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ON INTERTWINING DILATIONS. VIII

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0. INTRODUCTION

Since the existence of contractive intertwining dilations was proved in early 1968 ([25]), it became clear that the problem of describing all of them is (or will be) involved in several important topics of Operator Theory and other parts of Analysis ([13]). Many basic ideas concerning special cases of this problem appeared explicitely or implicitely in quite a few papers (for example: [1], [2], [3], [4], [5], [12], [21], [26], and so on).

There are now at hand some general descriptions of the set of all contractive intertwining dilations (see [10], [11]); among these we quote here the following:

(A) the labelling using choice sequences (see [10], or the next section), which deals with the "free" part of the problem, and

(B) the idea of Lemma 4.3 from [10], where the general problem is reduced to a very simple case. These facts are given, more or less, implicitely in [10]; it is the aim of this paper to make explicite these labellings and the connections between them.

We will heavily use the account given in [6] for the choice sequence labelling.

This paper has two main results:

(A′) an algorithm which connects a contractive intertwining dilation with its choice sequence (Theorem 4.1, below); this algorithm gives more workable algorithm in seismic exploration used in Geophisics.

(B′) an explicit formula for the connection (B) (Theorem 6.1, below). This formula turns out to be a generalization of both classical and operational extrapolation formulas ([1], [2], [3], [8], [9], [17], [18], [19], [22], or for example [16], Ch. IV, Th. 8.9 and Ch. V, § 4) and the characteristic function of a contraction ([24], Ch. VI).

Different consequences of these results for Operator Theory, Differential Equations and Geophysics as well as further developments connecting explicitely our results to [3], [7], [14], [15], [16], [26], [27] will be given elsewhere.

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1. LABELLINGS OF CONTRACTIVE INTERTWINING DILATIONS

In this section we will recall the basic definitions and results which will be used in the sequel.

Let $\mathscr{H}$ and $\mathscr{H}'$ be (complex) Hilbert spaces, and $\mathscr{L}(\mathscr{H},\mathscr{H}')$ be the set of all (bounded linear) operators from $\mathscr{H}$ to $\mathscr{H}'$; the set $\mathscr{L}(\mathscr{H},\mathscr{H})$ will be denoted by $\mathscr{L}(\mathscr{H})$ and I (resp. 0) will be the identity operator on any Hilbert space (resp. the zero operator on any Hilbert space, or the zero element of any vector space). For a contraction $C\in\mathscr{L}(\mathscr{H},\mathscr{H}')$ (i.e. $\|C\|\leqslant 1$), $D_C=(I-C^*C)^{1/2}$ and $\mathscr{D}_C=D_C(\mathscr{H})^-$ will be called the defect operator, resp. space, of C.

In the sequel we fix $T\in\mathscr{L}(\mathscr{H})$ and $T'\in\mathscr{L}(\mathscr{H}')$ two contractions, and let $U\in\mathscr{L}(\mathscr{K})$ and $U'\in\mathscr{L}(\mathscr{K}')$ be their minimal isometric dilations. (For the geometry of the minimal isometric dilation, see [24], [23].) Consider:

$$\mathscr{L}=(U-T)(\mathscr{H})^-$$

$$\mathscr{L}_*=(I-UT^*)(\mathscr{H})^-,$$

and define

$$\mathscr{H}_0=\mathscr{H}\,, \tag{1.1$_0$}$$

and for every $n\geqslant 1$,

$$\mathscr{H}_n=\mathscr{H}+\mathscr{L}+U\mathscr{L}+\ldots+U^{n-1}\mathscr{L}=\mathscr{L}_*+U\mathscr{L}_*+\ldots+U^{n-1}\mathscr{L}_*+U^n\mathscr{H} \tag{1.1$_n$}$$

(where the sums of subspaces are always orthogonal). We will use the following notation

$$P=P_0=P^{\mathscr{K}}_{\mathscr{H}}\,,^{(1)} \tag{1.2$_0$}$$

$$P_n=P^{\mathscr{K}}_{\mathscr{H}_n}\,, \qquad (n\geqslant 1), \tag{1.2$_n$}$$

and

$$T_n=P^{\mathscr{K}}_{\mathscr{H}_n}U|\mathscr{H}_n\,, \qquad (n\geqslant 0). \tag{1.3$_n$}$$

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(1) If $\mathscr{G}$ is a (closed linear) subspace of $\mathscr{H}$, $P^{\mathscr{H}}_{\mathscr{G}}$ will stand for the (orthogonal) projection of $\mathscr{H}$ onto $\mathscr{G}$; in this case $P^{\mathscr{H}}_{\mathscr{H}\ominus\mathscr{G}}$ will be denoted also by $1-P^{\mathscr{H}}_{\mathscr{G}}$.

It follows that

(1.4) $$\mathscr{K} = \bigvee_{n=0}^{\infty} \mathscr{H}_n, \text{ and}$$

(1.5) $$U = s\text{-}\lim_{n\to\infty} T_n P_n$$

(this means strong (operatorial) limit).

The notation $\mathscr{L}'$, $\mathscr{L}'_*$, $\mathscr{H}'_n$, P', P'_n, T'_n $(n \geqslant 0)$ are now clear.

Consider a fixed contraction $A \in \mathscr{I}(T', T)$ (i.e. $A \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ and $T'A = AT$; we say that A intertwines T and T'). A *contractive intertwining dilation* of A is a contraction $A_\infty \in \mathscr{I}(U', U)$, such that $P'A_\infty = AP$. The set of all contractive intertwining dilation of A will be denoted by CID(A); this is the main object of our study. One easy way to describe CID(A) is to use the "construction in steps" of the minimal isometric dilation (see $(1.1)_n$, $(1.3)_n$, $n \geqslant 0$, (1.4) and (1.5)). For $n \geqslant 0$, a *n-partial contractive intertwining dilation* of A is a contraction $A_n \in \mathscr{I}(T'_n, T_n)$, such that $P'A_n = AP|\mathscr{H}_n$ (we say that $A_n \in n$-PCID(A)). A sequence of contractions $\{A_n\}_{n=0}^{\infty}$ is a *chain of PCID* of A if for every $n \geqslant 0$, $A_n \in n$-PCID(A) and $P'_n A_{n+1} = A_n P_n|\mathscr{H}_{n+1}$. With this notation we have that the applications:

(1.6) $$A_\infty \to \{P'_n A_\infty | \mathscr{H}_n\}_{n=0}^{\infty}, \qquad \text{and}$$

(1.6)′ $$\{A_n\} \to A_\infty = \text{s-}\lim_{n\to\infty} A_n P_n$$

are reciprocal bijections between CID(A) and the set of all chains of PCID of A. The main point in this description is that in a chain $\{A_n\}_{n=0}^{\infty}$ of PCID of A, we have that $A_n \in$ 1-PCID(A_{n-1}) for every $n \geqslant 1$; this means that a careful study of 1-PCID(A) and an inductive argument will provide a way to settle the problem.

The main labelling of CID(A) which will be considered here uses "choice sequences" (see [10]). For this, let us define

(1.7) $$\begin{cases} \mathscr{F}_A(T) = \mathscr{F}_A = \{D_A Th + (U - T)h : \ h \in \mathscr{H}\}^- \subset \mathscr{D}_A + \mathscr{L} \\ \mathscr{R}_A(T) = \mathscr{R}_A = (\mathscr{D}_A + \mathscr{L}) \ominus \mathscr{F}_A, \end{cases}$$

(1.8) $$\begin{cases} \mathscr{F}^A(T') = \mathscr{F}^A = \{D_A h \oplus (U' - T')Ah : h \in \mathscr{H}\}^- \subset \mathscr{D}_A \oplus \mathscr{L}' \\ \mathscr{R}^A(T') = \mathscr{R}^A = (\mathscr{D}_A \oplus \mathscr{L}') \ominus \mathscr{F}^A. \end{cases}$$

These spaces appear in [5], following the result from [4] which gives the uniqueness condition for CID(A) in terms of regular factorizations ([24], Ch. VII). So, the factorization $A \cdot T$ (resp. $T' \cdot A$) is regular if and only if $\mathscr{R}_A = \{0\}$ (resp. $\mathscr{R}^A = \{0\}$) We will also use the projections:

(1.9) $$\begin{aligned} p_A(T) &= p_A = P_{\mathscr{F}_A}^{\mathscr{D}_A + \mathscr{L}} & p^A(T') &= p^A = P_{\mathscr{F}^A}^{\mathscr{D}_A \oplus \mathscr{L}'} \\ q_A(T) &= q_A = P_{\mathscr{L}}^{\mathscr{D}_A + \mathscr{L}} & q^A(T') &= q^A = (i_{\mathscr{D}_A \oplus \mathscr{L}'}^{\mathscr{L}'})^* \ P_{\{0\} \oplus \mathscr{L}'}^{\mathscr{D}_A \oplus \mathscr{L}'}, \end{aligned}$$

(where $i_{\mathscr{D}_A \oplus \mathscr{L}'}^{\mathscr{L}'} : \mathscr{L}' \to \{0\} \oplus \mathscr{L}' \subset \mathscr{D}_A \oplus \mathscr{L}'$ is the canonical injection of $\mathscr{L}'$ in $\{0\} \oplus \mathscr{L}' \subset \mathscr{D}_A \oplus \mathscr{L}'$; in this case $1 - q^A$ is in fact $(i_{\mathscr{D}_A \oplus \mathscr{L}'}^{\mathscr{D}_A})^* \ P_{\mathscr{D}_A \oplus \{0\}}^{\mathscr{D}_A \oplus \mathscr{L}'}$).

A sequence $\{\Gamma_n\}_{n=1}^{\infty}$ of contractions is called an *A-choice sequence*, if $\Gamma_1 \in \mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$, and for every $n \geqslant 2$, $\Gamma_n \in \mathscr{L}(\mathscr{D}_{\Gamma_{n-1}}, \mathscr{D}_{\Gamma^*_{n-1}})$. Combining Propositions 2.2 and 3.1 of [10], we obtain that there exists a bijection between CID(A) and the set of all A-choice sequences. The analysis of this bijection given in [6] follows the "stepwise approach" described above.

In this respect, the analysis of the "first step" gives an explicit bijection between 1-PCID(A) and the contractions of $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$. The "imposed" part of an $A_1 \in$ 1-PCID(A) is given by the unitary operator:

$$(1.10)\qquad \begin{cases} \sigma(A; T', T) = \sigma_A : \mathscr{F}_A \to \mathscr{F}^A \\ \sigma_A(D_A Th + (U - T)h) = D_A h \oplus (U' - T')Ah\,, \qquad h \in \mathscr{H}. \end{cases}$$

The one-to-one correspondence between $A_1 \in$ 1-PCID(A) and $\Gamma_1 \in \mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$ is given by (see [6], Lemma 2.1):

$$(1.11)\quad A_1(A, \Gamma_1) = A_1 = AP|\mathscr{H}_1 + q^A(\sigma_A p_A + \Gamma_1(1 - p_A))(D_A P + I - P)|\mathscr{H}_1\,,$$

and for $\Gamma_1(A, A_1) = \Gamma_1$

$$(1.12)\qquad q^A \Gamma_1(1 - p_A)(D_A P + I - P)|\mathscr{H}_1 = A_1 - A_1^0\,,$$

where $A_1^0 = A_1(A, 0) = AP|\mathscr{H}_1 + q^A \sigma_A p_A(D_A P + I - P)|\mathscr{H}_1$.
In order to use inductive arguments we must go further with the analysis of the "first step". We obtain then (see [6], (2.10), (4.12) and (4.15)) the unitary operators:

$$(1.13)\qquad \begin{cases} \omega_{A_1}(T_1', T_1) = \omega_{A_1} : \mathscr{R}_{A_1} \to \mathscr{D}_{\Gamma_1} \\ \omega_{A_1}(1 - p_{A_1})D_{A_1} = D_{\Gamma_1}(1 - p_A)(D_A P + I - P)|\mathscr{H}_1\,, \end{cases}$$

$$(1.14)\qquad \begin{cases} \omega^{A_1}(T_1', T_1) = \omega^{A_1} : \mathscr{R}^{A_1} \to \mathscr{D}_{\Gamma_1^*} \\ \omega^{A_1}(1 - p^{A_1})(0 \oplus U'l') = D_{\Gamma_1^*}(1 - p^A)(0 \oplus l'), \quad l' \in \mathscr{L}'\,, \end{cases}$$

which verify

$$(1.15)_1\qquad \Gamma_1 \omega_{A_1} = -\omega^{A_1}(1 - p^{A_1})|\mathscr{R}_{A_1}\,.$$

(Note that $\mathscr{R}_{A_1} = \mathscr{D}_{A_1} \ominus (D_{A_1} U(\mathscr{H}))^- \subset \mathscr{D}_{A_1}$.)

The operator ω_{A_1} from (1.13) is the restriction to $\mathscr{R}_{A_1}$ of a "larger" unitary operator (see [6], (2.7)), namely:

$$(1.16)\begin{cases} \widetilde{\omega}_{A_1}(T_1', T_1) = \widetilde{\omega}_{A_1} : \mathscr{D}_{A_1} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \\ \widetilde{\omega}_{A_1} D_{A_1} = [(1 - q^A)(\sigma_A p_A + \Gamma_1(1 - p_A)) \oplus D_{\Gamma_1}(1 - p_A)](D_A P + I - P)|\mathscr{H}_1\,. \end{cases}$$

We will need also that (see [6], (2.8))

$$(1.16)' \qquad \widetilde{\omega}_{A_1} D_{A_1} U h = D_A h \oplus 0 , \qquad (h \in \mathscr{H}).$$

The operators ω_{A_1} and ω^{A_1} suggest how to construct an A-choice sequence associated to an $A_\infty \in \mathrm{CID}(A)$; we define by induction:

$$(1.17)_1 \qquad \Gamma_1 = \Gamma_1(A, A_1) , \ \Omega_{A_1} = \omega_{A_1}, \ \Omega^{A_1} = \omega^{A_1} ,$$

and for $n \geqslant 2$

$$(1.17)_n \ \Gamma_n = \Omega^{A_{n-1}} \Gamma_1(A_{n-1}, A_n) \Omega^*_{A_{n-1}} , \Omega_{A_n} = \Omega_{A_{n-1}} \circ \omega_{A_n} , \Omega^{A_n} = \Omega^{A_{n-1}} \circ \omega^{A_n} ,$$

where $\Gamma_1(A_{n-1}, A_n)$, $\omega_{A_n} = \omega_{A_n}(T'_n, T_n)$, $\omega^{A_n} = \omega^{A_n}(T'_n, T_n)$ are associated to $A_n \in$ 1-PCID(A_{n-1}).

