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A POLAR AREA INEQUALITY FOR HYPONORMAL SPECTRA

C. R. PUTNAM

1. INTRODUCTION

Let $\mathfrak{H}$ be a separable Hilbert space and let T be a bounded operator with the rectangular representation

$$T = A + iB, \text{ where } A = \int t dE_t. \tag{1.1}$$

Then T is said to be hyponormal if

$$T^*T - TT^* = D \geqq 0, \text{ equivalently, } AB - BA = -\frac{1}{2} iD. \tag{1.2}$$

In this case, $\sigma(A)$ and $\sigma(B)$ are the real projections of $\sigma(T)$ onto the real and imaginary axes; see [7], p. 46. If Δ is any open interval of the real line then $E(\Delta)TE(\Delta)$ is hyponormal on $E(\Delta)\mathfrak{H}$ and hence its spectrum (as an operator on $E(\Delta)\mathfrak{H}$) lies in the closure of the strip $\{z: \operatorname{Re}(z) \in \Delta\}$.

It is known that

$$\sigma(E(\Delta)TE(\Delta)) \subset \sigma(T) \tag{1.3}$$

and that

$$\pi \|D\| \leqq \operatorname{meas}_2 \sigma(T). \tag{1.4}$$

These relations were proved in [8] and, in the special case when D is compact, in [1]. In [9] it was shown that

$$\sigma(E(\Delta)TE(\Delta)) \cap \{z: \operatorname{Re}(z) \in \Delta\} = \sigma(T) \cap \{z: \operatorname{Re}(z) \in \Delta\}. \tag{1.5}$$

It follows from (1.5), or even from (1.3), that if β is any Borel set of the real line and if $F(t)$ is the Lebesgue linear measure of the vertical cross section $\sigma(T) \cap$

$\cap \{z: \operatorname{Re}(z) = t\}$ of $\sigma(T)$ then

$$\pi \| E(\beta) D E(\beta) \| \leqq \int_{\beta} F(t) \, dt. \tag{1.6}$$

In particular, (1.4) follows by choosing $\beta = (-\infty, \infty)$.

As was shown in [9], p. 701, relation (1.3) can be generalized to

$$\sigma(E(\beta) T E(\beta)) \subset \sigma(T). \tag{1.7}$$

Several results were obtained in [12] relating the spectra of a hyponormal operator T and its polar factors in case T has a polar factorization

$$T = UP, \quad U = \text{unitary} = \int e^{it} \, dG_t \left(\int = \int_0^{2\pi - 0} \right), \quad P \geqq 0. \tag{1.8}$$

In this paper there will be proved certain polar analogues of (1.3) and (1.6). For later use, observe that, by (1.2),

$$T^*T - TT^* = P^2 - UP^2U^* = D \geqq 0. \tag{1.9}$$

It may be noted that if T is any bounded operator on $\mathfrak{H}$ and if $0 \notin \sigma(T)$ then T has a factorization (1.8); see Wintner [15]. A generalization was given by von Neumann [6], p. 307. Further, it was shown by Hartman [4], using von Neumann's result, that an arbitrary bounded T has a representation (1.8) if and only if the null spaces of T and T^* have the same dimension. Also, if this common dimension is zero, so that 0 is not in the point spectra of T and T^*, then the representation (1.8) is unique.

A hyponormal operator T is said to be completely hyponormal if it has no nontrivial reducing subspace on which it is normal. If T is completely hyponormal then $0 \notin \sigma(T)$, and so (1.8) holds if and only if $0 \notin \sigma_p(T^*)$.

THEOREM 1. *Let T be completely hyponormal with a polar factorization* (1.8) *For each t, $0 \leqslant t < 2\pi$, define*

$$\alpha_t = \sigma(T) \cap \{z: \arg z = t\} \textit{ and } M(t) = \int_{\alpha_t} r \, dr. \tag{1.10}$$

Then for any Borel set β of $[0, 2\pi)$,

$$\pi \| G(\beta) D G(\beta) \| \leqq \int_{\beta} M(t) \, dt. \tag{1.11}$$

The proof of Theorem 1 will depend upon some results in [8] and, in addition, will be similar to the argument given there. A number of preliminary lemmas will be given in Section 2 and the proof of Theorem 1 will be given in Section 3. In Section 4 there will be noted some consequences of the inequality (1.11).

