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SPECTRAL MULTIPLICITY FOR DIRECT INTEGRALS OF NORMAL OPERATORS

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INTRODUCTION

The theory of spectral multiplicity solves the unitary equivalence problem for normal operators. Briefly recalling the theory, one associates with each normal operator N a subset $\sigma(N)$ of $\mathbf{C}$, a measure ν on $\mathbf{C}$, and a multiplicity function m defined on $\mathbf{C}$ and taking values in the extended natural numbers. The ordered triple consisting of $\sigma(N)$, the equivalence class $[\nu]$, and the equivalence class $[m]_\nu$ then provide a complete set of unitary invariants for N. The statement that this set of invariants "solves" the unitary equivalence problem for normal operators requires these invariants to be "computable".

Suppose now that (X, μ) is a probability space and N is the operator of multiplication by a bounded Borel function $\varphi : X \to \mathbf{C}$. Then $\sigma(N)$ is the essential range of φ and the scalar spectral measure ν is $\mu \circ \varphi^{-1}$, both of which can be regarded as computable. In their work [1], [2], B. Abrahamse and T. Kriete consider the problem of computing the multiplicity function for N. The folk intuition is that $m(\lambda)$ should be the cardinality of the preimage $\varphi^{-1}(\lambda)$, but Abrahamse and Kriete provide several illuminating examples to show that this intuition is not correct even when μ is Lebesgue measure on [0, 1] and φ is smooth. They then introduce the notion of the essential preimage $\varphi_\mu^{-1}(\lambda)$ and show that the function $\lambda \to \operatorname{card} \varphi_\mu^{-1}(\lambda)$ is a multiplicity function for N. The definition of $\varphi_\mu^{-1}(\lambda)$ depends on a limiting process so that the Abrahamse-Kriete multiplicity function, though concrete, is somewhat removed from the original function φ.

The function $m_\varphi = \operatorname{card} \varphi^{-1}(\cdot)$ is in general too large to be a multiplicity function for N; however; it is always possible to find a Borel function φ_0 agreeing with φ almost everywhere so that m_{φ_0} does provide a multiplicity function. This result, which was independently derived by J. Howland [8], appears as Theorem 4.1 below. This description of multiplicity seems quite natural in view of the fact that φ and φ_0 induce the same multiplication operator.

Multiplication operators are direct integrals of operators acting on one-dimensional spaces. In Theorem 5.2 of this paper, we generalize Theorem 4.1 by showing how to compute the multiplicity function for a direct integral of normal operators in terms of the multiplicity functions of the direct integrands.

The structure of the remainder of the paper is as follows. The first three sections treat preliminaries in measure theory, direct integral theory, and multiplicity theory respectively; with the possible exception of Proposition 2.1, this material is well-known. In Section 4, we study multiplicity functions for multiplication operators. The main result of this section is Theorem 4.1 discussed above. Several applications of this result are then presented, including some guidance in identifying the modified function φ_0, and a discussion of the relationship between the unitary equivalence of multiplication operators and the quasi-equivalence of the underlying measure spaces.

The final section of the paper deals with direct integrals of normal operators. Given such a direct integral $N = \int_X^{\oplus} N(x)\mathrm{d}\mu$, we first construct a special model for N which simultaneously represents N and the $\{N(x)\}$ as multiplication operators; in a special case, this model is similar to one constructed by J. Ball in [3]. This model is then used to describe the scalar spectral measure and multiplicity function for N. As a simple application of this multiplicity theory we show that self-adjoint Toeplitz operators based on the half-plane always have uniform infinte multiplicity. The paper closes with an alternative derivation of a multiplicity function for direct integrals which was introduced by Ball [3].

1. MEASURE-THEORETIC PRELIMINARIES

Throughout this preliminary section X, Y will denote complete separable metric spaces (Polish spaces). A probability measure on X or Y will mean a (regular) Borel probability measure. The measure will automatically be completed.

Let μ be a probability measure on X. A field of probability measures $x \to \mu_x$ on Y will be said to be *μ-compatible* in case for each Borel set $B \subset Y$ the function $x \to \mu_x(B)$ is μ-measurable.

The following pair of propositions show how fields of compatible measures are related to measures on the product space $X \times Y$.

If E is a subset of $X \times Y$, then we will use the standard notations $E_x = \{y \in Y \mid (x, y) \in E\}$ and $E^y = \{x \in X \mid (x, y) \in E\}$. The projections of E onto X and Y will be denoted by $\pi_X(E)$ and $\pi_Y(E)$, respectively. If f is a function on $X \times Y$, we denote the corresponding functions $f(x, \cdot)$ and $f(\cdot, y)$ of one variable by f_x and f^y, respectively.

The first result is a generalization of Fubini's Theorem.

PROPOSITION 1.1. *Let $\{\mu_x\}_{x\in X}$ be μ-compatible field of probability measures on Y. There is a unique probability measure ρ on $X \times Y$ such that*

$$\int_{X\times Y} f\,\mathrm{d}\rho = \int_X \left[\int_Y f_x\,\mathrm{d}\mu_x\right]\mathrm{d}\mu \tag{1.1}$$

for every non-negative Borel function on $X \times Y$. Moreover, formula (1.1) *continues to hold for all complex-valued ρ-integrable functions on $X \times Y$.*

Proof. Let $E = A\times B \subset X\times Y$ where $A \subset X$ and $B \subset Y$ are Borel. Then $x \mapsto \mu_x(E_x)$ is μ-measurable. Standard techniques can be used to show that

$$\rho(E) = \int_X \mu_x(E_x)\,\mathrm{d}\mu$$

extends to a Borel probability measure on $X \times Y$. The fact that ρ satisfies (1.1) is established in a routine manner. We omit the details.

We will use the notation $\int_X^{\oplus} \mu_x\,\mathrm{d}\mu$ for the measure ρ constructed in the preceding proposition. The measure ρ will be called *the direct integral of the measures* $\{\mu_x\}_{x\in X}$ with respect to μ.

Let ρ be a probability measure on $X \times Y$. Then the probability measure $\mu(A) = \rho(A \times Y)$ is called the *X-marginal measure* of ρ.

The following proposition is a special case of the theorem on disintegration of measures (see, for example, [2], [5] in particular, Theorem V 8.1 in [5]). It should be regarded as a converse to Proposition 1.1.

PROPOSITION 1.2. *Let ρ be a probability measure on $X \times Y$ and μ the X-marginal measure of ρ. Then there exists a μ-compatible field of measures $\{\mu_x\}_{x\in X}$ such that $\rho = \int_X^{\oplus} \mu_x\,\mathrm{d}\mu$. Moreover, the measures μ_x are uniquely determined almost everywhere with respect to μ.*

The two following propositions will be applied in the sequel.

PROPOSITION 1.3. *Let μ be a probability measure on X and $\{\mu_x\}_{x\in X}$ a μ-compatible field of probability measures on Y. Then there is a Borel set $S \subset X \times Y$ such that for μ-almost all x, S_x is the closed support of μ_x.*

Proof. Let U_n, $n = 1, 2, \ldots$ be a base for the topology of Y. The set $E_n = \{x\in X : \mu_x(U_n) = 0\}$ is *μ-measurable*. Choose a Borel set $F_n\subset E_n$ with $\mu(E_n\setminus F_n) = 0$. Let $S = X\times Y\setminus[\bigcup_n (F_n\times U_n)]$. Obviously, S is Borel and with $\rho=\int_X^{\oplus} \mu_x\mathrm{d}\mu$, $\rho(S) = 1$. Thus $\mu(\pi_X(S)) = 1$ and for μ-almost all x, $\mu_x(S_x)=1$. It is easily verified that when $x \notin [X \setminus \pi_X(S)] \cup [\bigcup_n (E_n \setminus F_n)]$ and $\mu_x(S_x) = 1$, then S_x is the closed support of μ_x. This ends the proof.

PROPOSITION 1.4. (Principle of measurable choice). *Let μ be a measure on X and suppose E is a Borel set in $X \times Y$. Then $\pi_X(E)$ is μ-measurable, and there is a Borel set $F \subset E$ such that no F_x contains more than one point and F_x is non-empty for μ-almost all $x \in \pi_X(E)$. In particular if $\mu(\pi_X(E))=1$, there is a Borel function $f: X \to Y$ such that for μ-almost all x, $(x, f(x)) \in E$.*

A proof of Proposition 1.4 is given, for example, in [4, p. 332].

2. DIRECT INTEGRALS OF HILBERT SPACES

In this section we recall some terminology and background information concerning direct integral theory. The main reference is [4].

Let X be a Polish space and μ a probability measure on X. A *field of Hilbert spaces* is an association $x \to \mathscr{H}(x)$ of a non-trivial separable Hilbert space $\mathscr{H}(x)$ with each $x \in X$. A function (section) $f: X \to \bigcup_{x \in X} \mathscr{H}(x)$ satisfying $f(x) \in \mathscr{H}(x)$ is called a *field of vectors*. A *measurable field of Hilbert spaces* is a field $\{\mathscr{H}(x)\}_{x \in X}$ of Hilbert spaces and a given distinguished subspace $\mathscr{S}$ of fields of vectors satisfying:

(i) For each $f \in \mathscr{S}$, the function $x \to \|f(x)\|$ is μ-measurable.

(ii) If g is a field of vectors such that $x \to \langle f(x), g(x) \rangle$ is μ-measurable for all $f \in \mathscr{S}$, then $g \in \mathscr{S}$.