The sequence $\{\Gamma_n\}_{n=1}^{\infty}$ is called the *A-choice sequence* of A_∞. The sequences $\{\Omega_{A_n}\}_{n=1}^{\infty}$ and $\{\Omega^{A_n}\}_{n=1}^{\infty}$ are called the *sequences of identificators* of A_∞; for $n \geqslant 1$, Ω_{A_n} (resp. Ω^{A_n}) is a unitary operator from $\mathscr{R}_{A_n}$ (resp. $\mathscr{R}^{A_n}$) onto $\mathscr{D}_{\Gamma_n}$ (resp. $\mathscr{D}_{\Gamma^*_n}$). We have, of course, that for every $n \geqslant 1$

$$(1.15)_n \qquad \Gamma_1(A_{n-1}, A_n)\, \omega_{A_n} = -\omega^{A_n}(1 - p^{A_n}) | \mathscr{R}_{A_n}.$$

As in the case of ω_{A_1}, the operator Ω_{A_n} is the restriction to $\mathscr{R}_{A_n}$ of a larger unitary operator, namely

$$(1.18)_n \qquad \widetilde{\Omega}_{A_n} : \mathscr{D}_{A_n} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} ,$$

defined recurrently by

$$(1.18)'_1 \qquad \widetilde{\Omega}_{A_1} = \widetilde{\omega}_{A_1},$$

and for $n \geqslant 2$

$$(1.18)'_n \qquad \widetilde{\Omega}_{A_n} = (\widetilde{\Omega}_{A_{n-1}} \oplus \Omega_{A_{n-1}}) \circ \widetilde{\omega}_{A_n} .$$

2. OBSERVABLE SEQUENCES

The aim of Sections 2, 3 and 4 is to give an algorithmic variant of the one-to--one correspondence between CID(A) and A-choice sequences. From the point of view of Operator Theory this analysis produces new objects (connected also with [11]) which might be of interest. On the other hand, a very particular case of this algorithm is a variant of the Robinson's method (see for example [20]) used in Geophysics for seismic exploration; our algorithm seems to be more adequate for controlling small perturbations.

Consider again the typical situation of this paper: $T \in \mathscr{L}(\mathscr{H})$, $T' \in \mathscr{L}(\mathscr{H}')$ $A \in \mathscr{I}(T', T)$ are fixed contractions and A_∞ is a contractive intertwining dilation of A (i.e. $A_\infty \in \mathrm{CID}(A)$). Let $\{A_n\}_{n=0}^{\infty}$ be the chain of PCID of A_∞. In this section we will give an alternate description of A_∞. We will use for this the decompositions of $\mathscr{H}$ and $\mathscr{H}'$ given by $(1.1)_n$, $n \geqslant 0$.

DEFINITION 2.1. Let $n \geqslant 1$ and $1 \leqslant k \leqslant n$; the operator

$$S_k(A_n, A) = S_k(A_n) = (U'^*)^{k-1}(P'_k - P'_{k-1})A_n|\mathscr{L}_* : \mathscr{L}_* \to \mathscr{L}' \tag{2.1$_{n,k}$}$$

is called the *k-th observable operator of* A_n. The string $\{S_k(A_n)\}_{k=1}^{n}$ is called *the observable string of* A_n.

It is plain that $\{S_k(A_n)\}_{k=1}^{n}$ is uniquely determined by A_n. Conversely:

LEMMA 2.1. *For any* $n \geqslant 1$, *the observable string of* A_n *uniquely determines* A_n.

Proof. Let $n \geqslant 1$ be a fixed positive integer. The operator $A_n \in \mathscr{L}(\mathscr{H}_n, \mathscr{H}'_n)$ is uniquely determined by its matrix with respect to the decompositions:

$$\mathscr{H}_n = \mathscr{L}_* + U\mathscr{L}_* + \ldots + U^{n-1}\mathscr{L}_* + U^n\mathscr{H},$$

$$\mathscr{H}'_n = \mathscr{H}' + \mathscr{L}' + \ldots + U'^{n-1}\mathscr{L}'.$$

From the very definition of the set n-PCID(A), we have that

$$P'A_n = AP|\mathscr{H}_n,$$

which means that A_n is completely determined by the operator $(1 - P')A_n$. On the other hand, because $U^n(\mathscr{H}) \subset \mathscr{H}_n$,

$$A_nU^n|\mathscr{H} = A_nP_nU^n|\mathscr{H} = A_nT_n^n|\mathscr{H} =$$

$$= T_n'^nA_n|\mathscr{H} = P'_nU'^nA_n|\mathscr{H} =$$

$$= P'_nU'^nP'A_n|\mathscr{H} = T_n'^nA,$$

which implies that A_n is completely determined by the operator $(1 - P')A_n|(\mathscr{L}_* + U\mathscr{L}_* + \ldots + U^{n-1}\mathscr{L}_*)$. Finally, we have for every $0 \leqslant k \leqslant n-1$

$$(I - P')A_nU^k|\mathscr{L}_* = (I - P')A_nT_n^k|\mathscr{L}_* = (I - P')T_n'^kA_n|\mathscr{L}_* =$$

$$= (I - P')T_n'^k\left[AP|\mathscr{L}_* + \sum_{m=1}^{n} U'^{m-1}S_m(A_n)\right],$$

and therefore the observable string of A_n uniquely determines A_n.

REMARK 2.1. Let $n, m \geqslant 1$ and $1 \leqslant k \leqslant \min\{m, n\}$; then

$$S_k(A_n) = S_k(A_m)\,, \tag{2.2}$$

just because $\{A_p\}_{p=1}^{\infty}$ is a *chain* of PCID of A.

The relation (2.2) justifies the notation

$$S_n(A_\infty; A) = S_n(A_\infty) = S_n(A_m)\,, \qquad (n \geqslant 1), \tag{2.3$_n$}$$

where $m \geqslant n$ is arbitrary.

DEFINITION 2.2. The sequence $\{S_n(A_\infty)\}_{n=1}^{\infty}$ is called *the observable sequence of* A_∞.

From Lemma 2.1 it follows that

COROLLARY 2.1. *The observable sequence of A_∞ uniquely determines A_∞.*

REMARK 2.2. It will be of some interest (even in the numerical case) to find out (for given T and T') the conditions for a sequence $\{S_n\}_{n=1}^{\infty}$ (where $S_n : \mathscr{L}_* \to \mathscr{L}'$ for every $n \geqslant 1$) such that there exist $A \in \mathscr{I}(T', T)$ and $A_\infty \in \mathrm{CID}(A)$ with $S_n(A_\infty) = S_n$ for every $n \geqslant 1$.

Due to Corollary 2.1, the connection between A_∞ and its choice sequence will be clear when we will have an algorithm for calculating the observable sequence of A_∞ from the choice sequence of A_∞. This will be done in the sequel. To this end, we analyse now a little more the structure of the observable sequence of A_∞.

From (1.11) it follows that

$$S_1(A_\infty) = q^A\sigma_A p_A(D_A P + I - P)|\mathscr{L}_* + q^A\Gamma_1(1 - p_A)(D_A P + I - P)|\mathscr{L}_*\,. \tag{2.4}$$

Having in mind the connection between $S_1(A_\infty)$ and Γ_1, define the operators:

$$\begin{cases} R^0(A) : \mathscr{L}_* \to \mathscr{L}' \\ R^0(A) = q^A\sigma_A p_A(D_A P + I - P)|\mathscr{L}_*\,, \end{cases} \tag{2.5}$$

$$\begin{cases} R(A) : \mathscr{R}_A \to \mathscr{L}_* \\ R(A)^* = (1 - p_A)(D_A P + I - P)|\mathscr{L}_*\,, \end{cases} \tag{2.6}$$

and

$$\begin{cases} R'(A) : \mathscr{R}^A \to \mathscr{L}' \\ R'(A) = q^A|\mathscr{R}^A\,. \end{cases} \tag{2.7}$$

Using (2.5), (2.6) and (2.7) in (2.4), we have

$$S_1(A_\infty) = R^0(A) + R'(A)\,\Gamma_1 R(A)^*\,. \tag{2.8}$$

The previous considerations make sense for every A_n (considered in 1-PCID(A_{n-1})) instead of A, $n \geq 1$; we will define then:

$$R_0^0(A_\infty) = R^0(A)\,, \tag{2.9$_0$}$$

$$R_0(A_\infty) = R(A)\,, \tag{2.10$_0$}$$

$$R_0'(A_\infty) = R'(A)\,, \tag{2.11$_0$}$$

and for every $n \geq 1$,

$$R_n^0(A_\infty) = U'^{*n} R^0(A_n) : \mathscr{L}_* \to \mathscr{L}'\,, \tag{2.9$_n$}$$

$$R_n(A_\infty) = R(A_n)\ \Omega_{A_n}^* : \mathscr{D}_{\Gamma_n} \to \mathscr{L}_*\,, \tag{2.10$_n$}$$

$$R_n'(A_\infty) = U'^{*n} R'(A_n)(\Omega^{A_n})^* : \mathscr{D}_{\Gamma_n^*} \to \mathscr{L}'\,. \tag{2.11$_n$}$$

From (2.8), we have that for every $n \geq 1$,

$$S_1(A_\infty, A_{n-1}) = R^0(A_n) + R'(A_n)\,\Gamma_1(A_{n-1}, A_n)R(A_n)^*\,. \tag{2.12$_n$}$$

From the formulas (2.1), it follows that for every $n \geq 1$:

$$S_n(A_\infty, A) = (U'^*)^{n-1}\, S_1(A_\infty, A_{n-1})\,. \tag{2.13$_n$}$$

For every $n \geq 1$, using $(2.13)_n$, $(2.12)_n$, $(2.9)_n$, $(2.10)_n$ and $(2.11)_n$, we obtain

$$S_n(A_\infty) = R_{n-1}^0(A_\infty) + R_{n-1}'(A_\infty)\,\Gamma_n(R_{n-1}(A_\infty))^*. \tag{2.14$_n$}$$

We are now faced with the problem of studying the sequences $\{R_n^0(A_\infty)\}_{n=0}^\infty$, $\{R_n(A_\infty)\}_{n=0}^\infty$ and $\{R_n'(A_\infty)\}_{n=0}^\infty$. Before going into this, let us record the following facts.

LEMMA 2.2. *The operators $R_n(A_\infty)$ and $R_n'(A_\infty)$ are injective for every $n \geq 0$.*

Proof. Using $(2.10)_n$ and $(2.11)_n$, $(n \geq 0)$, the result will follow from the fact that $R(A)$ and $R'(A)$ are injective. For $l_* = (I - UT^*)h \in \mathscr{L}_*$, where $h \in \mathscr{H}$, we have that

$$R(A)^* l_* = (1 - p_A)(D_A P + I - P)(h - TT^*h - (U - T)\,T^*h) =$$

$$= (1 - p_A)(D_A h - (D_A TT^*h + (U - T)\,T^*h) = (1 - p_A)\,D_A h\,.$$

Lemma 1.1 (b) and (c) from [6] implies now that $R(A)^*$ and $R'(A)^*$ have dense range, which means that $R(A)$ and $R'(A)$ are injective.

COROLLARY 2.2. *Let $n \geq 0$. The factorization $A_n \cdot T_n$ (resp. $T_n' \cdot A_n$) is regular if and only if $R_n(A_\infty) = 0$ (resp. $R_n'(A_\infty) = 0$).*

Proof. The factorization $A_n \cdot T_n$ is regular if and only if $\mathscr{R}_{A_n} = \{0\}$, which is equivalent to $\mathscr{D}_{\Gamma_n} = \{0\}$, which in turn is equivalent by Lemma 2.2. to the fact that $R_n(A_\infty) = 0$. A similar argument based on Lemma 2.2 is valid for the factorization $T'_n \cdot A_n$.

3. THE ALGORITHM

This section is devoted to the study of the sequences of operators defined in $(2.9)_n$, $(2.10)_n$ and $(2.11)_n$, $n \geqslant 0$; this will represent the main part of the algorithm which connects the observable sequence with the choice sequence. The "first objects" involved in this algorithm will be studied in the next section, where the definite form of the algorithm will be given.

We fix $A_\infty \in \mathrm{CID}(A)$ and we denote

$$R_n^0 = R_n^0(A_\infty),$$

$$R_n = R_n(A_\infty),$$

$$R'_n = R'_n(A_\infty), \qquad \text{for } n \geqslant 0;$$

and

$$S_n = S_n(A_\infty), \qquad \text{for } n \geqslant 1.$$

Concerning the sequence $\{R_n\}_{n=0}^\infty$, we have the following

LEMMA 3.1. *For every* $n \geqslant 1$

$$R_n = R_0 D_{\Gamma_1} D_{\Gamma_2} \ldots D_{\Gamma_n}, \tag{3.1$_n$}$$

where

$$R_0^* = (1 - p_A)(D_A P + I - P)|\mathscr{L}_*. \tag{3.1$_0$}$$

Proof. We proceed by induction. From $(2.10)_1$ we have that

$$R_1 = R(A_1)\,\Omega_{A_1}^*. \tag{3.2}$$

From (2.6), we have that (because $\mathscr{L}_* \subset \mathscr{H}_1$)

$$R(A_1)^* = (1 - p_{A_1})(D_{A_1} P_1 + I - P_1)|\mathscr{L}_* = (1 - p_{A_1}) D_{A_1}|\mathscr{L}_*. \tag{3.3}$$

Using (3.2), the definition of Ω_{A_1}, (3.3), (1.13) and (2.6) we have that

$$R_1^* = \Omega_{A_1} R(A_1)^* = \omega_{A_1}(1 - p_{A_1})\, D_{A_1}|\mathscr{L}_* =$$
$$= D_{\Gamma_1}(1 - p_A)(D_A P + I - P)|\mathscr{L}_* = D_{\Gamma_1} R_0^*,$$

which means that $(3.1)_1$ is proved.

