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ON TOEPLITZ OPERATORS WITH LOOPS

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INTRODUCTION

For $F(e^{it})$ a bounded measurable function on $[0, 2\pi]$, define the Toeplitz operator T_F on H^2 by

$$T_F x = \mathbf{P}F(e^{it})x(e^{it})$$

where $\mathbf{P}$ is the orthogonal projection of L^2 on H^2. In a recent paper [1], we obtained a rather simple representation for T_F, up to similarity, in case $F(z)$ is rational and maps the unit circle $\mathbf{T}$ to a simple closed curve. In the present paper, more complicated behavior by the rational function $F(z)$ is permitted.

To be precise, let $F(z)$ be a rational function with no poles on $|z| = 1$. Label as $\mathscr{L}_i$ the bounded components of the complement $\mathbf{C}\setminus F(\mathbf{T})$ for which the index $(=N_i)$ of $T_F - \lambda I$ is positive and as ℓ_i those for which that index $(=\nu_i)$ is negative. The $\mathscr{L}_i$ and ℓ_i will be referred to as the *loops of F*. We make the following assumptions, which will hold throughout this paper.

(I) The intersection of the closures of any two loops consists of a finite set of points (called the *multiple points of F*).

(II) The boundary of each loop is an analytic curve except at the multiple points, where it is piecewise smooth, with inner angle θ satisfying $\theta \neq 0, \pi, 2\pi$. No (distinct) arcs of $\partial F(\mathbf{T})$ meet at angle $\theta = 0$.

(III) No multiple point of F is the image $F(z_0)$ of a point $z_0 \in \mathbf{T}$ where $F'(z_0) = 0$.

(IV) F never "backs up": if τ_i is the Riemann mapping function from $|z| < 1$ to ℓ_i, we assume the argument of $\tau_i^{-1}(F(e^{it}))$ is a monotone decreasing function and, if T_i is the Riemann mapping function to $\overline{\mathscr{L}}_i$, we assume the argument of $T_i^{-1}(\overline{F}(e^{it}))$ is monotone decreasing.

An arc of $\partial\mathscr{L}_i$ or $\partial\ell_i$, one endpoint of which is a multiple point, will be referred to as a *branch* of $F(z)$.

Throughout, $f(z)$ denotes $\overline{F}(\bar{z}^{-1})$, so that $T_f = T_F^*$. The sets σ_f and σ_F denote the unions

$$\sigma_f = \cup \ell_i \quad \sigma_F = \cup \mathscr{L}_i .$$

The main result is

THEOREM 1. *The operator T_F is similar to*

$$\sum \oplus T_{\tau_i} \oplus \sum \oplus T^*_{T_i} \text{ on } \sum \oplus H^2_{-\nu_i} \oplus \sum \oplus H^2_{N_i}$$

where H^2_ν is the vector H^2 space based on a Hilbert space of dimension ν.

Two immediate consequences of Theorem 1 are worth recording. Let $\mathscr{M}_i$ denote the closed span of the eigenvectors of T_F corresponding to eigenvalues in $\mathscr{L}_i$ and let m_i denote the closed span of the eigenvectors of T_f corresponding to eigenvalues in $\overline{\ell_i}$ (a bar over a set always denotes complex conjugation). It is easy to see that $\mathscr{M}_i \perp m_j$ for any (i, j).

COROLLARY 1. *The intersection of any two of the subspaces in $\{\mathscr{M}_i, m_j\}$ is $\{0\}$.*

COROLLARY 2. *The (direct) sum*

$$\sum \mathscr{M}_i + \sum m_i$$

is all of H^2.

We conclude the introduction by giving two examples of the theorem and corollaries.

EXAMPLE 1. $F(z) = a(z^{n+1} - z^{-(n-1)})$. $F(\mathbf{T})$ is the classical N leaved rose of analytic geometry. If n is odd, there are n leaves (loops) ℓ_i, each with $\nu_i = -2$ and if n is even, there are $2n$ leaves ℓ_i, each with $\nu_i = -1$. By Theorem 1, T_F is similar to

$$\begin{cases} T_{\tau_1} \oplus T_{\tau_1} \oplus T_{\tau_2} \oplus T_{\tau_2} \oplus \ldots \oplus T_{\tau_n} \oplus T_{\tau_n}, & n \text{ odd} \\ T_{\tau_1} \oplus T_{\tau_2} \oplus \ldots \oplus T_{\tau_{2n}}, & n \text{ even} \end{cases}$$

EXAMPLE 2. $F(z) = z^2/(z - \beta), \frac{1}{2} < |\beta| < 1$. In this case $F(\mathbf{T})$ is a "figure 8", having loops $\mathscr{L}_1$ (with $N_1 = 1$) and ℓ_1 (with $\nu_1 = -1$). By Theorem 1, T_F is similar to

$$T_{\tau_1} \oplus T^*_{T_1},$$

and the eigenvectors for T_F (for $\lambda \in \mathscr{L}_1$) together with those of T^*_F (for $\lambda \in \bar{\ell}_1$) span H^2. This example, and the question of determining the span of the eigenvectors of T_F and T^*_F, was considered by Gambler [3]. Although [3] contains a number of interesting results, Gambler did not obtain a proof of Corollary 2, even in this special case. The question solved in Corollary 2 was first posed to this author in a conversation with R. G. Douglas in January, 1975.

To give an idea of the proof of Theorem 1, we outline it in the case of Example 2 with $\beta > 0$. In this case, $F(z)$ has the two loops: ℓ_1, the image under F of the arc of the circle from $u_0 = (1 + \sqrt{1 - 4\beta^2})/2\beta$ to $\bar{u}_0$ containing -1, and $\mathscr{L}_1$, the image under F of the arc from $\bar{u}_0$ to u_0 containing 1. The only multiple point is $\beta^{-1} = F(u_0) = F(\bar{u}_0)$. The eigenvectors k_λ of T_F satisfying $k_\lambda(0) = 1$ are given by

$$k_\lambda(z) = (z^2/\lambda\beta - z/\beta + 1)^{-1} \qquad \text{for } \lambda \in \mathscr{L}_1 ,$$

and the eigenvectors $h_\lambda(z)$ for T_f are

$$h_\lambda(z) = (1 - \beta z)(1 - \lambda z + \lambda\beta z^2)^{-1} \quad \text{for } \lambda \in \bar{\ell}_1.$$

We prove similarity with the direct sum of multiplication by z on $H^2(\ell_1)$ and the adjoint of multiplication by z on $H^2(\mathscr{L}_1)$.

Define $L_{0f}: H^2 \to H^2(\ell_1)$ by

$$L_{0f}\, x = (x, h_{\bar{\lambda}}) \qquad \lambda \in \ell_1.$$

L_{0f} intertwines T_F and multiplication by λ, since

$$L_{0f} T_F x = (T_F x, h_{\bar{\lambda}}) = (x, T_f h_{\bar{\lambda}}) = \lambda L_{0f} x.$$

The proof that L_{0f} is bounded is obtained by writing

$$h_\lambda(z) = c_1(\lambda)(1 - d_1(\lambda)z)^{-1} + c_2(\lambda)(1 - d_2(\lambda)z)^{-1}$$

where $d_i(\lambda) = 2\beta\lambda/(\lambda + (-1)^{i+1}\sqrt{\lambda^2 - 4\beta\lambda})$ and where c_1 and c_2 are suitably chosen. Then

$$L_{0f} x = \bar{c}_1(\bar{\lambda}) x(\bar{d}_1(\bar{\lambda})) + \bar{c}_2(\bar{\lambda}) x(\bar{d}_2(\bar{\lambda})) = T_1 x + T_2 x.$$

It can be seen that $|d_1(\lambda)| = 1$ on $\partial\ell_1$, so T_1 consists of a change of variable followed by a multiplication. We have $|d_2(\lambda)| < 1$ except at $\lambda = 1/\beta$, where $d_2(\lambda) = u_0$. This leads to a boundedness problem somewhat different from that considered in [2], where $|d_2(\lambda)| < c < 1$, or [1], where $|d_2(\lambda)| = 1$ at a boundary point λ, if and only if $d_1(\lambda)$ and $d_2(\lambda)$ coincide. The boundedness of T_1 and T_2 (and hence of L_{0f}) is proved in § 2 below.