SECTION 2

LEMMA 1. *Let T be hyponormal with the rectangular representation* (1.1) *and let Δ be any open interval of the real line. If $\mathfrak{K}$ is a subspace of $E(\Delta)\mathfrak{H}$ reducing $E(\Delta)TE(\Delta)$ for which $E(\Delta)TE(\Delta)|\mathfrak{K}$ is normal, then $\mathfrak{K}$ reduces T and $T|\mathfrak{K}$ is normal.*

Proof. This is Lemma 5 of [8], p. 326.

LEMMA 2. *Let T be hyponormal and nonsingular, so that $0 \notin \sigma(T)$. Let T have the polar factorization* (1.8) *and for any open interval, Δ, of the real line let*

$$T(\Delta) = (G(\Delta)U)\,[G(\Delta)P^2\,G(\Delta)]^{\frac{1}{2}}. \tag{2.1}$$

Then $T(\Delta)$ is hyponormal and (if $G(\Delta) \neq 0$) nonsingular on $G(\Delta)\mathfrak{H}$. Further, if $T(\Delta)$ is reduced by $\mathfrak{K} \subset G(\Delta)\mathfrak{K}$ and if $T(\Delta)|\mathfrak{K}$ is normal then $\mathfrak{K}$ reduces T and $T|\mathfrak{K}$ is normal.

Proof. See Lemma 8 of [8], pp. 326-327.

LEMMA 3. *If T is completely hyponormal and if N is any open disk of the plane, then* $\text{meas}_2(\sigma(T) \cap N) > 0$ *whenever* $\sigma(T) \cap N \neq \emptyset$.

Proof. See Theorem 4 of [8], p. 324.

If U is unitary with the spectral resolution of (1.8) let $\mathfrak{H}_a(U)$ denote the absolutely continuous part of U, that is, the subspace of vectors x for which $\|G_t x\|^2$ is an absolutely continuous function of t.

LEMMA 4. *If T is completely hyponormal with the polar factorization* (1.8) *then $\mathfrak{H}_a(U) = \mathfrak{H}$.*

Proof. The result can be obtained as a consequence of Theorem 2.3.1 of [7], p. 21. If $J = P^2$ then, in view of (1.9),

$$J - UJU^* = D \geqq 0. \tag{2.2}$$

In case $\text{meas}\,\sigma(U) < 2\pi$ then, since T is completely hyponormal, Lemma 4 follows directly from the above mentioned theorem in [7]. In case $\text{meas}\,\sigma(U) = 2\pi$, the result can be obtained as follows. Since $\mathfrak{H}$ is separable, there exists some real t for which $e^{it} \notin \sigma_p(U)$. By replacing U by $e^{-it}\,U$ it can be supposed that $1 \notin \sigma_p(U)$. For each $n = 1,2, \ldots$, let $\alpha_n = (1/n,\, 2\pi - 1/n)$ and put $U_n = UG(\alpha_n)$, $J_n = G(\alpha_n)JG(\alpha_n)$ and $D_n = G(\alpha_n)DG(\alpha_n)$, all regarded as operators on $\mathfrak{H}_n = E(\alpha_n)\mathfrak{H} \subset \mathfrak{H}$. A multiplication by $G(\alpha_n)$ on the right and left in (2.2) yields

$$J_n - U_n J_n U_n^* = D_n \geqq 0, \tag{2.3}$$

and hence, by the theorem of [7] cited above, $(\mathfrak{H}_n)_a(U_n)$ contains the least subspace, $\mathfrak{M}_n$, of $\mathfrak{H}_n$ reducing both J_n and U_n and containing the range of D_n. Since $1 \notin \sigma_p(U)$ it is clear that if x belongs to the least subspace of $\mathfrak{H}$ reducing J and U and containing D then there exist $x_n \in \mathfrak{M}_n$ satisfying $x_n \to x$ as $n \to \infty$. Consequently, if Z is any Borel subset of the real line of measure 0, then $0 = G_n(Z)x_n = G(Z)x_n$ and hence $G(Z)x = 0$, so that $\mathfrak{H}_a(U) = \mathfrak{H}$.

