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SOME ASSORTED INEQUALITIES FOR POSITIVE LINEAR MAPS ON C^*-ALGEBRAS

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§ 1. INTRODUCTION

The general structure of positive linear maps on C^*-algebras is highly combinatorial (see Appendix B); yet its subtlety should surpass its complexity. In this paper we try to exhibit some special features of positive linear maps, by means of 2×2 operator-matrix manipulations. The main results here, concerning some assorted inequalities of the Schwarz type, appear non-trivial even if all C^*-algebras involved are limited to the finite-dimensional matrix algebras. It is noteworthy that some preliminary results presented here are well known, but we include the proofs for completeness; this article should thus be considered as a partly expository paper. The readers are also referred to [1; 4; 16] for some recent research on the subject.

Throughout the paper, C^*-algebras possess a unit I and are written in German type $\mathfrak{A}$, $\mathfrak{B}$. $\mathscr{B}(\mathscr{H})$ denotes the algebra of all bounded operators on the Hilbert space $\mathscr{H}$. $\mathscr{M}_n$ denotes the algebra of all $n \times n$ complex matrices. $\mathscr{M}_n(\mathfrak{A})$ is the algebra of $n \times n$ matrices over $\mathfrak{A}$.

For an operator $T \in \mathscr{B}(\mathscr{H})$, we denote by $C^*(T)$ for the unital C^*-algebra generated by T. T is said to be *invertible* if T has a bounded inverse T^{-1}. T is said to be *positive* (in notation $T \geqslant 0$) if T is hermitian and the spectrum of T is included in $[0, \infty)$ — in the matrix case, T is also said to be *positive semi-definite*; the unique positive square root of T is denoted by $T^{1/2}$. A linear map $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ between two C^*-algebras is said to be *unital* if $\Phi(I) = I$; Φ is said to be *positive* if $\Phi(A)$ is positive for every positive A in $\mathfrak{A}$; Φ is said to be *2-positive* if

$$\begin{bmatrix} \Phi(A_{11}) & \Phi(A_{12}) \\ \Phi(A_{21}) & \Phi(A_{22}) \end{bmatrix} \text{ is positive for every positive } \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

in $M_2(\mathfrak{A})$.

We present in § 2 some basic 2×2 operator-matrix manipulations for preliminaries.

In § 3, we are concerned with some Schwarz-type inequalities and the cognate conjectures for every unital positive linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$. The main theorem (Theorem 3.2) says that if $T \geqslant A^*A$ and $TA = AT$ then $\Phi(T) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(T) \geqslant \Phi(A)\Phi(A^*)$. This covers several known results in literature. In particular letting $T = I$, we get Russo-Dye's Theorem [12]: every unital positive linear map is contractive. Another special case is Kadison's inequality [10] with $A = A^*$ and $T = A^2$.

It has been indicated by Ando [1, Section 2] that each positive linear map possesses a sort of 2-positive effect. We will extend Ando's argument to the utmost in § 4. Along with other results, the following formulation (Theorem 4.6) may reveal the strongest effect that a general positive linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ can ever take: If $\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \in \mathscr{M}_2(\mathfrak{A})$ is positive, and if the four operators $\{T, S, S^*, R\}$ together are linearly dependent (in particular if $S = S^*$), then $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} \in \mathscr{M}_2(\mathfrak{B})$ is positive, too.

Finally, we supply two counter-examples to demonstrate the fallacy of some plausible conjectures. In Appendix A, we show that a unital linear map Φ satisfying $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ for all A need not be 2-positive. In Appendix B, we deal with a positive linear map that is not "decomposable".

§ 2. 2 × 2 OPERATOR-MATRIX MANIPULATIONS

We begin with a well-known fact (see e.g., [13, p. 385] for a more general result).

LEMMA 2.1. *Let $R, S, T \in \mathscr{B}(\mathscr{H})$ with T being positive and invertible. Then* $\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \geqslant 0$ *if and only if* $R \geqslant S^*T^{-1}S$.

Proof. Write $P = \begin{bmatrix} T & S \\ S^* & R \end{bmatrix}$, $Q = \begin{bmatrix} I & 0 \\ 0 & R - S^*T^{-1}S \end{bmatrix}$ and $X = \begin{bmatrix} T^{1/2} & T^{-1/2}S \\ 0 & I \end{bmatrix}$.