Next we define an operator L_{0F} from $H^2(\ell_1)$ to H^2 by

$$L_{0F} p(\lambda) = \frac{1}{2\pi i} \int_{\ell_1} p(\lambda)/[-2/(\lambda + \sqrt{\lambda^2 - 4\lambda\beta}) + z/\lambda\beta]\, d\lambda$$

for $p(\lambda)$ a polynomial. The proof of boundedness for L_{0F} is similar to that for L_{0f} (the adjoint of L_{0F} is analogous to T_1 above). L_{0F} is related to the inverse of L_{0f}. ($L_{0F} = L_{0f}^{-1}$ in the case of only one loop [1].)

For the loop $\mathscr{L}_1$, we define operators analogous to L_{0f} and L_{0F}, with f and F interchanged; let L_{2f} be the operator corresponding to L_{0F} and $L_{2F'}$ the operator corresponding to L_{0f}. The operators $L = L_{2f}^* + L_{0f}$ and $L_1 = L_{2F}^* \oplus L_{0F}$ are bounded and L intertwines the operators T_F and $M_z^* \oplus M_z$ on $H^2(\mathscr{L}_1) + H^2(\ell_1)$. The proof is completed by showing that L has dense range and satisfies $L_1L = I$. The last fact is proved by setting up

$$(L_1L(1 - \bar{v}z)^{-1}, (1 - \bar{u}z)^{-1}), \ |u|, |v| < 1,$$

as an integral over $\partial(\mathscr{L}_1 \cup \ell_1)$ and by observing that this integral is the same as the limit of

$$\oint ((\lambda I - T_f)^{-1}(1 - \bar{v}z)^{-1}, (1 - \bar{u}z)^{-1})\, d\lambda$$

as the path of integration approaches $\partial(\mathscr{L}_1 \cup \ell_1)$ from outside $\sigma(T_f)$ (§ 3). The last contour integral is, of course, equal to $(1 - \bar{v}u)^{-1}$.

The proof we give of Theorem 1 uses the methods of [1], although only Part I of [1] is a prerequisite. In Section 4, we outline a version allowing for "backing up", i.e., removing assumption (IV) on F.

1. PROPERTIES OF f NEAR A MULTIPLE POINT

We need to extend certain results of [1] to the case where $f(\mathbf{T})$ has multiple points. As in [1], write the rational functions

$$f(z)-\lambda=a(\lambda)\prod(1-d_i(\lambda)z)\prod(1-e_i(\lambda)z)\prod(1-g_i(\lambda)z)/\prod_{i=1}^{m}(z-\gamma_i)\prod_{i=1}^{r}(z-\delta_i) \tag{1.1}$$

where $|d_i| < 1 = |g_i| < |e_i|$ and $|\gamma_i| < 1 < |\delta_i|$; and

$$F(z)-\lambda=A(\lambda)\prod(1-D_i(\lambda)z)\prod(1-E_i(\lambda)z)\prod(1-G_i(\lambda)z)/\prod_{i=1}^{M}(z-\Gamma_i)\prod_{i=1}^{N}(z-\Delta_i), \tag{1.2}$$

where $|D_i| < 1 = |G_i| < |E_i|$ and $|\Gamma_i| < 1 < |\Delta_i|$. As noted in [1], the above functions of λ may be multiple valued, so we assume that we have renumbered them so that the functions in (1.1) [resp. (1.2)] are single-valued and piecewise continuous on $\partial\sigma_f \cup \partial\tilde{\sigma}_F$ [resp. $\partial\sigma_f \cup \partial\sigma_F$] and that they are all analytically continuable across all multiple points (of course on the branches, some d_i or e_i change to g_i, etc.).

Let ℓ be a loop of $\partial\tilde{\sigma}_f$. We begin with a characterization of the multiple points of $\partial\ell$.

LEMMA 1.1. *A point $\lambda_0 \in \partial\ell$ such that λ_0 is not the image under f of any point on the unit circle where $f' = 0$ is a multiple point if and only if some one of the functions d_i or e_i has $|d_i(\lambda)| \to 1$ or $|e_i(\lambda)| \to 1$ as $\lambda \to \lambda_0$ along a branch of $\partial\ell$.*

Proof. At each point of $\partial\ell$, there is at least one $g_i(\lambda)$ (since $\partial\ell \subset f(\mathbf{T})$). Let $g_1(\lambda)$ be a g_i on one of the branches of $\partial\ell$ near the multiple point λ_0. We have that (the analytic continuation of) g_1^{-1} is the inverse of $f(z)$ as a map from a neighborhood of $g_1(\lambda_0)$ to a neighborhood of λ_0. On the other branch of $\partial\ell$ near λ_0, this analytic continuation cannot have modulus 1, for then a part of $\partial\ell$ in a neighborhood of λ_0 would be the image under f of an arc of the unit circle, contradicting assumption (II) on $\partial\ell$. Thus some d_i or e_i (namely the continuation of $g_i(\lambda)$ on the other branch of $\partial\ell$) has the desired property.

Conversely, suppose $d_1(\lambda)$ has $|d_1(\lambda_n)| \to 1$, $\{\lambda_n\} \subset \partial\ell$, $\lambda_n \to \lambda_0$. Continuing $d_1(\lambda)$ into a neighborhood of λ_0, we see that the continuation must have modulus 1 on some curve through λ_0. This curve must be the image under f of an arc of $\mathbf{T}$ and it cannot coincide with both branches of $\partial\ell$ since (by definition of the d_i) $|d_1(\lambda)| < 1$ on at least one branch of $\partial\ell$. Thus λ_0 is a multiple point.

LEMMA 1.2. *Suppose λ_0 is a multiple point in $\partial\ell$, and $d_1(\lambda)$ is as in Lemma* 1.1 ($|d_1(\lambda)| \to 1$ *along a branch of $\partial\ell$ at λ_0). Then $d_1(\lambda)$ tends to the unit circle nontangentially along that branch.*

Proof. By hypothesis (and by (III)), $d_1(\lambda)$ is analytically continuable across λ_0. By the proof of Lemma 1.1, (the continuation of) d_1 has modulus 1 on some branch β of $\partial\bar{\sigma}_f \cup \partial\bar{\sigma}_F$ at λ_0. Thus (the continuation of) d_1 maps β to the unit circle and the branch β^1 of ℓ (on which $|d_1(\lambda)| < 1$) must meet β at an angle $\theta \neq 0, \pi, 2\pi$ (by (II)). Since d_1 maps β^1 to a curve making angle θ with the unit circle, this proves the lemma.

