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# EXACT SEQUENCES FOR $K$-GROUPS AND $EXT$-GROUPS OF CERTAIN CROSS-PRODUCT $C^*$-ALGEBRAS

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Recently M. A. Rieffel began studying $K_0$ and $Ext$ for the irrational rotation $C^*$-algebras $A_\theta$ i.e. the crossed product of the continuous functions on the circle by the automorphism corresponding to a rotation of angle $2\pi\theta$ where $\theta$ is an irrational number. Also $K_0$ of such algebras appeared as the range of an index map in the context of A. Connes'work [8] on operator algebras associated with foliations, the irrational rotation algebras corresponding to an extremely simple case: the Kronecker flows on the 2-torus.

The irrational rotation algebras have a unique trace state and the results of M. A. Rieffel [28] and of the present authors [26] taken together provided a determination of the range of the homomorphism induced by the trace from $K_0$ into $\mathbf{R}$. On the other hand S. Popa and M. A. Rieffel [27] solved the problem of computing $Ext$ for these algebras.

In the present paper we study $K$-groups and $Ext$-groups for $C^*$-algebras which are crossed products by a single automorphism i.e. the case of an automorphic action of the rational integers.

We obtain for the $K$-groups and $Ext$-groups six terms exact sequences involving only the groups for the initial algebra and the cross-product algebra. These exact sequences are derived from the cyclic six terms exact sequences of K-theory and respectively $Ext$-theory applied to what we shall call the Toeplitz-extension associated with a crossed product.

For the irrational rotation algebras these results have as an immediate consequence the fact that the homomorphism from $K_0$ into $\mathbf{R}$ given by the trace is injective, which confirms a fact conjectured by M. A. Rieffel and provides the missing part in the computation of $K_0$. Also for the same algebras, our general results immediately answer the problem of computing $K_1$ and describe its generators.

We show in an Appendix that our present results lead also to a new proof for the fact that the range of the homomorphism $K_0(A_\theta) \to \mathbf{R}$ induced by the trace is $\mathbf{Z} + \theta\mathbf{Z}$.

We would like to mention that our work appears also to be related to recent work of J. Cuntz [9] (see Remark 2.7 below).

The $K$-theory to which our results refer is the $K$-theory for $C^*$-algebras and is periodic. The usual reference for this $K$-theory is [30] with the mention that the commutative algebras can be replaced by non-commutative ones. The basic definitions can also be found in the papers [9] and [14]. The *Ext*-groups are the Brown-Douglas-Fillmore *Ext*-groups.

§ 1

This section consists of two lemmas about generators for $K_1$ of certain cross-products of $C^*$-algebras.

We shall denote by $\mathcal{M}_n$ the $C^*$-algebra of $n\times n$ complex matrices and by $1_n$ its unit.

If $B$ is a $C^*$-algebra with unit, elements of $B\otimes\mathcal{M}_n$ will be written either as elements of this tensor product or as $n\times n$-matrices with entries from $B$.

For $x\in B\otimes\mathcal{M}_n$ an idempotent, $[x]_0$ (or simply $[x]$ when this doesn't lead to confusions) will stand for the corresponding class in $K_0(B)$. Similarly for $x\in B\otimes\mathcal{M}_n$ an invertible element, $[x]_1$ (or simply $[x]$) will denote the corresponding class in $K_1(B)$.

1.1. LEMMA. *Let $B$ be a unital $C^*$-algebra, $A\subset B$ a $C^*$-subalgebra with $1_B\in A$ and let $u\in B$ be a unitary element. Assume $B$ is generated by $A$ and $u$ and $uAu^*=A$.*

*Then, the group $K_1(B)$ is generated by the classes of the invertible elements of the form $1_B\otimes 1_n+x(u^*\otimes 1_n)$ with $x\in A\otimes\mathcal{M}_n$ ($n\in\mathbf{N}$).*

*Proof.* Let $\Gamma\subset K_1(B)$ denote the subgroup generated by the elements $[1_B\otimes 1_n+x(u^*\otimes 1_n)]$. Remark that $[1_B+2u^*]=[u^*]$ so that $[u]\in\Gamma$. Since the elements of the form $\sum_{j=s}^{t} a_j(u^j\otimes 1_p)$, with $s,t\in\mathbf{Z}$, $s\leqslant t$ and $a_j\in A\otimes\mathcal{M}_p$, are dense in $B\otimes\mathcal{M}_p$ it will be sufficient to prove that the classes of invertible elements of this form are in $\Gamma$. Also, since $[u]\in\Gamma$ it will be sufficient to do this only for the case $s=0$. Thus, let $y=\sum_{j=0}^{t} a_j(u^j\otimes 1_p)$ be invertible, and consider as in the proof of the periodicity theorem ([4]), the matrix-identity:

$$\begin{pmatrix} a_0 & a_1 & \cdots & & a_t \\ -u_p & I & & 0 & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & -u_p & I \end{pmatrix} =$$

$$=\begin{pmatrix} I & y_1 & \cdots & y_t \\ & I & & 0 \\ & & \ddots & \\ 0 & & & I \end{pmatrix}\begin{pmatrix} y & & & \\ & I & & 0 \\ & & \ddots & \\ 0 & & & I \end{pmatrix}\begin{pmatrix} I & & & \\ -u_p & I & & 0 \\ & \ddots & \ddots & \\ 0 & & -u_p & I \end{pmatrix}$$

where $I$ denotes $1_B \otimes 1_p$, $u_p$ denotes $u \otimes 1_p$ and

$$y_k = \sum_{j=0}^{t-k} a_{j+k}(u^j \otimes 1_p) = y_{k+1}(u \otimes 1_p) + a_k \,.$$

The first and third matrix in the right hand side of this identity are of the form identity plus nilpotent so that their classes in $K_1$ are trivial.

Thus, the above identity shows that $[y]$ is equal to the class of the matrix in the left hand side. Defining

$$S_\varepsilon = \begin{pmatrix} 0 & 0 & \cdots & \varepsilon I \\ -I & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & -I & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} a_0 & a_1 & \cdots & a_t \\ & I & & 0 \\ & & \ddots & \\ & 0 & & I \end{pmatrix}$$

where $\varepsilon \in \mathbf{C}$, we have

$$[y] = [S_0(u \otimes 1_n) + T]$$

where $n = (t+1)p$. Now choosing $\varepsilon \neq 0$ small enough, we shall have

$$[y] = [S_\varepsilon(u \otimes 1_n) + T] = [(u \otimes 1_n) + S_\varepsilon^{-1}T] =$$

$$= n[u] + [1_B \otimes 1_n + S_\varepsilon^{-1}T(u^* \otimes 1_n)] \,. \qquad \text{Q.E.D.}$$

Let $B, A, u$ be as in Lemma 1.

From now on we shall make the additional hypothesis that $B$ is the crossed product of $A$ by the automorphism induced by $u$.

This will mean that $B$ will be the cross-product $C^*$-algebra $A \times_\alpha \mathbf{Z}$ where the action of $\mathbf{Z}$ on $A$ is given by $\alpha\colon \mathbf{Z} \to \operatorname{Aut} A$ defined by $(\alpha(n))x = u^n x u^{*n}$ for $n \in \mathbf{Z}$ the representation $n \mapsto u^n$ of $\mathbf{Z}$ in $B$, being that which is obtained via the isomorphism of $B$ and $A \times_\alpha \mathbf{Z}$.

By $\beta\colon \mathbf{T} \to \operatorname{Aut} B$ we shall denote the dual action, that is $(\beta(\gamma))x = x$ for $x \in A$ and $(\beta(\gamma))u = \gamma u$. It is well-known that the fixed-point algebra of $\beta$ is $A$.

1.2. Lemma. *Let $B, A, u$ be as in Lemma 1 and assume that $B$ is the crossed product of $A$ by the automorphism induced by $u$. Then the group $K_1(B)$ is generated* *by the classes of the unitary elements of the form*

$$(1_B \otimes 1_n - F) + Fx(u^* \otimes 1_n)F$$

*where* $F, x \in A \otimes \mathfrak{M}_n$ $(n \in \mathbf{N})$ *and* $F$ *is a selfadjoint projection.*

*Proof.* In view of Lemma 1.1 it will be sufficient to prove that for an invertible element

$$y = 1_B \otimes 1_n + x(u^* \otimes 1_n)$$

where $x \in A \otimes \mathfrak{M}_n$, there is an element of the form considered in the present lemma having the same class in $K_1(B)$.

To this end we shall study the spectrum of $x(u^* \otimes 1_n)$. The invertibility of $y$ means that $-1$ is not in the spectrum of $x(u^* \otimes 1_n)$. On the other hand the spectra of the elements $(\beta(\gamma) \otimes \mathrm{id}_n)(x(u^* \otimes 1_n))$ (where $\mathrm{id}_n$ is the identical automorphism of $\mathfrak{M}_n$ and $\gamma \in \mathbf{T} = \{z \in \mathbf{C} \mid |z| = 1\}$) are all equal. Since

$$(\beta(\gamma) \otimes \mathrm{id}_n)(x(u^* \otimes 1_n)) = \bar{\gamma}\, x(u^* \otimes 1_n)$$

it follows that $\{z \in \mathbf{C} \mid |z| = 1\}$ is in the resolvent of $x(u^* \otimes 1_n)$.