A direct computation yields $P = X^*QX$. Note further that X is invertible $\left(\text{namely, } X^{-1} = \begin{bmatrix} T^{-1/2} & -T^{-1}S \\ 0 & I \end{bmatrix}\right)$; therefore P and Q are *congruent* to each other and

$$P \geqslant 0 \Leftrightarrow Q \geqslant 0.$$

It is clear $Q \geqslant 0 \Leftrightarrow R \geqslant S^*T^{-1}S$. Thus the equivalence $P \geqslant 0 \Leftrightarrow R \geqslant S^*T^{-1}S$ follows. ▨

Next, we present a natural construction of certain 2×2 normal operator-matrices.

LEMMA 2.2. *Let* $T, A \in \mathscr{B}(\mathscr{H})$ *with* $T \geqslant A^*A$ *and* $TA = AT$. *Then* $T \geqslant AA^*$, *and* $\begin{bmatrix} A & (T - AA^*)^{1/2} \\ (T - A^*A)^{1/2} & -A^* \end{bmatrix}$ *is a normal operator in* $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$.

Proof. First of all, assume that T is invertible. Letting $X = AT^{-1/2} = T^{-1/2}A$, we derive that

$$T \geqslant A^*A \Rightarrow I \geqslant T^{-1/2}A^*AT^{-1/2} = X^*X$$

$$\Rightarrow X \text{ is a contraction}$$

$$\Rightarrow 1 \geqslant XX^* = T^{-1/2}AA^*T^{-1/2}$$

$$\Rightarrow T \geqslant AA^*.$$

For a general positive T, we have that $T + I/n$ is invertible for all positive integer n. Hence, the argument above shows that $T + I/n \geqslant AA^*$ for all n. Let n approach to infinity; we get $T \geqslant AA^*$ as desired.

Moreover, write $N = \begin{bmatrix} A & (T - AA^*)^{-1/2} \\ (T - A^*A)^{1/2} & -A^* \end{bmatrix}$. A direct 2×2 matrix computation shows that the condition $N^*N = NN^*$ is a consequence of the equality

$$A(T - A^*A)^{1/2} = (T - AA^*)^{1/2}A. \tag{*}$$

[Now an operator-theory specialist may recognize the verity of (*) immediately by means of functional calculus (cf. [9, p. 327]). However, we offer below an elementary proof which arises from communication with Chandler Davis.] Put $S = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$, $R = \begin{bmatrix} T - A^*A & 0 \\ 0 & T - AA^* \end{bmatrix}$. The given hypotheses on T and A yield $SR = RS$ and $R \geqslant 0$; thus $SR^{1/2} = R^{1/2}S$, i.e.,

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}\begin{bmatrix} (T - A^*A)^{1/2} & 0 \\ 0 & (T - AA^*)^{1/2} \end{bmatrix} = \begin{bmatrix} (T - A^*A)^{1/2} & 0 \\ 0 & (T - AA^*)^{1/2} \end{bmatrix}\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} 0 & A^*(T - AA^*)^{1/2} \\ A(T - A^*A)^{1/2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & (T - A^*A)^{1/2}A^* \\ (T - AA^*)^{1/2}A & 0 \end{bmatrix}.$$

Therefore the equality(*) follows; this implies $N^*N = NN^*$ (N is normal) as asserted. ▧

§ 3. SCHWARZ-TYPE INEQUALITIES

The following is a well-known Schwarz-type inequality covering Kadison's inequality [10, Theorem 1]. For completeness, the short proof is provided.

LEMMA 3.1. *Suppose that Φ: $\mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map between two C^*-algebras. Then $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(A^*A) \geqslant \Phi(A)\Phi(A^*)$ for every normal $A \in \mathfrak{A}$.*

Proof. Let $\mathscr{H}$ be the underlying Hilbert space of $\mathfrak{B}$ and let A be a fixed normal operator in $\mathfrak{A}$. Then $C^*(A)$ is a commutative C^*-algebra. By Neumark's Theorem (see [14, the Introduction Section or the combination of Theorems 1 and 4]), Φ restricted to $C^*(A)$ admits a decomposition $\Phi(X) = V^*\pi(X)V$ for all $X \in C^*(A)$, where π is a representation of $C^*(A)$ on a Hilbert space $\mathscr{K}$, and V is an isometry from $\mathscr{H}$ into $\mathscr{K}$. Hence

$$\Phi(A^*)\Phi(A) = V^*\pi(A^*)V\cdot V^*\pi(A)V \leqslant V^*\pi(A^*)\pi(A)V =$$

$$= V^*\pi(A^*A)V = \Phi(A^*A).$$

Replace A by A^*; the other inequality

$$\Phi(A)\Phi(A^*) \leqslant \Phi(AA^*) = \Phi(A^*A)$$

follows too. ▨

The theorem below is general enough to cover several known results about positive linear maps.