We use the preceding lemmas to give a generalization of a result on L_Γ operators from [1]. Recall that if

$$\frac{\prod(1-\delta_i^{-1}z)}{\prod(1-d_iz)\prod_\Gamma(1-g_iz)} = \sum c_i(1-d_iz)^{-1} + \sum_\Gamma \xi_i(1-g_iz)^{-1} \tag{1.3}$$

where Γ is some set of the g_i, then, for $x(z)$ a polynomial, we define

$$\begin{aligned} L_\Gamma x(e^{it}) &= \\ &= \rho(e^{it})\,[\textstyle\sum \bar{c}_i(a(e^{it}))x(\bar{d}_i(a(e^{it}))) + \sum_\Gamma \bar{\xi}_i(a(e^{it}))x(\bar{g}_i(a(e^{it})))] \end{aligned} \tag{1.4}$$

where ρ is a measurable function and a is a continuous function from $|z| = 1$ to $\partial\ell$, where ℓ is a loop of $\bar{\sigma}_f$.

LEMMA 1.3. *Let Γ be a set of the $g_i(\lambda)$ on $\partial\ell$. Suppose $a'(z)$ is continuous and non-zero except at those points u_0 such that $a(u_0)$ is a multiple point in $\partial\ell$, where*

$$a(z) - a(u_0) \sim (z-u_0)^\alpha, \quad |a'(z)| \leqslant |z-u_0|^{\alpha-1}$$

and suppose that a is one-to-one on $\mathbf{T}$. *Let $\rho(e^{it})$ be bounded except at the points z_0 where $f'(z_0) = \ldots = f^{(\beta)}(z_0) = 0$ and the points u_0 above, and suppose ρ satisfies*

$$|\rho(e^{it})| \leqslant c|t - z_0|^{-\frac{1}{2} + [2(p+q)-1]/2\beta}$$

near the z_0 (where p of the $d_i(\lambda)$ and q of the $g_i(\lambda)$ agree at $f(z_0)$), and

(1.5) $$|\rho(e^{it})| \leqslant c|e^{it} - u_0|^{(\alpha-1)/2}$$

near the u_0. Then L_Γ is bounded in L^2 norm.

Proof. We need only deal with $L_\Gamma x(e^{it})$ near a point u_0 where $\lambda_0 = a(u_0)$ is a multiple point; at all other points of $\partial\ell$, the results of [1, § 3] apply.

Near λ_0, the proof of boundedness for the terms including g_i is the same as the proof of the inequality (3.5) of [1]. By hypothesis (III), none of the $d_i(a(u_0))$, $g_i(a(u_0))$, $e_i(a(u_0))$ coincide, so (in [1, (2.6)])

$$2(p+q) - 1 = 2(0+1) - 1 = 1 = \beta$$

and [1, (2.6)] becomes (1.5) above.

Each term in (1.4) involving a $d_i(\lambda)$ such that $|d_i(\lambda)| < c < 1$ (i.e., those d_i not included in the conclusion of Lemma 1.1) represents an operator of finite Hilbert--Schmidt norm given by [1, (3.4)]. Thus, the methods of [1] prove such terms represent bounded operators.

It remains to prove the boundedness of the terms

$$x \to \rho(e^{it})\bar{c}_i(a(e^{it}))x(\bar{d}_i(a(e^{it})))$$

where d_i satisfies the conclusion of Lemma 1.1. To do this, we apply an argument identical to that following [1; Lemma 3.1]. Note that

$$c_i(\lambda) = \prod (1 - \delta_j^{-1} d_i^{-1}) / [\prod_{j \neq i} (1 - d_j/d_i) \prod_{\Gamma} (1 - g_j/d_i)]$$

is bounded, since none of the $d_j(\lambda)$, $g_j(\lambda)$ coincide with $d_i(a(e^{it}))$ near $a(u_0)$ (by III). Thus the bound on $\mu((r, 1))$ following [1, Lemma 3.1] is correct, with $p = 1$, $q = 0$ and $\beta = 1$. (Note that Lemma 1.2 above justifies the containment relations concerning the set S). This proves Lemma 1.3.

COROLLARY 1.1. *Suppose in Lemma* 1.3 *that*

$$a(z) = \tau(z)$$

the Riemann mapping function for the loop ℓ. Then (1.5) *may be replaced by*

$$\rho(e^{it}) = \tau'(e^{it})^{\frac{1}{2}} \rho_1(e^{it})$$

where ρ_1 is bounded near $a(u_0)$.

Proof. The Riemann mapping function $\tau(z)$ satisfies the hypothesis on $a(z)$ n Lemma 1.3 with $\alpha = \pi \div$ the inner angle of ℓ at $\tau(u_0)$. By [4, Theorem IX.8], $|\tau'(z)| \leqslant c|z - u_0|^{\alpha-1}$ near u_0.

2. L_{0f} AND L_{0F}

In this section, we deal with the part of T_F similar to a sum of analytic Toeplitz operators, T_{τ_j}. First note that T_{τ_j} on $H^2_{-\nu_j}$ is unitarily equivalent to multiplication by z on $H^2_{-\nu_j}(\ell_j)$, the closure of the $-\nu_j$ dimensional vector-polynomials in the norm

$$\|p\|^2 = \int_0^{2\pi} \|p(\tau_j(e^{it}))\|^2 |\tau_j'(e^{it})| \, dt =$$

$$= \int_0^{2\pi} \|p(\tau_j(e^{it}))\|^2 |d\, \tau_j|.$$

We will actually replace $\sum \oplus H^2_{-\nu_j}(\ell_j)$ by an even more convenient H^2 space. Let ν denote the maximum of the $-\nu_j$ and let $u_1, \ldots, u_\nu$ be orthonormal vectors in some auxiliary Hilbert space. Let $H^2(\cup \ell_i^{(\nu_i)})$ be the closure of the functions of the form $p(\lambda) = \sum_{k=1}^{\nu} p_k(\lambda)\, u_k$ in the norm

$$\|p\|_\nu^2 = \sum_i \sum_{k=1}^{-\nu_i} \int_0^{2\pi} |p_k(\tau_i(e^{it}))|^2 |\tau_i'(e^{it})| \, dt.$$

We will use for our similarity the space $H^2(\cup \ell_i^{(\nu_i)})$. Indeed

$$H^2(\cup \ell_i^{(\nu_i)}) = \sum \oplus H^2_{-\nu_i}(\ell_i). \tag{2.1}$$

To prove (2.1), all we need to show is that $H^2_{-\nu_i}(\ell_i) \subset H^2(\cup \ell_i^{(\nu_i)})$. Let $i = 1$ and note first that if p is any polynomial, then

$$p_1(\lambda) = \begin{cases} \prod(\lambda - \lambda_j)\, p(\lambda) & \lambda \in \ell_1 \\ 0 & \lambda \in \bigcup_{i>1} \ell_i \end{cases},$$

where the λ_j are the multiple points in $\partial\ell_1$, is continuous on the closure of σ_f and analytic in each ℓ_i. Thus p_1 is the uniform limit, on the closure of σ_f, of polynomials (by Mergelyan's Theorem). Clearly the functions $\prod(\lambda - \lambda_j)\, p(\lambda)$ are dense in the closure of the polynomials in the norm

$$\|p\|^2 = \int_0^{2\pi} |p(\tau_1(e^{it}))|^2 |\tau_1'(e^{it})| \, dt$$

on $\partial\ell_1$, and this proves (2.1).

Now let $L_{0f} : H^2 \to H^2(\cup_i \ell_i^{(\nu_i)})$ be defined by

$$L_{0f}x = \sum_k \rho_k(\lambda)\,(x, h_\lambda^{(k)})\,u_k$$

where $h_\lambda^{(k)}$ is the eigenvector of T_f defined by

$$h_\lambda^{(k)}(z) = \prod_{j \neq k}(z - z_j) \prod (1 - \delta_i^{-1}z)/\prod(1 - d_i(\lambda)z),$$

where $z_1, \ldots, z_\nu$ are fixed (distinct) complex numbers of modulus <1 and ρ_k are the (bounded) functions $\rho_k(\lambda) = \overline{h_\lambda^{(k)}}(z_k)^{-1}$.