Suppose now that for an $n \geqslant 1$, $(3.1)_n$ is verified. As in the case $n = 1$ (using also $(1.17)_{n+1}$), we have:

$$R^*_{n+1} = \Omega_{A_{n+1}} R(A_{n+1})^* = \Omega_{A_n} \circ \omega_{A_{n+1}}(1 - p_{A_{n+1}}) D_{A_{n+1}} | \mathscr{L}_* =$$

$$= \Omega_{A_n} D_{\Gamma_1(A_n, A_{n+1})}(1 - p_{A_n}) D_{A_n} | \mathscr{L}_* = D_{\Gamma_{n+1}} \Omega_{A_n} R(A_n)^* = D_{\Gamma_{n+1}} R^*_n .$$

From $(3.1)_n$, we have now that:

$$R^*_{n+1} = D_{\Gamma_{n+1}} D_{\Gamma_n} \ldots D_{\Gamma_1} R^*_0 ,$$

which means that $(3.1)_{n+1}$ is verified. The formulas $(3.1)_n$, $(n \geqslant 1)$ are therefore proved by induction. The formula $(3.1)_0$ follows from $(2.10)_0$ and (2.6).

Similar arguments can be used to obtain the structure of the sequence $\{R'_n\}_{n=0}^{\infty}$ we will not repeat the proof.

LEMMA 3.2. *For every $n \geqslant 1$*

$$(3.4)_n \qquad R'_n = R'_0 D_{\Gamma^*_1} \ldots D_{\Gamma^*_n} ,$$

where

$$(3.4)_0 \qquad R'_0 = q^A | \mathscr{R}^A .$$

We proved thus until now

PROPOSITION 3.1. *For every $n \geqslant 1$*

$$S_n = R^0_{n-1} + R'_{n-1} \Gamma_n R^*_{n-1} ,$$

where

$$R^*_0 = (1 - p_A)(D_A P + I - P) | \mathscr{L}_* , \qquad R_n = R_0 D_{\Gamma_1} \ldots D_{\Gamma_n} ,$$

$$R'_0 = q^A | \mathscr{R}^A \qquad , \qquad R'_n = R'_0 D_{\Gamma^*_1} \ldots D_{\Gamma^*_n},$$

and R^0_k depends only on A_k, $(k \geqslant 0)$.

The case of the sequence $\{R^0_n\}_{n=0}^{\infty}$ is far more complicated; the iterative formulas for it will give also iterative formulas of the same type for $\{S_n\}_{n=1}^{\infty}$. This is quite natural, because for the zero choice sequence, $S_n = R^0_{n-1}$ for every $n \geqslant 1$. Anyway, Proposition 3.1 will play its role in this matter.

The present formulas for the sequence $\{R^0_n\}_{n=0}^{\infty}$ are given by (2.5) and $(2.9)_n$, $(n \geqslant 0)$. We have

$$R^0_0 = q^A \sigma_A p_A (D_A P + I - P) | \mathscr{L}_* ,$$

and for $n \geqslant 1$

$$R_n^0 = U'^{*n} q^{A_n} \sigma_{A_n} p_{A_n} D_{A_n} \mid \mathscr{L}_* .$$

The whole idea for studying $\{R_n^0\}_{n=0}^{\infty}$ is to use the unitary operators $\{\tilde{\Omega}_n\}_{n=1}^{\infty}$ defined in $(1.18)_n$ and $(1.18)'_n$, $(n \geqslant 1)$. This means, in fact, to use the space $\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \mathscr{D}_{\Gamma_n}$, instead of $\mathscr{D}_{A_n}$, as intermediate space between $\mathscr{L}_*$ and $\mathscr{L}'$.

Define for every $n \geqslant 1$ (the case $n = 0$ will be discussed in the next section) the operators:

$$(3.5)_n \qquad \begin{cases} X_n : \mathscr{L}_* \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \\ X_n = \tilde{\Omega}_{A_n} D_{A_n} | \mathscr{L}_* . \end{cases}$$

$$(3.6)_n \qquad \begin{cases} Y_n : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to \mathscr{L}' \\ Y_n = U'^{*n} q^{A_n} (\sigma_{A_n} p_{A_n} + \Gamma_1(A_n, A_{n+1})(1 - p_{A_n})) \tilde{\Omega}^*_{A_n} . \end{cases}$$

In the matrix form the operators X_n and Y_n can be written as follows $(n \geqslant 1)$:

$$(3.7)_n \qquad \begin{cases} X_n = \begin{pmatrix} X_n^0 \\ X_n^1 \\ \cdot \\ \vdots \\ X_n^n \end{pmatrix}, & \text{where} \\ X_n^0 : \mathscr{L}_* \to \mathscr{D}_A , & \text{and} \\ X_n^j : \mathscr{L}_* \to \mathscr{D}_{\Gamma_j} , & (1 \leqslant j \leqslant n), \end{cases}$$

and

$$(3.8)_n \qquad \begin{cases} Y_n = (Y_n^0, Y_n^1, \ldots, Y_n^n) , & \text{where} \\ Y_n^0 = \mathscr{D}_A \to \mathscr{L}' , & \text{and} \\ Y_n^j : \mathscr{D}_{\Gamma_j} \to \mathscr{L}' , & (1 \leqslant j \leqslant n). \end{cases}$$

Taking into account $(2.13)_{n+1}$ and (1.11), it follows that

$$(3.9)_n \qquad S_{n+1} = Y_n X_n = Y_n^0 X_n^0 + Y_n^1 X_n^1 + \ldots + Y_n^n X_n^n , \qquad (n \geqslant 1).$$

Moreover for $n \geqslant 1$

$$X_n^n = P^{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}_{\{0\} \oplus \{0\} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}} \tilde{\Omega}_{A_n} D_{A_n} | \mathscr{L}_* = \Omega_{A_n}(1 - p_{A_n}) D_{A_n} \mid \mathscr{L}_* = R_n^* ,$$

where we used $(3.7)_n$, $(3.5)_n$, $(1.18)'_n$, $(1.17)_n$ and $(2.10)_n$. We have then

$$(3.10)_n \qquad X_n^n = R_n^* , \qquad (n \geqslant 1).$$

On the other hand, for $n \geqslant 1$

$$Y_n^n = U'^{*n} q^{A_n}(\sigma_{A_n} p_{A_n} + \Gamma_1(A_n, A_{n+1})(1 - p_{A_n}))\, \widetilde{\Omega}_{A_n}^{*}\, P_{\{0\} \oplus \{0\} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}^{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}} =$$

$$= U'^{*n} q^{A_n}(\Omega^{A_n})^* \Omega^{A_n}\, \Gamma_1(A_n, A_{n+1})\, \Omega_{A_n}^{*} = U'^{*n} q^{A_n} (\Omega^{A_n})^*\, \Gamma_{n+1} = R_n'\, \Gamma_{n+1},$$

where we used $(3.8)_n$, $(3.6)_n$, $(1.18)_n'$, $(1.17)_n$ and $(2.11)_n$. We have that

$$(3.11)_n \qquad Y_n^n = R_n' \Gamma_{n+1}, \qquad (n \geqslant 1).$$

Using $(3.9)_n$, $(3.10)_n$, $(3.11)_n$ and $(2.14)_{n+1}$, we obtain

$$(3.12)_n \qquad R_n^0 = Y_n^0 X_n^0 + Y_n^1 X_n^1 + \ldots + Y_n^{n-1} X_n^{n-1}, \qquad (n \geqslant 1).$$

The formulas $(3.12)_n$, $(n \geqslant 1)$ explain why we will study now the sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$; moreover, the formulas $(3.9)_n$ and $(3.12)_n$ explain the connection between the iterative formulas for $\{S_n\}_{n=1}^{\infty}$ and those of $\{R_n^0\}_{n=1}^{\infty}$.

We will start with the sequence $\{Y_n\}_{n=1}^{\infty}$; this will be unexpectedly simple.

LEMMA 3.3. (a) $Y_2^0 = Y_1^0$

(b) $Y_2^1 = Y_1^1$.

Proof. (a) Let $D_A h \in \mathscr{D}_A$, $(h \in \mathscr{H})$; we have

$$Y_1^0(D_A h) = U'^* q^{A_1}(\sigma_{A_1} p_{A_1} + \Gamma_1(A_1, A_2)(1 - p_{A_1}))\, \widetilde{\Omega}_{A_1}^{*}(D_A h) =$$

$$= U'^* q^{A_1}(\sigma_{A_1} p_{A_1} + \Gamma_1(A_1, A_2)(1 - p_{A_1}))\, D_{A_1} U h =$$

$$= U'^* q^{A_1} \sigma_{A_1}(D_{A_1} T_1 h + (U - T_1) h) = U'^* q^{A_1}(D_{A_1} h \oplus (U' - T_1')\, A_1 h) =$$

$$= U'^* U'(1 - P')\, A_1 h = (1 - P')\, A_1 h,$$

where we used in order $(3.6)_1$, $(1.18)_1'$, $(1.16)'$, the fact that $\mathscr{R}_{A_1} = \mathscr{D}_{A_1} \ominus D_{A_1} U(\mathscr{H})^-$, the equality $T_1 h = U h$ (for $h \in \mathscr{H}$), and (1.10) for σ_{A_1}. On the other hand

$$Y_2^0(D_A h) = U'^{*2} q^{A_2}(\sigma_{A_2} p_{A_2} + \Gamma_1(A_2, A_3)(1 - p_{A_2}))\, \widetilde{\Omega}_{A_2}^{*}(D_A h) =$$

$$= U'^{*2} q^{A_2}(\sigma_{A_2} p_{A_2} + \Gamma_1(A_2, A_3)(1 - p_{A_2}))\, \widetilde{\omega}_{A_2}^{*} \widetilde{\omega}_{A_1}^{*}(D_A h) =$$

$$= U'^{*2} q^{A_2}(\sigma_{A_2} p_{A_2} + \Gamma_1(A_2, A_3)(1 - p_{A_2}))\, D_{A_2} U^2 h =$$

$$= U'^{*2} q^{A_2} \sigma_{A_2}(D_{A_2} T_2(Uh) + (U - T_2) U h) =$$

$$= U'^{*2} q^{A_2}(D_{A_2} U h \oplus (U' - T_2')\, A_2 U h) =$$

$$= U'^{*2} U'(1 - P_1')\, A_2 U h = U'^*(1 - P_1')\, T_2' A_2 h =$$

$$= U'^*(1 - P_1')\, P_2' U' A_2 h = U'^*(1 - P_1')\, U' A_1 h = (1 - P')\, A_1 h,$$

where we used in order $(3.6)_2$, $(1.18)_2'$, $(1.18)_1'$, $(1.16)'$ for A_1 and A_2, the fact that $U h_1 = T_2 h_1$ (for $h_1 \in \mathscr{H}_1$), and (1.10) for σ_{A_2}. This concludes the proof of (a)

(b) Since

$$[D_{\Gamma_1}(1-p_A)(D_AP+I-P)(\mathscr{H}_1)]^- = \mathscr{D}_{\Gamma_1},$$

we have to prove that $Y_2^1\gamma_1 = Y_1^1\gamma_1$ for every $\gamma_1 \in D_{\Gamma_1}(1-p_A)(D_AP+I-P)(\mathscr{H}_1)$. Let $\gamma_1 = D_{\Gamma_1}(1-p_A)(D_AP+I-P)\,h_1$, where $h_1 \in \mathscr{H}_1$. From (1.13) it follows that

$$\gamma_1 = \omega_{A_1}(1-p_{A_1})\,D_{A_1}h_1\,.$$

Because $(1-p_{A_1})\,D_{A_1}h_1 \in \mathscr{R}_{A_1} \subset \mathscr{D}_{A_1}$, there exists a sequence $\{D_{A_1}h_1^n\}_{n=1}^{\infty}$, where $h_1^n \in \mathscr{H}_1$, $n \geqslant 1$, such that

$$\lim_{n\to\infty} D_{A_1}h_1^n = (1-p_{A_1})\,D_{A_1}h_1\,.$$

Then we have:

$$Y_2^1\gamma_1 = U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2} + \Gamma_1(A_2,A_3)\,(1-p_{A_2}))\,\widetilde{\Omega}_{A_2}^*\,\omega_{A_1}\,(1-p_{A_1})\,D_{A_1}h_1 =$$

$$= \lim_{n\to\infty} U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2} + \Gamma_1(A_2,A_3)(1-p_{A_2}))\,\widetilde{\omega}_{A_2}^*\,D_{A_1}h_1^n =$$

$$= \lim_{n\to\infty} U'^{*2}q^{A_2}\sigma_{A_2}D_{A_2}Uh_1^n = \lim_{n\to\infty} U'^{*2}(U'-T_2')\,A_2h_1^n =$$

$$= \lim_{n\to\infty} U'^{*}(1-P_1')\,A_2h_1^n = \lim_{n\to\infty} U'^{*}(1-P_1')(\sigma_{A_1}p_{A_1} + \Gamma_1(A_1,A_2)(1-p_{A_1}))\,D_{A_1}h_1^n =$$

$$= U'^{*}(1-P_1')(\omega^{A_1})^*\,\Gamma_2\omega_{A_1}(1-p_{A_1})\,D_{A_1}h_1 =$$

$$= U'^{*}q^{A_1}(\omega^{A_1})^*\,\Gamma_2\gamma_1 = R_1'\Gamma_2\gamma_1 = Y_1^1\gamma_1\,,$$

where we used the same machinery as in part (a) plus $(3.11)_1$. This concludes the proof of the lemma.

