace. In [4], an example satisfying this condition is given, and it verifies the first alternative; the above corollary gives another example verifying the second alternative.

3. Let $\mathscr{H} = L^2[0, 1]$ (with the Lebesgue measure μ). For any (essentially) bounded function φ on $[0, 1]$, M_φ denotes the multiplication operator induced by φ on $\mathscr{H}$, i.e., $M_\varphi f(t) = \varphi(t)f(t)$, $t \in [0, 1]$, $f \in \mathscr{H}$. For each α in $[0, 1]$, let $\mathscr{M}_\alpha$ denote the closed subspace of $\mathscr{H}$ consisting of all functions that vanish a.e. on $[0, \alpha]$. Let $\mathscr{A}$ be the algebra of operators on $\mathscr{H}$ leaving invariant every $\mathscr{M}_\alpha$, $\alpha \in [0, 1]$. It is known ([7]) that $\mathscr{A}$ is reflexive and is the weakly closed algebra generated by M_x (x denotes the function $t \to t$ on $[0, 1]$) and the Volterra operator V:

$$(Vf)(t) = \int_0^t f(s)\,ds, \quad t \in [0, 1], f \in \mathscr{H}.$$

We shall determine all operator ranges invariant under $\mathscr{A}$. Since this algebra is, in a sense, "thinner" than the algebra we considered earlier, the invariant operator ranges are harder to determine. We begin with the following measure theory lemma, which may be trivial to experts. (It is not trivial to us.)

LEMMA 4. *If $\varphi \in L^\infty[0, 1]$ is a non-negative function, then*

$$\varphi^*(t) = \operatorname{ess\,sup}\{\varphi(s) : s \in [0, t]\}, \quad t \in [0, 1],$$

defines a non-negative, non-decreasing, bounded function that is continuous from the left, and $\varphi^* \geqslant \varphi$ a.e..

Proof. Clearly, φ^* is a bounded non-negative and non-decreasing function.. It is not hard to see that it is left-continuous. We prove the inequality $\varphi^* \geqslant \varphi$ a.e. (This is not immediately clear, because of the almost-everywhere definition of φ.)

To prove $\varphi^* \geqslant \varphi$ a.e., we show that for each $n = 1, 2, \dots$ the set

$$E_n = \left\{t : t \in [0, 1],\ \varphi(t) \geqslant \varphi^*(t) + \frac{1}{n}\right\}$$

has measure zero. If E_n has positive measure, then $E_n \cap [0, s]$ has positive measure for some s with

$$\mu(E_n \cap [0, t]) < \mu(E_n \cap [0, s])$$

for all $t < s$. Now, since φ^* is left-continuous, there exists $t_0 \in (0, s)$ such that $\varphi^*(t) \geqslant \varphi^*(s) - \dfrac{1}{2n}$ for all t with $t_0 < t < s$. Also we have $\mu(E_n \cap [t_0, s]) > 0$, and

$$\begin{aligned}\varphi^*(s) &\geqslant \operatorname{ess\,sup}\{\varphi(t) : t \in E_n \cap [0, s]\} \geqslant \\ &\geqslant \operatorname{ess\,sup}\left\{\varphi^*(t) + \frac{1}{n} : t \in E_n \cap [0, s]\right\} \geqslant \\ &\geqslant \operatorname{ess\,sup}\left\{\varphi^*(t) + \frac{1}{n} : t \in E_n \cap [t_0, s]\right\} \geqslant \\ &\geqslant \varphi^*(s) - \frac{1}{2n} + \frac{1}{n} = \varphi^*(s) + \frac{1}{2n},\end{aligned}$$

which is a contradiction. $\mu(E_n) = 0$. $\varphi^* \geqslant \varphi$ a.e. .

Let Φ denote the set of all bounded, non-negative, non-decreasing, left-continuous functions on [0, 1].

THEOREM 5. $\mathrm{Lat}_{1/2}\mathscr{A} = \{M_\varphi \mathscr{H} : \varphi \in \Phi\}$.