To prove that L_{0f} is bounded, restrict to one loop of σ_f and follow the proof given in [1]. Indeed,

$$L_{0f}x = \sum_k \rho_k(\lambda)\,(x, h_{\bar\lambda}^{(k)})u_k = \sum_k \rho_k(\lambda)\,(y, \prod(1 - \delta_i^{-1}z)/\prod(1 - d_i(\bar\lambda)z))\,u_k$$

where $y = \mathbf{P}\prod_{j \neq k}(e^{-it} - \bar z_j)x(e^{it})$, so that

$$L_{0f}x = \sum_{k,j} \rho_k(\lambda)\bar c_i(\bar\lambda)y(\bar d_i(\bar\lambda))u_k$$

where the c_i are given by

$$\prod(1 - \delta_i^{-1}z)/\prod(1 - d_i(\lambda)z) = \sum c_i/(1 - d_i(\lambda)z).$$

On a loop ℓ_1, we have

$$\|L_{0f}x\|_{\ell_1}^2 \leqslant \sum_i \int |\rho_k(\bar\tau_1(e^{it}))|^2 |c_i(\bar\tau_1(e^{it}))|^2 |y(\bar d_i(\bar\tau_1(e^{it})))|^2 |\tau_1'|\,dt$$

which, by Corollary 1.1, is bounded if ρ_k is bounded. We have

LEMMA 2.1. *L_{0f} is bounded and satisfies*

$$(L_{0f}T_F y)(\lambda) = \lambda(L_{0f}y)(\lambda) \tag{2.2}$$

for $y \in H^2$.

Proof. We have shown boundedness. The proof of (2.2) is exactly as in [1]:

$$L_{0f}T_F y = \sum_k \rho_k(\lambda)\,(T_F y, h_{\bar\lambda}^{(k)})\,u_k =$$

$$= \sum_k \rho_k(\lambda)(y, T_f h_{\bar\lambda}^{(k)})u_k = \lambda \sum_k \rho_k(\lambda)(y, h_{\bar\lambda}^{(k)})u_k = \lambda\, L_{0f}y\,.$$

The motivation for the construction of L_{0f}^{-1} that follows is this: we would like to define

$$L_{0f}^{-1} \sum p_k(\lambda)u_k = \sum p_k(T_F)(1 - \bar z_k z)^{-1}.$$

A simple proof plugging this into L_{0f} shows how this gives a right inverse for L_{0f}. But if p_k is just in $H^2(\cup \ell_i)$, or even if p_k is continuous on $\cup \ell_i$, $p_k(T_F)$ may not make sense. Even the integral

$$\int_\gamma p_k(\lambda)(\lambda I - T_F)^{-1}(1 - \bar{z}_k z)^{-1} \, \mathrm{d}\lambda \tag{2.3}$$

for γ a curve surrounding $\cup \ell_i$ may not make sense because $\mathrm{sp}(T_F) = (\cup \ell_i) \cup (\cup \mathscr{L}_i)$ may be bigger than $\cup \ell_i$. To overcome this, we take the integral in (2.3), substitute for γ the union of the boundaries of the ℓ_i and write down the result formally, for $L^2(\cup \ell_i)$ functions $p_1, \ldots, p_\nu$, as

$$L_{0F}(\sum p_k(\lambda)u_k) = \sum_{j,m} \frac{1}{2\pi \mathrm{i}} \int A^{-1} \frac{\prod (z - \Delta_i) \prod (1 - \Gamma_i \bar{z}_m)(1 - \bar{z}_m z)^{-1} p_m(\tau_j) \, \mathrm{d}\tau_j}{\prod (1 - D_i(\tau_j)z) \prod (1 - G_i(\tau_j)z) \prod (\bar{z}_m - E_i(\tau_j))}.$$

LEMMA 2.2. *The operator S defined on $(1 - \bar{v}z)^{-1}$, $|v| < 1$, by*

$$S(1 - \bar{v}z)^{-1} = \frac{-1}{2\pi \mathrm{i}} \bar{A}^{-1} \frac{\prod (\bar{v} - \bar{\Delta}_i) \prod (1 - \bar{\Gamma}_i z_m)(1 - z_m \bar{v})^{-1}}{\prod (1 - \bar{D}_i(\lambda)\bar{v}) \prod (1 - \bar{G}_i(\lambda)\bar{v}) \prod (z_m - \bar{E}_i(\lambda))} \frac{|\tau_j'|}{\tau_j'}$$

has a unique bounded extension from H^2 into $L^2(|d\tau_j|)$.

Proof. Following the proof in [1, Lemma 7.1], we note that

$$\| S(1 - \bar{v}z)^{-1} \|^2 = \int \left| \frac{1}{2\pi} A^{-1} \frac{\prod (\bar{v} - \bar{\Delta}_i) \prod (1 - \bar{\Gamma}_i z_m)(1 - z_m \bar{v})^{-1} \tau_j'(\mathrm{e}^{it})^{\frac{1}{2}}}{\prod (1 - \bar{D}_i(\tau_j)\bar{v}) \prod (1 - \bar{G}_i(\tau_j)\bar{v}) \prod (z_m - \bar{E}_i(\tau_j))} \right|^2 \mathrm{d}t =$$

$$= \|T_1 L_\Gamma T_2 (1 - \bar{v}z)^{-1}\|^2.$$

Here T_2 is the (co-analytic) Toeplitz operator with symbol $(1 - z_m \mathrm{e}^{-it})^{-1}$, so that

$$T_2(1 - \bar{v}z)^{-1} = (1 - z_m \bar{v})^{-1}(1 - \bar{v}z)^{-1}.$$

L_Γ is the L_Γ operator with f replaced by F, with $a(z) = \tau_j(z)$, with $\Gamma = \{G_i\}$, and with $\rho(\mathrm{e}^{it}) = \tau_j'(\mathrm{e}^{it})^{\frac{1}{2}}$; so that

$$L_\Gamma(1 - \bar{v}z)^{-1} = \tau_j'^{\frac{1}{2}} [\sum \bar{C}_i(\tau_j(\mathrm{e}^{it}))(1 - \bar{v}\bar{D}_i(\tau_j(\mathrm{e}^{it})))^{-1} +$$

$$+ \sum \bar{\Xi}_i(\tau_j(\mathrm{e}^{it}))(1 - \bar{v}G_i(\tau_j(\mathrm{e}^{it})))^{-1}].$$

T_1 is the operator of multiplication by

$$\frac{1}{2\pi} A^{-1} \prod (\bar{\Delta}_i) \prod (1 - \bar{\Gamma}_i z_m) / \prod (z_m - \bar{E}_i(\tau_j)).$$

T_1 and T_2 are clearly bounded, and L_Γ is bounded, by Corollary 1.1. This proves Lemma 2.2.