REMARKS. It is clear that the hypothesis meas$\sigma(U) < 2\pi$ occurring in the statement of Theorem 2.3.1 of [7], p. 21, is not needed as far as concerns the assertion made there about $\mathfrak{H}_a(U)$. It is needed, though, in order to ensure the claim concerning $\mathfrak{H}_a(J)$. Further, it is seen that the conclusion of Theorem 2.3.2 of [7], p. 22, can be strengthened by noting that, in that theorem, $\mathfrak{H}_a(U)$ contains the least subspace of $\mathfrak{H}$ reducing U and J and containing the range of D. This strengthened version is precisely the statement of Lemma 4 above. Similarly, the assertion of Theorem 6 in [12], p. 424, concerning the absolute continuity of U (but *not* that of P), holds without the "wedge hypothesis" occurring there. See also the remarks in Section 7 of Kato [5].

LEMMA 5. *Let T be completely hyponormal and nonsingular with the polar factorization* (1.8) *and, for any open interval $\Delta = (a, b)$, $0 \leqq a < b \leqq 2\pi$, define $T(\Delta)$ by* (2.1). *Suppose that* $0 \neq z = |z|\, e^{i\theta}$, *where $a < \theta < b$, except that, in case $a = 0$, then $0 \leqq \theta < b$ is allowed, and if $b = 2\pi$, then $a < \theta \leqq 2\pi$ is allowed. Then*

$$\text{(2.4)} \qquad \textit{if } z \in \partial(\sigma(T)) \quad \textit{then } z \in \sigma(T(\Delta))$$

and

$$\text{(2.5)} \qquad \textit{if } z \in \partial(\sigma(T(\Delta))) \quad \textit{then } z \in \sigma(T).$$

Proof. Actually, only relation (2.5) will be needed later, but it is not essentially harder to prove both (2.4) and (2.5). In order to prove (2.4), suppose that $z \in \partial(\sigma(T))$. Then, by (1.2), $(T - z)^* (T - z) - (T - z)(T - z)^* = D \geqq 0$ and so there exist unit vectors x_n in $\mathfrak{H}$ satisfying both $(T - z)\, x_n \to 0$ and $(T^* - \bar{z})x_n \to 0$ (strongly). Since $T^*T = P^2$ it follows that

$$\text{(2.6)} \qquad (P^2 - |z|^2)x_n \to 0, \text{ and hence } (P - |z|)x_n \to 0.$$

But $(T - z)x_n = (U(T^*T)^{\frac{1}{2}} - z)x_n = (UP - |z|e^{i\theta})x_n$, so that, since $z \neq 0$,

$$\text{(2.7)} \qquad (U - e^{i\theta})x_n \to 0.$$

Since $\mathfrak{H}_a(U) = \mathfrak{H}$, by Lemma 4, then, in particular, $G_{t-0} = G_t = G_{t+0}, -\infty < t < \infty$. In view of the assumption on θ, it is clear that (2.7) implies that $x_n - G(\Delta)x_n \to 0$. Thus, by (2.6) and (2.7),

$$G(\Delta)(P^2 - |z|^2)G(\Delta)x_n \to 0,$$

hence

$$[(G(\Delta)P^2G(\Delta))^{\frac{1}{2}} - |z|]\, x_n \to 0,$$

and so by (2.1) and (2.7), $(T(\Delta) - z)x_n \to 0$, hence (2.4).

Next, suppose that $z \in \partial(\sigma(T(\Delta)))$. Then, corresponding to (2.6) and (2.7) for T, one now has for $T(\Delta)$,

$$(2.8) \qquad (G(\Delta)P^2G(\Delta) - |z|^2)x_n \to 0 \quad \text{and} \quad (U - e^{i\theta})x_n \to 0,$$

for unit vector $x_n = G(\Delta)x_n$ in $G(\Delta)\mathfrak{H}$. In view of (1.9),

$$(2.9) \qquad T(\Delta)^*T(\Delta) - T(\Delta)T(\Delta)^* = G(\Delta)\,(P^2 - UP^2U^*)G(\Delta) =$$

$$= G(\Delta)DG(\Delta) \geqq 0.$$

On applying (2.9) to x_n, then taking an inner product with x_n and noting that $D \geqq 0$, one sees that $Dx_n \to 0$. It follows from (1.9) that

$$(P^2 - |z|^2)x_n - U(P^2 - |z|^2)U^*x_n \to 0,$$

and hence, by the second relation of (2.8), that

$$(U - e^{i\theta})\,(P^2 - |z|^2)x_n \to 0.$$

Consequently, as above,

$$(2.10) \qquad (P^2 - |z|^2)x_n - G(\Delta)(P^2 - |z|^2)x_n \to 0.$$

Relation (2.10) and the first relation of (2.8) imply that $(P^2 - |z|^2)x_n \to 0$, hence $(P - |z|)x_n \to 0$. An application of the second relation of (2.8) then shows that $(T - z)x_n = (UP - z)x_n \to 0$, hence (2.5).