THEOREM 3.2. *Suppose that Φ: $\mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map between two C^*-algebras. If T, A are operators in $\mathfrak{A}$ with $T \geqslant A^*A$ and $TA = AT$, then $\Phi(T) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(T) \geqslant \Phi(A)\Phi(A^*)$.*

Proof. It follows from Lemma 2.2 that $T \geqslant AA^*$ and

$$N = \begin{bmatrix} A & (T - AA^*)^{1/2} \\ (T - A^*A)^{1/2} & -A^* \end{bmatrix}$$

is a normal operator in $\mathscr{M}_2(\mathfrak{A})$. Let Θ: $\mathscr{M}_2(\mathfrak{A}) \to \mathfrak{A}$ be the compression map $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \mapsto A_{11}$; then $\Phi \circ \Theta: \mathscr{M}_2(\mathfrak{A}) \to \mathfrak{B}$ is a unital positive linear map. By Lemma 3.1,

$$\Phi \circ \Theta(N^*N) \geqslant \Phi \circ \Theta(N^*) \cdot \Phi \circ \Theta(N), \ \Phi \circ \Theta(N^*N) \geqslant \Phi \circ \Theta(N) \cdot \Phi \circ \Theta(N^*).$$

Since $N^*N = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$, we have $\Theta(N^*N) = T$, $\Theta(N) = A$; thus the desired inequalities follow. ▨

REMARK 3.3. In case $T = I$, the theorem above reduces to a well-known fact discovered by Russo and Dye [12, Corollary 1]: Every unital positive linear map between two C^*-algebras is contractive. Incidentally, the short proof given in the preceding paragraph (for the very simple case $T = I$) appears to be more elementary than any other known proofs of Russo-Dye's Theorem.

Much of the theory of normal operators can be extended to a larger class of operators. A natural way is to weaken some conditions of normal operators. Recall that an operator A is said to be *quasinormal* if A^*A commutes with A; A is said to be *hyponormal* if $A^*A \geqslant AA^*$; $A \in \mathscr{B}(\mathscr{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and a normal operator $N \in \mathscr{B}(\mathscr{K})$ such that N is represented by an operator-matrix of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ corresponding to the decomposition $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^{\perp}$. It has been known (see [9, Chapter 16]) that the implications

$$\text{normal} \Rightarrow \text{quasinormal} \Rightarrow \text{subnormal} \Rightarrow \text{hyponormal}$$

hold in general, but all of the reverse implications are invalid in the infinite-dimensional case.

At this point, it is reasonable to formulate a plausible generalization of Lemma 3.1 as

CONJECTURE 3.4. *Suppose that $\Phi: \mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map between two C^*-algebras. Then $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(A^*A) \geqslant \Phi(A)\Phi(A^*)$ for every "hyponormal" operator $A \in \mathfrak{A}$.*

REMARK 3.5. From Theorem 3.2 (with $T = A^*A$), we see that the statement of Conjecture 3.4 is true if the word "hyponormal" is replaced by "quasinormal". Indeed, we proceed to establish further in Proposition 3.6 that the analogous result remains true for any "subnormal" operator. It may be pertinent to note here that there is an algebraic characterization for subnormal operators [2, p. 76 and p. 80]. Namely, an operator $A \in \mathscr{B}(\mathscr{H})$ is subnormal if and only if for each positive integer n, the associated $(n+1) \times (n+1)$ operator-matrix

$$\begin{bmatrix} I & A^* & \dots & A^{*n} \\ A & A^*A & \dots & \cdot \\ \vdots & & & \cdot \\ A^n & \cdot & \cdot & A^{*n}A^n \end{bmatrix} \in \mathscr{B}(\mathscr{H}^{n+1})$$

is positive. Consequently, if $A \in \mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$ is a subnormal operator and if π is a $*$-representation of the C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathscr{H}_\pi$, then $\pi(A) \in \mathscr{B}(\mathscr{H}_\pi)$ remains subnormal.