Moreover, the spectral projections $P_+$ and $P_-$ of $x(u^* \otimes 1_n)$ corresponding to $\{z \in \mathbf{C} \mid |z| > 1\}$ and $\{z \in \mathbf{C} \mid |z| < 1\}$ are invariant under $\beta(\gamma) \otimes \mathrm{id}_n$. Indeed $(\beta(\gamma) \otimes \mathrm{id}_n)(P_\pm)$ are the spectral projections for the same sets of $(\beta(\gamma) \otimes \mathrm{id}_n)(x(u^* \otimes \otimes 1_n)) = \bar{\gamma}x(u^* \otimes 1_n)$. But, it is easily seen that the spectral projections of $x(u^* \otimes 1_n)$ and $\bar{\gamma}x(u^* \otimes 1_n)$ for the above sets coincide. Thus $(\beta(\gamma) \otimes \mathrm{id}_n)(P_\pm) = P_\pm$ and hence $P_\pm \in A \otimes \mathfrak{M}_n$.

Consider now for $0 \leqslant \varepsilon \leqslant 1$ the elements

$$y_\varepsilon = (\varepsilon P_+ + x(u^* \otimes 1_n)P_+) + (P_- + \varepsilon x(u^* \otimes 1_n)P_-).$$

It is easily seen that the elements $y_\varepsilon$ for $0 \leqslant \varepsilon \leqslant 1$ are invertible.

We infer that $[y_0] = [y_1]$ and since $y = y_1$, we have $[y] = [y_0]$. Thus we have $[y] = [(1_B \otimes 1_n - P_+) + P_+x(u^* \otimes 1_n)P_+]$ and what is left to be done is to show that the idempotent $P_+$ can be replaced by a selfadjoint idempotent.

Now $P_+ = P_+^2 \in A \otimes \mathfrak{M}_n$ implies that

$$P_+ = F + FT(1_A \otimes 1_n - F)$$

with $T \in A \otimes \mathfrak{M}_n$ and $F = F^* = F^2 \in A \otimes \mathfrak{M}_n$ ($F$ is the orthogonal projection onto the range of $P_+$ in any representation of $A \otimes \mathfrak{M}_n$). Using the fact that $x(u^* \otimes 1_n)F = Fx(u^* \otimes 1_n)F$, we have

$$y_0 = (1_A \otimes 1_n - F) + Fx(u^* \otimes 1_n)F + FS(1_A \otimes 1_n - F)$$

where $S = -T + x(u^* \otimes 1_n)FT$.

It is easily seen that the invertibility of $y_0$ implies the invertibility of

$$(1_A \otimes 1_n - F) + Fx(u^* \otimes 1_n)F + \varepsilon FS(1_A \otimes 1_n - F)$$

for all $\varepsilon \in \mathbf{C}$ and hence

$$[y] = [y_0] = [(1_A \otimes 1_n - F) + Fx(u^* \otimes 1_n)F] .$$

To make $y_0$ unitary it is sufficient to replace $y_0$ by the unitary $(y_0 y_0^*)^{-1/2} y_0$ which is easily seen to be of the same form. Q.E.D.

## § 2

In this section we shall obtain a six-terms exact sequence for the $K$-groups of certain cross-product $C^*$-algebras. This exact sequence will be derived from the usual exact sequence of $K$-theory applied to what we shall call the Toeplitz extension associated with the crossed product.

So, the first thing we shall do, will be to construct the Toeplitz extension.

As in the preceding section, $A$ will be a unital $C^*$-algebra, $\alpha: \mathbf{Z} \to \operatorname{Aut} A$ and action of $\mathbf{Z}$ on $A$, $B$ will denote the cross-product $A \times_\alpha \mathbf{Z}$. Also, $A$ will be viewed as a subalgebra of $B$ and $u$ shall denote the unitary in $B$, corresponding to the cross-product structure, which together with $A$ generates $B$ and such that $\alpha(n)a = u^n a u^{*n}$. The unit of $B$ will be denoted by 1.

Let further $C^*(S)$ denote the $C^*$-algebra generated by a non-unitary isometry $S$, i.e. $S^*S = I$, $SS^* \neq I$ where $I$ denotes the unit element.

It is known that $C^*(S)$ does not depend on the choice of $S$. Consider the self-adjoint projection $P = I - SS^*$.

Let $K$ denote the $C^*$-algebra of compact operators on some complex separable Hilbert space with basis indexed by $\{0, 1, 2 \ldots\}$ and $(e_{ij})_{0 \leq i,j}$ the corresponding system of matrix-units for $K$. Then there is a homomorphism $\varphi: K \to C^*(S)$ given by $\varphi(e_{ij}) = S^i P S^{*j}$ and $\varphi(K)$ coincides with the closed two-sided ideal generated by $P$ in $C^*(S)$. As is well-known $C^*(S)/\varphi(K)$ is isomorphic with $C(\mathbf{T})$ (where $\mathbf{T} \simeq \{z \in \mathbf{C} \mid |z| = 1\}$) and the image of $S$ in $C(\mathbf{T})$ is the identical function $z$ on $\mathbf{T}$. This can be written as an exact sequence:

$$0 \longrightarrow K \xrightarrow{\varphi} C^*(S) \xrightarrow{p} C(\mathbf{T}) \longrightarrow 0 .$$

We define the Toeplitz algebra for $(A, \alpha)$ denoted by $\mathcal{T}(A, \alpha)$ or simply $\mathcal{T}$, as the $C^*$-subalgebra of $B \otimes C^*(S)$ generated by $A \otimes I$ and $u \otimes S$. Remark that $\mathcal{T}$ is invariant under the inner automorphism of $B \otimes C^*(S)$ given by $u \otimes I$. Thus, we can define $\tilde{\alpha}: \mathbf{Z} \to \operatorname{Aut}(\mathcal{T})$ by $\tilde{\alpha}(n)x = (u \otimes I)^n x (u \otimes I)^{*n}$. Clearly, the homomorphism $d: A \to \mathcal{T}$, $d(a) = a \otimes I$ is $\mathbf{Z}$-equivariant for the actions $\alpha$ and $\tilde{\alpha}$.

Consider now $J \subset \mathscr{T}$ the closed two-sided ideal generated by the selfadjoint projection $Q = 1 \otimes I - (u \otimes S)(u \otimes S)^* = 1 \otimes P$. It is easy to check that there is a homomorphism $\psi : A \otimes K \to \mathscr{T}$, defined by

$$\psi(a \otimes e_{ij}) = (u \otimes S)^i(a \otimes P)(u \otimes S)^{*j} = u^i a u^{*j} \otimes \varphi(e_{ij})$$

and moreover that $\psi(A \otimes K) = J$. Also, $(B \otimes \varphi(K)) \cap \mathscr{T} = J$. Indeed, $(B \otimes \varphi(K)) \cap \mathscr{T} \supset J$ is immediate and for $y \in (B \otimes \varphi(K)) \cap \mathscr{T}$ we have because of $y \in B \otimes \varphi(K)$

$$0 = \lim_{m \to \infty} \| y - (1 \otimes \varphi(e_{00} + \ldots + e_{mm}))y(1 \otimes \varphi(e_{00} + \ldots + e_{mm})) \|$$

and since $1 \otimes \varphi(e_{00} + \ldots + e_{mm}) = \psi(1 \otimes (e_{00} + \ldots + e_{mm})) \in J$ and $y \in \mathscr{T}$ we infer $y \in J$.

Tensoring by $B$ the exact sequence of $C^*(S)$ (which is nuclear), we obtain an exact sequence:

$$0 \longrightarrow B \otimes K \longrightarrow B \otimes C^*(S) \longrightarrow B \otimes C(\mathbf{T}) \longrightarrow 0 .$$

Since $\mathscr{T} \cap (B \otimes \varphi(K)) = J$, the algebra $\mathscr{T}/J$ identifies with the $C^*$-subalgebra of $B \otimes C(\mathbf{T})$ generated by $A \otimes 1$ and $u \otimes z$ (1 is the unit of $C(\mathbf{T})$ here and $z$ is the identical function on $\mathbf{T}$). But this $C^*$-algebra is clearly isomorphic with $B = A \times_\alpha \mathbf{Z}$ via the correspondences $a \mapsto a \otimes 1$, $u \mapsto u \otimes z$. Thus we have obtained an exact sequence:

$$0 \longrightarrow A \otimes K \xrightarrow{\psi} \mathscr{T} \xrightarrow{\pi} A \times_\alpha \mathbf{Z} \longrightarrow 0 ,$$

which we shall call the Toeplitz extension associated with $A \times_\alpha \mathbf{Z}$ .

Let us also remark that the diagram

$$\begin{tikzcd} \mathscr{T} \arrow[rr, "\pi"] & & A \times_\alpha \mathbf{Z} \\ & A \arrow[ul, "d"] \arrow[ur, "i"'] & \end{tikzcd}$$

is commutative (here $i$ is the inclusion of $A$ in $B = A \times_\alpha \mathbf{Z}$).

With these preparations, we can now pass to $K$-theory. The same notations as in § 1 will be used.

2.1. LEMMA. a) *The following diagram*

$$\begin{tikzcd} K_1(A \otimes K) \arrow[r, "\psi_*"] & K_1(\mathscr{T}) \\ K_1(A) \arrow[u, phantom, "\wr|"] \arrow[r, "(\mathrm{id}_A)_* - (\alpha(-1))_*"] & K_1(A) \arrow[u, "d_*"'] \end{tikzcd}$$

*is commutative.*

b) *The homomorphism $d_* : K_1(A) \to K_1(\mathscr{T})$ is injective.*

*Proof.* Both a) and b) are statements concerning elements of $K_1(A)$, i.e. concerning classes of unitaries from the algebras $A \otimes \mathfrak{M}_n$. Since replacing $(A, \alpha, \mathscr{T})$ by $(A \otimes \mathfrak{M}_n, \alpha \otimes \mathrm{id}_n, \mathscr{T} \otimes \mathfrak{M}_n)$ we have the same situation, it is easily seen that it is sufficient to prove a) and b) only for the classes of unitaries from $A$.

a) Let $v \in A$ be unitary. The isomorphism $K_1(A) \simeq K_1(A \otimes K)$ associates with $[v]$ the class $[v \otimes e_{00} + (\tilde{I} - 1 \otimes e_{00})]$ where $v \otimes e_{00} + (\tilde{I} - 1 \otimes e_{00})$ is a unitary in the $C^*$-algebra $\widetilde{A \otimes K}$ obtained by adjoining to $A \otimes K$ the unit $\tilde{I}$.