Proof. We first prove that all operator ranges of the form $M_\varphi\mathscr{H}$, $\varphi \in \Phi$, are invariant under $\mathscr{A}$. Let $\varphi \in \Phi$ and assume that φ is not identically zero and that $0 \leqslant \varphi \leqslant 1$. For each $k = 1, 2, \ldots$, let χ_k be the characteristic function of the set $E_k = \{t \in [0, 1] : 2^{-k} < \varphi(t) \leqslant 1\}$. Then M_{χ_k} is the spectral projection of M_φ corresponding to the set $(2^{-k}, 1]$. E_k is of the form $(\alpha, 1]$ since φ is non-decreasing and left-continuous. It is now clear that $M_{\chi_k}\mathscr{H} \in \mathrm{Lat}\,\mathscr{A}$, and hence $M_\varphi \mathscr{H} \in \mathrm{Lat}_{1/2}\mathscr{A}$ by Theorem B.

To prove the converse we proceed in steps.

Step (1). We show that for any $T\mathscr{H} \in \mathrm{Lat}_{1/2}\mathscr{A}$ there exists $\varphi^* \in \Phi$ such that $M_{\varphi^*}\mathscr{H} \supseteq T\mathscr{H}$.

Since $\mathscr{A}$ contains the maximal abelian self-adjoint algebra consisting of all M_ψ, $\psi \in L^\infty[0, 1]$, we may assume, by Theorem A, that $T = M_\varphi$, where $\varphi \in L^\infty[0, 1]$ and $0 \leqslant \varphi \leqslant 1$. Define φ^* as in Lemma 4. Then $\varphi^* \in \Phi$ and $M_{\varphi^*} \geqslant M_\varphi$ or $M^2_{\varphi^*} \geqslant M^2_\varphi$ so $M_{\varphi^*}\mathscr{H} \supseteq T\mathscr{H}$ by Douglas' Theorem.

Step (2). If $s \in (0, 1]$ and $\varphi > 0$ on a set E of positive measure contained in $(0, s]$, then $\varphi > 0$ a.e. on $(s, 1]$.

Let $F = \{t \in (s, 1] : \varphi(t) = 0\}$. We shall show that $\mu(F) = 0$. Define the operator K on $\mathscr{H}$ by

$$Kf = (f, \chi_E)\, \chi_F \qquad (f \in \mathscr{H}).$$

Then it is easily seen that K is in $\mathscr{A}$. Consider φ as an element of $\mathscr{H}$, $K\varphi = \left(\int_E \varphi\right) \chi_F$ is in $M_\varphi\mathscr{H}$; $K\varphi$ must vanish on F since $\int_E \varphi > 0$; hence $\chi_F = 0$ or $\mu(F) = 0$, i.e. $\varphi(t) > 0$ a.e. on $(s, 1]$.

Step (3). There exists a constant $c > 0$ such that

$$(2\delta\mu(E)^2)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2} \left(\int_E \varphi(t)\, \mathrm{d}t\right)^2 \mathrm{d}x \leqslant c^2$$

for all $s \in (0, 1)$, $\delta > 0$ with $(s - \delta,\ s + \delta) \leqslant (0,\ 1)$, $E \subseteq (0,\ s - \delta]$ of positive measure and $\varphi > 0$ a.e. on E. (Note. Step (2) assures $(\varphi(t))^{-1}$ is defined a.e. on $(s - \delta, s + \delta)$.)

To see this, let K be the operator defined by

$$Kf = (2\delta\mu(E))^{-1/2}(f, \chi_E)\chi_{(s-\delta,\, s+\delta)} \qquad (f \in \mathscr{H}).$$

A simple argument shows that $K \in \mathscr{A}$ and $\|K\| = 1$.