(iii) There is a countable set $P \subset \mathscr{S}$ such that for each $x \in X$, $\{f(x): f \in P\}$ spans $\mathscr{H}(x)$.

The elements in $\mathscr{S}$ are called *measurable fields of vectors*.

A field $f \in \mathscr{S}$ is called *square integrable* if $\int \|f(x)\|^2 \mathrm{d}\mu < \infty$. The collection of μ-equivalence classes of square-integrable measurable fields of vectors furnished with the inner product

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \, \mathrm{d}\mu$$

forms a Hilbert space denoted by $\int_X^{\oplus} \mathscr{H}(x) \mathrm{d}\mu$ and called the *direct integral* associated with the field $\{\mathscr{H}(x)\}_{x \in X}$. We give the usual caution that the notation $\int_X^{\oplus} \mathscr{H}(x) \, \mathrm{d}\mu$ does not make explicit the dependence of the direct integral on the subspace $\mathscr{S}$ of measurable fields of vectors.

Two examples of this construction are important in the sequel.

1. Assume all the spaces $\mathscr{H}(x)$ $(x \in X)$ coincide with a fixed Hilbert space $\mathscr{H}_0$. Let $\mathscr{S}_0$ be the set of vector fields f such that $x \to \langle f(x), g \rangle$ is μ-measurable $(g \in \mathscr{H}_0)$. Clearly, $\mathscr{S}_0$ satisfies (i)-(iii). In this case the measurable field of Hilbert spaces $\{\mathscr{H}(x)\}_{x\in X}$ will be said to be given by the *constant fields of vectors*. The associated direct integral space in this case is usually denoted by $L_2(\mu) \otimes \mathscr{H}_0$.

2. Let Y be another Polish space and $\{\mu_x\}_{x\in X}$ be a μ-compatible field of probability measures on Y. Set $\mathscr{H}(x) = L_2(\mathrm{d}\mu_x)$. Let $\mathscr{S}_1$ be the collection of fields of vectors $x \to f(x) \in L_2(\mathrm{d}\mu_x)$ such that for every Borel set $U \subset Y$,

$$x \to \langle f(x), \chi_U \rangle = \int_U f(x)(y)\, \mathrm{d}\mu_x(y)$$

is μ-measurable. Since $\{\mu_x\}_{x\in X}$ is μ-compatible, it is plain that $\mathscr{S}_1$ possesses property (ii). If $\{U_n\}_{n=1}^{\infty}$ is a base for the topology Y and $\{V_n\}_{n=1}^{\infty}$ is an enumeration of the collection of finite unions of the U_n's, then the measurable fields of vectors $g_n(x) = \chi_{V_n}$ have the property that for every $x \in X$, $\{g_n(x) | n = 1, 2, \ldots\}$ span $L^2(\mathrm{d}\mu_x)$. Thus $\mathscr{S}_1$ has property (iii). Further, if we apply the Gramm-Schmidt process pointwise to the $g_n(x)$, then we obtain a sequence $\{\varphi_n\}$ of measurable fields of vectors such that for each $x \in X$, $\{\varphi_n(x)\}$ is a basis in $L^2(\mathrm{d}\mu_x)$. These fields of vectors φ_n have the form

$$\varphi_n(x) = \sum_{j=1}^{n} a_j^{(n)}(x)\, \chi_{V_j},$$

where $a_1^{(n)}, \ldots, a_n^{(n)}$ are scalar μ-measurable functions. Clearly, if $f \in S$, then

$$\|f(x)\|^2 = \lim \sum_{j=1}^{n} |\langle f(x), \varphi_n(x) \rangle|^2$$

is μ-measurable. This shows $\mathscr{S}_1$ has the structure of a subspace of measurable fields of vectors. Accordingly, we have the direct integral Hilbert space $\int_X^{\oplus} L^2(\mathrm{d}\mu_x)\, \mathrm{d}\mu$.

The collection $\mathscr{S}_1$ of measurable fields of vectors described above is the same as the subspace of measurable fields of vectors employed in [2]. It is interesting to observe that the elements in $\mathscr{S}_1$ can be considered as sections of measurable functions on $X \times Y$. More precisely, we formulate the following:

PROPOSITION 2.1. *Let $\{\mu_x\}_{x\in X}$ be a μ-compatible field of probability measures on Y and let $\mathscr{S}_1$ be the subspace of measurable fields of vectors defined above. Denote by $\mathscr{S}_2$ the collection of fields of vectors f for which there is a Borel function g on $X \times Y$ such that for μ-almost all x, $f(x)$ equals the μ_x equivalence class $[g_x]_{\mu_x}$. Then $\mathscr{S}_1 = \mathscr{S}_2$.*

Proof. Suppose first that $f \in \mathscr{S}_2$ and chose a Borel function g on $X \times Y$ such that $f(x) = [g_x]_{\mu_x}$ for μ-almost all x. Thus if U is a Borel subset of Y, the function sending x to $\langle f(x), \chi_U \rangle$ agrees μ-almost everywhere with the function sending x to $\int_Y \chi_U\, g_x \mathrm{d}\mu_x$ which is μ-measurable by Proposition 1.1. Therefore $f \in \mathscr{S}_1$.

Conversely, suppose $f \in \mathscr{S}_1$. By changing $f(x)$ for x in some μ-null set, we can assume that the function $x \mapsto \|f(x)\|_{L^2(\mu_x)}$ is finite-valued and Borel. Since $\mathscr{S}_1$ and $\mathscr{S}_2$ are both invariant under multiplication by scalar-valued Borel functions on X, we can assume that $\|f(x)\|_{L^2(\mu_x)} \leqslant 1$ for all x. Define a complex measure $\tilde{\rho}$ on $X \times Y$ by

$$\tilde{\rho}(E) = \int_X \left[\int_Y \chi_E(x, y)\, f(x)(y)\, \mathrm{d}\mu_x(y) \right] \mathrm{d}\mu.$$

Note that $\tilde{\rho}$ is absolutely continuous with respect to $\int_X^{\oplus} \mu_x \mathrm{d}\mu$ and let $g : X \times Y \to \mathbf{C}$ be a Borel representative of $\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\rho}$. Fix a basis $\{U_n\}$ for the topology on Y. Then $f(x) \neq [g_x]_{\mu_x}$ if and only if

$$\int_{U_n} f(x)(y)\, \mathrm{d}\mu_x \neq \int_{U_n} g(x, y)\, \mathrm{d}\mu_x$$

for some integer n. The integrals appearing in the latter inequality are measurable functions of x. If the equation $f(x) = [g_x]_{\mu_x}$ did not hold almost everywhere with respect to μ, there would therefore exist an integer n and a Borel set $A \subseteq X$ for which

$$\int_A \left[\int_{U_n} f(x)(y) \mathrm{d}\mu_x \right] \mathrm{d}\mu \neq \int_A \left[\int_{U_n} g(x, y)\, \mathrm{d}\mu_x \right] \mathrm{d}\mu.$$

But the latter two integrals are $\tilde{\rho}(A \times U_n)$ and $\int_{A \times U_n} g\, \mathrm{d}\rho$ respectively which are equal by definition of g. Thus $f(x)$ does equal $[g_x]_{\mu_x}$ almost everywhere with respect to μ so $f \in \mathscr{S}_2$ and the proof is complete.

It follows immediately from the preceding proposition that $L^2(\rho) \left(\rho = \int_X^{\oplus} \mu_x \mathrm{d}\mu \right)$ and $\int_X^{\oplus} L^2(\mu_x) \mathrm{d}\mu$ are naturally isomorphic. Indeed, if $f \in L^2(\rho)$ and g is a Borel representative of f, the field of vectors $Uf(x) = [g_x]_{\mu_x}$ is square integrable and its equivalence class in $\int_X^{\oplus} L^2(\mu_x) \mathrm{d}\mu$, which we still denote by Uf, satisfies $\|Uf\| = \|f\|$. The result in Proposition 2.1 makes it clear that U is a unitary mapping from $L^2(\rho)$ onto $\int_X^{\oplus} L^2(\mu_x)\, \mathrm{d}\mu$.

We conclude this section with some remarks on decomposable operators.

Let $\{\mathscr{H}(x)\}_{x\in X}$ and $\{\tilde{\mathscr{H}}(x)\}_{x\in X}$ be two μ-measurable fields of Hilbert spaces given, respectively, by the subspaces $\mathscr{S}$ and $\tilde{\mathscr{S}}$ of measurable vectors. The associated direct integral spaces will be denoted by $\mathscr{H} = \int_X^{\oplus} \mathscr{H}(x)\mathrm{d}\mu$ and $\tilde{\mathscr{H}} = \int_X^{\oplus} \tilde{\mathscr{H}}(x)\,\mathrm{d}\mu$.

The fields of Hilbert spaces $\{\mathscr{H}(x)\}_{x\in X}$ and $\{\tilde{\mathscr{H}}(x)\}_{x\in X}$ are called *isomorphic* in case for each $x\in X$ there is a unitary operator $U(x)$ from $\mathscr{H}(x)$ to $\tilde{\mathscr{H}}(x)$ such that $\tilde{\mathscr{S}}=\{\{U(x)f(x)\}_{x\in X}|f\in\mathscr{S}\}$. A result in [4, p. 145] establishes that every measurable field of Hilbert spaces $\{\mathscr{H}(x)\}_{x\in X}$ satisfying $\dim\mathscr{H}(x) = \text{constant}$ is isomorphic to a measurable field of Hilbert spaces given by the constant fields of vectors.