We have

$$\psi_*[v \otimes e_{00} + (\tilde{I} - 1 \otimes e_{00})] =$$

$$= [v \otimes P + (1 \otimes I - 1 \otimes P)] =$$

$$= [v \otimes P + 1 \otimes SS^*] .$$

On the other hand

$$d_* \circ ((\mathrm{id}_A)_* - (\alpha(-1))_*)\,[v] = d_*([v] - [\alpha(-1)v]) = [v \otimes I] - [u^* vu \otimes I] .$$

Consider now the unitary

$$\Omega = \begin{pmatrix} u \otimes S & Q \\ 0 & u^* \otimes S^* \end{pmatrix} \in \mathscr{T} \otimes \mathfrak{M}_2 .$$

We have

$$[v \otimes I] - [u^* vu \otimes I] = [v \otimes I] - \left[\Omega \begin{pmatrix} u^* vu \otimes I & 0 \\ 0 & 1 \otimes I \end{pmatrix} \Omega^* \right] =$$

$$= [v \otimes I] - \left[\begin{pmatrix} v \otimes SS^* + Q & 0 \\ 0 & 1 \otimes I \end{pmatrix}\right] =$$

$$= [v \otimes I] - [v \otimes SS^* + 1 \otimes P] =$$

$$= [(v \otimes I)(v \otimes SS^* + 1 \otimes P)^{-1}] =$$

$$= [(v \otimes I)(v^* \otimes SS^* + 1 \otimes P)] =$$

$$= [1 \otimes SS^* + v \otimes P]$$

which concludes the proof of a).

b) Arguing twice as in the discussion at the beginning of the proof, we see that we must prove the following fact: if $v_0, v_1 \in A$ are unitaries and if $[0, 1] \ni t \mapsto w_t \in \mathscr{T}$ is a continuous function with values unitary elements of $\mathscr{T}$, such that $w_0 = v_0 \otimes I$, $w_1 = v_1 \otimes I$ then in $K_1(A)$ we have $[v_0] = [v_1]$.

Let us consider again the unitary

$$\Omega = \begin{pmatrix} u \otimes S & Q \\ 0 & u^* \otimes S^* \end{pmatrix}.$$

We have

$$\begin{pmatrix} w_t & 0 \\ 0 & 1 \otimes I \end{pmatrix} \Omega \begin{pmatrix} \tilde{\alpha}(-1)w_t^* & 0 \\ 0 & 1 \otimes I \end{pmatrix} \Omega^* =$$

$$= \begin{pmatrix} w_t(1 \otimes S)w_t^*(1 \otimes S^*) + w_t Q & 0 \\ 0 & 1 \otimes I \end{pmatrix}.$$

Denoting by $y_t$ the element $w_t(1 \otimes S)w_t^*(1 \otimes S^*) + w_t Q$ the preceding computation shows that $y_t \in \mathscr{T}$ is unitary and clearly $y_t$ depends continuously on $t \in [0, 1]$. Also, $y_0 = 1 \otimes SS^* + v_0 \otimes P$ and $y_1 = 1 \otimes SS^* + v_1 \otimes P$. Moreover, we shall prove that $y_t \in 1 \otimes I + J$.

Since

$$y_t = 1 \otimes I + (w_t - 1 \otimes I)Q +$$

$$+ w_t((1 \otimes S)w_t^* - w_t^*(1 \otimes S))(1 \otimes S^*)$$

and $J = (B \otimes \varphi(K)) \cap \mathscr{T}$ it follows that it will be sufficient to prove that $(1 \otimes S)w - w(1 \otimes S) \in B \otimes \varphi(K)$ for all $w \in \mathscr{T}$. Since the unstarred algebra generated by $A \otimes I$, $u \otimes S$ and $u^* \otimes S^*$ is dense in $\mathscr{T}$ it will be sufficient to check the last assertion for elements $w$ taken among these generators which is immediate.

Summing up the preceding discussion, we have proved that $y_t$ is a continuous function of $t$, with values unitaries in $1 \otimes I + J$ and such that $y_0 = 1 \otimes SS^* + v_0 \otimes P$, $y_1 = 1 \otimes SS^* + v_1 \otimes P$. Using the isomorphism of $\mathbf{C}(1 \otimes I) + J$ with $\widetilde{A \otimes K}$ given by $\psi$, this implies that the classes in $K_1(\widetilde{A \otimes K})$ of $(\tilde{I} - 1 \otimes e_{00}) + v_0 \otimes e_{00}$ and $(\tilde{I} - 1 \otimes e_{00}) + v_1 \otimes e_{00}$ coincide. But this is equivalent with $[v_0] = [v_1]$ in $K_1(A)$. Q.E.D.

2.2. LEMMA. *The following diagram:*

$$\begin{tikzcd} K_0(A \otimes K) \arrow[r, "\psi_*"] & K_0(\mathscr{T}) \\ \wr\!\! \mid & \\ K_0(A) \arrow[r, "(\mathrm{id}_A)_* - (\alpha(-1))_*"] & K_0(A) \arrow[uu, "d_*"'] \end{tikzcd}$$

*is commutative.*

*Proof.* As in the proof of Lemma 2.1 it will be sufficient to prove the lemma only for the classes of selfadjoint projections from $A$.

Let $q \in A$ be a selfadjoint projection. The isomorphism $K_0(A) \simeq K_0(A \otimes K)$ associates with $[q]$ the class $[q \otimes e_{00}]$.

We have

$$\psi_*[q \otimes e_{00}] = [q \otimes P].$$

On the other hand

$$d_* \circ ((\mathrm{id}_A)_* - (\alpha(-1))_*)\,[q] = [q \otimes I] - [(\alpha(-1)q) \otimes I].$$

Using the unitary operator

$$\Omega = \begin{pmatrix} u \otimes S & Q \\ 0 & u^* \otimes S^* \end{pmatrix}$$

we have

$$[(\alpha(-1)q) \otimes I] = \left[\Omega \begin{pmatrix} (\alpha(-1)q) \otimes I & 0 \\ 0 & 0 \end{pmatrix} \Omega^* \right] =$$

$$= \left[\begin{pmatrix} q \otimes SS^* & 0 \\ 0 & 0 \end{pmatrix}\right] = [q \otimes SS^*] =$$

$$= [q \otimes I] - [q \otimes P]. \qquad \text{Q.E.D.}$$

2.3. LEMMA. *The homomorphism*

$$d_* : K_1(A) \to K_1(\mathcal{T}) \text{ is an isomorphism.}$$

*Proof.* Consider the diagram:

$$\begin{tikzcd}
K_1(A \otimes K) \arrow[r, "\psi_*"] \arrow[d, phantom, "\wr"] & K_1(\mathcal{T}) \arrow[r, "\pi_*"] & K_1(A \times_\alpha \mathbf{Z}) \arrow[r, "\delta"] & K_0(A \otimes K) \\
K_1(A) \arrow[r, "(\mathrm{id}_A)_* - (\alpha(-1))_*"] & K_1(A) \arrow[u, "d_*"'] \arrow[ur, "i_*"'] & &
\end{tikzcd}$$

The top row of the diagram is an exact sequence, being a segment of the exact sequence of $K$-theory applied to the Toeplitz extension. Also, because of Lemma 2.1 the square in this diagram is commutative and $d_*$ is injective. Recall also that $i = \pi \circ d$ and hence $i_* = \pi_* \circ d_*$.

It is easily seen from these facts, that all we have to prove is that $\mathrm{Im}\, i_* \supset \mathrm{Ker}\, \delta$. We shall use to this end Lemma 1.2.

Since the sum of the classes of two generators of the form considered in Lemma 1.2 is again the class of such a generator, we infer that every element in $K_1(A\times_\alpha \mathbf{Z})$ is the difference of the classes of two generators. On the other hand we assert that for a generator $(1\otimes 1_n - F)+Fx(u^*\otimes 1_n)F$ as in Lemma 1.2 we have

$$\delta[(1\otimes 1_n - F)+Fx(u^*\otimes 1_n)F]_1 = [F\otimes e_{00}]_0\ ,$$

where $F\otimes e_{00}$ is viewed as an $n\times n$ matrix over $A\otimes K$. Using again the argument that $(A,\alpha)$ can be replaced by $(A\otimes \mathcal{M}_n, \alpha\otimes \mathrm{id}_n)$ we see that there is no loss of generality in assuming that $n=1$. Remark that the unitary:

$$(1-F)+Fxu^*F\in A\times_\alpha \mathbf{Z}$$

lifts to the co-isometric element $w=(1-F)\otimes I + Fx\,u^*F\otimes S^*\in \mathcal{T}$. But for such elements, $\delta$ is computed as an index of the lifting $w$.