PROPOSITION 3.6. *Suppose that $\Phi: \mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map between two C^*-algebras. Then $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(A^*A) \geqslant \Phi(A)\Phi(A^*)$ for every subnormal operator $A \in \mathfrak{A}$.*

Proof. Assume that $A \in \mathfrak{A} \subseteq \mathscr{B}(\mathscr{H})$ and A is subnormal; i.e., there is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and there is a normal operator $N \in \mathscr{B}(\mathscr{K})$ of the expression $N = \begin{bmatrix} A & \# \\ 0 & \# \end{bmatrix}$, corresponding to the decomposition $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^{\perp}$. [Here, $\#$'s stand for entries we do not have to specify.] Let $\Theta : C^*(N) \to \mathscr{B}(\mathscr{H})$ be the compression map $\begin{bmatrix} X & \# \\ \# & \# \end{bmatrix} \mapsto X$. We *claim* that $\Theta(C^*(N)) \subseteq \mathfrak{A}$. Thus $\Phi \circ \Theta$ is a well-defined unital positive linear map from $C^*(N)$ into $\mathfrak{B}$. From Lemma 3.1, we get

$$\Phi \circ \Theta(N^*N) \geqslant \Phi \circ \Theta(N^*) \cdot \Phi \circ \Theta(N), \quad \Phi \circ \Theta(N^*N) \geqslant \Phi \circ \Theta(N) \cdot \Phi \circ \Theta(N^*).$$

Since $N^*N = \begin{bmatrix} A^*A & \# \\ \# & \# \end{bmatrix}$, it follows that $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(A^*A) \geqslant \Phi(A)\Phi(A^*)$ as desired.

To see the *claim*, note that every element in the commutative C^*-algebra $C^*(N)$ can be approximated uniformly by finite sums $\sum_{n,m} \alpha_{nm} N^{*m} N^n$. Since $N^{*m}N^n$ is of the form $\begin{bmatrix} A^{*m} A^n & \# \\ \# & \# \end{bmatrix}$, the claim follows immediately. ▨

Another apparently stronger conjecture has been proposed in [18]:

WORONOWICZ'S CONJECTURE. *Suppose that $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a unital positive linear map between two C^*-algebras. If T, A are operators in $\mathfrak{A}$ with $T \geqslant A^*A$ and $T \geqslant AA^*$, then $\Phi(T) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(T) \geqslant \Phi(A)\Phi(A^*)$.*

PROPOSITION 3.7. *Conjecture* 3.4 *and Woronowicz's Conjecture are equivalent.*

Proof. Conjecture 3.4 is the special case of Woronowicz's Conjecture with $T = A^*A$.

Conversely, suppose that we are given a unital positive linear map $\Phi : \mathfrak{A} \to \mathfrak{B}$ and $T \geqslant A^*A$, $T \geqslant AA^*$. Granting the truth of Conjecture 3.4 in general, we wish to show that $\Phi(T) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(T) \geqslant \Phi(A)\Phi(A^*)$.

Without loss of generality, we may assume further that $T - AA^*$ is invertible. (Otherwise, replace T by $T + I/n$; any conclusion drawn on $T + I/n$ can be carried forward to T, by letting n approach to infinity.) Let $\mathscr{H}$ be the underlying Hilbert space of $\mathfrak{A}$, and let $\mathscr{H}^\infty = \mathscr{H} \oplus \mathscr{H} \oplus \ldots$ be the direct sum of countably infinite many copies of $\mathscr{H}$. Put

$$M = \begin{bmatrix} A & 0 & & & \\ (T - A^*A)^{1/2} & 0 & & & \\ 0 & mI & 0 & & \\ \cdot & & mI & \cdot & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \in \mathscr{B}(\mathscr{H}^\infty),$$

and

$$V = \begin{bmatrix} I \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \text{the natural isometric imbedding of } \mathscr{H} \text{ into } \mathscr{H}^{\infty}.$$