Now to see that L_{0F} is bounded and maps into H^2, let

$$L(z) = L_{0F}(p)(z)$$

for $p = \sum p_k(\lambda)u_k$ with $p_1, \ldots, p_\nu \in H^2(\cup \ell_i)$. We have

$$L(v) = (p, S(1 - \bar{v}z)^{-1}) = (S^* p, (1 - \bar{v}z)^{-1}) .$$

Since S is bounded, by Lemma 2.2, this implies that (there is a function) $\mathbf{P}S^*p \in H^2$ such that

$$L(v) = (\mathbf{P}S^*p)(v). \tag{2.4}$$

Thus, as a function of z, $L(z) = L_{0F}(p)(z) \in H^2$. It also follows from (2.4) that

$$\|L_{0F}p\| \leqslant \|L(z)\| \leqslant \|S^*\| \, \|p\|$$

and hence that L_{0F} has a unique bounded extension to all of $H^2(\cup \ell_i^{(\nu_i)})$. We have

LEMMA 2.3. *L_{0F} is bounded in L^2 norm and satisfies*

$$T_F L_{0F} x = L_{0F} \lambda x \tag{2.5}$$

and

$$L_{0f} L_{0F} = I . \tag{2.6}$$

In particular, L_{0f} has dense range.

Proof. We have shown that L_{0F} is bounded. To prove (2.5), let $x(\lambda) = \sum p_k(\lambda)u_k$;

$$FL_{0F}x(\lambda) - L_{0F}\lambda x(\lambda) = L_{0F}\Big(\sum_k (F(z) - \lambda)p_k(\lambda)u_k\Big) =$$

$$= \frac{1}{2\pi i}\int AA^{-1} \sum_{m,j} \frac{\prod(1-D_i(\tau_j)z)\prod(1-G_i(\tau_j)z)\prod(1-E_i(\tau_j)z)\prod(z-\Delta_i)\prod(1-\Gamma_i\bar{z}_m)p_m(\tau_j)d\tau_j}{\prod(z-\Delta_i)\prod(z-\Gamma_i)\prod(1-D_i(\tau_j)z)\prod(1-G_i(\tau_j)z)\prod'(\bar{z}_m-E_i(\tau_j))(1-\bar{z}_m z)} =$$

$$= \frac{1}{2\pi i} \sum_m \frac{\prod(1 - \Gamma_i \bar{z}_m)}{\prod(z - \Gamma_i)(1 - \bar{z}_m z)} \sum_j \int \frac{\prod(1 - E_i(\tau_j)z)p_m(\tau_j)}{\prod(\bar{z}_m - E_i(\tau_j))} d\tau_j .$$

Note that the sum of the last integrals is a polynomial $r(z)$ of degree s_2 (= the number of $E_i(\lambda)$, λ outside σ_f). At $\bar{z}_m^{-1}$, its value is

$$r(\bar{z}_m^{-1}) = \bar{z}_m^{-s_2} \sum_j \int_0^{2\pi} p_m(\tau_j)d\tau_j = \bar{z}_m^{-s_2} \sum_j \int_{\partial\ell_j} p_m(w)\,dw = 0$$

if p_m is, say, a polynomial, and therefore

$$\mathbf{P} \prod (z - \Gamma_i)^{-1} r(z)/(1 - \bar{z}_m z) = 0$$

(since $T_F - \lambda I$ invertible for λ outside $\operatorname{sp}(T_F)$ implies the number of $\Gamma_i =$ the number of E_i; [2], Lemma 2.1.). Thus

$$\mathbf{P} F L_{0F} x - L_{0F} \lambda x = 0$$

and (2.5) follows.

Now we compute

$$L_{0f} L_{0F} x = \sum_k L_{0f} L_{0F} p_k(\lambda) u_k =$$

$$= \sum_k L_{0f} p_k(T_F) L_{0F} u_k =$$

$$= \sum_k p_k(\lambda) L_{0f} L_{0F} u_k$$

(at least if the p_m are polynomials and hence, by taking limits, for $x \in \sum H^2(\cup \ell_i^{(\nu_i)})$). Hence

$$L_{0f} L_{0F} x = \sum_{k,m} \rho_k(\lambda) p_k(\lambda) (L_{0F} u_m, h_{\bar{\lambda}}^{(k)}) u_k.$$

We need to show that $\rho_k = \bar{h}_{\bar{\lambda}}^{(k)}(z_k)$ satisfies

$$\rho_k(\lambda)\,(L_{0F} u_m, h_{\bar{\lambda}}^{(k)}) = \delta_{km}. \tag{2.7}$$

To do this, we first prove

LEMMA 2.4. *If g is a rational function, analytic in $|z| < 1$ and if, for some $\mu \in \mathbf{C}$, $(F(e^{it}) - \mu) g(e^{it}) \in L^2$, then*

$$\sum \bar{c}_\varepsilon \mathbf{P}[(F - \mu) g](\bar{d}_\varepsilon) = (\lambda - \mu) \sum \bar{c}_\varepsilon g(\bar{d}_\varepsilon) \tag{2.8}$$

where the c_ε are chosen so that

$$h_{\bar{\lambda}}^{(k)}(z) = \sum c_\varepsilon(\lambda)\,(1 - d_\varepsilon(\lambda) z)^{-1}$$

for $\lambda \in \ell_i$.

Proof. If $g \in H^2$, the conclusion is clear; viz:

$$\sum \bar{c}_\varepsilon \mathbf{P}(F - \mu) g(\bar{d}_\varepsilon) = (T_{F-\mu} g, h_{\bar{\lambda}}^{(k)}) = (g, T_{F-\mu}^* h_{\bar{\lambda}}^{(k)}) =$$

$$= (g, (\bar{\lambda} - \bar{\mu}) h_{\bar{\lambda}}^{(k)}) = (\lambda - \mu)\,(g, h_{\bar{\lambda}}^{(k)}) = (\lambda - \mu) \sum \bar{c}_\varepsilon g(\bar{d}_\varepsilon)\,.$$

For general g, let $g_r(z) = g(rz)$, $0 < r < 1$. By the above, (2.8) holds with g replaced by g_r. Furthermore, as $r \to 1$,

$$(\mathbf{P}(F - \mu)g_r)(\bar{d}_\varepsilon) \to (\mathbf{P}(F - \mu)g)(\bar{d}_\varepsilon),$$

because if s is chosen so that F and g are analytic in $1 > |z| > s$ and so that $|d_\varepsilon| < s$, we have

$$\mathbf{P}(F - \mu)g_r(\bar{d}_\varepsilon) = \int_{|z|=s} \frac{(F - \mu)g_r(z)}{z - \bar{d}_\varepsilon} \mathrm{d}z \to \int_{|z|=s} \frac{(F - \mu)g(z)}{z - \bar{d}_\varepsilon} = \mathbf{P}(F - \mu)g(\bar{d}_\varepsilon).$$

Since obviously $g_r(\bar{d}_\varepsilon) \to g(\bar{d}_\varepsilon)$, the lemma is established.