LEMMA 3.4. *Let $n, m \geqslant 1$ and $0 \leqslant j \leqslant \min\{n, m\}$. Then we have that*

$$Y_n^j = Y_m^j\,.$$

Proof. It is sufficient to prove that for every $n \geqslant 1$ and every $0 \leqslant j \leqslant n$,

$$Y_n^j = Y_{n+1}^j\,.$$

For this, we choose $n \geqslant 1$ and we apply Lemma 3.3 for $A_{n-1} \in \mathscr{I}(T_{n-1}', T_{n-1})$ instead of $A \in \mathscr{I}(T', T)$; in this case A_{n+k} will play the role of A_{k+1} for every $k = 0, 1, 2, \ldots$. We denote by $Y_i^j(n)$, $(i = 1, 2;\ 0 \leqslant j \leqslant i)$, the operators defined by $(3.6)_1$, $(3.8)_1$ $(3.6)_2$, $(3.8)_2$ for A_{n-1} instead of A. Looking at $(3.6)_n$ it is clear that only the first and the last factor in the expression of Y_n and, respectively, $Y_1(n)$ are different. Using $(1.18)_1'$ with A replaced by A_{n-1} and $(1.18)_n'$, we have that

$$Y_1(n) = U'^{n-1}\,Y_n(\widetilde{\Omega}_{A_{n-1}} \oplus \Omega_{A_{n-1}})\,.$$

which means that

$$(3.13)_n \qquad Y_1^0(n) = U'^{n-1}(Y_n^0, Y_n^1, \ldots, Y_n^{n-1})\, \widetilde{\Omega}_{A_{n-1}},$$

$$(3.14)_n \qquad Y_1^1(n) = U'^{n-1} Y_n^n\, \Omega_{A_{n-1}}.$$

A similar argument shows that

$$(3.15)_n \qquad Y_2^0(n) = U'^{n-1}(Y_{n+1}^0, \ldots, Y_{n+1}^{n-1})\, \widetilde{\Omega}_{A_{n-1}},$$

$$(3.16)_n \qquad Y_2^1(n) = U'^{n-1} Y_{n+1}^n\, \Omega_{A_{n-1}}.$$

From Lemma 3.3 it follows that $Y_1^0(n) = Y_2^0(n)$ and that $Y_1^1(n) = Y_2^1(n)$. Combining these equalities with $(3.13)_n$, $(3.15)_n$, $(3.14)_n$ and $(3.16)_n$, we have that

$$Y_n^j = Y_{n+1}^j \qquad \text{for every } 0 \leqslant j \leqslant n,$$

and the lemma is completely proved.

This enables us to define for every $n \geqslant 0$

$$(3.17)_n \qquad Y^n = Y_m^n,$$

where $m > n$ is arbitrary.

PROPOSITION 3.2. *For every $m \geqslant n \geqslant 1$,*

$$(3.18)_n \qquad Y^n = Y_m^n = R_0' D_{\Gamma_1^*} \ldots D_{\Gamma_n^*} \Gamma_{n+1}.$$

Proof. Lemma 3.4 implies that $Y^n = Y_n^n$ and the proposition follows from $(3.11)_n$ and $(3.4)_n$.

We concentrate now on the sequence $\{X_n\}_{n=1}^{\infty}$. The study of the connection between X_n and X_{n+1} (for $n \geqslant 1$) necessitates the following considerations.

Let $n \geqslant 1$ be fixed for a while. For $h_{n-1} \in \mathscr{H}_{n-1}$, the formula (1.11) — for $A_n \in$ 1-PCID(A_{n-1}) — implies that

$$\|A_n h_{n-1}\|^2 \geqslant \|A_{n-1} h_{n-1}\|^2,$$

so

$$\|D_{A_n} h_{n-1}\|^2 = \|h_{n-1}\|^2 - \|A_n h_{n-1}\|^2 \leqslant$$
$$\leqslant \|h_{n-1}\|^2 - \|A_{n-1} h_{n-1}\|^2 = \|D_{A_{n-1}} h_{n-1}\|^2.$$

This means that there exists a contraction

$$(3.19)_n \qquad \begin{cases} B_n : \mathscr{D}_{A_{n-1}} \to \mathscr{D}_{A_n} \\ B_n D_{A_{n-1}} = D_{A_n} | \mathscr{H}_{A_{n-1}}. \end{cases}$$

Using $(3.5)_{n+1}$, $(3.19)_{n+1}$ and $(3.5)_n$, we have that

$$(3.20)_{n+1} \qquad X_{n+1} = \widetilde{B}_{n+1} X_n,$$

where

$$(3.21)_{n+1} \qquad \begin{cases} \widetilde{B}_{n+1} : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n+1}} \\ \widetilde{B}_{n+1} = \widetilde{\Omega}_{A_{n+1}} B_{n+1} \widetilde{\Omega}_{A_n}^*. \end{cases}$$

The operator

$$(3.21)_1 \qquad \begin{cases} \widetilde{B}_1 : \mathscr{D}_A \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \\ \widetilde{B}_1 = \widetilde{\Omega}_{A_1} B_1 \end{cases}$$

will be also useful.

Because of $(3.20)_n$, $(n \geqslant 2)$, we must firstly study the sequence $\{\widetilde{B}_n\}_{n=1}^{\infty}$. For this, let us denote for every $n \geqslant 1$

$$Q_n = (i_{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}^{\mathscr{D}_{\Gamma_n}})^* \; P_{\{0\} \oplus \{0\} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}^{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}} ;$$

in this case $1 - Q_n$ is in fact the operator

$$1 - Q_n = (i_{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}^{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}})^* \; P_{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}} \oplus \{0\}}^{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}}$$

The key step in understanding $\{\widetilde{B}_n\}_{n=1}^{\infty}$ is the following

LEMMA 3.5. *For every* $n \geqslant 1$

$$(3.22)_{n+1} \qquad Q_{n+1}\widetilde{B}_{n+1} = D_{\Gamma_{n+1}} Q_n ,$$

and

$$(3.23)_{n+1} \quad (1 - Q_{n+1})\widetilde{B}_{n+1} = \widetilde{B}_n(1 - Q_n) + \widetilde{\Omega}_{A_n}(1 - q^{A_n})(\Omega^{A_n})^* \; \Gamma_{n+1} Q_n .$$

Proof. Using the same arguments as in the proof of Lemma 3.4, the case $n \geqslant 2$ can be reduced to the case $n = 2$. Thus, it will be sufficient to consider the cases $n = 1$ and $n = 2$. Since the proof for $n = 1$ is similar (in fact also easier) to the proof for $n = 2$, we shall consider only the case $n = 2$.

We will verify now $(3.22)_3$ and $(3.23)_3$. For this, let $x \in \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}$. Choose a sequence $\{D_{A_1} h_1^n\}_{n=1}^{\infty}$, $(h_1^n \in \mathscr{H}_1 , n \geqslant 1)$, such that

$$\lim_{n \to \infty} D_{A_1} h_1^n = \widetilde{\Omega}_{A_1}^* x .$$

Then

$$\widetilde{B}_3(x \oplus 0) = \widetilde{\Omega}_{A_3} B_3 \widetilde{\Omega}_{A_2}^*(x \oplus 0) = \widetilde{\Omega}_{A_3} B_3 \widetilde{\omega}_{A_2}^*(\widetilde{\Omega}_{A_1}^* x \oplus 0) =$$

$$= \lim_{n \to \infty} \widetilde{\Omega}_{A_3} B_3 \widetilde{\omega}_{A_2}^*(D_{A_1} h_1^n \oplus 0) = \lim_{n \to \infty} \widetilde{\Omega}_{A_3} B_3 D_{A_2} U h_1^n =$$

$$= \lim_{n \to \infty} \widetilde{\Omega}_{A_3} D_{A_3} U h_1^n = \lim_{n \to \infty} (\widetilde{\Omega}_{A_2} D_{A_2} h_1^n) \oplus 0 =$$

$$= \lim_{n \to \infty} (\widetilde{\Omega}_{A_2} B_2 D_{A_1} h_1^n) \oplus 0 = (\widetilde{\Omega}_{A_2} B_2 \widetilde{\Omega}_{A_1}^* x) \oplus 0 = \widetilde{B}_2 x \oplus 0 ,$$

where we used in order $(3.21)_3$, $(1.18)_2'$, $(1.18)_1'$, $(1.16)'$, $(3.19)_3$, $(1.18)_3'$, $(1.16)'$, $(3.19)_2$, $(3.21)_2$. This means that

$$(3.24) \qquad (1 - Q_3)\, \widetilde{B}_3(1 - Q_2) = \widetilde{B}_2(1 - Q_2) ,$$

and

$$(3.25) \qquad Q_3\widetilde{B}_3 = Q_3\widetilde{B}_3 Q_2.$$

Let now $h_1 \in \mathscr{H}_1$ and consider $\gamma_2 = \Omega_{A_2}(1-p_{A_2})D_{A_2}h_1 \in \mathscr{D}_{\Gamma_2}$. (From [6], Lemma 1.1 (a) it follows that the set of such γ_2 is dense in $\mathscr{D}_{\Gamma_2}$.) We have

$$Q_3\widetilde{B}_3(0 \oplus 0 \oplus \gamma_2) = Q_3\widetilde{\Omega}_{A_3}B_3\widetilde{\Omega}^*_{A_2}(0 \oplus 0 \oplus \gamma_2) = Q_3\widetilde{\Omega}_{A_3}B_3(1-p_{A_2})D_{A_2}h_1 =$$

$$= \Omega_{A_3}(1-p_{A_3})B_3(1-p_{A_2})D_{A_2}h_1 = \Omega_{A_3}(1-p_{A_3})B_3D_{A_2}h_1 =$$

$$= \Omega_{A_3}\omega_{A_3}(1-p_{A_3})\,D_{A_2}h_1 = \Omega_{A_2}D_{\Gamma_1(A_2,\,A_3)}\,(1-p_{A_2})D_{A_2}h_1 =$$

$$= D_{\Gamma_3}\Omega_{A_2}(1-p_{A_2})D_{A_2}h_1 = D_{\Gamma_3}\gamma_2\,,$$

where we used $(3.21)_3$, definition of γ_2, the structure of $\widetilde{\Omega}_{A_3}$, the fact that $B_3(D_{A_3}U\mathscr{H}_1)^- \subset (D_{A_3}U\mathscr{H}_1)^-$, $(1.17)_3$ and (1.13). This relation and (3.25) imply that

$$Q_3\widetilde{B}_3 = D_{\Gamma_3}Q_2\,,$$

so $(3.22)_3$ is proved. From (3.24) it follows that $(3.23)_3$ will be proved if we verify that

$$(1-Q_3)\widetilde{B}_3Q_2 = \widetilde{\Omega}_{A_3}(1-q^{A_2})(\Omega^{A_2})^*\Gamma_3Q_2\,.$$

For this, take again $h_1 \in \mathscr{H}_1$ and $\gamma_2 = \Omega_{A_2}(1-p_{A_2})D_{A_2}h_1 \in \mathscr{D}_{\Gamma_2}$. Because $(1-p_{A_2})D_{A_2}h_1 \in \mathscr{D}_{A_2}$, there exists a sequence $\{h_2^n\}_{n=1}^{\infty}$, where $h_2^n \in \mathscr{H}_2$ for every $n \geqslant 1$, such that

$$\lim_{n\to\infty} D_{A_2}h_2^n = (1-p_{A_2})D_{A_2}h_1\,.$$

Then

$$(1-Q_3)\widetilde{B}_3(0 \oplus 0 \oplus \gamma_2) = (1-Q_3)\widetilde{\Omega}_{A_3}B_3\Omega^*_{A_2}(0 \oplus 0 \oplus \gamma_2) =$$

$$= \lim_{n\to\infty}(1-Q_3)\widetilde{\Omega}_{A_3}B_3D_{A_2}h_2^n = \lim_{n\to\infty}\Omega_{A_2}P^{\mathscr{D}_{A_2}\oplus\mathscr{D}_{\Gamma_1(A_2,\,A_3)}}_{\mathscr{D}_{A_2}\oplus\{0\}}\widetilde{\omega}_{A_3}D_{A_3}h_2^n =$$

$$= \lim_{n\to\infty}\widetilde{\Omega}_{A_3}(1-q^{A_2})(\sigma_{A_3}p_{A_2} + \Gamma_1(A_2\,,A_3)(1-p_{A_2}))D_{A_2}h_2^n =$$

$$= \widetilde{\Omega}_{A_3}(1-q^{A_2})\Gamma_1(A_2\,,A_3)(1-p_{A_2})D_{A_2}h_1 =$$

$$= \widetilde{\Omega}_{A_3}(1-q^{A_2})(\Omega^{A_2})^*\Gamma_3\gamma_2\,,$$

where we used $(3.21)_3$, the properties of γ_2, $(1.18)_3'$, (1.16) and $(1.17)_3$. This concludes the proof of the lemma.

For completing Lemma 3.5, we see that we will have to study the operators

$$(3.26)_n \qquad \begin{cases} C_n : \mathscr{D}_{\Gamma_n^*} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \\ C_n = -\widetilde{\Omega}_{A_n}(1 - q^{A_n})(\Omega^{A_n})^* . \end{cases}$$

for $n \geqslant 1$.

LEMMA 3.6. *For every $n \geqslant 2$,*

$$(3.27)_n \qquad Q_n C_n = \Gamma_n^* | \mathscr{D}_{\Gamma_n^*} ,$$

$$(3.28)_n \qquad (1 - Q_n) C_n = C_{n-1} D_{\Gamma_n^*} .$$

Proof. Let $n \geqslant 2$ be fixed. From $(1.15)_n$ it follows that

$$(\Omega^{A_{n-1}})^* \Gamma_n \Omega_{A_{n-1}} \omega_{A_n} = -\omega^{A_n}(1 - p^{A_n}) | \mathscr{R}_{A_n} .$$

Thus,

$$\Gamma_n | \mathscr{D}_{\Gamma_n} = -\Omega^{A_n}(1 - p^{A_n}) \Omega_{A_n}^* ,$$

so

$$\Gamma_n^* | \mathscr{D}_{\Gamma_n^*} = -\Omega_{A_n}(1 - q^{A_n})(\Omega^{A_n})^* = Q_n C_n ,$$

and $(3.27)_n$ is proved. (We used here that $\mathscr{R}_{A_n} \subset \mathscr{D}_{A_n}$.)