Then

$$M^*M - MM^* = \begin{bmatrix} T - AA^* & S & 0 & \\ S^* & m^2I - (T - A^*A) & 0 & \\ 0 & 0 & 0 & \\ & & & \ddots \end{bmatrix},$$

with $S = -A(T - A^*A)^{1/2}$. When m^2I is sufficiently large, we apply Lemma 2.1 (the fact $T - AA^*$ being invertible is needed here) to deduce that $M^*M - MM^*$ is positive, i.e., M is hyponormal. Now consider the compression map $\Theta\colon C^*(M) \to \mathscr{B}(\mathscr{H})$ defined by $\Theta(X) = V^*XV$. It is evident that $\Theta(C^*(M)) \subseteq \mathfrak{A}$. Thus $\Phi \circ \Theta\colon C^*(M) \to \mathfrak{B}$ is a unital positive linear map. Taking Conjecture 3.4 for granted, we get

$$\Phi \circ \Theta(M^*M) \geqslant \Phi \circ \Theta(M^*) \cdot \Phi \circ \Theta(M),$$

$$\Phi \circ \Theta(M^*M) \geqslant \Phi \circ \Theta(M) \cdot \Phi \circ \Theta(M^*).$$

As $\Theta(M^*M) = T$ and $\Theta(M) = A$, it follows that $\Phi(T) \geqslant \Phi(A^*)\Phi(A)$ and $\Phi(T) \geqslant \Phi(A)\Phi(A^*)$; therefore Conjecture 3.4 implies Woronowicz's Conjecture. ▨

§ 4. TWO-POSITIVITY

We are going to show that every positive linear map must possess the 2-positive effect to a considerable extent. The readers are referred to [3, Theorem 4, p. 524] for a precise distinction between the class of positive linear maps and the subclass of 2-positive linear maps.

In the present section, positive linear maps are not required to be unital. Indeed, in some cases, we assume that $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a positive linear map with $\Phi(I)$ being a positive invertible operator in $\mathfrak{B}$. Thus, if T is a positive invertible operator in $\mathfrak{A}$ (i.e., $T \geqslant \varepsilon I$ for some $\varepsilon > 0$), then $\Phi(T) \geqslant \varepsilon\Phi(I)$ must also be positive and invertible.

For the sake of contrast, we give below a simple characterization of 2-positive linear maps.

PROPOSITION 4.1. *Suppose that $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a positive linear map between two C^*-algebras with $\Phi(I)$ being positive and invertible. Then the following are equivalent:*

(i) *Φ is 2-positive,*

(ii) $\Phi(R) \geqslant \Phi(S^*)\Phi(T)^{-1}\Phi(S)$ *whenever* R, S, T *are operators in* $\mathfrak{A}$ with T *being positive and invertible, and* $R \geqslant S^*T^{-1}S$.

(iii) $\Phi(S^*T^{-1}S) \geqslant \Phi(S^*)\Phi(T)^{-1}\Phi(S)$ *whenever* S, T *are operators in* $\mathfrak{A}$ *with* T *being positive and invertible.*

Proof. (ii) $\Leftrightarrow$ (iii) is trivial.

By Lemma 2.1, it follows immediately that (ii) is equivalent to

(ii′) $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} \geqslant 0$ *whenever* T *is positive and invertible and* $\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \geqslant 0$.

Obviously (i) $\Rightarrow$ (ii′). It remains to show (ii′) $\Rightarrow$ (i). Suppose that Φ satisfies (ii′) and $\begin{bmatrix} A & S \\ S^* & R \end{bmatrix}$ is a given positive operator in $\mathcal{M}_2(\mathfrak{A})$. Thus

$$\begin{bmatrix} A + I/n & S \\ S^* & R \end{bmatrix} \geqslant 0$$

and $A + I/n$ is invertible for each n, and an application of (ii′) leads to

$$\begin{bmatrix} \Phi(A + I/n) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} \geqslant 0$$

for each n. Therefore

$$\begin{bmatrix} \Phi(A) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} \geqslant 0$$

and Φ is 2-positive. This proves (ii′) $\Rightarrow$ (i). ▨

In view of Proposition 4.1, we describe some analogous properties of positive linear maps in Propositions 4.2-4.3.

PROPOSITION 4.2. *Suppose that* $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ *is a positive linear map between two* C^**-algebras with* $\Phi(I)$ *positive invertible. If* T, S *are operators in* $\mathfrak{A}$ *with* T *being positive and invertible, and* $T \geqslant S^*T^{-1}S$, *then* $\Phi(T) \geqslant \Phi(S^*)\Phi(T)^{-1}\Phi(S)$.