Since the projection onto the kernel of $w$ is $F\otimes P=\psi(F\otimes e_{00})$ and the cokernel of $w$ is 0, we infer that

$$\delta[(1-F)+Fxu^*F]_1=[F\otimes e_{00}]_0\ .$$

Returning now to matrices, we have

$$[(1\otimes 1_n - F_1)+F_1x_1(u^*\otimes 1_n)F_1]_1 -$$

$$-[(1\otimes 1_m - F_2)+F_2x_2(u^*\otimes 1_m)F_2]_1\in \operatorname{Ker}\delta$$

if and only if in $K_0(A)$ we have $[F_1]_0=[F_2]_0$. Since

$$[(1\otimes 1_n - F_1)+F_1x_1(u^*\otimes 1_n)F_1]_1 =$$

$$=[(1\otimes 1_{n+p} - F_1')+F_1'x_1'(u^*\otimes 1_{n+p})F_1']_1$$

where $F_1'=(0\otimes 1_p)\oplus F_1$, $x_1'=(1\otimes 1_p)\oplus x_1$ there is no loss of generality in assuming $m=n$. Also, because of

$$[(1\otimes 1_n - F_1)+F_1x_1(u^*\otimes 1_n)F_1]_1 -$$

$$-[(1\otimes 1_n - F_2)+F_2x_2(u^*\otimes 1_n)F_2]_1 =$$

$$=[(1\otimes 1_{n+p} - F_1'')+F_1''x_1''(u^*\otimes 1_{n+p})F_1'']_1 -$$

$$-[(1\otimes 1_{n+p} - F_2'')+F_2''x_2''(u^*\otimes 1_{n+p})F_2'']_1$$

where $F_k''=F_k\oplus(1\otimes 1_p)$, $x_k''=x_k\oplus(1\otimes 1_{n+p})$ $(k=1,2)$, we see that every element $\omega$ of $\operatorname{Ker}\delta$ can be written as:

$$\omega=[(1\otimes 1_n - F_1)+F_1x_1(u^*\otimes 1_n)F_1]_1 -$$

$$-[(1\otimes 1_n - F_2)+F_2x_2(u^*\otimes 1_n)F_2]_1$$

where $F_1 = vF_2v^*$ for some unitary $v \in A \otimes \mathfrak{M}_n$. But, then

$$\omega = [(1 \otimes 1_n - F_1) + F_1x_1(u^* \otimes 1_n)F_1]_1 -$$

$$- [(1 \otimes 1_n - F_1) + F_1x_3(u^* \otimes 1_n)F_1]_1$$

where $x_3 = vx_2(\alpha(-1) \otimes \mathrm{id}_n)v^* \in A \otimes \mathfrak{M}_n$.

This in turn gives:

$$\omega = [(1 \otimes 1_n - F_1) + F_1x_1(u^* \otimes 1_n)F_1)((1 \otimes 1_n - F_1) + F_1x_3(u^* \otimes 1_n)F_1)^*]_1 =$$

$$= [(1 \otimes 1_n - F_1) + F_1x_1((\alpha(-1) \otimes \mathrm{id}_n)F_1)x_3^*F_1]_1 .$$

This last equality shows that $\omega \in \mathrm{Im}\, i_*$. Thus we have proved that $\mathrm{Ker}\delta \subset \mathrm{Im}\, i_*$.

Q.E.D.

2.4. THEOREM. *The diagram*

$$\begin{tikzcd}
K_0(A) \arrow[r, "(\mathrm{id}_A)_* - (\alpha(-1))_*"] & K_0(A) \arrow[r, "i_*"] & K_0(A \times_\alpha \mathbf{Z}) \arrow[d] \\
K_1(A \times_\alpha \mathbf{Z}) \arrow[u] & K_1(A) \arrow[l, "i_*"'] & K_1(A) \arrow[l, "(\mathrm{id}_A)_* - (\alpha(-1))_*"']
\end{tikzcd}$$

*where the vertical arrows correspond to the connecting homomorphisms in the exact sequence for the Toeplitz extension (modulo the isomorphisms $K_0(A) \simeq K_0(A \otimes K)$, $K_1(A) \simeq K_1(A \otimes K)$) is an exact sequence.*

*Proof.* Looking at the diagram in the proof of Lemma 2.3 and using Lemma 2.1 a) and Lemma 2.3 we have that the sequence

$$K_0(A \times_\alpha \mathbf{Z}) \longrightarrow K_1(A) \xrightarrow{(\mathrm{id}_A)_* - (\alpha(-1))_*} K_1(A) \xrightarrow{i_*} K_1(A \times_\alpha \mathbf{Z}) \longrightarrow K_0(A)$$

is exact. This shows also, that the theorem will be proved if we can establish the analogue of Lemma 2.3 with $K_1$ replaced by $K_0$ (the analogue of Lemma 2.1 a) is just Lemma 2.2).

To this end, remark first that

$$(\mathrm{id}_{C(\mathbf{T})} \otimes d)_* : K_1(C(\mathbf{T}) \otimes A) \to K_1(C(\mathbf{T}) \otimes \mathcal{T})$$

is an isomorphism. Indeed, applying Lemma 2.3 to $(C(\mathbf{T}) \otimes A, \mathrm{id}_{C(\mathbf{T})} \otimes \alpha)$ instead of $(A, \alpha)$ and remarking that $\mathcal{T}(C(\mathbf{T}) \otimes A, \mathrm{id}_{C(\mathbf{T})} \otimes \alpha) \simeq C(\mathbf{T}) \otimes \mathcal{T}(A, \alpha)$ we obtain exactly this fact.

Now, related to the periodicity theorem, there is a natural isomorphism

$$K_1(C(\mathbf{T}) \otimes M) \simeq K_0(M) \oplus K_1(M)$$

for any unital $C^*$-algebra $M$. This, together with the fact that

$$(\mathrm{id}_{C(\mathbf{T})} \otimes d)_* : K_1(C(\mathbf{T}) \otimes A) \to K_1(C(\mathbf{T}) \otimes \mathscr{T})$$

is an isomorphism, immediately gives that $d_* : K_0(A) \to K_0(\mathscr{T})$ is an isomorphism.

Q.E.D.

We conclude this section with applications of the results obtained, to the irrational rotation $C^*$-algebra. For $0 < \theta < 1$ an irrational number, the irrational rotation $C^*$-algebra $A_\theta$ is the crossed product of $C(\mathbf{T})$ by the automorphism corresponding to a rotation of angle $2\pi\theta$. Equivalently $A_\theta$ is the $C^*$-algebra generated by two unitaries $u$ and $v$, such that $uvu^* = e^{2\pi i\theta} v$.

Applying Theorem 2.4, we obtain that $K_1(A_\theta)$ is a free abelian group with two generators. Taking into account also Lemma 1.2 and the description of the map $K_1(A \times_\alpha \mathbf{Z}) \to K_0(A)$, given in the proof of Lemma 2.3, one easily obtains

2.5. COROLLARY. *$K_1(A_\theta)$ is isomorphic with $\mathbf{Z} \oplus \mathbf{Z}$ the generators being the classes* $[u]$ and $[v]$.

Also, from Theorem 2.4 we obtain that $K_0(A_\theta)$ is also a free abelian group with two generators. The unique trace-state of $A_\theta$ induces a homomorphism $K_0(A_\theta) \to \mathbf{R}$ the range of which, by results of M. A. Rieffel [28] and of the present authors [26], taken together, coincides with the subgroup $\mathbf{Z} + \theta\mathbf{Z}$ of $\mathbf{R}$. Since $K_0(A_\theta)$ is a free abelian group with two generators, we infer, that this map is an isomorphism of $K_0(A_\theta)$ onto the group $\mathbf{Z} + \theta\mathbf{Z}$.

2.6. COROLLARY. *The unique trace-state of $A_\theta$ induces an isomorphism of $K_0(A_\theta)$ with the subgroup $\mathbf{Z} + \theta\mathbf{Z}$ of $\mathbf{R}$.*

2.7. REMARK. J. Cuntz has pointed out to us that it is possible to use Theorem 2.4 also for studying $K_0(A \times {}_\alpha\mathbf{Z})$ in case $A$ is not unital and that this may be useful in studying $K_0(O_A)$ for the algebras $O_A$ considered in [9] and [10] since by [10] the algebras $O_A \otimes K$ can be represented as crossed products of certain non-unital $C^*$-algebras by single automorphisms. For $A$ non-unital one should use Theorem 2.4 for $\tilde{A} \times_{\tilde{\alpha}} \mathbf{Z}$ and the split exact sequence

$$0 \to A \times_\alpha \mathbf{Z} \to \tilde{A} \times_{\tilde{\alpha}} \mathbf{Z} \to C(\mathbf{T}) \to 0 .$$

## § 3.

In this section we shall obtain a six-terms exact sequence for the weak *Ext*-groups of certain cross-product $C^*$-algebras. The proof of this fact is closely patterned after the proof of the corresponding result for $K$-groups, given in § 2.

We shall use the same notations concerning the crossed product $A \times_\alpha \mathbf{Z} = B$ and the associated Toeplitz algebra $\mathscr{T}$ as in § 2.

3.1. LEMMA. *Let $M$ be a unital $C^*$-algebra, $\rho : A \to M$ a unital $*$-homomorphism and $s \in M$ such that*

$$s^*\rho(a)s = \rho(\alpha(-1)a)$$

*for all $a \in A$. Then there exists a unital $*$-homomorphism $\sigma : \mathscr{T} \to M$, which is also unique, such that:*

$$\sigma(d(a)) = \rho(a), \quad \sigma(u \otimes S) = s .$$

*Proof.* The uniqueness of $\sigma$ is obvious, so we shall concentrate on the existence. Representing $M$ on some Hilbert space it is easily seen that there is no loss of generality if we treat only the special case when $M$ is the algebra $L(H)$ of all operators on the Hilbert space $H$. Since $s^*s = s^*\rho(1)s = \rho(\alpha(-1)1) = I_H$ we see that $s$ is an isometry. Moreover

$$((I_H - s^n s^{*n})\rho(a)s^n)^* ((I_H - s^n s^{*n})\rho(a)s^n) =$$

$$= s^{*n}\rho(a^*a)s^n - (s^{*n}\rho(a)s^n)^* (s^{*n}\rho(a)s^n) =$$

$$= \rho(\alpha(-n)(a^*a)) - (\rho(\alpha(-n)a))^* (\rho(\alpha(-n)a)) = 0$$

so that

$$\rho(a)s^n = s^n s^{*n}\rho(a)s^n = s^n\rho(\alpha(-n)a) .$$

Thus $s^n$ interwines $\rho \circ \alpha(-n)$ and $\rho$ .