On the other hand, take $h_{n-1} \in \mathscr{H}_{n-1}$ and $x = \widetilde{\Omega}_{A_{n-1}} D_{A_{n-1}} h_{n-1} \in \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}$. Then:

$$C_n^*(x \oplus 0) = -\Omega^{A_n}(1 - p^{A_n}) \widetilde{\Omega}_{A_n}^* (\widetilde{\Omega}_{A_{n-1}} D_{A_{n-1}} h_{n-1} \oplus 0) =$$

$$= -\Omega^{A_n}(1 - p^{A_n}) \widetilde{\omega}_{A_n}^* (D_{A_{n-1}} h_{n-1} \oplus 0) = -\Omega^{A_n}(1 - p^{A_n}) D_{A_n} U h_{n-1} =$$

$$= -\Omega^{A_n}(1 - p^{A_n})[(D_{A_n} T_n h_{n-1} \oplus (U' - T_n') A_n T_n h_{n-1}) - (0 \oplus (U' - T_n') A_n T_n h_{n-1})] =$$

$$= \Omega^{A_n}(1 - p^{A_n})(0 \oplus (U' - T_n') A_n T_n h_{n-1}) =$$

$$= \Omega^{A_{n-1}} \omega^{A_n}(1 - p^{A_n})(0 \oplus U'(1 - P_{n-1}') T_n' A_n h_{n-1}) =$$

$$= \Omega^{A_{n-1}} D_{\Gamma_1^*(A_{n-1}, A_n)}(1 - p^{A_{n-1}})(0 \oplus (1 - P_{n-1}') T_n' A_n h_{n-1}) =$$

$$= D_{\Gamma_n^*} \Omega^{A_{n-1}}(1 - p^{A_{n-1}})(0 \oplus (U' - T_{n-1}') A_{n-1} h_{n-1}) =$$

$$= -D_{\Gamma_n^*} \Omega^{A_{n-1}}(1 - p^{A_{n-1}}) D_{A_{n-1}} h_{n-1} = D_{\Gamma_n^*} C_{n-1}^* x ,$$

where we used $(3.26)_n$, $(1.18)'_n$, $(1.16)'$, (1.8), (1.14), again (1.8) and $(3.26)_{n-1}$. This relation is exactly $(3.28)_n$ and the lemma is proved.

Lemmas 3.5 and 3.6 give us the possibility to compute the sequence $\{B_n\}_{n=2}^{\infty}$. The analysis of formulas $(3.22)_n$, $(3.23)_n$, $(3.27)_n$ and $(3.28)_n$, for $n \geqslant 2$, leads us to consider the following operators.

First, for an arbitrary contraction $\Gamma : G \to G'$, (where G and G' are Hilbert spaces), consider

$$(3.29) \qquad \begin{cases} J(\Gamma) : G \oplus \mathscr{D}_{\Gamma^*} \to G' \oplus \mathscr{D}_{\Gamma} \\ J(\Gamma) = \begin{pmatrix} -\Gamma & D_{\Gamma^*} \\ D_{\Gamma} & \Gamma^* \end{pmatrix}. \end{cases}$$

A matrix computation shows that $J(\Gamma)$ is unitary; this unitary $J(\Gamma)$ is known and used for a long time in the literature.

Define now for $n \geqslant 2$ and $2 \leqslant k \leqslant n$ the operator

$$(3.30)_n^k \quad J_n(\Gamma_k) : \mathscr{F}_A \oplus \mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \boxed{\mathscr{D}_{\Gamma_{k-1}} \oplus \mathscr{D}_{\Gamma_k^*}} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to$$

$$\to \mathscr{F}_A \oplus \mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \boxed{\mathscr{D}_{\Gamma_{k-1}^*} \oplus \mathscr{D}_{\Gamma_k}} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n},$$

such that $J_n(\Gamma_k)$ acts like the identity on each component of the direct sum, except the framed ones on which it acts like $J(\Gamma_k)$. For $n \geqslant 1$, we will need also the operator

$$(3.30)_n^1 \qquad J_n(\Gamma_1) : \mathscr{F}_A \oplus \boxed{\mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1^*}} \oplus \mathscr{D}_{\Gamma_2} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to$$

$$\to \mathscr{F}_A \oplus \boxed{\mathscr{R}^A \oplus \mathscr{D}_{\Gamma_1}} \oplus \mathscr{D}_{\Gamma_2} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}$$

defined analogously. These operators were used also in [11] in connection with the indexing of CID(A) by A-adequate isometries.

Finally consider also the operator

$$(3.31)_1 \qquad \Delta_1 : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1^*} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1},$$

such that $\Delta_1 i^{\mathscr{D}_A}_{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}} = \widetilde{B}_1$ and $\Delta_1 i^{\mathscr{D}_{\Gamma_1}}_{\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}} = C_1$, and for $n \geqslant 2$, the operators

$$(3.31)_n \qquad \Delta_1^n : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1^*} \oplus \mathscr{D}_{\Gamma_2} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n},$$

where Δ_1^n acts like the identity on each component, except on the first two on which it acts like Δ_1. We denote also for $n \geq 2$ by

$$(3.32)_n \quad i_n : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}} \oplus \mathscr{D}_{\Gamma_n^*}$$

the canonical inclusion.

PROPOSITION 3.3. *For every* $n \geq 2$,

$$(3.33)_n \qquad \widetilde{B}_n = \Delta_1^n J_n(\Gamma_2) \ldots J_n(\Gamma_n) i_n .$$

Proof. Consider for any $n \geq 2$ the operator

$$M_n = \Delta_1^n J_n(\Gamma_2) \ldots J_n(\Gamma_n) \mid \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}} \oplus \mathscr{D}_{\Gamma_n^*} .$$

Writing the domain of M_n as $(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}) \oplus (\mathscr{D}_{\Gamma_n^*})$, we will prove by induction that for every $n \geq 2$

$$(3.34)_n \qquad M_n = (\widetilde{B}_n, C_n) .$$

The relation $(3.33)_n$ will follow immediately from $(3.34)_n$, $n \geq 2$. For $n = 2$, we have (if we write the range of M_2 as $(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}) \oplus (\mathscr{D}_{\Gamma_2})$)

$$M_2 = \Delta_1^2 J_2(\Gamma_2) \mid \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \mathscr{D}_{\Gamma_2^*} = \begin{pmatrix} \widetilde{B}_1 & C_1 & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I_{\mathscr{D}_A} & 0 & 0 \\ 0 & -\Gamma_2 & D_{\Gamma_2^*} \\ 0 & D_{\Gamma_2} & \Gamma_2^* \end{pmatrix} =$$

$$= \begin{pmatrix} \widetilde{B}_1 & -C_1\Gamma_2 & C_1 D_{\Gamma_2^*} \\ 0 & D_{\Gamma_2} & \Gamma_2^* \end{pmatrix}.$$

From $(3.22)_2$, $(3.23)_2$ and $(3.26)_1$ it follows that

$$(3.35)_2' \qquad \widetilde{B}_2 = \begin{pmatrix} \widetilde{B}_1 & -C_1\Gamma_2 \\ 0 & D_{\Gamma_2} \end{pmatrix}$$

where the matrix is written with respect to the decompositions $(\mathscr{D}_A) \oplus (\mathscr{D}_{\Gamma_1})$ and $(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}) \oplus (\mathscr{D}_{\Gamma_2})$. On the other hand, from $(3.27)_2$ and $(3.28)_2$ it follows that

$$(3.35)_2'' \qquad C_2 = \begin{pmatrix} C_1 D_{\Gamma_2^*} \\ \Gamma_2^* \end{pmatrix},$$

where the range of C_2 is written as $(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}) \oplus (\mathscr{D}_{\Gamma_2})$. The relations $(3.35)_2$, $(3.35)_2'$ and $(3.35)_2''$ imply $(3.34)_2$. Suppose now that $(3.34)_n$ is verified for an $n \geq 2$. Then $M_n = (\widetilde{B}_n, C_n)$, so

$$(3.35)_{n+1} \quad M_{n+1} = \begin{pmatrix} \widetilde{B}_n & C_n & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & -\Gamma_{n+1} & D_{\Gamma_{n+1}^*} \\ 0 & D_{\Gamma_{n+1}} & \Gamma_{n+1}^* \end{pmatrix} = \begin{pmatrix} \widetilde{B}_n & -C_n\Gamma_{n+1} & C_n D_{\Gamma_{n+1}^*} \\ 0 & D_{\Gamma_{n+1}} & \Gamma_{n+1}^* \end{pmatrix}$$

where the matrixes are written with respect to the decompositions

$$(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}) \oplus (\mathscr{D}_{\Gamma_n}) \oplus (\mathscr{D}_{\Gamma^*_{n+1}})$$

$$(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_{n-1}}) \oplus (\mathscr{D}_{\Gamma^*_n}) \oplus (\mathscr{D}_{\Gamma_{n+1}})$$

$$(\mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n}) \oplus (\mathscr{D}_{\Gamma_{n+1}}).$$

From $(3.22)_{n+1}$, $(3.23)_{n+1}$ and $(3.26)_n$ if follows that

$$(3.35)'_{n+1} \qquad \widetilde{B}_{n+1} = \begin{pmatrix} \widetilde{B}_n & -C_n\Gamma_{n+1} \\ 0 & D_{\Gamma_{n+1}} \end{pmatrix};$$

from $(3.27)_{n+1}$ and $(3.28)_{n+1}$ we have

$$(3.35)''_{n+1} \qquad C_{n+1} = \begin{pmatrix} C_n D_{\Gamma^*_{n+1}} \\ \Gamma_{n+1} \end{pmatrix}.$$

The relations $(3.35)_{n+1}$, $(3.35)'_{n+1}$ and $(3.35)''_{n+1}$ imply $(3.34)_{n+1}$, and the proposition is completely proved.

From $(3.20)_n$ and Proposition 3.3 we have

COROLLARY 3.1. *For every $n \geqslant 2$,*

$$(3.36)_n \qquad \begin{pmatrix} X_n^0 \\ X_n^1 \\ \vdots \\ \\ X_n^n \end{pmatrix} = \Delta_1^n J_n(\Gamma_2) \ldots J_n(\Gamma_n) \begin{pmatrix} X_{n-1}^0 \\ X_{n-1}^1 \\ \vdots \\ X_{n-1}^{n-1} \\ 0 \end{pmatrix}$$

(the operator 0 in the right hand side being between $\mathscr{L}_*$ and $\mathscr{D}_{\Gamma^*_1}$).

Proposition 3.2 and Corollary 3.1 show that everything is clear in $(3.9)_n$, $(n \geqslant 1)$, except the first object of the process. These will be studied in the next section.

4. THE ALGORITHM (continued)

The aim of this section is to include in the iterative formulas of Section 3 the connection between S_1 and S_2. This will also slightly modify the formulas $(3.9)_n$, $(3.18)_n$ and $(3.36)_{n+1}$, $(n \geqslant 1)$. This separate treatment of the "first step" is quite natural: it is required even by the definition of choice sequences. Indeed, every Y^n (for $n \geqslant 1$) is defined on the domain of Γ_{n+1} (which is $\mathscr{D}_{\Gamma_n}$), but Y^0 is defined on $\mathscr{D}_A$ while the domain of Γ_1 is $\mathscr{R}_A$. This fact is connected with the condition $n \geqslant 1$ which appears in Proposition 3.2.

From $(2.14)_1$ we have that

$$S_1 = R_0^0 + R_0'\Gamma_1 R_0^* = q^A\sigma_A p_A(D_A P + I - P)|\mathscr{L}_* + q^A\Gamma_1(1 - p_A)(D_A P + I - P)|\mathscr{L}_* .$$

We define

$$(4.1)\qquad \begin{cases} X_0' = p_A(D_A P + I - P)|\mathscr{L}_* : \mathscr{L}_* \to \mathscr{F}_A \\ Y' = q^A\sigma_A : \mathscr{F}_A \to \mathscr{L}' , \end{cases}$$

and

$$(4.2)\qquad \begin{cases} X_0'' = (1 - p_A)(D_A P + I - P)|\mathscr{L}_* : \mathscr{L}_* \to \mathscr{R}_A \\ Y'' = q^A\Gamma_1 = R_0'\Gamma_1 : \mathscr{R}_A \to \mathscr{L}' ; \end{cases}$$

with these definition we have that $R_0^0 = Y'X'$ and

$$(4.3)\qquad S_1 = Y'X_0' + Y''X_0'' .$$

From $(3.9)_n$ and Lemma 3.4 it follows that

$$(4.4)_n\qquad S_{n+1} = Y^0X_n^0 + Y^1X_n^1 + \ldots + Y^nX_n^n ,$$

for every $n \geq 1$.

In order to connect (4.3) with $(4.4)_n$, $(n \geq 1)$, define

$$(4.5)_n\qquad \begin{cases} X_n' = p_A X_n^0 \\ X_n'' = (1 - p_A)X_n^0 , \end{cases} \qquad (n \geq 1).$$

From (4.1), (4.2), $(4.5)_n$ and the proof of Lemma 3.3, we have, for $n \geq 1$

$$Y'X_n' + Y''X_n'' = q^A(\sigma_A p_A + \Gamma_1(1 - p_A))X_n^0 = Y^0X_n^0 ;$$

thus $(4.4)_n$, $(n \geq 1)$, and (4.3) can be written as

$$(4.6)_n\qquad S_{n+1} = Y'X_n' + Y''X_n'' + Y^1X_n^1 + \ldots + Y^nX_n^n ,$$

where $n \geq 0$.

It remains to include in the iterative relations $(3.36)_n$, $(n \geq 2)$, the operators X_0', X_0'' and the splitting of X_m^0 into X_m' and X_m'', $(m \geq 1)$. To this end consider the operator

$$(4.7)_0\qquad \begin{cases} \Delta_0 : \mathscr{F}_A \oplus \mathscr{R}^A \to \mathscr{F}_A \oplus \mathscr{R}_A \\ \Delta_0 = \begin{pmatrix} p_A(1 - q^A)\sigma_A & -p_A(1 - q^A) \\ (1 - p_A)(1 - q^A)\sigma_A & -(1 - p_A)(1 - q^A) \end{pmatrix}, \end{cases}$$

and for every $n \geq 1$,

$$(4.7)_n\qquad \Delta_0^n : \mathscr{F}_A \oplus \mathscr{R}^A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} \to \mathscr{F}_A \oplus \mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1} \oplus \ldots \oplus \mathscr{D}_{\Gamma_n} ,$$

where Δ_0^n acts identically on all components, except on the first two, on which it acts like Δ_0.