An operator $A : \mathscr{H} \to \tilde{\mathscr{H}}$ will be called *decomposable* in case for each x there is an operator $A(x) : \mathscr{H}(x) \to \tilde{\mathscr{H}}(x)$ such that

(i) For $f = \{f(x)\}_{x\in X}$ in $\mathscr{S}$, $\{A(x)f(x)\}_{x\in X}$ is in $\tilde{\mathscr{S}}$,

(ii) μ-ess sup $\|A(x)\| < \infty$, and

(iii) for $f\in\mathscr{H}$, Af is the μ-equivalence class of $\{A(x)f(x)\}_{x\in X}$ in $\mathscr{H}$.

We write in this case $A = \int_X^{\oplus} A(x)\,\mathrm{d}\mu$.

Two decomposable operators $A=\int_X^{\oplus} A(x)\mathrm{d}\mu$ on $\mathscr{H}$ and $B=\int_X^{\oplus} B(x)\,\mathrm{d}\mu$ on $\tilde{\mathscr{H}}$ are called *isomorphic* in case there is a decomposable unitary transformation $U : \mathscr{H} \to \tilde{\mathscr{H}}$ such that $UA=BU$. It is a useful result that the pointwise unitary equivalence of $A(x)$ and $B(x)$ insures that A and B are isomorphic. This is the content of the following well-known proposition. We include a proof for completeness.

PROPOSITION 2.2. *Let $A = \int_X^{\oplus} A(x)\mathrm{d}\mu$ and $B = \int_X^{\oplus} B(x)\mathrm{d}\mu$ be decomposable operators on $\mathscr{H} = \int_X^{\oplus} \mathscr{H}(x)\mathrm{d}\mu$ and $\tilde{\mathscr{H}} = \int_X^{\oplus} \tilde{\mathscr{H}}(x)\mathrm{d}\mu$ respectively. If for μ-almost all x, the operator $A(x)$ is unitarily equivalent to $B(x)$, then A and B are isomorphic.*

Proof. Consider first the case where for all $x\in X$, $\tilde{\mathscr{H}}(x) = \mathscr{H}(x) = \mathscr{H}$ and the subspace $\mathscr{S}$ of measurable fields of vectors in both $\{\mathscr{H}(x)\}_{x\in X}$, $\{\tilde{\mathscr{H}}(x)\}_{x\in X}$ is given by the constant field of vectors. If we equip the unit ball in the algebra $\mathscr{L}(\mathscr{H}_0)$ of bounded operators on $\mathscr{H}_0$ with the strong operator topology, then $\mathrm{Ball}[\mathscr{L}(\mathscr{H}_0)]$ is a Polish space. We may assume the fields of operators $\{A(x)\}_{x\in X}$, $\{B(x)\}_{x\in X}$ are Borel functions from X to $\mathrm{Ball}[\mathscr{L}(\mathscr{H}_0)]$. Then the set

$$E = \{(x, U)|\,UA(x) = B(x)U,\ U\text{-unitary}\}$$

is a Borel set in $X\times \mathrm{Ball}[\mathscr{L}(\mathscr{H}_0)]$. Applying Proposition 1.4, we obtain a Borel mapping $U : X \to \mathrm{Ball}[\mathscr{L}(\mathscr{H}_0)]$ such that for μ-almost all x, $(x, U(x))\in E$. Clearly, $\int_X^{\oplus} U(x)\mathrm{d}\mu(x)$ is the desired decomposable unitary.

The general case can be reduced to the case of constant fields by appealing to the result in [4, Prop. 3, p. 145]. This completes the proof.

3. THE SPECTRAL MULTIPLICITY THEOREM

In this section we briefly formulate the spectral multiplicity theorem for a single normal operator. This section contains nothing new. It is presented only to prescribe standard notations and terminology.

Let N be a normal operator on the separable Hilbert space $\mathscr{H}$ and let $N = \int \lambda \mathrm{d}E(\lambda)$ be the spectral resolution of N. A probability measure ν on $\mathbf{C}$ is called a *scalar spectral measure* for N in case $E(S) = 0$ if and only if $\nu(S) = 0$, for S a Borel set in $\mathbf{C}$.

Let ν be a probability measure on $\mathbf{C}$ with compact support. Let $\{\mathscr{H}(\lambda)\}_{\lambda\in\mathbf{C}}$ be a ν-measurable field of Hilbert spaces and $\int_{\mathbf{C}}^{\oplus} \mathscr{H}(\lambda)\mathrm{d}\nu$ the associated direct integral space. There is a natural decomposable normal operator M_λ defined on $\int_{\mathbf{C}}^{\oplus} \mathscr{H}(\lambda)\,\mathrm{d}\nu$ by $M_\lambda f(\lambda) = \lambda f(\lambda)$. The measure ν is a scalar spectral measure for M_λ.

THEOREM 3.1. (Spectral Multiplicity Theorem). *Let N be a normal operator on $\mathscr{H}$. There is a probability measure ν supported on the spectrum of N, and a ν-measurable field of Hilbert spaces $\{\mathscr{H}(\lambda)\}_{\lambda\in\mathbf{C}}$ such that N is unitarily equivalent to M_λ on $\int_{\mathbf{C}}^{\oplus} \mathscr{H}(\lambda)\mathrm{d}\nu$. Moreover, the measure ν is unique modulo mutual absolute continuity and the ν-measurable field $\{\mathscr{H}(\lambda)\}_{\lambda\in\mathbf{C}}$ is unique up to isomorphism.*

The spectral multiplicity theorem provides a complete set of unitary invariants for normal operators. This set of invariants is the pair $([\nu], [m]_\nu)$, where $[\nu]$ is the equivalence class of scalar spectral measures for N and $[m]_\nu$ is the $[\nu]$-equivalence class of the function $m(\lambda) = \dim \mathscr{H}(\lambda)$ such that N is unitarily equivalent to M_λ on $\int_{\mathbf{C}}^{\oplus} \mathscr{H}(\lambda)\mathrm{d}\nu$. We will refer to $[m]_\nu$ as the *multiplicity class* of N. A representative of the multiplicity class will be called a *multiplicity function* for N.

We outline a proof of the existence assertion in Theorem 3.1; this construction generalizes the usual proof that every normal operator having a cyclic vector is unitarily equivalent to a position operator, and will be used in Section 5.

Choose an orthonormal set $\{e_q\}_{q=1}^{p}$ $(1 \leqslant p \leqslant \infty)$ in $\mathscr{H}$ such that the smallest N-cyclic subspaces $\mathscr{M}_q$ containing e_q are pairwise orthogonal with $\mathscr{H} = \sum_{q=1}^{p} \oplus \mathscr{M}_q$. We define a measure μ on the Borel subsets S of $\mathbf{C} \times \mathbf{N}$ as follows

If $p < \infty$, then

$$\mu(S) = \frac{1}{p} \sum_{q=1}^{p} (E(S_q)\, e_q, e_q)$$

and when $p = \infty$

$$\mu(S) = \sum_{q=1}^{\infty} \frac{1}{2^q} (E(S^q)\, e_q, e_q).$$

It is easily seen that the operator N is unitarily equivalent to M_λ on $L^2(\mu)$, where $M_\lambda f(\lambda, n) = \lambda f(\lambda, n)$. Further the **C**-marginal measure ν of μ is a scalar spectral measure for N and the disintegration $\mu = \int_{\mathbf{C}}^{\oplus} \theta_\lambda \, d\nu$ provides a ν-measurable field of Hilbert spaces $\{L^2(\theta_\lambda)\}_{\lambda \in \mathbf{C}}$ such that N is unitarily equivalent to M_λ on $\int_{\mathbf{C}}^{\oplus} L^2(\theta_\lambda) d\nu$ as described in Theorem 3.1. For a proof of the uniqueness of the field $\{H(\lambda)\}_{\lambda \in \mathbf{C}}$ the reader is referred to [4].

Finally we remark for later use that if ν is a given scalar spectral measure and m a given multiplicity function for N, then it is possible to construct a model for N more directly. Indeed, for $1 \leqslant p \leqslant \infty$, let $E_p = \{\lambda \in \mathbf{C} \mid m(\lambda) = p\}$ and define a measure $\tilde{\mu}_p$ on $\mathbf{C} \times \mathbf{N}$ by

$$\tilde{\mu}_p(S) = \begin{cases} \dfrac{1}{p} \displaystyle\sum_{q=1}^{p} \nu(S^q \cap E_p), & 1 \leqslant p < \infty \\ \displaystyle\sum_{q=1}^{\infty} \dfrac{1}{2^q} \nu(S^q \cap E_\infty), & p = \infty. \end{cases}$$

Take $\tilde{\mu} = \sum_{1 \leqslant p \leqslant \infty} \tilde{\mu}_p$. The operator N is unitarily equivalent to M_λ operating on $L^2(\tilde{\mu})$

4. MULTIPLICITY FUNCTIONS FOR MULTIPLICATION OPERATORS

Let X be a Polish space and μ a probability measure on X. As usual $L^\infty(\mu)$ will denote the collection of μ-equivalence classes of essentially bounded complex valued functions on X. Each element $\Phi \in L^\infty(\mu)$ gives rise to a bounded normal operator M_Φ defined on $L^2(\mu)$ by $M_\Phi [f]_\mu = [\varphi f]_\mu$, where we have used the notation φ for any representative of $\Phi \in L^\infty(\mu)$. Usually, it is not crucial to distinguish between an element in $L^\infty(\mu)$ and its various representatives, however, here this distinction will be most important.