LEMMA 6. *Let* U_n $(n = 1,2, \dots)$ *and* U *be unitary operators on a Hilbert space* $\mathfrak{H}$ *with the spectral resolutions* $U_n = \int e^{it}\, dG_{nt}$ *and* $U = \int e^{it} dG_t$ $\left(\int = \int_0^{2\pi - 0}\right)$ *and suppose that* $U_n \to U$ *strongly (hence also* $U_n^* \to U^*$*) as* $n \to \infty$. *If* $s \notin \sigma_p(U)$ *then* $G_{ns} \to G_s$ *(strongly) as* $n \to \infty$.

Proof. This is a result of Rellich; see Lemma 6 or [8], p. 326, and the reference there to Sz.-Nagy [14], p. 56. It is usually stated for self-adjoint rather than unitary operators but the situations are similar. In fact, an analogous result holds even for normal operators; see Dunford and Schwartz [2], p. 923.

3. PROOF OF THEOREM 1

Let Δ be any open interval of the real line. It follows from (2.9) and an application of (1.4) with T and $\mathfrak{H}$ replaced by $T(\Delta)$ and $G(\Delta)\mathfrak{H}$ that

$$\pi \, \| G(\Delta)D^{\frac{1}{2}} \|^2 \leqq \operatorname{meas}_2 \sigma(T(\Delta)). \tag{3.1}$$

Since T is completely hyponormal it follows from Lemma 4 that

$$\mathfrak{H}_a(U) = \mathfrak{H}. \tag{3.2}$$

Also, since

$$\|G(\beta)DG(\beta)\| = \|G(\beta)D^{\frac{1}{2}}\|^2,$$

it is clear that in order to establish (1.11) it is sufficient to show that

$$\sigma(T(\Delta)) \subset \sigma(T) \cap w_{\Delta}^{-}, \tag{3.3}$$

where w_{Δ}^{-} denotes the closure of the wedge

$$w_{\Delta} = \{z : z = re^{it}, r > 0, a < t < b\}, \; \Delta = (a, b) \subset [0, 2\pi]. \tag{3.4}$$

(For a similar argument but involving the rectangular representation, see [8], p. 324.) It is seen from Theorem 3 of [12], p. 422, that $\sigma(T(\Delta)) \subset w_{\Delta}^{-}$, so that relation (3.3) becomes simply

$$\sigma(T(\Delta)) \subset \sigma(T). \tag{3.5}$$

Compare relation (1.9) of [8], p. 324.)

Thus, in order to complete the proof, it is enough to show that

$$\text{if } q = c + ie \notin \sigma(T) \text{ then } q \notin \sigma(T(\Delta)). \tag{3.6}$$

In order to prove (3.6), first suppose that

$$0 \notin \sigma(T). \tag{3.7}$$

Let T have the rectangular representation (1.1), and define for each $n = 1, 2, \ldots,$ the hyponormal operator T_n on $E(\delta_n)\mathfrak{H}$ by

$$T_n = E(\delta_n)\, TE(\delta_n), \text{ where } \delta_n = (-\infty, c - 1/n) \cup (c + 1/n, \infty). \tag{3.8}$$

Then, by (3.7) and (1.7), $0 \notin \sigma(T_n)$ and, in particular, T_n has a polar factorization

$$T_n = U_n P_n, \; U_n = \int e^{it} \, dG_{nt} \; \left(\int = \int_0^{2\pi - 0} \right). \tag{3.9}$$

Define the hyponormal operator S_n on $G_n((-\infty, b))\mathfrak{H}$ (cf. (2.1)) by

$$S_n = G_n((-\infty, b))U_n\,[G_n((-\infty, b))P_n^2G_n((-\infty, b))]^{\frac{1}{2}}. \tag{3.10}$$