We infer that $H_0 = \bigcap_{n>0} s^n H$ and $H_1 = H \ominus H_0$ are reducing subspaces fot $\rho(A)$ and the Wold decomposition asserts that they are also reducing for $s$. Now $s|H_0$ is unitary and hence there is a representation $\tilde{\sigma}_0$ of the crossed product $A \times_\alpha \mathbf{Z}$ on $H_0$ such that $\tilde{\sigma}_0(a) = \rho(a)|H_0$ and $\tilde{\sigma}_0(u) = s|H_0$. Defining $\sigma_0 : \mathscr{T} \to L(H_0)$ by $\sigma_0 = \tilde{\sigma}_0 \circ \pi$ we have $\sigma_0 \circ d = \rho|H_0$ and $\sigma_0(u \otimes S) = s|H_0$ .

To complete the proof, it will be sufficient to show that there is a representation $\sigma_1$ of $\mathscr{T}$ on $H_1$ such that $\sigma_1 \circ d = \rho|H_1$ and $\sigma_1(u \otimes S) = s|H_1$ . Denoting $s|H_1$ by $s_1$ we define on $H_1$ a representation $\tilde{\sigma}_1$ of $J = \psi(A \otimes K)$ by

$$\tilde{\sigma}_1(\psi(a \otimes e_{ij})) = s_1^i(\rho(a)|H_1)(I_{H_1} - s_1 s_1^*)s_1^{*j} .$$

It is not difficult to check that this is a representation; we mention only that the projection $(I_{H_1} - s_1 s_1^*)$ commutes with $\rho(A)$ and leave the details to the reader.

Now, $\tilde{\sigma}_1$ is non-degenerate and hence there is a unique representation $\sigma_1$ of $\mathscr{T}$ on $H_1$ extending $\tilde{\sigma}_1$ and which can be defined by

$$\sigma_1(x) = s\text{-}\lim_{m\to\infty} \tilde{\sigma}_1(x\psi(1 \otimes e_{00} + \ldots + 1 \otimes e_{mm})) .$$

For $x = a \otimes I = d(a)$ $(a \in A)$ this gives

$$\sigma_1(d(a)) = s\text{-}\lim_{m\to\infty} \tilde{\sigma}_1(\psi(a \otimes e_{00} + (\alpha(-1)a) \otimes e_{11} + \ldots$$

$$\ldots + (\alpha(-m)a) \otimes e_{mm})) =$$

$$= s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} s_1^k(\rho(\alpha(-k)a)|H_1)(I_{H_1} - s_1s_1^*)s_1^{*k} =$$

$$= s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} (\rho(a)|H_1)s_1^k(I_{H_1} - s_1s_1^*)s_1^{*k} = \rho(a)|H_1 .$$

Also,

$$\sigma_1(u \otimes S) = s\text{-}\lim_{m\to\infty} \tilde{\sigma}_1(\psi(1 \otimes e_{10} + \ldots + 1 \otimes e_{m+1,\, m})) =$$

$$= s\text{-}\lim_{m\to\infty} \sum_{k=0}^{m} s_1^{k+1}(I_{H_1} - s_1s_1^*)s_1^{*k} = s_1 .$$

Thus $\sigma_1 \circ d = \rho|H_1$ and $\sigma_1(u \otimes S) = s_1$ which concludes the proof. Q.E.D.

We pass now to *Ext*-groups. For $H$ a complex separable Hilbert space of infinite dimension, we shall denote by $K(H), L(H), L/K(H)$ the compact operators on $H$, the bounded operators on $H$ and respectively the Calkin algebra. The canonical map $L(H) \to L/K(H)$ will be denoted by $q$. When this will not lead to confusions, we shall write simply $K, L$ and $L/K$.

In order that the considered *Ext*'s be groups *we shall assume in the results involving Ext-groups (Lemma* 3.2, *Lemma* 3.3, *Theorem* 3.5 *below) that A is separable and nuclear*. We remark that this implies that $A \times_\alpha \mathbf{Z}$ and $\mathscr{T}$ are also nuclear and hence their *Ext*'s are groups.

By $Ext\, A$ and $Ext_w A$ we shall denote the *Ext*-groups with respect to strong equivalence and respectively weak equivalence. For a $*$-monomorphism $\sigma : A \to L/K$ the corresponding classes in $Ext\, A$ and $Ext_w A$ will be denoted by $[\sigma]$ and respectively $[\sigma]_w$.

Remark that if $\sigma : A \to L/K$ is a $*$-monomorphism and $e \in L/K$ is a projection in the commutant of $\sigma(A)$, and $e \neq 0$, then there is an isomorphism $eL/Ke \simeq L/K$ which gives us the possibility of considering the "restriction" of $\sigma$ to $e$, however the class of this "restriction" is determined in general only up to weak equivalence.

We shall say that a unital separable nuclear $C^*$-algebra $M$ has the homotopy-invariance property if the following holds: if $\sigma_0, \sigma_1 : M \to L/K$ are unital $*$-monomorphisms, which can be joined by a continuous curve in the space of unital $*$-homomorphisms from $M$ to $L/K$ endowed with the topology of point-norm-convergence, then $[\sigma_0] = [\sigma_1]$.

This property is somewhat weaker than the homotopy-invariance property for homogeneous extensions considered in [24]. Further, from the sharpening of Salinas result [29] obtained in [24] we have that generalized quasidiagonal (abbreviated g.q.d.) $C^*$-algebras have the homotopy-invariance property. Generalized quasidiagonal means that there is a composition series with quasidiagonal quotients and $M$ is quasidiagonal if it has a faithful representation $\rho : M \to L(H)$ ($H$ separable) which is almost diagonal, i.e. so that there are finite rank orthogonal projections $P_j \uparrow I$, $P_j \in L(H)$ for which $\lim_{j\to\infty} \|[\rho(y), P_j]\| = 0$ for all $y \in M$.

We shall also need the six-terms exact sequence for $Ext$. This sequence proved by Brown-Douglas-Fillmore for commutative $C^*$-algebras, has been generalized to wider classes of $C^*$-algebras, together with other parts of the Brown-Douglas-Fillmore as the result of the work of several authors ([2], [3], [5], [11], [24], [25], [29], [31], [32]).

On the other hand the exact sequence for the weak $Ext$-groups should also follow (without any quasidiagonality assumption) from the results announced in [19]. We shall give here the result from [25]. Let $A$ be a unital $C^*$-algebra, $I$ a closed two-sided ideal. The diagram:

$$\begin{array}{ccccc} Ext\,(A/I) & \longrightarrow & Ext\,(A) & \longrightarrow & Ext\,(\tilde{I}) \\ \uparrow & & & & \downarrow \\ Ext\,(\bar{S}\tilde{I}) & \longleftarrow & Ext\,(\bar{S}A) & \longleftarrow & Ext\,(\bar{S}A/I) \end{array}$$

is an exact sequence, where $A$ (and hence all the other algebras) is nuclear, moreover $I$, $A/I$ are g.q.d. (and hence all the other algebras) and it is also assumed that there is a unital $*$-homomorphism $\chi : A/I \to \mathbf{C}$. Here $\bar{S}A$ is the reduced suspension of the "pointed" $C^*$-algebra $(A, \chi)$ i.e.

$$\bar{S}A = \{f \in C([0, 1], A) \mid \chi f(t) = 0 \ \ (\forall)\ t \in [0, 1],\ f(0) = f(1) = 0\}^{\sim}.$$

Let now $A$ be unital, nuclear and let $I \subset A$ be a closed two-sided ideal and assume that $\tilde{I}$ and $A/I$ are g.q.d.

Then without assuming $A/I$ "pointed", we can pass to

$$0 \longrightarrow I \longrightarrow \tilde{A} \longrightarrow \widetilde{A/I} \longrightarrow 0$$

where $\widetilde{A/I}$ is clearly "pointed".

Now, there are isomorphisms $Ext\,(\tilde{M}) \simeq Ext_w\,(M)$ and $Ext\,(\bar{S}\tilde{M}) \simeq Ext_w(\Omega M)$ where $\Omega M = \{f \in C(\mathbf{T}, M) \mid f(1) = 0\}^{\sim}$.

Then the six-terms exact sequence applied to $0 \to I \to \tilde{A} \to \widetilde{A/I} \to 0$ gives an exact sequence:

$$\begin{array}{ccccc} Ext_w\,(A/I) & \longrightarrow & Ext_w\,(A) & \longrightarrow & Ext_w\,(\tilde{I}) \\ \uparrow & & & & \downarrow \\ Ext_w\,(\Omega I) & \longleftarrow & Ext_w\,(\Omega A) & \longleftarrow & Ext_w\,(\Omega A/I)\,. \end{array}$$

Moreover when $M$ is unital nuclear, g.q.d., there is a natural isomorphism

$$Ext_w(C(\mathbf{T}) \otimes M) \simeq Ext_w(\Omega M) \oplus Ext_w(M)$$

and when $M$ is not unital:

$$Ext_w(\widetilde{C(\mathbf{T}) \otimes M}) \simeq Ext_w(\Omega M) \oplus Ext_w(\widetilde{M})$$

(use for instance Lemma 8.5 of [25]).