Proof. Define $\Psi\colon \mathfrak{A} \to \mathfrak{B}$ by

$$\Psi(A) = \Phi(T)^{-1/2}\Phi(T^{1/2}AT^{1/2})\Phi(T)^{-1/2}.$$

Thus Ψ is a unital positive linear map. By Russo-Dye's Theorem [12, Corollary 1, p. 415] (see also Remark 3.3), it follows that $I \geqslant \Psi(X^*)\Psi(X)$ for every contractive X in $\mathfrak{A}$. From the given hypothesis $T \geqslant S^*T^{-1}S$, we have that

$$I \geqslant T^{-1/2}S^*T^{-1}ST^{-1/2} = (T^{-1/2}ST^{-1/2})^* \ (T^{-1/2}ST^{-1/2}),$$

i.e., $T^{-1/2}ST^{-1/2}$ is contractive. Plugging in $X = T^{-1/2}ST^{-1/2}$, we get

$$I \geqslant \Psi(T^{-1/2}S^*T^{-1/2})\Psi(T^{-1/2}ST^{-1/2}) =$$

$$= \Phi(T)^{-1/2}\Phi(S^*)\Phi(T)^{-1}\Phi(S)\Phi(T)^{-1/2}.$$

Therefore, $\Phi(T) \geqslant \Phi(S^*)\Phi(T)^{-1}\Phi(S)$ as desired. ▨

It is well known that each unital positive linear map $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ has the following effect: $\Phi(T^{-1}) \geqslant \Phi(T)^{-1}$ for every positive invertible T [4, Corollary 2.3], and $\Phi(S^2) \geqslant \Phi(S)^2$ for every hermitian S (Kadison's inequality [10]). The next proposition is the unification of these results without requiring $\Phi(I) = I$ (cf. [1, Corollary 3.1]).

PROPOSITION 4.3. *Suppose that $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a positive linear map between two C^*-algebras with $\Phi(I)$ being a positive invertible operator in $\mathfrak{B}$. Then $\Phi(ST^{-1}S) \geqslant \Phi(S)\Phi(T)^{-1}\Phi(S)$ for every hermitian $S \in \mathfrak{A}$ and positive invertible $T \in \mathfrak{A}$.*

Proof. Again, define $\Psi\colon \mathfrak{A} \to \mathfrak{B}$ by

$$\Psi(A) = \Phi(T)^{-1/2}\Phi(T^{1/2}AT^{1/2})\Phi(T)^{-1/2}.$$

Thus Ψ is a unital positive linear map. By Kadison's inequality, it follows that $\Psi(X^2) \geqslant \Psi(X)^2$ for every hermitian $X \in \mathfrak{A}$. Hence $X = T^{-1/2}ST^{-1/2}$ leads to

$$\Phi(T)^{-1/2}\Phi(ST^{-1}S)\,\Phi(T)^{-1/2} \geqslant \Phi(T)^{-1/2}\Phi(S)\Phi(T)^{-1}\Phi(S)\Phi(T)^{-1/2},$$

i.e.,

$$\Phi(ST^{-1}S) \geqslant \Phi(S)\Phi(T)^{-1}\Phi(S)$$

as desired. ▨

In the next corollary (cf. [18, Theorem 5.3]), we restate Propositions 4.2-4.3 without assuming $\Phi(I)$ invertible.

COROLLARY 4.4. *Suppose $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is a positive linear map between two C^*-algebras. Then we have the following:*

(i) $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(T) \end{bmatrix} \geqslant 0$ *whenever* $\begin{bmatrix} T & S \\ S^* & T \end{bmatrix}$ *is a positive operator in* $\mathcal{M}_2(\mathfrak{A})$;

(ii) $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S) & \Phi(R) \end{bmatrix} \geqslant 0$ *whenever* $\begin{bmatrix} T & S \\ S & R \end{bmatrix}$ *is a positive operator in* $\mathcal{M}_2(\mathfrak{A})$.

Proof. Assume further that $\Phi(I)$ is positive invertible and T is invertible. By virtue of Lemma 2.1, the assertions follow immediately from Propositions 4.2-4.3.

In general, let ρ be a positive linear functional on $\mathfrak{A}$ with $\rho(I) = 1$. Define $\Phi_n\colon \mathfrak{A} \to \mathfrak{B}$ by $\Phi_n(A) = \Phi(A) + \rho(A)I/n$; then $\Phi_n(I)$ is positive invertible. Thus, with Φ_n in place of Φ, and $T + I/n$ in place of T, the argument of the preceding paragraph applies. Letting n approach to infinity, we are done. ▨

The next theorem may reveal the strongest effect that a general positive linear map can ever take.