Now to prove (2.7), write $(L_{0F}u_m, h_\lambda^{(k)})$

$$(2.9) \quad = \frac{1}{2\pi \mathrm{i}} \sum_j \left(\int \frac{\prod (z - \Delta_i) \prod (1 - \bar{z}_m \Gamma_i)\, \mathrm{d}\tau_j}{\prod (1 - D_i(\tau_j)z) \prod (1 - G_i(\tau_j)z) \prod (\bar{z}_m - E_i(\tau_j))(1 - \bar{z}_m z)}, h_\lambda^{(k)}(z) \right) =$$

$$(2.10) \quad = \frac{1}{2\pi \mathrm{i}} \sum_{j,\varepsilon} \bar{c}_\varepsilon \int \frac{\prod (\bar{d}_\varepsilon - \Delta_i) \prod (1 - \bar{z}_m \Gamma_i)\, \mathrm{d}\tau_j}{\prod (1 - D_i(\tau_j)\bar{d}_\varepsilon) \prod (1 - G_i(\tau_j)\bar{d}_\varepsilon) \prod (\bar{z}_m - E_i(\tau_j))(1 - \bar{z}_m \bar{d}_\varepsilon)}.$$

Note that $F - \tau_j$ times the integrand in (2.9) equals

$$r_{jm}(z) = \frac{\prod (1 - E_i(\tau_j)z) \prod (1 - \Gamma_i \bar{z}_m)}{\prod (z - \Gamma_i) \prod (\bar{z}_m - E_i(\tau_j))(1 - \bar{z}_m z)}$$

which is a rational function (of z) with $r_{jm}(\mathrm{e}^{\mathrm{i}t}) \in L^2$. Thus Lemma 2.4 applies to the integral in (2.10) and yields

$$(L_{0F}u_m, h_\lambda^{(k)}) = \frac{1}{2\pi \mathrm{i}} \sum_{j,\varepsilon} \bar{c}_\varepsilon \int (\lambda - \tau_j)^{-1} (\mathbf{P} r_{jm})(\bar{d}_\varepsilon)\, \mathrm{d}\tau_j.$$

We claim that

$$r_{jm}(\mathrm{e}^{\mathrm{i}t}) = \bar{R}_{jm}(\mathrm{e}^{\mathrm{i}t})(1 - \bar{z}_m \mathrm{e}^{\mathrm{i}t})^{-1}$$

where $R_{jm}(\mathrm{e}^{\mathrm{i}t}) \in H^2$. This follows, once again, by the fact that the number of $E_i(\lambda)$ (for λ outside $\mathrm{sp}(T_F)$) equals the number of Γ_i; [2, Lemma 2.1]. Furthermore

$$R_{jm}(z) = \frac{\prod (z - \bar{E}_i(\tau_j)) \prod (1 - \bar{\Gamma}_i z_m)}{\prod (1 - \bar{\Gamma}_i z) \prod (z_m - \bar{E}_i(\tau_j))}.$$

It follows that $\mathbf{P} r_{jm}$ is just the (co-analytic) Toeplitz operator $T^*_{R_{jm}}$ applied to (its eigenvector) $(1 - \bar{z}_m z)^{-1}$, yielding

$$\mathbf{P} r_{jm} = \bar{R}_{jm}(z_m)(1 - \bar{z}_m z)^{-1} = (1 - \bar{z}_m z)^{-1}.$$

Thus

$$(L_{0F}u_m, h_\lambda^{(k)}) = \frac{1}{2\pi i} \sum_{j,\varepsilon} \bar{c}_\varepsilon \int (\lambda - \tau_j)^{-1} (1 - \bar{z}_m \bar{d}_\varepsilon)^{-1} \, d\tau_j =$$

$$= \frac{1}{2\pi i} \bar{h}_{\bar\lambda}^{(k)}(z_m) \int_{\partial \dot\sigma_f} (\lambda - \tau_j)^{-1} \, d\tau_j = \delta_{km} \bar{h}_{\bar\lambda}^{(m)}(z_m),$$

and Lemma 2.3 is proved.

3. PROOF OF THEOREM 1

We begin with a computation involving the resolvent of T_F. Let $|u|, |v| < 1$; we have, by standard Wiener-Hopf factorization,

$$((T_F - \lambda I)^{-1}(1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) =$$
$$= A^{-1} \prod (v - \Delta_i) \prod (1 - D_i(\lambda)v)^{-1} \prod (1 - \Gamma_i \bar{u} \prod) (\bar{u} - E_i(\lambda))^{-1} (1 - \bar{u}v^{-1}).$$

By spectral theory, if γ is any simple closed positively oriented curve containing $\mathrm{sp}(T_F)$ in its interior,

$$(1 - \bar{u}v)^{-1} = ((1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) =$$

$$= \frac{1}{2\pi i} \int_\gamma ((\lambda I - T_F)^{-1}(1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) \, d\lambda = \tag{3.1}$$

$$= \frac{1}{2\pi i} \int_\gamma A^{-1} \frac{\prod (v - \Delta_i)}{\prod (1 - D_i(\lambda)v)} \frac{\prod (1 - \Gamma_i \bar{u})}{\prod (\bar{u} - E_i(\lambda))} (1 - \bar{u}v)^{-1} d\lambda .$$

Since the integrand is bounded, we can let the curve γ approach $\partial\mathrm{sp}(T_F)$ and take the limit. By (IV), [1, Lemma 1.3] implies that, as λ approaches a loop ℓ_i from the outside, $\prod(1 - D_i(\lambda)v)$ approaches

$$\prod (1 - D_i(\lambda)v) \prod (1 - G_i(\lambda)v)$$

and each $|E_i(\lambda)|$ remains >1. Similarly as λ approaches a loop $\mathscr{L}_i$ from the outside, $\prod (u - E_i(\lambda))$ approaches

$$\prod (u - E_i(\lambda)) \prod (u - G_i(\lambda))$$

and each $|D_i(\lambda)|$ remains <1. Thus, taking the limit in (3.1), we get

$$(1 - \bar{u}v)^{-1} = \frac{1}{2\pi i} \sum_j \int_{\partial \ell_j} A^{-1} \frac{\prod(v - \Delta_i) \prod (1 - \Gamma_i \bar{u}) \, d\lambda}{\prod (1 - D_i(\lambda)v) \prod (1 - G_i(\lambda)v) \prod (\bar{u} - E_i(\lambda))(1 - \bar{u}v)} +$$

$$+ \frac{1}{2\pi i} \sum_j \int_{\partial \mathscr{L}_j} A^{-1} \frac{\prod (v - \Delta_i) \prod (1 - \Gamma_i \bar{u}) \, d\lambda}{\prod (1 - D_i(\lambda)v) \prod (\bar{u} - E_i(\lambda)) \prod (\bar{u} - G_i(\lambda))(1 - \bar{u}v)} \tag{3.2}$$

where each $\partial \ell_j, \partial \mathscr{L}_j$ is oriented counterclockwise.

Now we deal with $L_{0F}L_{0f}$. We have

$$(L_{0F}L_{0f}(1-\bar{u}z)^{-1},(1-\bar{v}z)^{-1})=$$

$$=(L_{0F}\sum_k h_\lambda^{(k)}(z_k)^{-1}\bar{h}_\lambda^{(k)}(u),\ (1-\bar{v}z)^{-1})=$$

$$=\sum_{j,m}\frac{1}{2\pi i}\int_0^{2\pi}A^{-1}\frac{\prod(v-\Delta_i)\prod(1-\Gamma_i\bar{z}_m)(1-\bar{z}_m v)^{-1}}{\prod(1-D_i(\tau_j)v)\prod(1-G_i(\tau_j)v)\prod(\bar{z}_m-E_i(\tau_j))}h_{\tau_j}^{(m)}(z_m)^{-1}h_{\tau_j}^{(m)}(u)\,d\tau_j.$$

Summing up the terms depending upon m, we get

$$\sum_m\frac{\prod(1-\Gamma_i z_m)}{\prod(z_m-E_i(\tau_j))(1-\bar{z}_m v)}h_{\bar{\tau}_j}^{(j)}(z_m)^{-1}\prod_{i\neq m}(u-z_i)=\tag{3.3}$$

$$=\sum_m\frac{\prod(1-\Gamma_i z_m)\prod(1-\bar{d}_i(\bar{\tau}_j)\bar{z}_m)\prod(1-g_i(\bar{\tau}_j)z_m)\prod_{i\neq m}(\bar{u}-\bar{z}_i)}{\prod(\bar{z}_m-E_i(\tau_j))(1-\bar{z}_m v)\prod_{j\neq m}(\bar{z}_m-\bar{z}_j)\prod(1-\delta_i^{-1}z_m)}$$

where we have used the fact that, as z tends to $|z|=1$ from inside $|z|<1$, $\prod(1-\bar{d}_i(\tau_j(z))\bar{z}_m)$ tends to

$$\prod(1-\bar{d}_i(\bar{\tau}_j(z))\bar{z}_m)\prod(1-\bar{g}_i(\bar{\tau}_j(z))\bar{z}_m)\ ;$$

([1]; Lemma 1.3).