As discussed in [2], it is easy to obtain a direct integral decomposition for M_Φ. The following proposition presents this construction from a slightly different perspective.

PROPOSITION 4.1. *Let φ be a representative of the element $\Phi \in L^\infty(\mu)$. Let ρ be the measure* $\rho = \int_X^\oplus \delta_{\varphi(x)} d\mu$ *on* $X \times \mathbf{C}$.

(1) *The operator M_Φ is unitarily equivalent to the operator M_λ defined on $L^2(\rho)$ by $M_\lambda f(x, \lambda) = \lambda f(x, \lambda)$.*

(2) *The measure ν defined for Borel sets $B \subset \mathbf{C}$ by $\nu(B) = \rho(X \times B) = \mu(\varphi^{-1}(B))$ is a scalar spectral measure for M_Φ.*

(3) *If $\rho = \int_{\mathbf{C}}^\oplus \nu_\lambda d\nu(\lambda)$ is the disintegration of ρ with respect to ν, then $m(\lambda) = \dim L^2(\nu_\lambda)$ is a multiplicity function for M_Φ.*

Proof. (1) As we observed in the discussion following the proof of Proposition 2.1 there is a natural unitary map carrying $L^2(\rho)$ onto $\int_X^\oplus L^2(\delta_{\varphi(x)}) d\mu(x)$ which shows that M_λ is pointwise unitarily equivalent to the decomposable operator M_Φ on $L^2(\mu)$. As a consequence of Proposition 2.2, the operators M_Φ and M_λ are unitarily equivalent.

(2), (3). The natural unitary map between $L^2(\rho)$ and $\int_{\mathbf{C}}^\oplus L^2(\nu_\lambda) d\nu(\lambda)$ sends M_λ into the operator M_λ on $\int_{\mathbf{C}}^\oplus L^2(\nu_\lambda) d\nu(\lambda)$. Thus (2) and (3) are obvious. This completes the proof.

We remark that the measures ρ, ν appearing in Proposition 4.1 are clearly independent of the representative φ.

The result in Proposition 4.1 cannot be regarded as a computation of the multiplicity function of M_Φ since disintegration of measures is not a constructive process. Nevertheless, it does provide information in certain cases. For example, if μ is totally atomic, it is not difficult to see that each ν_λ is totally atomic with $\dim L^2(\nu_\lambda)$ equal to the number of atoms in $\varphi^{-1}(\lambda)$. In particular, M_Φ will have uniform multiplicity one if and only if some (hence every) representative of Φ is one-to-one off a set of measure zero. The next theorem shows that a similar analysis can be made in the general case.

Suppose $\varphi : X \to \mathbf{C}$ is a Borel function. We will write $m_\varphi(\lambda)$ for the number of points in $\varphi^{-1}(\lambda)$. We use the convention $m_\varphi(\lambda) = \infty$ unless $\varphi^{-1}(\lambda)$ is a finite set. The following lemma when applied to the graph of φ shows that m_φ is universally measurable.

LEMMA 4.1. *Let X and Y be Polish spaces and ν a measure on X. Suppose E is a Borel set in $X \times Y$. For each integer n, the set of $x \in X$ such that E_x contains exactly n points is ν-measurable.*

Proof. Let $E_n = \{x \in X | E_x$ contains at least n points$\}$. It clearly suffices to show that each E_n is ν-measurable. That $E_1 = \pi_X(E)$ is ν-measurable follows from

Proposition 1.4. Let $F \subseteq X$ be a Borel set chosen as in Proposition 1.4, and set $G = E \setminus F$. Then for each $n \geqslant 1$, the sets E_{n+1} and $G_n = \{x \in X | G_x$ contains at least n points$\}$ differ by a ν-null set. Thus an easy induction argument shows E_n is ν-measurable for all n and the proof is complete.

THEOREM 4.1. *Let $\Phi \in L^\infty(\mu)$, let ν be the scalar spectral measure $\nu = \mu \circ \Phi^{-1}$ of the multiplication operator M_Φ on $L^2(\mu)$, and let m_0 be one of its multiplicity functions. If φ is any Borel representative of Φ, then for ν-almost all λ, $m_0(\lambda) \leqslant m_\varphi(\lambda)$. There is a Borel representative φ_0 of Φ such that m_{φ_0} is a multiplicity function for M_Φ.*

Proof. Let φ be any Borel representative of Φ. Define $\rho = \int_X^\oplus \delta_{\varphi(x)} \mathrm{d}\mu$ as in Proposition 4.1 and disintegrate ρ with respect to ν. Thus $\rho = \int_{\mathbf{C}}^\oplus \nu_\lambda \mathrm{d}\nu$. Since ρ is supported on the graph of φ, for ν-almost all λ, the measure ν_λ is supported on $\varphi^{-1}(\lambda)$. Since $\dim L^2(\nu_\lambda)$ equals the cardinality of the closed support of ν_λ, we have for ν-almost all λ, $m_0(\lambda) \leqslant m_\varphi(\lambda)$.

To complete the proof it suffices to construct a Borel representative φ_0 of Φ, such that for ν-almost all λ, $m_0(\lambda) = m_{\varphi_0}(\lambda)$. Let φ be a fixed Borel representative and define $\rho = \int_{\mathbf{C}}^\oplus \nu_\lambda \mathrm{d}\nu$ as above. Applying Proposition 1.3, we know there is a Borel set $E \subset X \times \mathbf{C}$ such that for ν-almost all λ, the set E^λ is the closed support of ν_λ. Let G be the intersection of E and the graph of φ. Since G is Borel and $\pi_X : G \to X$ is one-to-one, G is the graph of a Borel function, namely, φ restricted to $D = \pi_X(G)$. As ρ is supported on G, we have $\mu(X \setminus D) = 0$. Choose a $\lambda_0 \in \mathbf{C}$ with $\nu(\lambda_0) = 0$ (any λ_0 with $|\lambda_0| > \|\Phi\|_\infty$ will do). Define $\varphi_0 : X \to \mathbf{C}$ by

$$\varphi_0(x) = \begin{cases} \varphi(x), & x \in D \\ \lambda_0, & x \in X \setminus D. \end{cases}$$

Then φ_0 is a Borel representative of Φ. Moreover, for ν-almost all λ, the closed support of ν_λ contains $\varphi_0^{-1}(\lambda)$. This shows $m_0(\lambda) \geqslant m_{\varphi_0}(\lambda)$ for ν-almost all λ, and completes the proof.

COROLLARY 4.1. *Let Φ be in $L^\infty(\mu)$. The following are equivalent:*

(1) *The multiplication operator M_Φ on $L^2(\mu)$ has uniform multiplicity one.*

(2) *Every representative of Φ has a one-to-one restriction off a set of μ-measure zero.*

(3) *There is a one-to-one Borel representative of Φ.*

Proof. The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are trivial consequences of Theorem 4.1. We show (2) $\Rightarrow$ (3).

Let φ be a Borel representative of Φ. From (2) we learn there is a Borel set $A \subset X$ with $\mu(A) = 0$ off of which φ is one-to-one. Let $B = \{\lambda \in \mathbf{C} | \, |\lambda| > \| \Phi\|_\infty\}$

and $C = A \cup \varphi^{-1}(B)$. Since $\nu(B) = 0$, we have $\mu(C) = 0$. There is an one-to-one Borel map ψ of C into B. (See e.g. [5, Theorem 2.12, p. 14]). Let φ_0 agree with ψ on C and φ off C. Then φ_0 is a one-to-one Borel representative of Φ. This completes the proof.

REMARKS (1) If φ is merely assumed to be a μ-measurable representative, the function m_φ can fail to be ν-measurable. However the inequality in Theorem 4.1 still holds, i.e., $\{\lambda | m_0(\lambda) \leqslant m_\varphi(\lambda)\}$ is ν-measurable of measure 1.

(2) If ν is not totally atomic, every Borel multiplicity function for M_Φ is of the form m_φ for a suitable Borel representative φ of Φ.

(3) The usual well-ordering on $\mathbf{N}_\infty = \{0, 1, 2, \ldots \infty\}$ induces an order $\prec$ on the collection of ν-equivalence classes of ν-measurable functions from $\mathbf{C}$ to $\mathbf{N}_\infty$. Theorem 4.1 expresses the multiplicity class of M_Φ as

$$\inf_{\prec}\{[m_\varphi]_\nu | \varphi \text{ is a Borel representative of } \Phi\}$$

with the infimum in question being attained. The interplay between functions and equivalence classes is critical here; the pointwise infimum of $\{m_\varphi | \varphi$ is a Borel representative of $\Phi\}$ is almost never a multiplicity function for M_Φ.

We now turn to a couple of applications of Theorem 4.1. The following definition and theorem make this easier.

Let Φ be in $L^\infty(\mu)$ and m a multiplicity function for M_Φ. The Borel representative φ of Φ is called *distinguished* provided whenever E is a Borel set in X with $\mu(E) = 0$, then $\{\lambda \in \varphi(E) | m(\lambda) < \infty\}$ has ν-measure zero.