THEOREM 4.5. *Suppose Φ: $\mathfrak{A} \to \mathfrak{B}$ is a positive linear map between two C^*-algebras. If* $\begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \in \mathcal{M}_2(\mathfrak{A})$ *is positive, and if the four operators* $\{T, S, S^*, R\}$ *together are linearly dependent, then* $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} \in \mathcal{M}_2(\mathfrak{B})$ *is also positive.*

Proof. There are four possible (overlapping) cases.

(i) *S and S^* are linearly dependent*; i.e., there is a complex number λ_0 of modulus 1 such that $\lambda_0 S = S^*$. Choose a complex number λ with $\lambda^2 = \lambda_0$; then $\lambda S = \bar{\lambda} S^* = S_0$ say. Thus

$$\begin{bmatrix} T & S_0 \\ S_0 & R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \bar{\lambda} I \end{bmatrix} \begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix} \geqslant 0.$$

From Corollary 4.4 (ii), $\begin{bmatrix} \Phi(T) & \Phi(S_0) \\ \Phi(S_0) & \Phi(R) \end{bmatrix} \geqslant 0.$

Hence, $\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \lambda I \end{bmatrix} \begin{bmatrix} \Phi(T) & \Phi(S_0) \\ \Phi(S_0) & \Phi(R) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \bar{\lambda} I \end{bmatrix} \geqslant 0.$

(ii) *T and R are linearly dependent.* Ignoring the trivial case with $T = 0$ or $R = 0$, we may assume that there is a positive number a such that $R = a^2 T$. Then

$$\begin{bmatrix} T & S/a \\ S^*/a & T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I/a \end{bmatrix} \begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I/a \end{bmatrix} \geqslant 0.$$

From Corollary 4.4 (i), $\begin{bmatrix} \Phi(T) & \Phi(S)/a \\ \Phi(S^*)/a & \Phi(T) \end{bmatrix} \geqslant 0.$ Hence

$$\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & aI \end{bmatrix} \begin{bmatrix} \Phi(T) & \Phi(S)/a \\ \Phi(S^*)/a & \Phi(T) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & aI \end{bmatrix} \geqslant 0.$$

(iii) *R is a linear combination of T, S, and S^**; i.e., $R = \alpha_1 T + \alpha_2 S + \alpha_3 S^*$. Thus

$$R = \frac{1}{2}(R + R^*) = \alpha T + \beta S + \bar{\beta} S^*,$$

with $\alpha = \operatorname{Re} \alpha_1$, $\beta = \dfrac{1}{2}(\alpha_2 + \bar{\alpha}_3)$. Write $S_1 = S - \bar{\beta} T$, $R_1 = (\alpha + |\beta|^2) T$; it follows

$$\begin{bmatrix} T & S_1 \\ S_1^* & R_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\beta I & I \end{bmatrix} \begin{bmatrix} T & S \\ S^* & R \end{bmatrix} \begin{bmatrix} I & -\bar{\beta} I \\ 0 & I \end{bmatrix} \geqslant 0.$$

From the case (ii), we get $\begin{bmatrix} \Phi(T) & \Phi(S_1) \\ \Phi(S_1^*) & \Phi(R_1) \end{bmatrix} \geqslant 0$. Hence

$$\begin{bmatrix} \Phi(T) & \Phi(S) \\ \Phi(S^*) & \Phi(R) \end{bmatrix} = \begin{bmatrix} I & 0 \\ \bar{\beta} I & I \end{bmatrix} \begin{bmatrix} \Phi(T) & \Phi(S_1) \\ \Phi(S_1^*) & \Phi(R_1) \end{bmatrix} \begin{bmatrix} I & \beta I \\ 0 & I \end{bmatrix} \geqslant 0.$$

(iv) *T is a linear combination of R, S, S^*.* This case is similar to (iii). ▨

APPENDIX A

In view of Propositions 4.1 and 4.3, we might conjecture that a unital positive linear map $\Phi: \mathfrak{A} \to \mathfrak{B}$ satisfying the inequality $\Phi(A^*A) \geqslant \Phi(A^*)\Phi(A)$ for all A in $\mathfrak{A}$ must be 2-positive. The example below shows that the conjecture is false.