Now from the fact that $F(z)=\bar{f}(\bar{z}^{-1})$, we have

$$D_i(\lambda)=\bar{e}_i(\bar{\lambda})^{-1},\ E_i(\lambda)=\bar{d}_i(\bar{\lambda})^{-1},\ \ G_i(\lambda)=\bar{g}_i(\bar{\lambda})$$

$$\Gamma_i=\bar{\delta}_i^{-1}\text{ and }\Delta_i=\bar{\gamma}_i^{-1}.$$

Using these relations, (3.3) becomes

$$\prod(\bar{u}-\bar{z}_i)\prod(-E_i(\tau_j))^{-1}\prod(-G_i(\tau_j))^{-1}\sum_m\frac{\prod(\bar{z}_m-G_i(\tau_j))}{(1-\bar{z}_m v)\prod_{i\neq m}(\bar{z}_m-\bar{z}_i)(u-z_m)}.$$

From the partial fractions expansion

$$\frac{\prod(z-G_i(\tau_j))}{(1-vz)\prod(z-\bar{z}_i)}=\sum_m\frac{\prod(\bar{z}_m-G_i(\tau_j))}{(1-v\bar{z}_m)\prod_{i\neq m}(\bar{z}_m-\bar{z}_i)(z-\bar{z}_m)}+\frac{\prod(v^{-1}-G_i(\tau_j))}{\prod(v^{-1}-\bar{z}_i)(1-vz)},$$

in which we set $z=\bar{u}$, we get for (3.3)

$$\prod(-E_i(\tau_j))^{-1}\prod(-G_i(\tau_j))^{-1}\left[\frac{\prod(\bar{u}-G_i(\tau_j))}{(1-v\bar{u})}-\frac{\prod(\bar{u}-\bar{z}_i)\prod(1-G_i(\tau_j)v)}{\prod(1-\bar{z}_i v)(1-v\bar{u})}\right].$$

Putting this in our expression for $L_{0F}L_{0f}$, together with

$$h_{\tau_j}^{(m)}(u) = \prod_{j\neq m} (\bar{u} - \bar{z}_j) \prod (1 - \bar{\delta}_i^{-1}\bar{u})/[\prod (1 - d_i(\bar{\tau}_j)\bar{u}) \prod (1 - \bar{g}_i(\bar{\tau}_j)\bar{u})] =$$

$$= \prod (-E_i) \prod (-G_i) \prod_{j\neq m} (\bar{u} - \bar{z}_j) \prod (1 - \Gamma_i\bar{u})/[\prod (\bar{u} - E_i(\bar{\tau}_j)) \prod (\bar{u} - G_i(\bar{\tau}_j))]$$

we get

$$(L_{0F}L_{0f}(1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) =$$

$$= \sum_j \frac{1}{2\pi i}\left[\int_0^{2\pi} A^{-1} \frac{\prod(v - \Delta_i)\prod(1 - \Gamma_i\bar{u})}{\prod(1 - D_i(\tau_j)v)\prod(1 - G_i(\tau_j)v)\prod(\bar{u} - E_i(\tau_j))(1 - \bar{u}v)} d\tau_j - \right.$$

$$\left. - \int_0^{2\pi} A^{-1} \frac{\prod (v - \Delta_i) \prod (1 - \Gamma_i\bar{u}) \prod(\bar{u} - \bar{z}_i)\, d\tau_j}{\prod(1 - D_i(\tau_j)v\prod(\bar{u} - E_i(\tau_j))\prod(\bar{u} - G_i(\tau_j))\prod(1 - \bar{z}_iv)(1 - \bar{u}v)}\right] =$$

$$= \sum_j \frac{1}{2\pi i}\left[\int_{\partial\ell_j} A^{-1} \frac{\prod(v - \Delta_i) \prod (1 - \Gamma_i\bar{u})}{\prod(1 - D_i(\lambda)v) \prod(1 - G_i(\lambda)v) \prod (\bar{u} - E_i(\lambda))(1 - \bar{u}v)} d\lambda - \right.$$

$$\left. - \frac{\prod(v - \Delta_i)\prod(1 - \Gamma_i\bar{u})\prod(\bar{u} - \bar{z}_i)}{\prod(1 - \bar{z}_iv)(1 - \bar{u}v)} \int_{\partial\ell_i} A^{-1} \frac{d\lambda}{\prod(1-D_i(\lambda)v)\prod(\bar{u}-E_i(\lambda))\prod(\bar{u}-G_i(\lambda))}\right].$$

The last integral is 0 for every i, since the integrand is the boundary function of

$$A^{-1} \prod (1 - D_i(\lambda)v)^{-1} \prod (\bar{u} - E_i(\lambda))^{-1}$$

which is analytic inside each ℓ_i (since continuation around an algebraic singularity only serves to permute the $\{D_i\}$ or the $\{E_i\}$). Thus the sum of the first integrals is equal to the sum of integrals over the $\partial\ell_i$ in (3.2).

To complete the proof, we note that we may repeat the constructions of § 2 above with f and F interchanged, and working on the loops $\bar{\mathscr{L}}_i$ instead of ℓ_i. The result is a pair of operators L_{2f} and L_{2F} (corresponding to L_{0F} and L_{0f}, respectively), from $H^2(\cup \bar{\mathscr{L}}_i^{(N_i)})$ to H^2 and from H^2 to $H^2(\cup \bar{\mathscr{L}}_i^{(N_i)})$, respectively. The analogue of (2.5) is the intertwining relation

$$T_f L_{2f} x = L_{2f}\, \lambda x\,,$$

from which it follows that

(3.4) $$L_{2f}^* T_F x = \bar{\lambda}\, L_{2f}^* x\,.$$

The analogue of (2.6) is

(3.5) $$L_{2F}L_{2f} = I$$

from which it follows that L_{2f} has 0 kernel, and hence that L_{2f}^* has dense range.

Now let $L = L_{2f}^* + L_{0f}$. From (2.2) and (3.4), we have

$$LT_F x = \bar{\lambda} L_{2f}^* x + \lambda L_{0f} x\,.$$

We claim L has dense range (in $H^2(\cup \mathscr{L}_i^{(N_i)}) \oplus H^2(\cup \ell_i^{(\nu_i)})$). For this we use the orthogonality of the manifolds $\mathscr{M}_i$ and m_i in the introduction. [To prove this orthogonality, suppose $T_F k_\lambda = \lambda k_\lambda$ and $T_f h_\mu = \mu h_\mu$. Then

$$(k_\lambda, h_\mu) = \lambda^{-1}(T_F k_\lambda, h_\mu) = \lambda^{-1}(k_\lambda, T_f h_\mu) = \bar{\mu}\lambda^{-1}(k_\lambda, h_\mu)\,,$$

and $\bar{\mu}\lambda^{-1} \neq 1$, since $\lambda \in \mathscr{L}_i$ and $\mu \in \ell_i$.] By definition the kernel of L_{0f} is the orthogonal complement of $\sum m_i$. Analogously, the kernel of L_{2F} is $(\sum \mathscr{M}_i)^\perp$. Thus $\sum \mathscr{M}_i$ is the range of the partial isometry (c.f. (3.5)) $V = L_{2F}^* L_{2f}^*$. Clearly the range $R(L_{2f}^*) = R(L_{2f}^{*}{}'_{R(V)})$ and $R(V) \subset \ker L_{0f}$. By applying L to vectors in $R(V)$, we can now see that $R(L_{2f}^*) \subset R(L)$. To see that $R(L)$ is dense, let $x \oplus y \in H^2(\cup \mathscr{L}_i^{(N_i)}) \oplus$ $\oplus H^2(\cup \ell_i^{(\nu_i)})$. Approximate y with vectors $L_{0f} y_n$ and then approximate each $x - L_{2f}^* y_n$ by $L_{2f}^* x_n$ $(x_n \in R(V))$. It follows that

$$L(x_n + y_n) = (L_{2f}^* x_n + L_{2f}^* y_n) \oplus L_{0f} y_n \to x \oplus y\,.$$