THEOREM 4.2. *Let $\Phi \in L^\infty(\mu)$ and suppose φ is a Borel representative of Φ. Then φ is distinguished if and only if m_φ is a multiplicity function for M_Φ.*

Proof. Suppose φ is distinguished. Choose a Borel representative φ_0 of Φ such that m_{φ_0} is a multiplicity function for M_Φ. Then the Borel set $E = \{x \in X | \varphi(x) \neq \varphi_0(x)\}$ has μ-measure zero. Thus $\{\lambda \in \varphi(E) | m_{\varphi_0}(\lambda) < \infty)$ has ν-measure zero. If $\lambda \notin \varphi(E)$, then $\varphi^{-1}(\lambda) \subset \varphi_0^{-1}(\lambda)$. Thus $m_\varphi(\lambda) \leqslant m_{\varphi_0}(\lambda)$ off a set of ν-measure zero. This shows m_φ is a multiplicity function of M_Φ.

Suppose conversely that m_φ is a multiplicity function for M_Φ. Let E be a Borel set in X with $\mu(E) = 0$ and choose $\lambda_0 \in \mathbf{C}$ with $\nu(\{\lambda_0\}) = 0$. Let φ_0 be the Borel function which sends E to λ_0 and agrees with φ off E. Then $\varphi_0^{-1}(\lambda) \subsetneqq \varphi^{-1}(\lambda)$ for all $\lambda \neq \lambda_0$ in $\varphi(E)$. Thus every $\lambda (\neq \lambda_0)$ in $\varphi(E)$ for which $\varphi^{-1}(\lambda)$ is a finite set has $m_{\varphi_0}(\lambda) < m_\varphi(\lambda)$. On the other hand $\{\lambda | m_{\varphi_0}(\lambda) < m_\varphi(\lambda)\}$ must have ν-measure zero. This completes the proof.

Let X_1, X_2 be Polish spaces and μ_1, μ_2 probability measures on X_1, X_2, respectively. The measure spaces (X_1, μ_1) and (X_2, μ_2) will be said to be *quasi-equivalent* in case there are Borel sets $A \subset X_1$, $B \subset X_2$ with $\mu_1(X_1 \setminus A) = \mu_2(X_2 \setminus B) = 0$

and a Borel isomorphism $\eta : A \to B$ such that $E \subset A$ is a Borel set with $\mu_1(E) = 0$ if and only if $\mu_2(\eta(E)) = 0$. The mapping $\eta : A \to B$ will be referred to as a *quasi-equivalence* between (X_1, μ_1) and (X_2, μ_2).

PROPOSITION 4.2. *Let $\Phi_1 \in L^\infty(\mu_1)$, $\Phi_2 \in L^\infty(\mu_2)$ and suppose the multiplication operators M_{Φ_1} and M_{Φ_2} have finite spectral multiplicity. Then the operators M_{Φ_1} and M_{Φ_2} are unitarily equivalent if and only if there is a quasi-equivalence $\eta : A \to B$ between (X_1, μ_1) and (X_2, μ_2) with $f_2(\eta(x)) = f_1(x)$ for μ_1-almost all $x \in A$ and f_1, f_2 arbitrary representatives of Φ_1, Φ_2 respectively.*

Proof. Given the quasi-equivalence $\eta : A \to B$, the map $U : L^2(\mu_1) \to L^2(\mu_2)$ defined by

$$Uf = f \circ \eta^{-1}\left(\frac{d\mu_1 \circ \eta^{-1}}{d\mu_2}\right)^{1/2}$$

defines a unitary equivalence between M_{Φ_1} and M_{Φ_2}.

Suppose next that the operators M_{Φ_1} and M_{Φ_2} are unitarily equivalent. Let ν_1 and ν_2 be the scalar spectral measures for M_{Φ_1} and M_{Φ_2} as described in Proposition 4.1. These measures are mutually absolutely continuous. Let φ_1, φ_2 be distinguished representatives of Φ_1, Φ_2 with m_{φ_1}, m_{φ_2} Borel. By Theorem 4.2 we know that m_{φ_1}, m_{φ_2} are representatives of the multiplicity classes of M_{Φ_1}, M_{Φ_2}.

For each integer n, the set $G_n = \{\lambda | \, m_{\varphi_1}(\lambda) = m_{\varphi_2}(\lambda) = n\}$ is Borel and $\bigcup_{n=1}^{\infty} G_n$ is a set of full ν-measure. If for each n, we can construct a quasi-equivalence η_n between $\Phi_1^{-1}(G_n)$ and $\Phi_2^{-1}(G_n)$ (equipped with the corresponding restrictions of μ_1 and μ_2 respectively), we can put the $\{\eta_n\}$ together to yield a quasi-equivalence η between X_1 and X_2. It follows that there is no loss of generality in assuming M_{Φ_1}, M_{Φ_2} to have uniform multiplicity $n < \infty$.

Applying Proposition 1.4 (several times) to the graph of φ_1 we can find disjoint Borel sets $A_1, \ldots, A_n$ whose union is almost all of X_1 such that each restricted function $\varphi_1|_{A_j}$ is one-to-one. Choose $B_1, \ldots, B_n$ similarly for φ_2. Let $\eta_j : A_j \to B_j$ be the composition of φ_1 and the inverse of $\varphi_2|_{B_j}$ and set $\eta = \bigcup_{j=1}^{n} \eta_j$. If E is a Borel set in A_j with $\mu_1(E) = 0$, then since φ_1 is distinguished, $\nu_1(\varphi_1(E)) = 0$. Thus $\nu_2(\varphi_1(E)) = \mu_2(\varphi_2^{-1} \circ \varphi_1(E)) = 0$. Since $\eta(E) \subset \varphi_2^{-1} \circ \varphi_1(E)$, we conclude that $\mu_2(\eta(E)) = 0$. Thus if E is a Borel μ_1-null set, then $\eta(E)$ is a μ_2-null set. Similarly, $\mu_2(E) = 0$ implies $\mu_1(\eta^{-1}(E)) = 0$ and it follows easily that η is the desired map. This completes the proof.

REMARKS. (4) Let $X_1 = X_2 = [0, 1]$, μ_1 be non-atomic and μ_2 have countably many atoms. Let M_i be the operator of multiplication by the constant function 1 on $L^2(\mu_i)$ $(i = 1, 2)$. Then N_1 and N_2 are unitarily equivalent; however, X_1 and X_2 are not quasi-equivalent. Thus the hypothesis that M_{Φ_1} and M_{Φ_2} have finite multiplicity is necessary in the preceding proposition.

(5) The quasi-equivalence of (X_1, μ_1) and (X_2, μ_2) is necessary and sufficient for a star-isomorphism between the algebras $L^\infty(\mu_1)$ and $L^\infty(\mu_2)$. In fact, every star-isomorphism $\Psi : L^\infty(\mu_2) \to L^\infty(\mu_1)$ is given by $\Psi f(x) = f(\eta(x))$, for some quasi-equivalence η from (X_1, μ_1) to (X_2, μ_2). This result is due to J. von Neumann and a proof is given in Appendix IV of [4]. It is a corollary of Proposition 4.2 that if two unitarily equivalent multiplication operators M_{Φ_1}, $\Phi_1 \in L^\infty(\mu_1)$ and M_{Φ_2}, $\Phi_2 \in L^\infty(\mu_2)$ have finite spectral multiplicity, then the algebras $L^\infty(\mu_1)$ and $L^\infty(\mu_2)$ are star-isomorphic.

(6) In general, the quasi-equivalence η between (X_1, μ_1) and (X_2, μ_2) which appears in Proposition 4.2 is not a Borel isomorphism of X_1 onto X_2. However, if μ_1 and μ_2 are non-atomic then η can be modified to be a Borel isomorphism of X_1 onto X_2.

As a further illustration of the use of Theorem 4.1, we apply it to the discussion in Section 5 of [2]. Recalling the setting, $X = [0, 1]$, μ is Lebesgue measure, φ is a continuously differentiable real valued function on $[0, 1]$ and Φ is the equivalence class of φ in $L^\infty(\mu)$. Set $Z = \{x \in [0, 1] : \varphi'(x) = 0\}$.

PROPOSITION 4.3. *If the boundary of Z has Lebesgue measure zero or if $\varphi(Z)$ is countable, then m_φ is a multiplicity function for M_Φ.*

Proof. Suppose first the boundary ∂Z of Z has Lebesgue measure zero. Express the complement of ∂Z as the disjoint union of relatively open intervals $\{I_n\}$ and $\{J_n\}$ such that each I_n is contained in the complement of Z and each J_n is in the interior of Z.

Assume E is a Borel set in $[0, 1]$ with $\mu(E) = 0$. Note that φ is constant on each J_n so that for $y \in \varphi(J_n)$, we have $m_\varphi(y) = \infty$. Write $\varphi_n = \varphi|_{I_n}$ and let $B = \{y \in \varphi(E) | m_\varphi(y) < \infty\}$. Then

$$\nu(B) = \mu(\varphi^{-1}(B)) = \sum \mu(\varphi_n^{-1}(B)) + \mu(\varphi^{-1}(B) \cap \partial Z).$$

Since φ is absolutely continuous, B has Lebesgue measure zero. Each φ_n^{-1} is absolutely continuous, so $\mu(\varphi_n^{-1}(B)) = 0$. The fact that m_φ is a multiplicity function for M_Φ follows from Theorem 4.2.

If $\varphi(Z)$ is countable, express the complement of Z as the disjoint union of relatively open intervals $\{I_n\}$ as above and again set $\varphi_n = \varphi|_{I_n}$. If E is a Borel set in $[0, 1]$ and $\mu(E) = 0$, we set

$$B = \{y \in \varphi(E) \mid m_\varphi(y) < \infty\}.$$

Then B is essentially disjoint from $\varphi(Z)$. Therefore, $\nu(\varphi^{-1}(B)) = \sum \nu(\varphi_n^{-1}(B)) = 0$ as above. This completes the proof.