LEMMA 4.1. *With the above notation*

$$(4.8)\qquad \begin{pmatrix} X_1' \\ X_1'' \\ X_1^1 \end{pmatrix} = \Delta_0^1 J_1(\Gamma_1) \begin{pmatrix} X_0' \\ X_0'' \\ 0_{\mathscr{D}_{\Gamma_1^*}} \end{pmatrix}.$$

Proof. The operators in (4.8) being defined on $\mathscr{L}_*$, consider $x \in \mathscr{L}_*$. We have:

$$\Delta_0^1 J_1(\Gamma_1) \begin{pmatrix} X_0' \\ X_1'' \\ 0_{\mathscr{D}_{\Gamma_1^*}} \end{pmatrix} x = \Delta_0^1 \begin{pmatrix} I & 0 & 0 \\ 0 & -\Gamma_1 & D_{\Gamma_1^*} \\ 0 & D_{\Gamma_1} & \Gamma_1^* \end{pmatrix} \begin{pmatrix} p_A(D_A P + I - P)x \\ (1-p_A)(D_A P + I - P)x \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q^A)\sigma_A & -p_A(1-q^A) & 0 \\ (1-p_A)(1-q^A)\sigma_A & -(1-p_A)(1-q^A) & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} p_A(D_A P + I - P)x \\ -\Gamma_1(1-p_A)(D_A P + I - P)x \\ D_{\Gamma_1}(1-p_A)(D_A P + I - P)x \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q^A)(\sigma_A p_A + \Gamma_1(1-p_A))(D_A P + I - P)x \\ (1-p_A)(1-q^A)(\sigma_A p_A + \Gamma_1(1-p_A))(D_A P + I - P)x \\ D_{\Gamma_1}(1-p_A)(D_A P + I - P)x \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-Q_1)\tilde{\omega}_{A_1} D_{A_1} x \\ (1-p_A)(1-Q_1)\tilde{\omega}_{A_1} D_{A_1} x \\ Q_1 \tilde{\omega}_{A_1} D_{A_1} x \end{pmatrix} = \begin{pmatrix} X_1' \\ X_1'' \\ X_1^1 \end{pmatrix} x,$$

which concludes the proof of the lemma. (We used here (1.16), $(4.5)_1$, $(3.5)_1$ and $(3.7)_1$).)

The relation (4.8) "looks like" relations $(3.36)_n$, $(n \geqslant 2)$. There remains only one step: to connect Δ_1 with Δ_0.

LEMMA 4.2.

$$(4.9)\qquad \Delta_1 = \Delta_0^1 J_1(\Gamma_1) | \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1^*}.$$

Proof. We know from $(3.31)_1$ that

$$\Delta_1 : \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1^*} \to \mathscr{D}_A \oplus \mathscr{D}_{\Gamma_1}.$$

The operator $\Delta_0^1 J_1(\Gamma_1)$ acts between $\mathscr{F}_A \oplus \mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1^*} (= (\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1^*})$ and $\mathscr{F}_A \oplus \mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1} (= (\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1})$, which justifies the restriction on the right side of (4.9). We have

$$\Delta_0^1 J_1(\Gamma_1) = \begin{pmatrix} p_A(1-q^A)\sigma_A & -p_A(1-q^A) & 0 \\ (1-p_A)(1-q^A)\sigma_A & -(1-p_A)(1-q^A) & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & -\Gamma_1 & D_{\Gamma_1^*} \\ 0 & D_{\Gamma_1} & \Gamma_1^* \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q^A)\sigma_A & p_A(1-q^A)\Gamma_1 & -p_A(1-q^A) D_{\Gamma_1^*} \\ 1-p_A)(1-q^A)\sigma_A & (1-p_A)(1-q^A)\Gamma_1 & -(1-p_A)(1-q^A) D_{\Gamma_1^*} \\ 0 & D_{\Gamma_1} & \Gamma_1^* \end{pmatrix}.$$

Writing this matrix with respect to the decompositions $(\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1^*}$ and $(\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1}$, we obtain

$$\text{(4.10)} \qquad \Delta_0^1 J_1(\Gamma_1) = \begin{pmatrix} (1-q^A)(\sigma_A p_A + \Gamma_1(1-p_A)) & -(1-q^A) D_{\Gamma_1^*} \\ D_{\Gamma_1}(1-p_A) & \Gamma_1^* \end{pmatrix}.$$

On the other hand, from $(3.21)_1$, $(3.19)_1$, $(1.18)_1'$ and (1.16) we have

$$\tilde{B}_1 D_A h = \tilde{\Omega}_{A_1} B_1 D_A h = \tilde{\Omega}_{A_1} D_{A_1} h =$$

$$= [(1-q^A)(\sigma_A p_A + \Gamma_1(1-p_A)) \oplus D_{\Gamma_1}(1-p_A)] D_A h,$$

for every $h \in \mathscr{H}$, which means that (see 4.10)

$$\text{(4.11)} \qquad \Delta_0^1 J_1(\Gamma_1)\, i^{\mathscr{D}_A}_{(\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1^*}} = \tilde{B}_1 .$$

The relation $(3.27)_1$ shows that

$$\text{(4.12)} \qquad Q_1 C_1 = \Gamma_1^* | \mathscr{D}_{\Gamma_1^*} .$$

Finally, for $h \in \mathscr{H}$,

$$C_1^*(D_A h \oplus 0_{\mathscr{D}_{\Gamma_1}}) = -\omega^{A_1}(1-p^{A_1})(\tilde{\omega}_{A_1})^*(D_A h \oplus 0) =$$

$$= -\omega^{A_1}(1-p^{A_1}) D_{A_1} U h = \omega^{A_1}(1-p^{A_1})(0 \oplus (U' - T_1')A_1 T_1 h) =$$

$$= \omega^{A_1}(1-p^{A_1})(0 \oplus U'(1-P')T_1' A_1 h = D_{\Gamma_1^*}(1-p^A)(0 \oplus (U' - T')Ah) =$$

$$= -D_{\Gamma_1^*}(1-p^A)(D_A h \oplus 0_{\mathscr{L}'}),$$

(see also the proof of Lemma 3.6) which means that:

$$(1 - Q_1)C_1 = -(1 - q^A)\, D_{\Gamma_1^*} | \mathscr{D}_{\Gamma_1^*} . \tag{4.13}$$

From (4.12), (4.13) and (4.10), it follows that

$$\Delta_0^1 J_1(\Gamma_1)\, i^{\mathscr{D}_{\Gamma_1^*}}_{(\mathscr{D}_A + \mathscr{L}) \oplus \mathscr{D}_{\Gamma_1^*}} = C_1 . \tag{4.14}$$

The lemma follows now from (4.11), (4.13) and $(3.31)_1$.

COROLLARY 4.1. *For every* $n \geqslant 2$,

$$\Delta_1^n J_n(\Gamma_2) \ldots J_n(\Gamma_n) = \Delta_0^n J_n(\Gamma_1) J_n(\Gamma_2) \ldots J_n(\Gamma_n) .$$

Putting together all the pieces we have: (4.3), $(4.6)_{n=1}^{\infty}$, (4.1), (4.2), $(3.18)_{n=1}^{\infty}$, $(3.4)_0$, $(3.36)_{n=2}^{\infty}$, (4.8) and Corollary 4.1, we dispose now of the algorithm for obtaining the sequence $\{S_n\}_{n=2}^{\infty}$ from the sequence $\{\Gamma_n\}_{n=1}^{\infty}$. We have:

THEOREM 4.1. *For every* $n \geqslant 1$,

$$S_n = Y' X'_{n-1} + Y'' X''_{n-1} + Y^1 X^1_{n-1} + \ldots + Y^{n-1} X^{n-1}_{n-1} ,$$

where

$$Y' = q^A \sigma_A ,$$

$$Y'' = q^A \Gamma_1 , \qquad \textit{and}$$

$$Y^k = q^A D_{\Gamma_1^*} \ldots D_{\Gamma_k^*} \Gamma_{k+1} , \qquad \textit{for} \quad k \geqslant 1,$$

and

$$X'_0 = p_A (D_A P + I - P) | \mathscr{L}_* ,$$

$$X''_0 = (1 - p_A)(D_A P + I - P) | \mathscr{L}_* , \qquad \textit{and}$$

$$\begin{pmatrix} X'_k \\ X''_k \\ X^1_k \\ \vdots \\ X^k_k \end{pmatrix} = \Delta_k \begin{pmatrix} X'_{k-1} \\ X''_{k-1} \\ X^1_{k-1} \\ \vdots \\ X^{k-1}_{k-1} \\ 0_{\mathscr{D}_{\Gamma_1^*}} \end{pmatrix} \tag{4.15$_k$}$$

where if Δ_0^k *is defined by* $(4.7)_k$, *then*

$$\Delta_k = \Delta_0^k J_k(\Gamma_1) \ldots J_k(\Gamma_k) , \qquad \textit{for } k \geqslant 1. \tag{4.16$_k$}$$

REMARK 4.1. It is easy to note that the factor $(D_A P + I - P)$ can be "omitted" in the iterative relations for the X_n's. More precisely, there exist the operators

$$Z'_n : \mathscr{D}_A + \mathscr{L} \to \mathscr{F}_A ,$$

$$Z''_n : \mathscr{D}_A + \mathscr{L} \to \mathscr{R}_A ,$$

for every $n \geqslant 0$, and

$$Z_n^k : \mathscr{D}_A + \mathscr{L} \to \mathscr{D}_{\Gamma_k},$$

for $n \geqslant 1$ and $1 \leqslant k \leqslant n$, such that

$$(4.17)_n \qquad \begin{pmatrix} X_n' \\ X_n'' \\ X_n^1 \\ \vdots \\ X_n^n \end{pmatrix} = \begin{pmatrix} Z_n' \\ Z_n'' \\ Z_n^1 \\ \vdots \\ Z_n^n \end{pmatrix} (D_A P + I - P)|\mathscr{L}_*,$$

for $n \geqslant 0$, where

$$(4.18)_n \qquad \begin{pmatrix} Z_n' \\ Z_n'' \\ Z_n^1 \\ \vdots \\ \vdots \\ Z_n^n \end{pmatrix} = \Delta_n \begin{pmatrix} Z_{n-1}' \\ Z_{n-1}'' \\ Z_{n-1}^1 \\ \vdots \\ Z_{n-1}^{n-1} \\ 0_{\mathscr{D}_{\Gamma_1^*}} \end{pmatrix},$$

for $n \geqslant 1$.

REMARK 4.2. The exact transcription of this algorithm for the case $\dim \mathscr{H} = \dim \mathscr{H}' = 1$ and $T = T' = 0$ provides a numerical method which seems to be useful in detecting the reflection coefficients of layered media by seismic exploration.

5. SCHUR-TYPE FORMULA

The aim of Sections 5 and 6 is to give an explicit formula for the one-to-one correspondence between CID(A) and the contractive analytic $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$-valued functions (see [10], Proposition 4.1).

Let us recall first the construction made in [10], Section 4. Fix $A_\infty \in$ CID(A) with the A-choice sequence $\{\Gamma_n\}_{n=1}^{\infty}$. In the following we consider $\Gamma_1 \in \mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$ as being in $\mathscr{I}(0_{\mathscr{R}^A}, 0_{\mathscr{R}_A})$. Lemma 4.1 from [10] establishes a one-to-one correspondence between CID(Γ_1) and all contractive analytic $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$-valued functions $s(z)$, such that $s(0) = \Gamma_1$. For this, the set CID(Γ_1) is analysed as a particular case of the general theory with $T = 0_{\mathscr{R}_A}$, $T' = 0_{\mathscr{R}^A}$ and $A = \Gamma_1$. We choose as minimal isometric dilation for $0_{\mathscr{R}_A}$(resp. $0_{\mathscr{R}^A}$) the unilateral shift of multiplicity $\mathscr{R}_A$ (resp. $\mathscr{R}^A$), which will be denoted by V_A (resp. V^A). Any element Γ_∞ of CID(Γ_1) is a contraction intertwining $V_{\mathscr{R}_A}$ and $V_{\mathscr{R}^A}$, so it is the multiplication by a contractive analytic $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$--valued function $s(z)$ (see [24], Ch. V, Sec. 3), and, of course, $s(0) = \Gamma_1$. There is a simple connection between this description of $\mathrm{CID}(\Gamma_1)$ and the observable sequences. Indeed, in this particular situation the role of $\mathscr{L}_*$ is played by $(I - V_A 0^*_{\mathscr{R}_A})\mathscr{R}_A^- = \mathscr{R}_A$ and the role of $\mathscr{L}'$ is played by $(V^A - 0_{\mathscr{R}_A})\mathscr{R}^{A-} = V^A\mathscr{R}^A$. Let $\Gamma_\infty \in \mathrm{CID}(\Gamma_1)$ and

$$s(z) = s_0 + zs_1 + z^2 s_2 + \dots \qquad |z| < 1$$

be its contractive analytic $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$-valued function associated as above. Then, taking into consideration Definition 2.1 and the construction made in [24], Ch. V, Sec. 3, it follows that the observable sequence of Γ_∞, namely $\{\tilde{s}_n\}_{n=1}^\infty$ verifies

$$(5.1)_n \qquad \tilde{s}_n = V^A s_n, \qquad (n \geqslant 1).$$

Note also that

$$(5.1)_0 \qquad s_0 = \Gamma_1 .$$

Next, Lemmas 4.2 and 4.3 from [10] establish a one-to-one correspondence between all A-choice sequences and all pairs formed by a contraction $\Gamma_1 \in \mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$ and a Γ_1-choice sequence. This is done in the following way. Note that using (1.7), (1.8) and (1.10) we have

$$(5.2) \qquad \begin{cases} \mathscr{F}_{\Gamma_1} = V_A \mathscr{R}_A \\ \mathscr{R}_{\Gamma_1} = \mathscr{D}_{\Gamma_1}, \end{cases}$$

$$(5.3) \qquad \begin{cases} \mathscr{F}^{\Gamma_1} = \{D_{\Gamma_1} r \oplus V^A \Gamma_1 r : r \in \mathscr{R}_A\} \\ \mathscr{R}^{\Gamma_1} = (\mathscr{D}_{\Gamma_1} \oplus V^A \mathscr{R}^A) \ominus \mathscr{F}^{\Gamma_1}, \end{cases}$$

and

$$(5.4) \qquad \sigma_{\Gamma_1}(V_A r) = D_{\Gamma_1} r \oplus V^A \Gamma_1 r, \qquad (r \in \mathscr{R}_A).$$

Define now

$$(5.5) \qquad \begin{cases} j(\Gamma_1) : \mathscr{D}_{\Gamma_1^*} \to \mathscr{R}^{\Gamma_1} \\ j(\Gamma_1) = (I \oplus V^A) J [J(-\Gamma)] i^{\mathscr{D}_{\Gamma_1^*}}_{\mathscr{R}_A \oplus \mathscr{D}_{\Gamma_1^*}}, \end{cases}$$

where J is the operator which permutes the sumands in a direct sum and $J(-\Gamma)$ is given by (3.29). Then, it is easy to see that

$$(5.5)' \qquad j(\Gamma_1)(d_*) = (-\Gamma_1^* d_*) \oplus V^A(D_{\Gamma_1^*} d_*)$$

is unitary. Define for every $n \geqslant 1$

$$(5.6)_n \qquad \gamma_n = j(\Gamma_1)\Gamma_{n+1} .$$

The sequence $\{\gamma_n\}_{n=1}^\infty$ is then a Γ_1-choice sequence (see [10], Lemma 4.3).