3.2. LEMMA. a) *The following diagram*

$$\begin{array}{ccc} Ext_w(\widetilde{A \otimes K}) & \xleftarrow{\quad \psi^* \quad} & Ext_w(\mathscr{T}) \\ \wr\| & & \Big\downarrow d^* \\ Ext_w(A) & \xleftarrow{(\mathrm{id}_A)^* - (\alpha(-1))^*} & Ext_w(A) \end{array}$$

*is commutative.*

b) *The homomorphism*

$$d : Ext_w(\mathscr{T}) \to Ext_w(A)$$

*is surjective.*

*Proof.* a) Let $\tau : \mathscr{T} \to L/K$ be a unital $*$-monomorphism. Then the projection $\tau(\psi(1 \otimes e_{00}))$ commutes with $(\tau \circ d)(A)$. The "restrictions" of $\tau \circ d$ to $\tau(\psi(1 \otimes e_{00}))$ and $1_{L/K} - \tau(\psi(1 \otimes e_{00}))$ determine two classes $[\sigma_0]_w$ and respectively $[\sigma_1]_w$ in $Ext_w(A)$. We have $[\sigma_0]_w + [\sigma_1]_w = d^*[\tau]_w$. Moreover the class in $Ext_w(A)$ corresponding via the isomorphism $Ext_w(\widetilde{A \otimes K}) \simeq Ext_w(A)$ to $\psi^*[\tau]_w$ is just the class $[\sigma_0]_w$ .

On the other hand consider $s = \tau(u \otimes S)$. We have $s^*s = 1_{L/H}$ and $ss^* = 1_{L/H} - \tau(\psi(1 \otimes e_{00}))$. Also for $a \in A$ we have

$$s\tau(d(a)) = \tau((u \otimes S)(a \otimes I)) = \tau((\alpha(1)a) \otimes I)s = (\tau \circ d)(\alpha(1)a)s$$

which shows that $(\alpha(1))^*[\sigma_1]_w = d^*[\tau]_w$ .

It follows

$$(\alpha(-1))^* d^*[\tau]_w = [\sigma_1]_w = d^*[\tau]_w - [\sigma_0]_w$$

which is the desired result.

b) Let $\rho : A \to L/K$ be a unital $*$-monomorphism. Since $Ext_w(A)$ is a group there is $[\eta]_w \in Ext_w(A)$ such that $[\rho]_w = [\eta]_w + (\alpha(-1))^*[\rho]_w$. This implies the existence of a selfadjoint projections $e$ in $L/K$ commuting with $\rho(A)$, such that the "restriction" of $\rho$ to $e$ defines an extension weakly equivalent with $(\alpha(-1))^*[\rho]_w$ .

But this means that there is $s \in L/K$ such that $s^*s = 1_{L/K} ss^* = e$ and $s^*\rho(a)s = \rho(\alpha(-1)a)$ .

In view of Lemma 3.1 there is a unital $*$-monomorphism $\sigma : \mathscr{T} \to L/K$ such that $\sigma \circ d = \rho$ and $\sigma(u \otimes S) = s$. Clearly $d^*[\sigma] = [\rho]$ and a fortiori $d^*[\sigma]_w = [\rho]_w$.

Q.E.D.

3.3. Lemma. *Assume $\mathscr{T}$ has the homotopy-invariance property. Then, the homomorphism*

$$d^* : Ext_w(\mathscr{T}) \longrightarrow Ext_w(A)$$

*is an isomorphism.*

*Proof.* By Lemma 3.2 b) we know that $d^*$ is surjective, so that it will be sufficient to prove that $d^*$ is also injective.

Consider the diagram

$$\begin{tikzcd}
Ext_w(\widetilde{A \otimes K}) \arrow[d, "\wr"] & Ext_w(\mathscr{T}) \arrow[l, "\psi^*"'] \arrow[d, "d^*"] & Ext_w(\widetilde{A \times_\alpha \mathbf{Z}}) \arrow[l, "\pi^*"'] \arrow[dl, "i^*"] \\
Ext_w(A) & Ext_w(A) \arrow[l, "(\mathrm{id}_A)^* - (\alpha(-1))^*"'] &
\end{tikzcd}$$

the top row of which is an exact sequence and in which the square and the triangle are commutative. If $d^*[\sigma]_w = 0$ for $[\sigma]_w \in Ext_w(\mathscr{T})$ then $\psi^*[\sigma]_w = 0$ and hence $[\sigma]_w = \pi^*[\eta]_w$ for some $[\eta]_w \in Ext_w(A \times_\alpha \mathbf{Z})$. Moreover $i^*[\eta]_w = d^*\pi^*[\eta]_w = d^*[\sigma]_w = 0$.

Thus it will be sufficient to prove that if $[\eta]_w \in Ext_w(A \times_\alpha \mathbf{Z})$ is such that $i^*[\eta]_w = 0$, then $\pi^*[\eta]_w = 0$. A slight further reduction is possible; it is sufficient to prove that if $[\eta] \in Ext(A \times_\alpha \mathbf{Z})$ is such that $i^*[\eta] = 0$ then $\pi^*[\eta]_w = 0$. Indeed, if $i^*[\eta]_w = 0$ then there is $[\eta']$ such that $i^*[\eta'] = 0$ and $[\eta']_w = [\eta]_w$.

Thus consider $[\eta] \in Ext(A \times_\alpha \mathbf{Z})$ such that $i^*[\eta] = 0$ and let us construct a special element of the class $\pi^*[\eta]$. Since trivial extensions are equivalent, we can find $\mu : A \times_\alpha \mathbf{Z} \to L(H)$, such that $\mathrm{Ker} q \circ \mu = 0$ and $q \circ \mu \circ i = \eta \circ i$. Let further $H_1$ be a Hilbert space with an orthogonal basis $\{f_0, f_1, \dots\}$, let $V$ be the unilateral shift on this basis and let $v$ be the representation of $\mathscr{T}$ on $H_2 = H \otimes H_1$ defined by $v(d(a)) = \mu(a) \otimes I_{H_1}$ and $v(u \otimes S) = \mu(u) \otimes V$.

Then we have a $*$-homomorphism:

$$(q \circ v) \oplus (\eta \circ \pi) : \mathscr{T} \to L/K(H_2) \oplus L/K(H) \subset L/K(H_2 \oplus H)$$

Taking $\mu$ of infinite multiplicity it is easy to insure that $q \circ v$ is a monomorphism, and hence we have

$$\pi^*[\eta] = [(q \circ v) \oplus (\eta \circ \pi)].$$

So, what must be proved, is that $[(q \circ v) \oplus (\eta \circ \pi)]_w = 0$.

Let $v \in L/K(H)$ denote the unitary $((q \circ \mu)(u))^*\eta(u)$ so that $\eta(u)v^* = (q \circ \mu)(u)$ and $v$ commutes with $(q \circ \mu \circ i)(A) = (\eta \circ i)(A)$ since

$$v\eta(a)v^* = ((q \circ \mu)(u))^*\eta(\alpha(1)a)((q \circ \mu)(u)) =$$

$$= ((q \circ \mu)(u))^*((q \circ \mu)(\alpha(1)a))((q \circ \mu)(u)) = (q \circ \mu)(a) = \eta(a).$$

Consider now the decomposition

$$H_2 \oplus H = (H_2 \ominus (H \otimes f_0)) \oplus ((H \otimes f_0) \oplus H)$$

which we shall write as

$$H_2 \oplus H = H_3 \oplus (H \oplus H)$$

after identifying $H \otimes f_0$ and $H$ via $h \otimes f_0 \mapsto h$ and denoting $H_2 \ominus (H \otimes f_0)$ by $H_3$. Corresponding to this decomposition we shall view

$$L/K(H_3) \oplus (L/K(H) \otimes \mathfrak{M}_2)$$

as a subalgebra of $L/K(H_2 \oplus H)$. In this subalgebra of $L/K(H_2 \oplus H)$ we consider the following unitaries:

$$v_0 = I \oplus \begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix}$$

$$v_1 = I \oplus \begin{pmatrix} v & 0 \\ 0 & I \end{pmatrix}$$

$$w_t = I \oplus \left( \begin{pmatrix} v & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & v^* \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right).$$

The following relations are easily obtained:

$$v_0^* v_1 = w_0, \quad w_{\pi/2} = I$$

and

$$v_1^*((q \circ v) \oplus (q \circ \mu \circ \pi))(u \otimes S) = ((q \circ v) \oplus (q \circ \mu \circ \pi))(u \otimes S) .$$

Moreover $v_0, v_1, w_t$ are in the commutant of $(((q \circ v) \oplus (\eta \circ \pi)) \circ d)(A)$.

Using Lemma 3.1 we now define $*$-homomorphisms:

$$\sigma_t : \mathscr{T} \longrightarrow L/K(H_2 \oplus H)$$

by

$$\sigma_t \circ d = ((q \circ v) \oplus (\eta \circ \pi)) \circ d \qquad \text{and}$$

$$\sigma_t(u \otimes S) = (((q \circ v) \oplus (\eta \circ \pi))(u \otimes S))w_t .$$

Clearly $\sigma_t$, in the topology of point-norm-convergence, depends continuously on $t$. By homotopy-invariance, we infer $[\sigma_0] = [\sigma_{\pi/2}]$. Now $[\sigma_{\pi/2}] = [(q \circ v) \oplus (\eta \circ \pi)]$ and hence to conclude the proof of the lemma it will be sufficient to prove that $[\sigma_0]_w = 0$.