REMARK. (7) In Example D of [2], a continuous function φ on $[0, 1]$ is constructed such that m_φ is not a multiplicity function for the multiplication operator induced by φ on $L^2(\mu)$, with μ equal to Lebesgue measure on $[0, 1]$. We note that with the aid of Theorem 4.2 it is possible to take φ to be C^∞.

5. SPECTRAL MULTIPLICIY FOR DIRECT INTEGRALS OF NORMAL OPERATORS

Let X be a Polish space and μ a probability measure on X. Suppose for each $x \in X$, $\mathscr{H}(x)$ equals a fixed separable Hilbert space $\mathscr{H}_0$ and $\{\mathscr{H}(x)\}_{x\in X}$ is the μ-measurable field of Hilbert spaces given by the constant fields of vectors. A decomposable operator $N = \int_X^\oplus N(x)\,\mathrm{d}\mu$ acting on the associated direct integral space $\mathscr{H} = \int_X^\oplus \mathscr{H}(x)\,\mathrm{d}\mu$ is normal if and only if for μ-almost all x the operator $N(x)$ is normal on $\mathscr{H}(x)$. In this section we will give a description of the spectral multiplicity for N in terms of the μ-measurable field of normal operators $\{N(x)\}_{x\in X}$.

The results will be in two directions. First we provide a model (Theorem 5.1) for N as a multiplication operator on $X \times \mathbf{C} \times \mathbf{N}$ ($\mathbf{N} = \{1, 2, \ldots\}$). Second we investigate (Theorem 5.2) the relation between the spectral multiplicity functions of N and $\{N(x)\}_{x\in X}$.

The "simplest" examples of direct integrals of normal operators are multiplication operators M_Φ, $\Phi \in L^\infty(\mu)$, on $L^2(\mu)$. The results of this section are generalizations of those obtained in Section 4 for multiplication operators.

Throughout this section $x \to N(x)$, $(x \in X)$ will denote a fixed bounded Borel mapping of X into the algebra $\mathscr{L}(\mathscr{H}_0)$ such that each $N(x)$ is a normal operator. The operator $N = \int_X^\oplus N(x)\,\mathrm{d}\mu$ will be considered on $\mathscr{H} = \int_X^\oplus \mathscr{H}(x)\,\mathrm{d}\mu$. The spectral resolution of N will be denoted by $N = \int \lambda \mathrm{d}E(\lambda)$ and $N(x) = \int \lambda \mathrm{d}E_x(\lambda)$ will denote the spectral resolution of $N(x)$ $(x \in X)$. The following simple lemma is proved by appealing to Proposition 4 on page 160 of [4].

LEMMA 5.1. *Let B be a Borel set in* $\mathbf{C}$. *Then $x \mapsto E_x(B)$ is a Borel mapping of X to $\mathscr{L}(\mathscr{H}_0)$ and* $E(B) = \int_X^\oplus E_x(B)\,\mathrm{d}\mu$.

LEMMA 5.2. *It is possible to choose a sequence $e_n = \{e_n(x)\}_{x\in X}$ of Borel measurable fields of vectors satisfying the following conditions:*

(1) *The vectors $e_n(x)$, $x \in X$; $n = 1,2, \ldots$ are in the unit ball of $\mathscr{H}_0$.*

(2) *For each* x, *the set* $\{e_n(x)\}_{n=1}^{\infty}$ *is cyclic for* $N(x)$.

(3) *For each* x, *the* $N(x)$*-cyclic subspaces generated by* $e_1(x), e_2(x), \ldots$ *are mutually orthogonal.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a complete orthonormal basis in $\mathscr{H}_0$. Let $e_1(x) = f_1$, $x \in X$. Suppose Borel measurable fields of vectors $e_1, \ldots, e_n$ satisfying (1), (3) have been chosen. The collection of fields of vectors of the form

$$p(N, N^*)e_k = \{p(N(x), N^*(x))e_k(x)\}_{x \in X}, \quad k = 1, \ldots, n$$

where $p(z, w)$ is a polynomial with (complex) rational coefficients form a countable set. The Gramm-Schmidt process may be applied (pointwise) to this collection of fields of vectors to obtain a sequence $\{g_n\}_{n=1}^{\infty}$ of Borel measurable fields of vectors. For all $x \in X$, the sequence $\{g_n(x)\}_{n=1}^{\infty}$ is orthonormal (except for zeros) and span the $N(x)$-cyclic subspace generated by $e_1(x), \ldots, e_n(x)$. We set

$$e_{n+1}(x) = \frac{f_{n+1} - \sum_k \langle f_{n+1}, g_k(x)\rangle g_k(x)}{\| f_{n+1} - \sum_k \langle f_{n+1}, g_k(x)\rangle g_k(x)\|}$$

where this makes sense and zero otherwise.

The sequence $\{e_n\}_{n=1}^{\infty}$ of Borel measurable fields of vectors which is inductively constructed as above clearly satisfies (1) and (3). That it satisfies (2) is due to the fact that f_n belongs to the span of $e_1(x), \ldots, e_n(x)$, for all $x \in X$. This completes the proof.

Let $\{e_n\}_{n=1}^{\infty}$ be a collection of fields of vectors constructed as in the preceding lemma. The sets X_p $(1 \leqslant p \leqslant \infty)$ defined by

$$X_p = \{x \in X | \operatorname{span}\{e_n(x)\}_{n=1}^{\infty} \text{ is } p\text{-dimensional}\}$$

are Borel sets in X with $\bigcup_{1 \leqslant p \leqslant \infty} X_p = X$. By a slight redefinition of the $\{e_n\}$, we can arrange it so that for $x \in X_p$, we have $e_n(x) = 0$ iff $n > p$.

For each $x \in X$ we define a measure μ_x on $\mathbf{C} \times \mathbf{N}$ as follows: If $S \subset \mathbf{C} \times \mathbf{N}$ is a Borel set, we will write $S^q = \{\lambda \in \mathbf{C} \,|\, (\lambda, q) \in S\}$. For $x \in X_p$, $(1 \leqslant p < \infty)$, we define

$$\mu_x(S) = \frac{1}{p} \sum_{q=1}^{p} \langle E_x(S^q)e_q(x), e_q(x)\rangle. \tag{5.1}$$

For $x \in X_\infty$, we define

$$\mu_x(S) = \sum_{q=1}^{\infty} \frac{1}{2^q} \langle E_x(S^q)e_q(x), e_q(x)\rangle. \tag{5.2}$$

It is clear from Lemma 5.1 that for every Borel set $S \subset \mathbf{C} \times \mathbf{N}$, the function $x \to \mu_x(S)$ is Borel measurable.

The next proposition describes a scalar spectral measure for $N = \int_X^{\oplus} N(x)\,\mathrm{d}\mu$ which is obtained from a field of scalar spectral measures for $\{N(x)\}_{x \in X}$.

PROPOSITION 5.1. *For $x \in X$ define the Borel probability measure ν_x on $\mathbf{C}$ by $\nu_x(B) = \mu_x(B \times \mathbf{N})$, where $B \subset \mathbf{C}$ is Borel and μ_x is the measure defined on $\mathbf{C} \times \mathbf{N}$ as in* (5.1) *and* (5.2). *Then $\{\nu_x\}_{x \in X}$ is a μ-compatible field of measures on $\mathbf{C}$ such that for $x \in X$, ν_x is a scalar spectral measure for $N(x)$ and $\nu(\cdot) = \int_X \nu_x(\cdot)\mathrm{d}\mu$ is a scalar spectral measure for $N = \int_X^{\oplus} N(x)\,\mathrm{d}\mu$.*

Proof. The fact that $\{\mu_x\}_{x \in X}$ is μ-compatible makes it clear that $\{\nu_x\}_{x \in X}$ is μ-compatible. For every $x \in X$, ν_x is a scalar spectral measure for $N(x)$ by construction. There remains only to see that the measure defined for $B \subset \mathbf{C}$ Borel by $\nu(B) = \int_X \nu_x(B)\,\mathrm{d}\mu$ is a scalar spectral measure for N.

If B is a Borel set in $\mathbf{C}$ such that $E(B) = 0$, then by Lemma 5.1, $E_x(B) = 0$ for μ-almost all $x \in X$. As a consequence, $\nu_x(B) = 0$ for μ-almost all $x \in X$. This implies $\nu(B) = \int_X \nu_x(B)\mathrm{d}\mu = 0$. On the other hand if $\nu(B) = \int_X \nu_x(B)\,\mathrm{d}\mu = 0$, then there is a Borel set $A \subset X$ with $\mu(A) = 0$ such that $\nu_x(B) = 0$ for all $x \notin A$. Thus $E_x(B) = 0$ for all $x \notin A$. By Lemma 5.1, $E(B) = 0$. This completes the proof.

The following corollary is obvious.