We are able to explain now what "the Schur-type formula" means. Consider $A_\infty \in \mathrm{CID}(A)$ with the A-choice sequence $\{\Gamma_n\}_{n=1}^\infty$ and let $\Gamma_\infty \in \mathrm{CID}(\Gamma_1)$ be defined by the Γ_1-choice sequence $\{\gamma_n\}_{n=1}^\infty$ (see $(5.6)_n$). Let further the corresponding contractive analytic $\mathscr{L}(\mathscr{R}_A, \mathscr{R}^A)$-valued function be denoted by $s(z)$. The Schur-type formula gives the expression of the observable sequence of A_∞ in terms of $s(z)$. The rest of this section is devoted to the study of $s(z)$ in terms of the objects involved in the description of A_∞; the Schur-type formula will be given in the next section.

The formulas $(5.1)_n$, $(n \geqslant 1)$, show that we can study the coefficients $\{s_n\}_{n=1}^\infty$, of $s(z)$ by means of Theorem 4.1. To this end we will identify the objects which appear in this theorem for the case $\Gamma_\infty \in \mathrm{CID}(\Gamma_1)$. Denote by y', y'', $\{y^n\}_{n=1}^\infty$, $\{x_n'\}_{n=0}^\infty$, $\{x_n''\}_{n=0}^\infty$, $\{\{x_n^k\}_{k=1}^n\}_{n=1}^\infty$, δ_0, $\{\delta_0^n\}_{n=1}^\infty$, $\{\delta_n\}_{n=1}^\infty$ the operators associated to $\Gamma_\infty \in \mathrm{CID}(\Gamma_1)$ by Theorem 4.1, which correspond to the operators denoted there with the corresponding capital letters. From (4.1) it follows that

$$(5.7)' \qquad y' = q^{\Gamma_1}\sigma_{\Gamma_1} = V^A\Gamma_1(V_A)^*,$$

so, using (4.2) we have

$$(5.8)' \qquad Y'' = q^A(V^A)^* y' V_A .$$

(The notation Y', Y'' etc., are those of Theorem 4.1 for $A_\infty \in \mathrm{CID}(A)$.) From (2.4) we infer

$$(5.7)'' \qquad y'' = q^{\Gamma_1}\gamma_1 = q^{\Gamma_1} j(\Gamma_1)\Gamma_2 = V^A D_{\Gamma_1^*}\Gamma_2,$$

so, using $(3.18)_1$ and $(3.4)_0$

$$(5.8)'' \qquad Y^1 = q^A(V^A)^* y'' .$$

From $(3.18)_n$ it follows that

$$(5.7)_n \qquad \begin{aligned} y^n &= q^{\Gamma_1} D_{\gamma_1^*} D_{\gamma_2^*} \dots D_{\gamma_n^*}\gamma_{n+1} = \\ &= q^{\Gamma_1} j(\Gamma_1) D_{\Gamma_2^*} D_{\Gamma_3^*} \dots D_{\Gamma_{n+1}^*}\Gamma_{n+2} = \\ &= V^A D_{\Gamma_1^*} D_{\Gamma_2^*} \dots D_{\Gamma_{n+1}^*}\Gamma_{n+2} ; \end{aligned}$$

so, using again $(3.18)_{n+1}$,

$$(5.8)_n \qquad Y^{n+1} = q^A(V^A)^* y^n , \qquad (n \geqslant 1).$$

The relations (4.1), (4.2) and (5.2) imply that

$$(5.9)_0' \qquad x_0' = 0 ,$$

and

$$(5.9)_0'' \qquad x_0'' = D_{\Gamma_1}.$$

From $(5.1)_n$, $(4.6)_{n-1}$—for s_n instead of S_n—, $(5.8)''$, $(5.8)_{k=1}^n$ and the formula $(5.9)_n'$ to be proved immediately, we have

$$(5.10)_n \qquad \begin{aligned} q^A s_n &= q_A(V^A)^*[y'' x_{n-1}'' + y^1 x_{n-1}^1 + \dots + y^{n-1} x_{n-1}^{n-1}] = \\ &= Y^1 x_{n-1}'' + Y^2 x_{n-1}^1 + \dots + Y^n x_{n-1}^{n-1} , \qquad (n \geqslant 1). \end{aligned}$$

Moreover, from $(4.7)_0$, (5.2) and (5.4) it follows that

$$\delta_0 = \begin{pmatrix} 0 & 0 \\ D_{\Gamma_1}V_A^* | V_A\mathscr{R}_A & -(1-q^{\Gamma_1})|\mathscr{R}^{\Gamma_1} \end{pmatrix} = \tag{5.11}$$

$$= [V_A \oplus I]\begin{pmatrix} 0 & 0 \\ D_{\Gamma_1} & \Gamma_1^* \end{pmatrix}[V_A^* \oplus j(\Gamma_1)^*] =$$

$$= [0 \oplus I]J(\Gamma_1)[V_A^* \oplus j(\Gamma_1)^*] .$$

In order to calculate $\{\delta_n\}_{n=1}^{\infty}$, note that from (3.29) and $(5.6)_n$ it follows that for every $n \geqslant 1$

$$J(\gamma_n) = [j(\Gamma_1) \oplus I]\, J(\Gamma_{n+1})[I \oplus j(\Gamma_1)^*] . \tag{5.12$_n$}$$

For every $n \geqslant 1$ and $1 \leqslant k \leqslant n$, the operator $J_n(\Gamma_k)$ acts as the identity on the first component (see $(3.30)_n^k$); denote by $J_n'(\Gamma_k)$ the operator obtained from $J_n(\Gamma_k)$ by eliminating the first components of the domain and of the range. Then we have for $n \geqslant 1$ and $1 \leqslant k \leqslant n$,

$$J_n(\gamma_k) = [V_A \oplus I_{k-1} \oplus j(\Gamma_1) \oplus I_{n-k+1}]J'_{n+1}(\Gamma_{k+1})[V_A^* \oplus I_k \oplus j(\Gamma_1^*) \oplus I_{n-k}], \tag{5.13$_n^k$}$$

where the subscript of I indicates the number of components of the direct sum on which it acts. Also, from (5.11) it follows that

$$\delta_0^n = [0 \oplus I_{n+1}]J'_{n+1}(\Gamma_1)[V_A^* \oplus j(\Gamma_1)^* \oplus I_n] \qquad \text{for every } n \geqslant 1. \tag{5.14$_n$}$$

Using $(4.16)_n$ in our particular situation it follows that for every $n \geqslant 1$,

$$\delta_n = [0 \oplus I_{n+1}]J'_{n+1}(\Gamma_1)J'_{n+1}(\Gamma_2) \ldots J'_{n+1}(\Gamma_{n+1})[V_A^* \oplus I_n \oplus j(\Gamma_1)^*] . \tag{5.15$_n$}$$

From $(4.15)_n$, $(5.9)_0'$ and $(5.15)_n$ we have

$$x_n' = 0 , \qquad \text{for every } n \geqslant 0. \tag{5.9$_n'$}$$

An important step in obtaining the Schur-type formula is

PROPOSITION 5.1. *For every $n \geqslant 1$*

$$J'_{n+1}(\Gamma_1)J'_{n+1}(\Gamma_2) \ldots J'_{n+1}(\Gamma_{n+1})[V_A^* \oplus I_n \oplus j(\Gamma_1)^*] \begin{pmatrix} 0 \\ x''_{n-2} \\ x^1_{n-1} \\ \vdots \\ x^{n-1}_{n-1} \\ 0 \end{pmatrix} = \tag{5.16$_n$}$$

$$= \begin{pmatrix} -s_n \\ x_n'' \\ x_n^1 \\ \vdots \\ x_n^n \end{pmatrix} [V_A^* \oplus I_{n+1}] .$$

Proof. The equality of the last $(n+1)$ components follows from $(4.15)_n$ (applied to $\Gamma_\infty \in \mathrm{CID}(\Gamma_1)$), $(5.9)'_{n-1}$ and $(5.15)_n$. There remains the first component to be taken care of. Computing the first component in the left hand side of $(5.16)_n$ we obtain

$$-(D_{\Gamma_1^*}\Gamma_2 x''_{n-1} + D_{\Gamma_1^*}D_{\Gamma_2^*}\Gamma_3 x^1_{n-1} + \ldots + D_{\Gamma_1^*}D_{\Gamma_2^*} \ldots D_{\Gamma_n^*}\Gamma_{n+1}x^{n-1}_{n-1}).$$

Note that from $(5.1)_n$, $(4.6)_{n-1}$ and $(5.9)'_{n-1}$, we have

$$(5.17)_n \qquad \tilde{s}_n = V^A s_n = y'' x''_{n-1} + y^1 x^1_{n-1} + \ldots + y^{n-1}x^{n-1}_{n-1}, \qquad n \geqslant 1.$$

The formula $(5.16)_n$ is now completely proved if we look at $(5.7)''$ and $(5.7)_k$, $(1 \leqslant k \leqslant n-1)$.

REMARK 5.1. The formulas $(5.16)_n$, $(n \geqslant 1)$ are in fact

$$(5.16)'_n \qquad J_{n+1}(\Gamma_1)J_{n+1}(\Gamma_2)\ldots J_{n+1}(\Gamma_{n+1})\begin{pmatrix} 0 \\ 0 \\ x''_{n-1} \\ x^1_{n-1} \\ \vdots \\ x^{n-1}_{n-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -s_n \\ x''_n \\ x^1_n \\ \vdots \\ \\ x^n_n \end{pmatrix},$$

the zero-operators acting between convenient spaces.

6. SCHUR-TYPE FORMULA (continued).

Consider for every $n \geqslant 1$ the operator

$$(6.1)_{n+1} \qquad \Psi_{n+1} = Y'Z'_n + Y''Z''_n + Y^1Z^1_n + \ldots + Y^nZ^n_n$$

and

$$(6.1)_1 \qquad \Psi_1 = Y'Z'_0 + Y''Z''_0 .$$

From Remark 4.1 it follows that

$$(6.2)_n \qquad S_n = \Psi_n(D_A P + I - P)|\mathscr{L}_*, \qquad (n \geqslant 1).$$

Define now the analytic $\mathcal{L}(\mathcal{D}_A + \mathcal{L}, \mathcal{L}')$-valued function

$$\Psi(z) = \Psi_1 + z\Psi_2 + \ldots + z^n\Psi_{n+1} + \ldots , \qquad (|z| < 1)$$

and the analytic $\mathcal{L}(\mathcal{L}_*, \mathcal{L}')$-valued function

$$S(z) = S_1 + zS_2 + \ldots + z^n S_{n+1} + \ldots , \qquad (|z| < 1);$$

then $(6.2)_n$, $(n \geqslant 1)$ imply that

$$S(z) = \Psi(z)(D_A P + I - P)|\mathcal{L}_* . \tag{6.3}$$

The key step in this section is

PROPOSITION 6.1. *For every* $n \geqslant 2$,

$$\Psi_n = [\Psi(z)(1 - q^A)s(z)(1 - p_A)]_{n-2} + \Psi_{n-1}(1 - q^A)\sigma_A p_A + q^A s_{n-1}(1 - p_A) , \tag{6.4$_n$}$$

where the first term in the right hand side means the coefficient of z^{n-2} *of the analytic function in the brackets.*

Proof. We will prove first that for every $n \geqslant 2$

$$\Psi_n p_A = \Psi_{n-1}(1 - q^A)\sigma_A p_A . \tag{6.5$_n$}$$

For this, we will prove by induction that for every $n \geqslant 1$

$$\begin{pmatrix} Z_n' \\ Z_n'' \\ Z_n^1 \\ \vdots \\ \\ Z_n^n \end{pmatrix} p_A = \begin{pmatrix} Z_{n-1}' \\ Z_{n-1}'' \\ Z_{n-1}^1 \\ \vdots \\ Z_{n-1}^{n-1} \\ 0 \end{pmatrix} (1 - q^A)\sigma_A p_A . \tag{6.5$'_n$}$$

These relations imply immediately $(6.5)_n$ for $n \geqslant 2$.

First, we have that

$$\begin{pmatrix} Z_1' \\ Z_1'' \\ Z_1^1 \end{pmatrix} p_A = \Delta_1 \begin{pmatrix} Z_0' \\ Z_0'' \\ 0_{\mathcal{D}_{T_1^*}} \end{pmatrix} p_A = \begin{pmatrix} p_A(1 - q^A)\sigma_A p_A \\ (1 - p_A)(1 - q^A)\sigma_A p_A \\ 0 \end{pmatrix} = \begin{pmatrix} Z_0' \\ Z_0'' \\ 0 \end{pmatrix} (1 - q^A)\sigma_A p_A$$

where we used $(4.18)_1$ and (4.10). This is exactly $(6.5)_2'$.

Suppose now that $(6.5)'_{n-1}$ is verified, where $n \geqslant 2$ is fixed. Then:

$$\begin{pmatrix} Z'_n \\ Z''_n \\ Z^1_n \\ \vdots \\ \\ Z^n_n \end{pmatrix} p_A = \Delta_n \begin{pmatrix} Z'_{n-1} \\ Z''_{n-1} \\ Z^1_{n-1} \\ \vdots \\ Z^{n-1}_{n-1} \\ 0 \end{pmatrix} p_A = [\Delta_{n-1} \oplus I_2] \begin{pmatrix} Z'_{n-2} \\ Z''_{n-2} \\ Z^1_{n-2} \\ \vdots \\ Z^{n-2}_{n-2} \\ 0 \\ 0 \end{pmatrix} (1-q^A)\sigma_A p_A =$$

$$= \begin{pmatrix} Z'_{n-1} \\ Z''_{n-1} \\ Z^1_{n-1} \\ \vdots \\ Z^{n-1}_{n-1} \\ 0 \end{pmatrix} (1-q^A)\sigma_A p_A ,$$

which is $(6.5)'_n$.