We have

$$(\sigma_0 \circ d)(a) = (((q \circ v) \oplus (\eta \circ \pi)) \circ d)(a) =$$

$$= (((q \circ v) \oplus (q \circ \mu \circ \pi)) \circ d)(a) =$$

$$= v_1^*(((q \circ v) \oplus (q \circ \mu \circ \pi)) \circ d)(a) v_1$$

$$\sigma_0(u \otimes S) = (((q \circ v) \oplus (\eta \circ \pi))(u \otimes S)) w_0 =$$

$$= (((q \circ v) \oplus (q \circ \mu \circ \pi))(u \otimes S)) v_1 =$$

$$= v_1^*(((q \circ v) \oplus (q \circ \mu \circ \pi))(u \otimes S)) v_1 .$$

It follows that

$$[\sigma_0]_w = [(q \circ v) \oplus (q \circ \mu \circ \pi)]_w = 0 . \qquad \text{Q.E.D.}$$

3.4. Remark. Concerning the proof of Lemma 3.3 we would like to mention that the ideas of constructing the unitary $v$ and then of getting rid of this unitary by means of a homotopy were inspired by [27], where similar arguments were used in the computation of the *Ext*-groups of the irrational rotation $C^*$-algebras.

3.5. Theorem. *Assume $A$ is unital separable and nuclear and assume that $A \times_\alpha \mathbf{Z}$ and $A$ are g.q.d. Then the diagram*

$$\begin{array}{ccccc}
Ext_w(A \times_\alpha \mathbf{Z}) & \xrightarrow{\quad i^* \quad} & Ext_w(A) & \xrightarrow{(\mathrm{id}_A)^* - (\alpha(-1))^*} & Ext_w(A) \\
\uparrow & & & & \downarrow \\
Ext_w(\Omega A) & \xleftarrow{(\mathrm{id}_{\Omega A})^* - (\Omega\alpha(-1))^*} & Ext_w(\Omega A) & \xleftarrow{\quad (\Omega i)^* \quad} & Ext_w(\Omega(A \times_\alpha \mathbf{Z}))
\end{array}$$

*is an exact sequence (the vertical arrows are defined from the exact sequence for the Toeplitz extension).*

*Proof.* Consider the diagram

$$\begin{array}{ccccc}
Ext_w(A \times_\alpha \mathbf{Z}) & \xrightarrow{\quad \pi^* \quad} & Ext_w(\mathscr{T}) & \xrightarrow{\quad \psi^* \quad} & Ext_w(\widetilde{A \otimes K}) \\
 & \searrow^{i^*} & \downarrow{d^*} & & \wr\!\! \mid \\
 & & Ext_w(A) & \xrightarrow{(\mathrm{id}_A)^* - (\alpha(-1))^*} & Ext_w(A)
\end{array}$$

where the top row is a segment of the six terms exact sequence for the Toeplitz extension, the triangle and the square are commutative (Lemma 3.2) and $d^*$ is an isomorphism (Lemma 3.3). This gives us the possibility to replace in the six terms exact sequence for the Toeplitz extension the segment

$$Ext_w(A\times_\alpha \mathbf{Z}) \xrightarrow{\pi^*} Ext_w(\mathscr{T}) \xrightarrow{\psi^*} Ext_w(\widetilde{A\otimes K})$$

by

$$Ext_w(A\times_\alpha \mathbf{Z}) \xrightarrow{i^*} Ext_w(A) \xrightarrow{(\mathrm{id}_A)^*-(\alpha(-1))^*} Ext_w(A)\,.$$

Thus to complete the proof of the theorem it will be sufficient to obtain the results corresponding to Lemma 3.2 a) and Lemma 3.3 with $Ext_w(?)$ replaced by $Ext_w(\Omega\, ?)$.

As for $K$-theory, this is achieved by remarking that the Toeplitz extension for $(A\otimes C(\mathbf{T}), \alpha\otimes \mathrm{id}_{C(\mathbf{T})})$ is

$$0 \longrightarrow A\otimes K\otimes C(\mathbf{T}) \longrightarrow \mathscr{T}\otimes C(\mathbf{T}) \longrightarrow (A\times_\alpha \mathbf{Z})\otimes C(\mathbf{T}) \longrightarrow 0$$

and then using the natural isomorphism

$$Ext_w(C(\mathbf{T})\otimes M) \simeq Ext_w(\Omega M)\oplus Ext_w(M)\,. \qquad \text{Q.E.D.}$$

In order to be able to use the preceding theorem it is necessary to have criteria for the generalized quasidiagonality of $A\times_\alpha\mathbf{Z}$. The authors have learned that in case $A$ is commutative results about quasidiagonality have been recently obtained by D. Hadwin.

Here we shall give a result we obtained for non-commutative $A$ when the action $\alpha:\mathbf{Z}\to \mathrm{Aut}\,A$ satisfies a certain almost-periodicity condition.

3.6. LEMMA. *Let $A$ be a unital separable $C^*$-algebra and let $\alpha:\mathbf{Z}\to\mathrm{Aut}\,A$ be a homomorphism, such that there exists a sequence of integers $0\leqslant n_1<n_2<\ldots$, so that*

$$\lim_{j\to\infty}\|\alpha(n_j)a-a\|=0$$

*for all $a\in A$. Assume, moreover, $A$ is quasidiagonal. Then $A\times_\alpha\mathbf{Z}$ is also quasidiagonal.*

*Proof.* Let $U$ be the bilateral shift on a Hilbert space $H_0$ with basis $\{e_n\}_{n\in\mathbf{Z}}$ and let $\rho:A\to L(H_1)$ be an almost-diagonal representation, so that the representation $\sigma$ of $A\times_\alpha\mathbf{Z}$ on $H_1\otimes H_0$ defined by

$$\sigma(u)=I_{H_1}\otimes U$$

$$\sigma(a)(h\otimes e_j)=(\rho(\alpha(-j)a)h)\otimes e_j$$

be a faithful representation of $A\times_\alpha\mathbf{Z}$. We shall prove that $\sigma$ is also almost diagonal. Let $(a_j)_{j=1}^\infty\subset A$ be a total sequence of selfadjoint elements. Replacing $n_1<n_2<\ldots$ by some subsequence, we may assume $\|\alpha(n_j)a_k-a_k\|<1/j$ for $1\leqslant k\leqslant j$. Let further $P_j\in L(H_1)$ be finite rank projections, such that $P_j\uparrow I$ and $\|[\rho(\alpha(k)a_i),P_j]\|<1/j$ for $1\leqslant i\leqslant j$, $|k|\leqslant n_j$.

Consider in $H_0$ the pairwise orthogonal vectors of norm 1

$$f_{k,j} = \cos \frac{k\pi}{2n_j} e_k + \sin \frac{k\pi}{2n_j} e_{k-n_j}$$

where $0 \leqslant k \leqslant n_j$. Remark that $\|Uf_{k,j} - f_{k+1,j}\| \leqslant \frac{\pi}{n_j}$ for $0 \leqslant k \leqslant n_j - 1$ and $f_{n_j,j} = f_{0,j}$. Since, moreover the vectors $Uf_{k,j} - f_{k+1,j}$ for $0 \leqslant k \leqslant n_j - 1$ are pairwise orthogonal, this easily gives $\| UQ_jU^* - Q_j\| \leqslant \frac{2\pi}{n_j}$ where $Q_j$ denotes the projection onto the linear span of $\{f_{0,j}, \ldots, f_{n_j-1,j}\}$.

Denoting by $R_j$ the projections $P_j \otimes Q_j$, it is clear that $R_j \nearrow I$ and $\lim\limits_{j\to\infty} \|[\sigma(u), R_j]\| = \lim\limits_{j\to\infty} \|\sigma(u)R_j\sigma(u^*) - R_j\| = 0$.

Thus, it will be sufficient to prove that $\lim\limits_{j\to\infty} \|[\sigma(a_k), R_j]\| = 0$ for $k = 1, 2, \ldots$. To this end, write $R_j = R_{0,j} + \ldots + R_{n_j-1,j}$ where $R_{i,j}$ is the orthogonal projection onto the subspace $(P_jH_1) \otimes f_{i,j}$. Since for $i \neq k$ $(0 \leqslant i, k \leqslant n_{j-1})$ we have

$$([\sigma(a), R_{i,j}])^*[\sigma(a), R_{k,j}] = [\sigma(a), R_{i,j}]([\sigma(a), R_{k,j}])^* = 0$$

for $a \in A$, we infer that

$$\|[\sigma(a), R_j]\| = \max_{1 \leqslant k \leqslant n_j - 1} \|[\sigma(a), R_{k,j}]\| .$$

Now, for $1 \leqslant i \leqslant j$, consider $a_i$ (which is selfadjoint) and remark that $\|[\sigma(a_i), R_{k,j}]\| \leqslant 2 \|(I - R_{k,j})\sigma(a_i)R_{k,j}\|$.

Consider $h \in P_jH_1$ with $\|h\| = 1$. We have

$$\|(I - R_{k,j})\sigma(a_i)R_{k,j}(h \otimes f_{k,j})\| \leqslant$$

$$\leqslant \|\sigma(a_i)(h \otimes f_{k,j}) - (P_j\rho(\alpha(-k)a_i)h) \otimes f_{k,j}\| \leqslant$$

$$\leqslant 2\|\rho(\alpha(-k)a_i)h - P_j\rho(\alpha(-k)a_ih\| +$$

$$+ \|\rho(\alpha(-k + n_j)a_i)h - \rho(\alpha(-k)a_i)h\| \leqslant$$

$$\leqslant 2\|[P_j, \rho(\alpha(-k)a_i)]\| +$$

$$+ \|\alpha(-k + n_j)a_i - \alpha(-k)a_i\| \leqslant$$

$$\leqslant 2/j + \|\alpha(n_j)a_i - a_i\| \leqslant 3/j .$$

Thus for $0 \leqslant k \leqslant n_j - 1$, $1 \leqslant i \leqslant j$ we have

$$\|(I - R_{k,j})\sigma(a_i)R_{k,j}\| \leqslant 3/j$$

and hence for $1 \leqslant i \leqslant j$

$$\|[\sigma(a_i), R_j]\| \leqslant 6/j .$$

This gives

$$\lim_{j\to\infty} \|[\sigma(a_i), R_j]\| = 0 .$$

Q.E.D.