Finally, we construct a left inverse for L. We have

$$(L_{2f} L_{2F} (1 - vz)^{-1}, (1 - \bar{u}z)^{-1}) =$$

$$= \sum_j \frac{1}{2\pi i} \int_{\partial \mathscr{L}_j} a^{-1} \frac{\prod (u - \delta_i) \prod (1 - \gamma_i \bar{v})}{\prod (1 - d_i(\lambda) u) \prod (1 - g_i(\lambda) u) \prod (\bar{v} - e_i(\lambda))(1 - \bar{v}u)}\, d\lambda.$$

Replacing the d_i, e_i, etc. with appropriate $\bar{E}_i^{-1}$, $\bar{D}_i^{-1}$, etc. and conjugating, we obtain the sum of integrals in (3.2) over $\partial \mathscr{L}_i$, so

$$((1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) = (L_{0F} L_{0f} (1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1}) +$$

$$+ (L_{2F}^* L_{2f}^* (1 - \bar{u}z)^{-1}, (1 - \bar{v}z)^{-1})$$

and thus $L_{2F}^* \oplus L_{0F}$ is the inverse of L. This completes the proof.

4. THE CASE WHEN F BACKS UP

Let σ be the set of points on $\mathbf{T}$ where F backs up. That is $e^{i\varphi} \in \sigma$ satisfies either

(4.1) $\quad F(e^{i\varphi}) \in \partial \ell_i$ and $\arg \tau_i^{-1} F(e^{it})$ is increasing at $t = \varphi$

or

(4.2) $\quad F(e^{i\varphi}) \in \partial \mathscr{L}_i$ and $\arg T_i^{-1} f(e^{it})$ is increasing at $t = \varphi$.

In case condition (IV) on F is lifted, we have the following generalization of Theorem 1.

THEOREM 2. *Let F satisfy* (I), (II) *and* (III). *Then T_F is similar to*

$$\sum \oplus T_{\tau_i} \oplus \sum \oplus T^*_{T_i} \oplus V$$

where T_{τ_i} acts on $H^2_{-\nu_i}$, T_{T_i} acts on $H^2_{N_i}$ and V is a normal operator with spectrum $F(\sigma)$. V is absolutely continuous with the spectral multiplicity of a point λ in the spectrum of V equal to the number of points $e^{i\varphi}$ where F backs up and $F(e^{i\varphi}) = \lambda$.

We include a brief sketch of how the results of the present paper and of [1, Part II] can be applied to prove Theorem 2.

First, the operators L_{0f}, L_{0F}, L_{2f} and L_{2F} must be modified as in [1, § 6]. For example, $\rho_k(\lambda)$, in the definition of L_{0f}, must be multiplied by

$$\rho'_k(\lambda) = \prod (\lambda - \lambda_i)^{-q_i/(2\beta_i)}$$

as in [1, § 6], and the m-th term in L_{0F} must be multiplied by ρ'^{-1}_m and the denominator altered as in [1]. On the loops ℓ_i, L_{1f} and L_{1F} are defined as in [1], and, on $\mathscr{L}_i$, L_{3f} and L_{3F} are defined so that L_{3f} [resp. L_{3F}] corresponds to L^*_{1F} [resp. L^*_{1f}], with F and f interchanged. Lemma 1.3 above is used to show that the operators so defined are bounded.

Now let

$$L = (L_{2f} + L_{3f})^* + L_{0f} + L_{1f}$$

and

$$L' = L^*_{2F} + L_{3F} + L_{0F} + L^*_{1F}\,.$$

Direct computation shows that $L'L = I$, so it suffices to show L has dense range.

From

$$(L^*_{3F} + L_{1F})T_f = \lambda(\overline{L}^*_{3F} + L_{1F})$$

it follows that, if k_μ is an eigenvector of T_f,

$$(L^*_{3F} + L_{1F})k_\mu = \bar{\lambda}\mu^{-1}(L^*_{3F} + L_{1F})k_\mu$$

so that $(L^*_{3F} + L_{1F})k_\mu = 0$ $(\bar{\lambda}\mu^{-1} \neq 1$, since $\bar{\mu} \in \operatorname{int}\ell_i$ and $\lambda \in \partial\ell_i \cup \partial\mathscr{L}_i)$. Thus $\sum m_i \subset \ker(L^*_{3F} + L_{1F})$ or

(4.3) $$R(L_{3F} + L^*_{1F}) \subset (\sum m_i)^\perp.$$

Similarly

(4.4) $$\sum \mathscr{M}_i \subset \ker(L^*_{3f} + L_{1f})\,.$$

Of course

(4.5) $$\sum \mathscr{M}_i \subset \ker L_{0f}$$

by definition of L_{0f} and since $\sum m_i \perp \sum \mathscr{M}_i$.

As a consequence, we can show that

(4.6) $$R(L^*_{2f}) \subset R(L)\,.$$

Indeed, for $x \in H^2(\cup \overline{\mathscr{L}}_i^{(N_i)})$, $L_{2F}^* x \in \sum \mathscr{M}_i$ and $L_{2f}^* L_{2F}^* x = x$ follow as [1, (7.5), and (7.6)], and then we have

$$x = L_{2f}^* L_{2F}^* x = L L_{2F}^* x \in R(L)$$

by (4.4) and (4.5).

It is also true that

$$R(L_{3f}^* + L_{1f}) \subset R(L) . \tag{4.7}$$

From (4.6), it suffices to prove

$$R(L_{2f}^* + L_{3f}^* + L_{1f}) \subset R(L) . \tag{4.8}$$

If x belongs to the left member of (4.8), we have

$$y = (L_{2F}^* + L_{3F} + L_{1F}^*) x \in (\sum m_i)^\perp$$

by (4.3) and the fact that $R(L_{2F}^*) \subset \sum \mathscr{M}_i \subset (\sum m_i)^\perp$. Thus

$$y = L'Ly = L'(L_{2f}^* + L_{3f}^* + L_{1f}) y =$$
$$= (L_{2F}^* + L_{3F} + L_{1F}^*)(L_{2f}^* + L_{3f}^* + L_{1f}) y$$

and, since $L_{2F}^* + L_{3F} + L_{1F}^*$ is one-to-one,

$$x = (L_{2f}^* + L_{3f}^* + L_{1f}) y = Ly \in R(L) .$$

This proves (4.8) and hence (4.7). $R(L_{3f}^* + L_{1f})$ is dense, by a modification of the proof of [1, Lemma 5.3].

From (4.6) and (4.7), it follows that $R(L_{0f}) \subset R(L)$ and the density of $R(L)$ is proved.

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