Thus all we have to prove is that for every $n \geqslant 2$,

$$(6.6)_n \quad \Psi_n(1-p_A) = \Psi_{n-1}(1-q^A)s_0(1-p_A) + \Psi_{n-2}(1-q^A)s_1(1-p_A) + \\ + \ldots + \Psi_1(1-q^A)s_{n-2}(1-p_A) + q^A s_{n-1}(1-p_A) .$$

Taking into account $(6.1)^n_{k=1}$ and $(5.10)^{n-2}_{k=1}$, it follows that $(6.6)_n$ is implied by

$$(6.7)_n \quad \begin{pmatrix} Z'_{n-1} \\ Z''_{n-1} \\ Z^1_{n-1} \\ \vdots \\ \\ Z^{n-1}_{n-1} \end{pmatrix} (1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x''_{n-2} \\ \vdots \\ \\ x^{n-2}_{n-2} \end{pmatrix} (1-p_A) = \begin{pmatrix} Z'_{n-2} \\ Z''_{n-2} \\ Z^1_{n-2} \\ \vdots \\ Z^{n-2}_{n-2} \\ 0 \end{pmatrix} (1-q^A)s_0(1-p_A) +$$

$$+ \begin{pmatrix} Z'_{n-3} \\ Z''_{n-3} \\ Z^1_{n-3} \\ \vdots \\ Z^{n-3}_{n-3} \\ 0 \\ \\ \end{pmatrix} (1-q^A)s_1(1-p_A) + \ldots + \begin{pmatrix} Z'_0 \\ Z''_0 \\ 0 \\ \vdots \\ \\ 0 \\ 0 \end{pmatrix} (1-q^A)s_{n-2}(1-p_A).$$

We will prove $(6.7)_n$, $(n \geqslant 2)$, by induction. For $n = 2$ we have to prove that

$$\begin{pmatrix} Z_1' \\ Z_1'' \\ Z_1^1 \end{pmatrix}(1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x_0'' \end{pmatrix}(1-p_A) = \begin{pmatrix} Z_0' \\ Z_0'' \\ 0 \end{pmatrix}(1-q^A)s_0(1-p_A),$$

which follows immediately from $(4.18)_1$, (4.10), $(5.9)_0''$ and $(5.1)_0$.

Suppose now that $(6.7)_n$ is verified, where $n \geqslant 2$ is fixed. Then

$$\begin{pmatrix} Z_n' \\ Z_n'' \\ Z_n^1 \\ \vdots \\ Z_n^n \end{pmatrix}(1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x_{n-1}'' \\ \vdots \\ x_{n-1}^{n-1} \end{pmatrix}(1-p_A) = \Delta_n \begin{pmatrix} Z_{n-1}' \\ Z_{n-1}'' \\ Z_{n-1}^1 \\ \vdots \\ Z_{n-1}^{n-1} \\ 0 \end{pmatrix}(1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x_{n-1}'' \\ \vdots \\ x_{n-1}^{n-1} \end{pmatrix}(1-p_A) =$$

$$= \Delta_n \begin{pmatrix} Z_{n-1}' \\ Z_{n-1}'' \\ Z_{n-1}^1 - x_{n-2}'' \\ \vdots \\ Z_{n-1}^{n-1} - x_{n-2}^{n-2} \\ 0 \end{pmatrix}(1-p_A) + \Delta_0^n J_n(\Gamma_1)\ldots J_n(\Gamma_n) \begin{pmatrix} 0 \\ 0 \\ x_{n-2}'' \\ \vdots \\ x_{n-2}^{n-2} \\ 0 \end{pmatrix}(1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x_{n-1}'' \\ \vdots \\ x_{n-1}^{n-1} \end{pmatrix}(1-p_A) =$$

$$= \Delta_n \left[\begin{pmatrix} Z_{n-2}' \\ Z_{n-2}'' \\ Z_{n-2}^1 \\ \vdots \\ Z_{n-2}^{n-2} \\ 0 \\ 0 \end{pmatrix}(1-q^A)s_0(1-p_A) + \ldots + \begin{pmatrix} Z_0' \\ Z_0'' \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}(1-q^A)s_{n-2}(1-p_A) \right] +$$

$$+ \Delta_0^n \begin{pmatrix} 0 \\ -s_{n-1} \\ x_{n-1}'' \\ \vdots \\ x_{n-1}^{n-1} \end{pmatrix}(1-p_A) - \begin{pmatrix} 0 \\ 0 \\ x_{n-1}'' \\ \vdots \\ x_{n-1}^{n-1} \end{pmatrix}(1-p_A) = \begin{pmatrix} Z_{n-1}' \\ Z_{n-1}'' \\ Z_{n-1}^1 \\ \vdots \\ Z_{n-1}^{n-1} \\ 0 \end{pmatrix}(1-q^A)s_0(1-p_A) + \ldots +$$

$$+\begin{pmatrix} Z_1' \\ Z_1'' \\ Z_1^1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}(1-q^A)s_{n-2}(1-p_A)+\begin{pmatrix} Z_0' \\ Z_0'' \\ 0 \\ \\ \vdots \\ 0 \\ 0 \end{pmatrix}(1-q^A)s_{n-1}(1-p_A),$$

which is exactly $(6.7)_{n+1}$. We used here $(4.18)_n$, $(4.16)_n$, $(6.7)_{n-1}$, $(5.16)'_{n-1}$, $(4.18)_{k=1}^{n-1}$ and $(4.7)_n$. The proof of $(6.7)_{n=2}^{\infty}$ concludes the proof of the proposition.

We are now ready to prove

THEOREM 6.1. (Schur-Type Formula). *The analytic operator-valued functions $S(z)$ and $s(z)$, ($|z| < 1$), are connected by*

$$S(z) = q^A(\sigma_A p_A + s(z)(1-p_A))\,[I - z(1-q^A)(\sigma_A p_A + \\ + s(z)(1-p_A))]^{-1}(D_A P + I - P)|\mathscr{L}_* . \tag{6.8}$$

Proof. The relations $(6.4)_{n=2}^{\infty}$ imply that

$$\Psi(z) = \Psi_1 + z\Psi_2 + z^2\Psi_3 + \ldots =$$

$$= \Psi_1 + z([\Psi(z)(1-q^A)s(z)(1-p_A)]_0 + \Psi_1(1-q^A)\sigma_A p_A + q^A s_1(1-p_A)) +$$

$$+ z^2([\Psi(z)(1-q^A)s(z)(1-p_A)]_1 + \Psi_2(1-q^A)\sigma_A p_A + q^A s_2(1-p_A)) + \ldots =$$

$$= \Psi_1 + z([\Psi(z)(1-q^A)s(z)(1-p_A)]_0 + z[\Psi(z)(1-q^A)s(z)(1-p_A)]_1 + \ldots) +$$

$$+ z(\Psi_1 + z\Psi_2 + \ldots)(1-q^A)\sigma_A p_A + q^A(zs_1 + z^2 s_2 + \ldots)(1-p_A) =$$

$$= \Psi_1 + z\Psi(z)(1-q^A)s(z)(1-p_A) + z\Psi(z)(1-q^A)\sigma_A p_A + q^A s(z)(1-p_A) -$$

$$- q^A s_0(1-p_A) = q^A(\sigma_A p_A + \Gamma_1(1-p_A)) - q^A\Gamma_1(1-p_A) +$$

$$+ \Psi(z)[z(1-q^A)s(z)(1-p_A) + z(1-q^A)\sigma_A p_A] + q^A s(z)(1-p_A) .$$

This means that

$$\Psi(z)\,[I - z(1-q^A)(\sigma_A p_A + s(z)(1-p_A))] = q^A s(z)(1-p_A) + q^A\sigma_A p_A ,$$

which implies

$$(6.9)\quad \Psi(z) = q^A(\sigma_A p_A + s(z)(1 - p_A))[I - z(1 - q^A)(\sigma_A p_A + s(z)(1 - p_A))]^{-1}.$$

The formula (6.8) follows now from (6.3) and (6.9) and the theorem is completely proved.

REMARK 6.1. The formula (6.8) shows that $S(z)$ is a *cascade transform* (see [15]) of $s(z)$. We will explicitate this now.

First, if A, B and $(A + B)$ are strict contraction, we have that

$$(6.10)\qquad (I - A - B)^{-1} = (I - (I - A)^{-1}B)^{-1}(I - A)^{-1}.$$

Using (6.10) in (6.9), we obtain

$$\Psi(z) = q^A(\sigma_A p_A + s(z)(1 - p_A))[I - (I - z(1 - q^A)\sigma_A p_A)^{-1} z(1 - q^A)s(z)(1 - p_A)]^{-1} \cdot$$
$$\cdot [I - z(1 - q^A)\sigma_A p_A]^{-1}.$$

By power series formula, we infer that

$$(6.11)\quad \Psi(z) = q^A\sigma_A p_A[I - z(1 - q^A)\sigma_A p_A]^{-1} + q^A s(z)(1 - p_A)[I - z(1 - q^A)\sigma_A p_A]^{-1} +$$
$$+ q^A(\sigma_A p_A + s(z)(1 - p_A))\left(\sum_{n=1}^{\infty} z^n[(I - z(1 - q^A)\sigma_A p_A)^{-1}(1 - q^A)s(z)(1 - p_A)]^n\right)\cdot$$
$$\cdot [I - z(1 - q^A)\sigma_A p_A]^{-1}.$$

It is plain that

$$(6.12)\qquad \sum_{n=1}^{\infty} z^n[(I - z(1 - q^A)\sigma_A p_A)^{-1}(1 - q^A)s(z)(1 - p_A)]^n =$$
$$= z[I - z(1 - q^A)\sigma_A p_A]^{-1}(1 - q^A)s(z)\{I - z(1 - p_A)[I -$$
$$- z(1 - q^A)\sigma_A p_A]^{-1}(1 - q^A)s(z)\}^{-1}(1 - p_A).$$

We will introduce now some notation

$$(6.13)\qquad a(z) = q^A\sigma_A p_A[I - z(1 - q^A)\sigma_A p_A]^{-1}(D_A P + I - P)|\mathscr{L}_*, \qquad |z| < 1,$$

$$(6.14)\qquad c(z) = (1 - p_A)[I - z(1 - q^A)\sigma_A p_A]^{-1}(D_A P + I - P)|\mathscr{L}_*, \qquad |z| < 1,$$

$$(6.15)\qquad d(z) = z(1 - p_A)[I - z(1 - q^A)\sigma_A p_A]^{-1}(1 - q^A)|\mathscr{R}^A, \qquad |z| < 1.$$

Then $a(z)$, $c(z)$ and $d(z)$ are analytic functions on $\{z;|z|<1\}$ with values in $\mathscr{L}(\mathscr{L}_*, \mathscr{L}')$, respectively $\mathscr{L}(\mathscr{L}_*, \mathscr{R}_A)$, $\mathscr{L}(\mathscr{R}^A, \mathscr{R}_A)$. Moreover, from (6.3), (6.11), (6.12), (6.13), (6.14) and (6.15), we infer that

$$S(z) = a(z) + q^A s(z)c(z) + q^A(z\sigma_A p_A + zs(z)(1-p_A))[I - z(1-q^A)\sigma_A p_A]^{-1}\cdot$$
$$\cdot(1-q^A)s(z)[I - d(z)s(z)]^{-1}c(z)\,,$$

which implies that

$$(6.16)\quad S(z) = a(z) + q^A[z\sigma_A p_A(I - z(1-q^A)\sigma_A p_A)^{-1}(1-q^A)s(z)][I - d(z)s(z)]^{-1}\cdot$$
$$\cdot c(z) + q^A s(z)[I - z(1-p_A)(I - z(1-q^A)\sigma_A p_A)^{-1}(1-q^A)s(z) +$$
$$+ z(1-p_A)(I - z(1-q^A)\sigma_A p_A)^{-1}(1-q^A)s(z)][I - d(z)s(z)]^{-1}c(z) =$$
$$= a(z) + [q^A(I + z\sigma_A p_A(I - z(1-q^A)\sigma_A p_A)^{-1}(1-q^A)]s(z)[I - d(z)s(z)]^{-1}\cdot c(z)\,.$$

Let us introduce one more notation

$$(6.17)\qquad b(z) = q^A(I + z\sigma_A p_A[I - z(1-q^A)\sigma_A p_A]^{-1}(1-q^A))|\mathscr{R}^A\,,$$

which is an analytic $\mathscr{L}(\mathscr{R}^A, \mathscr{L}')$-valued function for $|z|<1$.

Thus we have proved

COROLLARY 6.1. *The Schur-type formula can be written as*

$$S(z) = a(z) + b(z)s(z)\,[I - d(z)s(z)]^{-1}c(z)\,,$$

where the entries of the matrix

$$\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} : \begin{matrix} \mathscr{L}_* \\ \oplus \\ \mathscr{R}^A \end{matrix} \to \begin{matrix} \mathscr{L}' \\ \oplus \\ \mathscr{R}_A \end{matrix}$$

are analytic functions (on $\{z;\,|z|<1\}$) with $d(0)=0$ and both $b(0)$ and $c(0)^*$ injective

REMARK 6.2. The formula (6.8) can be viewed also as a generalization of the characteristic function of a contraction (see [24], ch. VI, (1.1)). Indeed, in the typical situation of this paper, take $\mathscr{H} = \mathscr{H}'$, $T = T' = 0$, and A an arbitrary *strict* contraction in $\mathscr{L}(\mathscr{H})$. Then $\mathscr{D}_A = \mathscr{D}_{A^*} = \mathscr{H}$, and from (5.2) and (5.5) we have that in this case $\mathscr{R}_A = \mathscr{H}$ and $\mathscr{R}^A = j(A)(\mathscr{H})$. Consider the A-choice sequence $\{\Gamma_n\}_{n=1}^{\infty}$, where

$$\Gamma_1 = -j(A)\,,$$

and for every $n \geqslant 2$,

$$\Gamma_n = 0 .$$

From $(5.17)_{n=1}^{\infty}$, $(5.7)''$ and $(5.7)_{n=1}^{\infty}$ it follows that $s(z)$ is constant (and equal with Γ_1). Denote by $V^{\mathscr{H}}$ the unilateral shift of multiplicity $\mathscr{H}$. Using (5.5)′, the formula (6.8) becomes

$$(V^{\mathscr{H}})^* S(z) = -D_{A^*}(I - zA^*)^{-1} D_A .$$

If $\Theta_A(z)$ is the characteristic function of A, then

$$\Theta_A(z) = -(A + z(V^{\mathscr{H}})^* S(z)).$$

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