Since $M_\varphi \mathscr{H} \in \mathrm{Lat}_{1/2}\mathscr{A}$, the homomorphism π, induced by M_φ, is bounded. Let $c = \|\pi\|$. For the unit vector $g = \mu(E)^{-1/2}\chi_E$, $K(\varphi g)$ is in $M_\varphi \mathscr{H}$, and vanishes outside of $(s - \delta, s + \delta)$. By Step (2), $\varphi > 0$ a.e. on $(s - \delta, s + \delta)$. If we define $\frac{0}{0} = 0$, then $\frac{1}{\varphi} K(\varphi g)$ is a function in $L^2[0, 1]$ and is, in fact, $\pi(K)g$. Hence, we have

$$c^2 = \|\pi\|^2 \geqslant \|\pi(K)\|^2 \geqslant \|\pi(K)\, g\|^2 = \left\|\frac{1}{\varphi} K(\varphi g)\right\|^2 =$$

$$= \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2} \left(\int_E (2\delta\mu(E)^2\right)^{-1/2} \varphi(t)\, \mathrm{d}t)^2\, \mathrm{d}x =$$

$$= (2\delta\mu(E)^2)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2} \left(\int_E \varphi(t)\, \mathrm{d}t\right)^2 \mathrm{d}x$$

as desired.

Step (4). Let $0 < \varepsilon < \varphi^*(1)$, so that

$$I_\varepsilon = \{t \in (0, 1) : \varphi^*(t) > \varepsilon\}$$

is of positive measure; and $s \in I_\varepsilon$. We show that there exists a positive δ_s (depending on both ε and s) such that $(s - \delta_s, s + \delta_s) \subseteq I_\varepsilon$ and that

$$(2\delta)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2}\, \mathrm{d}x \leqslant c^2(\varphi^*(s) - \varepsilon)^{-2}$$

for all $\delta \in (0, \delta_s]$.

Given such an s, there exists a measurable set $E \subseteq (0, s)$ with $\mu(E) > 0$ such that $\varphi|E \geqslant \varphi^*(s) - \varepsilon > 0$. There is some $\delta_s > 0$ such that $(s - \delta_s, s + \delta_s) \subseteq I$ and $E \cap (0, s - \delta_s)$ is of positive measure. Replacing E by $E \cap (0, s - \delta_s)$, we find that for any $\delta \in (0, \delta_s]$, s, δ, E satisfy all conditions in Step (3). We conclude, therefore, by Step (3), that

$$(2\delta\mu(E)^2)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2} \left(\int_E \varphi(t)\, \mathrm{d}t\right)^2 \mathrm{d}x \leqslant c^2 .$$

Furthermore, by the choice of E, we have

$$c^2 \geqslant (2\delta\mu(E)^2)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2} \left[\int_E (\varphi^*(s) - \varepsilon)\, \mathrm{d}t\right]^2 \mathrm{d}x =$$

$$= (2\delta)^{-1} (\varphi^*(s) - \varepsilon)^2 \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2}\, \mathrm{d}x .$$

Since $\varphi^*(s) - \varepsilon > 0$, by dividing both sides in the above inequality by $(\varphi^*(s) - \varepsilon)^2$, we obtain

$$(2\delta)^{-1} \int_{s-\delta}^{s+\delta} (\varphi(x))^{-2}\, dx \leqslant c^2(\varphi^*(s) - \varepsilon)^{-2},$$

and this is true for all $\delta \in (0, \delta_s]$.

Step (5). It is now clear that the function $(\varphi(t))^{-2}$ is locally integrable on I_ε. By the Lebesgue differentiation theorem

$$(\varphi(t))^{-2} \leqslant c^2(\varphi^*(t) - \varepsilon)^{-2} \quad \text{a.e.}$$

on I_ε, or $\varphi^*(t) - \varepsilon \leqslant c\varphi(t)$ a.e. on I_ε. Since $\varphi \geqslant 0$ on $[0, 1]$, $\varphi^*(t) - \varepsilon \leqslant c\varphi(t)$ a.e. on $[0, 1]$. By the arbitrariness of ε, we obtain $\varphi^*(t) \leqslant c\varphi(t)$ a.e. on $[0, 1]$; and hence $M_\varphi \leqslant M_{\varphi^*} \leqslant cM_\varphi$. A theorem of Douglas ([1]) with a routine argument show that $M_\varphi \mathscr{H} = M_{\varphi^*} \mathscr{H}$. This completes the the proof.

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SING-CHEONG ONG
University of Victoria
Victoria, B.C.
Canada V8W 2Y2

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