COROLLARY 5.1. *Let $N = \int_X^{\oplus} N(x)\mathrm{d}\mu(x)$. If for μ-almost all x, the scalar spectral measure (class) of $N(x)$ is absolutely continuous with respect to some fixed measure ν_0, then the scalar spectral measure (class) of N is absolutely continuous with respect to ν_0.*

THEOREM 5.1. *Let $N = \int_X^{\oplus} N(x)\mathrm{d}\mu$ and $\{\mu_x\}_{x \in X}$ be the μ-compatible fields of measures defined in* (5.1) *and* (5.2). *Denote by ρ the measure $\rho = \int_X^{\oplus} \mu_x d\mu$ on $X \times \mathbf{C} \times \mathbf{N}$. Let M_λ denote the operator of multiplication by the function $\varphi(x, \lambda, n) = \lambda$ on $L^2(\rho)$. Then N is unitarily equivalent to M_λ.*

Proof. For every $x \in X$ the operator $N(x)$ is unitarily equivalent to the multiplication operator M_{φ_x} on $L^2(\mu_x)$ where $\varphi_x(\lambda, n) = \lambda$ (see Section 3). As a consequence of Proposition 2.2 the operator N is unitarily equivalent to $\int_X^{\oplus} M_{\varphi_x}\mathrm{d}\mu$ on $\int_X^{\oplus} L^2(\mathrm{d}\mu_x)\mathrm{d}\mu$. The natural isomorphism between $L^2(\rho)$ and $\int_X^{\oplus} L^2(\mathrm{d}\mu_x)\mathrm{d}\mu$ described after the proof of Proposition 2.1 implements a unitary equivalence between M_λ and $\int_X^{\oplus} M_{\varphi_x}\mathrm{d}\mu$. This completes the proof.

REMARK. (1) When X is the interval $[0, l]$, μ is Lebesgue measure and N is a direct integral of unitary operators, Theorem 5.1 becomes Theorem 8.1 of [3]. There is an error in the proof of the latter. Namely, the set function $\mu_S[0, 2\pi)$ defined on page 51 of [3] need not be additive as a function of S. In fact, if $U(t) \equiv I$, then $\mu_S[0, 2\pi) = 1$ for every set S of positive Lebesgue measure. The argument given in [3] can easily be patched by deleting Lemma 8.2, defining the measure ν_t as

$$\nu_t(S) = \sum_{i=1}^{\infty} \frac{\langle E_t(S)e_i(t), e_i(t)\rangle}{2^i[\|e_i(t)\|^2 + 1]}$$

and finally replacing Γ_S by Γ in the definition of $n_{i,j}$.

We now turn to a discussion of spectral multiplicity functions for direct integrals of normal operators.

We begin with a result on "double disintegrations" of measures.

LEMMA 5.2. *Assume X, Y, Z are Polish spaces and ρ is a probability measure on $X \times Y \times Z$. Let μ, σ denote, respectively, the X and $X \times Y$ marginal measures of ρ. Let $\rho = \int_X^{\oplus} \mu_x \, d\mu$, $\rho = \int_{X \times Y}^{\oplus} \sigma_{x,y} \, d\sigma$ be the respective disintegrations of ρ. Finally, assume $\sigma = \int_X^{\oplus} \nu_x \, d\mu$ is the disintegration of σ with respect to μ. Then for μ-almost all x, $\{\sigma_{x,y}\}_{y \in Y}$ is a ν_x-compatible field of measures on Z such that $\mu_x = \int_Y^{\oplus} \sigma_{x,y} \, d\nu_x$.*

Proof. Let f be a bounded Borel function on $X \times Y \times Z$. By Proposition 1.1

$$\int_Z f(x, y, z) \, d\sigma_{x,y}$$

is σ-measurable. Two applications of formula (1.1) imply

$$\begin{aligned} \int_{X \times Y \times Z} f \, d\rho &= \int_{X \times Y} \left[\int_Z f(x, y, z) d\sigma_{x,y} \right] d\sigma = \\ &= \int_X \left[\int_Y \left[\int_Z f(x, y, z) d\sigma_{x,y} \right] d\nu_x \right] d\mu. \end{aligned} \tag{5.3}$$

The fact that

$$\int_Y \left[\int_Z f(x, y, z) \, d\sigma_{x,y} \right] d\nu_x$$

is μ-measurable is implicit in this last identity. In particular if $B \subset Y \times Z$ is a Borel set, the measures

$$\tilde{\mu}_x(B) = \int_Y \left[\int_Z \chi_B(y, z)\, d\sigma_{x,y}\right] d\nu_x$$

define a μ-compatible field of measures $\{\tilde{\mu}_x\}_{x\in X}$ on $Y\times Z$. For every bounded Borel function f on $X \times Y \times Z$, it follows from (5.3) that

$$\int_{X\times Y\times Z} f\, d\rho = \int_X \left[\int_{Y\times Z} f(x, y, z)\, d\tilde{\mu}_x\right] d\mu =$$
$$= \int_X \left[\int_{Y\times Z} f(x, y, z) d\mu_x\right] d\mu.$$

By uniqueness of disintegration of measures for μ-almost all x, $\mu_x = \tilde{\mu}_x$, and consequently, $\mu_x = \int_Y^{\oplus} \sigma_{x,y}\, d\nu_x$. This ends the proof.

We introduce the following definition. A Borel function $n(x, \lambda) : X \times \mathbf{C} \to \mathbf{N}_\infty$ recall $\mathbf{N}_\infty = \{0, 1, \ldots, \infty\}$) will be said to be a *field of multiplicity functions* for $\int_X^{\oplus} N(x) d\mu$ in case for μ-almost all x, $n(x, \cdot)$ is a multiplicity function for $N(x)$. If $n(x, \lambda)$ is a field of multiplicity functions for $\int_X^{\oplus} N(x) d\mu$, then we define $n_*(\lambda) = \sum_{x\in X} n(x, \lambda)$. It follows from Lemma 4.1 that the function $n_* : \mathbf{C} \to \mathbf{N}_\infty$ is ν-measurable with respect to any Borel measure ν on $\mathbf{C}$.

The following result generalizes Theorem 4.1.

THEOREM 5.2. *Let $n(x, \lambda)$ be a field of multiplicity functions for $N = \int_X^{\oplus} N(x) d\mu$. If ν is a scalar spectral measure for N and m_0 is a multiplicity function for N, then for ν-almost all λ, $m_0(\lambda) \leqslant n_*(\lambda)$. There is a field of multiplicity functions $m(x, \lambda)$ such that m_* is a multiplicity function for N.*

Proof. We will assume the model and notations of Theorem 5.1. We let σ be the $X \times \mathbf{C}$-marginal measure of ρ and $\rho = \int_{X\times\mathbf{C}}^{\oplus} \sigma_{x,\lambda}\, d\sigma$ the disintegration of ρ with respect to σ. Also, we take ν to be the scalar spectral measure of N defined for a Borel set $B \subset \mathbf{C}$ by $\nu(B) = \int_X \nu_x(B) d\mu(x)$ and let $\rho = \int_{\mathbf{C}}^{\oplus} \theta_\lambda\, d\nu$ be the disintegration of ρ with respect to ν. Note that ν is the $\mathbf{C}$-marginal measure of σ and $m_0(\lambda) \overset{\text{Def.}}{=} \dim L^2(\theta_\lambda)$ is a representative of the multiplicity class for N.

Suppose first that $n(x, \lambda)$ is a field of multiplicity functions for N. For each $x \in X$, follow the construction described at the end of Section 3 to build a measure $\tilde{\mu}_x$ on $\mathbf{C} \times \mathbf{N}$ from the scalar spectral measure ν_x and the multiplicity function $n(x, \cdot)$. Form the measure $\tilde{\rho} = \int_X^{\oplus} \tilde{\mu}_x \, d\mu$ on $X \times \mathbf{C} \times \mathbf{N}$, and let M_Φ be the operator of multiplication by $\Phi(x, \lambda, n) = \lambda$ on $L^2(\tilde{\rho})$. By construction, for μ-almost all x, $N(x)$ is unitarily equivalent to M_{Φ_x} acting on $L^2(\tilde{\mu}_x)$. Applying Proposition 2.2 and using the isomorphism of $L^2(\tilde{\rho})$ with $\int_X^{\oplus} L^2(\tilde{\mu}_x) d\mu$ (Proposition 2.1), we obtain that N is unitarily equivalent to M_Φ on $L^2(\tilde{\rho})$.

The set

$$\{(x, \lambda, q) \in X \times \mathbf{C} \times \mathbf{N} \,;\, q \leqslant n(x, \lambda)\}$$

is a support set for $\tilde{\rho}$. It is thus easy to choose a Borel representative φ of Φ such that for ν-almost all λ, $m_\varphi(\lambda) = n_*(\lambda)$. Applying Theorem 4.1, we conclude that $m_0(\lambda) \leqslant n_*(\lambda)$ for ν-almost all λ.

Finally, we construct a field of multiplicity functions $m(x, \lambda)$ with m_* a multiplicity function for N. Choose a Borel set $S \subset X \times \mathbf{C} \times \mathbf{N}$ such that for σ-almost all (x, λ), the section $S_{(x, \lambda)} = \{n \,(x, \lambda, n) \in S\}$ is a support set for $\sigma_{x, \lambda}$. Similarly, let R be a Borel set in $X \times \mathbf{C} \times \mathbf{N}$ such that for ν-almost every λ, $R^\lambda = \{(x, n) \,(x, \lambda, n) \in R\}$ is a support set for θ_λ. The existence of S, R is ensured by Proposition 1.3. Set $Q = S \cap R$. Let $m(x, \lambda)$ be the number of points in $Q_{(x, \lambda)}$, if this latter set is finite, otherwise $m(x, \lambda) = \infty$. The function $m : X \times \mathbf{C} \to \mathbf{N}_\infty$ is Borel. It is easy to conclude that for ν-almost all λ, $Q^\lambda \subset R^\lambda$ satisfies $\theta_\lambda(Q^\lambda) = 1$. Thus for ν-almost all λ, $m_*(\lambda)$, which equals the number of points in Q^λ, also equals $\dim L^2(\theta_\lambda)$. Thus m_* is a multiplicity function for N.