Since T is completely hyponormal on $\mathfrak{H}$ it follows from Lemma 1 that T_n is completely hyponormal on $E(\delta_n)\mathfrak{H}$. Hence, by Lemma 4, $\mathfrak{H}_a(U_n) = E(\delta_n)\mathfrak{H}$ and $G_n((-\infty, b)) = G_n((0, b))$. Further, by Theorem 3 of [12], p. 422,

$$\sigma(S_n) \subset \{z\colon z = re^{it},\ r \geqq 0,\ 0 \leqq t \leqq b\}. \tag{3.11}$$

Suppose that (cf. (3.6))

$$\operatorname{dist}(q, \sigma(T)) \geqq d > 0. \tag{3.12}$$

In view of (1.7), $\sigma(T_n) \subset \sigma(T)$ and hence

$$\operatorname{dist}(q, \sigma(T_n)) \geqq d. \tag{3.13}$$

It will next be shown that

$$\sigma(S_n) \cap \gamma_n = \emptyset, \tag{3.14}$$

where $\gamma_n = \{z\colon c - 1/n < \operatorname{Re}(z) < c + 1/n\} \cup \{z\colon |z - q| < d\}$.

If (3.14) fails to hold there exists some $z_0 \in \partial(\sigma)S_n)) \cap \gamma_n$. In case $z_0 \neq 0$ and $z_0 = |z_0|e^{i\theta}$ with $0 \leqq \theta < b$ then by (2.5) of Lemma 5, $z_0 \in \sigma(T_n) \cap \gamma_n$, a contradiction, in view of (3.13) and the fact (cf. the beginning of Section 1) that $\sigma(T_n)$ lies outside the strip $\{z\colon c - 1/n < \operatorname{Re}(z) < c + 1/n\}$. It is seen from (3.11) that the only remaining possibility is that $\partial(\sigma(S_n)) \cap \gamma_n \subset L$, where $L = \gamma_n \cap \{z\colon z = re^{ib},\ r \geqq 0\}$, and hence also $\sigma(S_n) \cap \gamma_n \subset L$. Since L has zero planar measure, it follows from Lemma 3 that S_n on $G_n((-\infty, b))\,\mathfrak{H}$ has a normal reducing subspace, hence, by Lemma 2, so does T_n on $E(\delta_n)\mathfrak{H}$, and hence, by Lemma 1, so does T on $\mathfrak{H}$, a contradiction. This proves (3.14).

In (3.10) let $V_n = G_n((-\infty, b))\,U_n$ and $Q_n = [\ldots]^{\frac{1}{2}}$, so that $S_n = V_nQ_n$, and then define the hyponormal operator R_n on $G_n((a, \infty))\,G_n((-\infty, b))\mathfrak{H} = G_n(\Delta)\mathfrak{H}$ by

$$R_n = G_n((a, \infty))\,V_n[G_n((a, \infty))\,Q_n^2G_n((a, \infty))]^{\frac{1}{2}} = G_n(\Delta)\,U_n[G_n(\Delta)\,P_n^2G_n(\Delta)]^{\frac{1}{2}}.$$

Clearly, R_n is related to S_n as S_n is to T_n. An argument similar to that of the preceding paragraph then shows that

$$\sigma(R_n) \cap \gamma_n = \emptyset. \tag{3.15}$$

Since $T = A + iB$ is completely hyponormal, then $\mathfrak{H}_a(A) = \mathfrak{H}$ (see [7], p. 42) and, by (3.8), $T_n \to T$ and $T_n^* \to T^*$ (strongly) as $n \to \infty$. Consequently, $P_n^2 = T_n^* T_n \to T^*T = P^2$ and so $P_n \to P$, and hence, by (3.7), also $U_n \to U$ (hence $U_n^* \to U^*$). By (3.2) and Lemma 6, $G_n(\Delta) \to G(\Delta)$ (strongly), and so, by (2.1), also $R_n \to T(\Delta)$ (strongly).