3.7. PROPOSITION. *Let $A$ be a unital separable $C^*$-algebra with a composition series $(J_\rho)_{0\leqslant\rho\leqslant\beta}$ such that $\tilde{J}_{\rho+1}/J_\rho$ are quasidiagonal. Let further $\alpha : \mathbf{Z} \to \mathrm{Aut}A$ be a homomorphism such that $\alpha(n)J_\rho = J_\rho$ for all $n \in \mathbf{Z}$ and $0 \leqslant \rho \leqslant \beta$. Assume moreover that there is a sequence of integers $0 \leqslant n_1 < n_2 < \ldots$ such that*

$$\lim_{j\to\infty} \|\alpha(n_j)a - a\| = 0$$

*for all $a \in A$. Then $A \times_\alpha \mathbf{Z}$ is g.q.d.*

*Proof.* Remark that $(J_\rho \times_\alpha \mathbf{Z})_{0\leqslant\rho\leqslant\beta}$ is a composition series for $A \times_\alpha \mathbf{Z}$. The quotients $\widetilde{(J_{\rho+1} \times_\alpha \mathbf{Z})/(J_\rho \times_\alpha \mathbf{Z})}$ can be identified with subalgebras of $(\tilde{J}_{\rho+1}/J_\rho) \times_\alpha \mathbf{Z}$ which are quasidiagonal by Lemma 3.6. Q.E.D.

## APPENDIX

In this appendix we shall give a new proof for the result obtained in [26] concerning the range of the homomorphism $K_0(A_\theta) \to \mathbf{R}$ induced by the trace. In fact what we shall do will be to give a method for verifying that $K_0(A \times_\alpha \mathbf{Z})$ is generated by certain projections in case there are sufficiently many what we shall call Rieffel-projections, i.e. self-adjoint idempotents of the form:

$$p = u^* x_1^* + x_0 + x_1 u \qquad (x_0, x_1 \in A)$$

in $A \times_\alpha \mathbf{Z}$. This amounts to finding a way of computing the image via the homomorphism $\delta : K_0(A \times_\alpha \mathbf{Z}) \to K_1(A)$, appearing in the exact sequence, of the classes of Rieffel-projections.

Let us begin with a few remarks about elements $x_0, x_1 \in A$, $x_0 = x_0^*$ such that

$$p = u^* x_1^* + x_0 + x_1 u$$

is a projection. From $p^2 = p$, we immediately get the following relations:

(i) $x_0 = x_0^2 + \alpha(-1)(x^* x_1) + x_1 x_1^*$

(ii) $x_1 = x_0 x_1 + x_1(\alpha(1)x_0)$

(iii) $0 = (\alpha(-1)x_1)x_1$ .

Consider $\Delta$ the left support projection of $x_1$ in the enveloping von Neumann algebra of $A$. Then (iii) gives:

$$(\alpha(-1)x_1)\Delta = 0$$

and (i) gives:

$$x_1x_1^* = \Delta(x_0 - x_0^2) = (x_0 - x_0^2)\Delta .$$

Finally from (ii) it follows that

$$\Delta x_0 x_1 = x_0 x_1$$

which together with $x_0 = x_0^*$ implies that

$$[x_0, \Delta] = 0 .$$

Let us also record for further use that these relations imply:

$$(\alpha(-1)x_1)(x_0^n - x_0)\Delta = 0 .$$

PROPOSITION. *Let* $p = u^*x_1^* + x_0 + x_1u \in A \times_\alpha \mathbf{Z}$ *be a Rieffel-projection and* $\Delta$ *the left support projection of* $x_1$ *in the enveloping von Neumann algebra of* $A$. *Then the unitary* $\exp(2\pi i x_0 \Delta)$ *is in* $A$ *and*

$$\delta[p]_0 = [\exp(2\pi i x_0\Delta)]_1 .$$

*Proof.* The homomorphism $\delta$ is the composition of the connecting homomorphism

$$\partial : K_0(A \times_\alpha \mathbf{Z}) \longrightarrow K_1(A \otimes K)$$

associated with the Toeplitz extension for $A \times_\alpha \mathbf{Z}$ and the isomorphism $K_0(A \otimes K) \simeq$ $\simeq K_0(A)$. So, it will be sufficient to prove that

$$\exp(2\pi i x_0\Delta) \otimes e_{00} + 1 \otimes (1 - e_{00}) \in \widetilde{A \otimes K}$$

and

$$\partial[p]_0 = [\exp(2\pi i x_0\Delta) \otimes e_{00} + 1 \otimes (1 - e_{00})]_1 .$$

Now in general if

$$0 \longrightarrow I \longrightarrow T \longrightarrow T/I \longrightarrow 0$$

is an exact sequence of $C^*$-algebras, then $\partial[p]_0$, where $p \in T/I$ is a selfadjoint projection, is computed as follows. Take a selfadjoint lifting $a \in T$ for $p$. Then

$$\partial[p]_0 = [\exp(2\pi i a)]_1 .$$

Thus, both assertions of the proposition will follow once we shall show that taking the selfadjoint lifting

$$a = u^*x_1 \otimes S^* + x_0 \otimes I + x_1u \otimes S$$

for the Rieffel-projection $p$, we have

$$\exp(2\pi i a) = \exp(2\pi i x_0 \Delta) \otimes P + 1 \otimes (I - P) =$$

$$= \tilde{\psi}(\exp\ (2\pi i x_0 \Delta) \otimes e_{00} + 1 \otimes (1 - e_{00}))\ .$$

This will follow from computations.

Note that

$$a^2 = a - x_1 x_1^* \otimes P = a + (x_0^2 - x_0)\Delta \otimes P\ .$$

We shall prove by induction that

$$a^n = a + (x_0^n - x_0)\Delta \otimes P\ .$$

Suppose this is true for $n$. Then

$$a^{n+1} = a^2 + a((x_0^n - x_0)\Delta \otimes P) =$$

$$= a + (x_0^2 - x_0)\Delta \otimes P + x_0(x_0^n - x_0)\Delta \otimes P +$$

$$+ x_1 u(x_0^n - x_0)\Delta \otimes SP =$$

$$= a + (x_0^{n+1} - x_0)\Delta \otimes P + u(\alpha(-1)x_1)(x_0^n - x_0)\Delta \otimes SP\ .$$

Since the last term vanishes, the assertion is proved.

Summing up it follows that:

$$\exp(2\pi i a) =$$

$$= (\exp(2\pi i) - 1)(a - x_0\Delta \otimes P) +$$

$$+ \exp(2\pi i x_0 \Delta) \otimes P + 1 \otimes (I - P) =$$

$$= \exp(2\pi i x_0 \Delta) \otimes P + 1 \otimes (I - P)\ . \qquad \text{Q.E.D.}$$

Now for $A_\theta$, we have

$$K_0(C(\mathbf{T})) \longrightarrow K_0(A_\theta) \xrightarrow{\delta} K_1(C(\mathbf{T}))\ .$$

Since $K_0(C(\mathbf{T})) \simeq \mathbf{Z}$ is generated by $[1]$ and $K_1(C(\mathbf{T})) \simeq \mathbf{Z}$ is generated by $[v]$, where $v$ is the identic function on $\mathbf{T} \simeq \{z \in \mathbf{C} \mid |z| = 1\}$, it follows that $K_0(A_\theta)$ is generated by $[1]_0$ and $[p]_0$ in case there is a Rieffel-projection $p$ such that $\delta[p]_0 = [v]_1$. The existence of such a projection is immediate by applying the Proposition we have proved, to the projections constructed by Rieffel [28]. Having found generators for $K_0(A_\theta)$ it is easily verified that the range of the homomorphism $K_0(A_\theta) \to \mathbf{R}$ induced by the trace is $\mathbf{Z} + \theta\mathbf{Z}$ .

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*Added in Proof.*

1. Among the predecessors of the Toeplitz extension we would like to mention Coburn's work on the $C^*$-algebra of an isometry, O'Donovan's work on weighted shifts (*Trans. Amer. Math. Soc.*, **208** (1975), 1–25) and the work of P. Muhly, Mc. Asey and K. S. Saito on non-selfadjoint crossed products.

2. After this paper has been circulated as a preprint (without the Appendix) several advances have been made:

a) J. Cuntz in "$K$-theory for certain $C^*$-algebras. II" has obtained a generalization of our Theorem 2.4 based on connecting our work to his previous work [9]. He also obtains an extremely nice and much simpler proof for the range of $K_0(A_\theta) \xrightarrow{\mathrm{Tr}} \mathbf{R}$.

b) Two preprints of A. Connes "An analogue of the Thom isomorphisms for cross products of a $C^*$-algebra by an action of $\mathbf{R}$" and "$C^*$-algèbres et géometrie differentielle" contain several fundamental results (Thom-isomorphism and index-theorem for Lie-group actions on $C^*$-algebras) which among many other things also provide a new proof for our Theorem 2.4 and give a non-commutative differential-geometric insight into the structure of $A_\theta$.