By similar considerations, for σ-almost all (x, λ), $m(x,\lambda)$ equals $\dim L^2(\sigma_{x, \lambda})$. Therefore, for μ-almost all x, $m(x, \lambda) = \dim L^2(\sigma_{x,\lambda})$, for ν_x-almost all λ. This last remark uses the fact that $\sigma = \int_X^{\oplus} \nu_x \, d\mu$. By Lemma 5.2, for μ-almost all x, $\mu_x = \int_{\mathbf{C}}^{\oplus} \sigma_{x, \lambda} \, d\nu_x$. Thus for μ-almost all x, $m(x, \cdot)$ is a multiplicity function for $N(x)$. This completes the proof.

The following proposition is a simple but useful result on spectral multiplicity functions for direct integrals of normal operators.

PROPOSITION 5.2. *Let $N = \int_X^{\oplus} N(x) d\mu$ be a direct integral of normal operators. Suppose there is a fixed non-atomic probability measure ν_0 on $\mathbf{C}$ such that for μ-almost all x, the scalar spectral measure ν_x is absolutely continuous with respect to ν_0. If μ is non-atomic, then N has uniform infinite multiplicity.*

Proof. We assume N is given by the model as described in Theorem 5.1 and we use the notations of the proof of Theorem 5.2. In addition let $\sigma = \int_{\mathbb{C}}^{\oplus} \tau_\lambda \, d\nu$ be the disintegration of σ with respect to ν.

From the representation $\sigma = \int_X^{\oplus} \nu_x \, d\mu$ it is clear that σ is absolutely continuous with respect to $\sigma_0 = \mu \times \nu_0$. Let $f = \frac{d\sigma}{d\sigma_0}$. Further, ν is absolutely continuous with respect to ν_0 (see Corollary 5.1). Set $g = \frac{d\nu}{d\nu_0}$ and let $G = \{\lambda \in \mathbb{C} : g(\lambda) \neq 0\}$.

Then $\sigma = \int_{\mathbb{C}}^{\oplus} g(\lambda)\tau_\lambda d\nu_0 = \int_{\mathbb{C}}^{\oplus} f_\lambda \mu d\nu_0$. By uniqueness of disintegration of measures, there is a set $A \subset \mathbb{C}$ with $\nu_0(X \setminus A) = 0$ such that $g(\lambda)\tau_\lambda = f_\lambda \mu$, for all $\lambda \in A$. Then for $\lambda \in G \cap A$ we have $\tau_\lambda = \frac{1}{g(\lambda)} f_\lambda \mu$. Since $\nu(G \cap A) = 1$, we learn that for ν-almost all λ, the measure τ_λ is non-atomic. From Lemma 5.1, for ν-almost all λ, $\theta_\lambda = \int_X^{\oplus} \sigma_{x,\lambda} d\tau_\lambda$ and for such a λ, $\dim L^2(\theta_\lambda) = \infty$. This completes the proof.

We give one application of Proposition 5.2. Let $\mathbb{T}$ denote the unit circle and μ_0 denote normalized Lebesgue measure on $\mathbb{T}$. Let $P : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ denote the projection onto the usual Hardy space $H^2(\mathbb{T})$. Each $\Phi \in L^\infty(\mathbb{T}^2)$ induces a Toeplitz operator T_Φ defined on $L^2(\mathbb{T}) \otimes H^2(\mathbb{T}) = \int_{\mathbb{T}}^{\oplus} H^2(\mathbb{T}) \, d\mu_0$ according to the formula

$$T_\Phi = (I \otimes P) M_\Phi.$$

Note that $T_\Phi = \int_{\mathbb{T}}^{\oplus} PM_{\Phi^\alpha} \, d\mu_0(\alpha)$, where for $|\alpha| = 1$, Φ^α is the usual section $\Phi^\alpha(e^{it}) = \Phi(e^{it}, \alpha)$. Thus T_Φ is a direct integral of the Toeplitz operators T_{Φ^α} on $H^2(\mathbb{T})$.

COROLLARY 5.2. *If Φ is a real valued function in $L^\infty(\mathbb{T}^2)$, then the self-adjoint Toeplitz operator T_Φ on $L^2(\mathbb{T}) \otimes H^2(\mathbb{T})$ has uniform infinite spectral multiplicity.*

Proof. A theorem of Rosenblum [6] implies that each T_{Φ^α} on $H^2(\mathbb{T})$ which is not a scalar multiple of the identity has a scalar spectral measure which is absolutely continuous with respect to Lebesgue measure. The result now follows from Proposition 5.2.

REMARK. (2) The conclusion of Proposition 5.2 does not hold if the ν_x are assumed to be only non-atomic. For example, let I^2 be the unit square

$$I^2 = \{\lambda = x + iy \mid 0 \leqslant x \leqslant 1, \ 0 \leqslant y \leqslant 1\}$$

and let ρ denote Lebesgue measure on I^2. If N is the operator $M_\lambda f(\lambda) = \lambda f(\lambda)$ on $L^2(\rho)$, then N clearly has uniform multiplicity one. However, in the decomposition

$N = \int_{[0,1]}^{\oplus} N(x) \mathrm{d}\mu$ (μ = Lebesgue measure on [0, 1]) relative to the identification of $L^2(I^2)$ with $\int_{[0,1]}^{\oplus} L^2(\mu)\mathrm{d}\mu$, the operator $N(x)$ is the operator $N(x)f(y) = (x + \mathrm{i}y)f(y)$ on $L^2(\mu)$. The operator $N(x)$ has scalar spectral measure absolutely continuous with respect to linear Lebesgue measure on the line segment $\{\lambda \in \mathbf{C} \mid \lambda = x + \mathrm{i}y, y \in [0, 1]\}$. Appropriate modifications of this example in [7] show that the construction of a multiplicity function for the tensor product of two normal operators can be a delicate matter; in particular, the conjecture in §7 of [2] is false.

We close with a generalization of a result of Ball [3] which provides a multiplicity function for certain direct integrals of normal operators.

Assume X is an interval $[0, l]$ in the real line and μ is normalized Lebesgue measure restricted to X. We will only discuss the case where the direct integral $N = \int_{[0,l]}^{\oplus} N(x)\mathrm{d}\mu$ has the property that μ-almost everywhere $N(x)$ has uniform multiplicity one.

In the situation described above the model in Theorem 5.1 reduces to M_λ on $L^2(\sigma)$ where $\sigma = \int_{[0,l]}^{\oplus} \nu_x \mathrm{d}\mu$ and $\{\nu_x\}_{x\in[0,l]}$ is a μ-compatible field of scalar spectral measures for $\{N(x)\}_{x\in[0,l]}$.

Following Ball [3] we define for $s \in [0, l]$ the measure τ_s for Borel $E \subset \mathbf{C}$ by

$$\tau_s(E) = \int_{[0,s]} \nu_x(E)\mathrm{d}\mu.$$

Note that for all $s \in [0, l]$, the measure τ_s is absolutely continuous with respect to ν and $\tau_l = \nu$, where ν is the scalar spectral measure for N described in Proposition 5.1.

For each rational $r \in [0, l]$ let $\frac{\mathrm{d}\tau_r}{\mathrm{d}\nu} \geqslant 0$ be a representative of the Radon-Nikodym derivative of τ_r with respect to ν. These representatives are chosen so that $\frac{\mathrm{d}\tau_r}{\mathrm{d}\nu} \leqslant \frac{\mathrm{d}\tau_{r'}}{\mathrm{d}\nu}$ when $r \leqslant r'$. For s in $[0, l]$, set $\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu} = \lim_{r \downarrow s} \frac{\mathrm{d}\tau_r}{\mathrm{d}\nu}$ where the limit is taken over rational r decreasing to s.

The following proposition provides a generalization (for the case of normal operators) of Theorem 8.3 in Ball [3].

PROPOSITION 5.3. *Let $m(\lambda)$ denote the number of jumps in $\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu}(\lambda)$ as a function of s if the set of values $\left\{\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu}(\lambda)\right\}_{s\in[0,l]}$ is finite, otherwise $m(\lambda) = \infty$. The function m is a multiplicity function for N.*

Proof. Let $\sigma = \int_{\mathbf{C}}^{\oplus} \theta_\lambda \mathrm{d}\nu$ be a disintegration of σ with respect to ν. Then for E a Borel set in $\mathbf{C}$

$$\tau_s(E) = \sigma([0, s] \times E) = \int_E \theta_\lambda([0, s])\, \mathrm{d}\nu.$$

The choices $\frac{\mathrm{d}\tau_r}{\mathrm{d}\nu}(\lambda)$ made for rational $r \in [0, l]$ will agree for ν-almost all λ with $\theta_\lambda([0, r])$. Consequently, for ν-almost all λ and $s \in [0, l]$, $\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu}(\lambda) = \theta_\lambda([0, s])$. The function $m_0(\lambda) = \dim[L^2(\theta_\lambda)]$ is a multiplicity function for N. This function equals the number of jumps in $\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu}(\lambda)$ if $\left\{\frac{\mathrm{d}\tau_s}{\mathrm{d}\nu}(\lambda)\right\}_{s \in [0, l]}$ is finite and is otherwise infinite. This completes the proof.

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