In view of (3.15),

$$\|(R_n - q)\, G_n(\Delta)x\| \geqq \|(R_n - q)^* G_n(\Delta)x\| \geqq d\|G_n(\Delta)x\|,$$

and hence

$$\|(T(\Delta) - q)G(\Delta)\, x\| \geqq \|(T(\Delta) - q)^* G(\Delta)\, x\| \geqq d\|G(\Delta)\, x\|. \tag{3.16}$$

Thus, (3.6) is proved under the hypothesis (3.7).

Next, let the special restriction (3.7) be omitted and for $h > 0$ define $T_h = U(P^2 + h)^{\frac{1}{2}}$. Clearly, T_h is nonsingular. Since

$$T_h^* T_h - T_h T_h^* = P^2 + h - U(P^2 + h)\, U^* = P^2 - UP^2U^* = D,$$

then T_h is also hyponormal. Further, since the least space reducing T_h (that is, the least space reducing U and $P^2 + h$ or, equivalently, reducing U and P) and containing the range of D is precisely the least space reducing T and containing the range of D, then T_h is completely hyponormal. In order to prove (3.5), then, as above, it is sufficient to prove (3.6). So, as before, suppose (3.12). Since $\|T_h - T\| \to 0$ as $h \to 0_+$, standard upper semicontinuity properties of spectra (cf. [3], p. 53) imply that there exists a function d_h for $h > 0$ satisfying $\operatorname{dist}(q, \sigma(T_h)) \geqq d_h$ and $d_h \to d$ as $h \to 0_+$. If

$$T_h(\Delta) = (G(\Delta)\, U)\, [G(\Delta)\, (P^2 + h)\; G(\Delta)]^{\frac{1}{2}}$$

(cf. (2.1)) then, corresponding to an earlier result,

$$\|(T_h(\Delta) - q)\; G(\Delta)x\| \geqq \|(T_h(\Delta) - q)^* G(\Delta)\, x\| \geqq d_h\|G(\Delta)\, x\|.$$

On letting $h \to 0_+$, one obtains (3.16) and hence also (3.6). This completes the proof of Theorem 1.

4. REMARKS

If T is completely hyponormal on $\mathfrak{H}$ with the rectangular representation (1.1) and if $n(t)$ denotes the spectral multiplicity function of $A = \operatorname{Re}(T) = \int t\, dE_t$, then for any Borel set β of the real line one has

$$\pi \operatorname{tr}[E(\beta)\, DE(\beta)] \leqq \int_\beta n(t)\; F(t)\, dt, \tag{4.1}$$

an inequality corresponding to (1.6). Relation (4.1) was proved in [10] and [11] using (1.6) together with a result of Kato [5]. This latter result was needed in order to guarantee that the self-commutator of a hyponormal operator be of trace class whenever its real part has bounded spectral multiplicity. In [13] it was shown that (4.1) and, in fact, a certain generalization follows from the inequality (1.6) without using the aforementioned Kato result. More precisely, if $\mathfrak{H}_1$ denotes the least subspace of $\mathfrak{H}$ invariant under A and containing the range of D of (1.2) and if $n_1(t)$ denotes the spectral multiplicity function of $A|\mathfrak{H}_1$ then $n_1(t) \leqq n(t)$ and

$$\pi \operatorname{tr}[E(\beta)\, DE(\beta)] \leqq \int_\beta n_1(t)\, F(t)\, \mathrm{d}t. \tag{4.2}$$

Suppose next that T has a polar factorization (1.8). Let $m(t)$ denote the spectral multiplicity function of the corresponding unitary operator U and let $m_2(t)$ denote the spectral multiplicity function of $U|\mathfrak{H}_2$, where $\mathfrak{H}_2$ is the least subspace of $\mathfrak{H}$ reducing U and containing the range of D of (1.2).

THEOREM 2. *If T is completely hyponormal with the polar factorization* (1.8) *then $m_2(t) \leqq m(t)$ and for any Borel set β of* $[0, 2\pi)$,

$$\pi \operatorname{tr}[G(\beta)\, DG(\beta)] \leqq \int_\beta m_2(t)\, M(t)\, \mathrm{d}t, \tag{4.3}$$

where $M(t)$ is defined in (1.10).

The proof will be omitted since it is virtually identical with that of (4.2) in [13] except that (1.11) now assumes the role formerly played by (1.6).

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C. R. PUTNAM
Mathematics Department
Purdue University
West Lafayette, IN. 47907
U.S.A.

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