EXAMPLE. Define $\Phi: \mathcal{M}_2 \to \mathcal{M}_2$ by

$$\Phi(A) = A^{\mathrm{tr}}/2 + \tau(A)\, I/4$$

where A^{tr} stands for the transpose of A and $\tau(A) =$ the usual trace of A. Namely, if $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then

$$\Phi(A) = \frac{1}{2}\begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \alpha+\delta & 0 \\ 0 & \alpha+\delta \end{bmatrix};$$

in particular, $\Phi(I) = I$.

Obviously, Φ is positive. Φ is not 2-positive because

$$\begin{bmatrix} \Phi\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \Phi\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \Phi\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \Phi\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \left[\begin{array}{cc|cc} \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{3}{4} \end{array}\right]$$

is clearly not a positive semidefinite matrix in $\mathcal{M}_4$.

Next, letting $\Delta(A) = \Phi(A^*A) - \Phi(A^*)\Phi(A)$, we can carry out the direct hard computation to get $\Delta(A) \geqslant 0$ for every A. Alternatively, observe that $\Delta(A) =$

$= \Lambda(A + \lambda I)$ for every complex number λ (because $\Phi(I) = I$). To show $\Lambda(A) \geqslant 0$, it suffices to assume that the trace of A is zero. Hence $\tau(A) = 0$ leads to

$$\Lambda(A) = \Phi(A^*A) - \Phi(A^*)\,\Phi(A) =$$

$$= (A^*A)^{\mathrm{tr}}/2 + \tau(A^*A)\, I/4 - A^{*\mathrm{tr}}A^{\mathrm{tr}}/4 \geqslant$$

$$\geqslant (A^*A)^{\mathrm{tr}}/2 \geqslant 0.$$

Note in the above, we use the fact $\tau(A^*A) = \tau(AA^*) = \tau(A^{*\mathrm{tr}}A^{\mathrm{tr}}) =$ the sum of eigenvalues of $A^{*\mathrm{tr}}A^{\mathrm{tr}}$ which, in turn, implies that $\tau(A^*A)\, I \geqslant A^{*\mathrm{tr}}A^{\mathrm{tr}}$.

APPENDIX B

The global structure of positive linear maps is highly combinatorial, even in the finite-dimensional case (see e.g. [15, Chapter 8; 5, Theorem 7; 6, Theorem 2; 18]).

For a tractable structure theory, it may be worthwhile to restrict the investigation to the subclass of "decomposable" positive linear maps (see [15, p. 267; 17]). By one of several equivalent definitions, we say that a positive linear map $\Phi: \mathcal{M}_n \to \mathcal{M}_k$ is *decomposable* if there exist $n \times k$ matrices V_i and U_j such that $\Phi(A) = \sum V_i^* A V_i + \sum U_j^* A^{\mathrm{tr}} U_j$ for all $A \in \mathcal{M}_n$, where A^{tr} stands for the transpose of A.

There arises the natural question: Must every positive linear map be decomposable? It turns out the answer is negative as indicated in [6, Theorem 2]. Woronowicz [18; 19] has later provided other counter-examples which are supported by indirect proof as well as lengthy computation. To clarify the situation, we reproduce below the counter-example in [6] — which is believable to be the simplest one — with a complete verification.

EXAMPLE. *The linear map $\Phi: \mathcal{M}_3 \to \mathcal{M}_3$ defined by*

$$\Phi\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{bmatrix} + \mu \begin{bmatrix} \alpha_{33} & 0 & 0 \\ 0 & \alpha_{11} & 0 \\ 0 & 0 & \alpha_{22} \end{bmatrix} \quad \text{with} \quad \mu \geqslant 1$$

is positive but not decomposable.

(Note: Here $\Phi(I) = (\mu + 1)\, I$; to get a *unital* map, we may consider $(\mu + 1)^{-1}\Phi$ instead of Φ.)

Proof. Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$ be free vectors in $\mathbf{R}^3$. Associate the linear map Φ with the real biquadratic form

$$F(\mathbf{x}; \mathbf{y}) = F(x_1, x_2, x_3; y_1, y_2, y_3) = [y_1\ y_2\ y_3] \cdot \Phi \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

$$= x_1y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1x_2y_1y_2 + x_2x_3y_2y_3 + x_3x_1y_3y_1) +$$

$$+ \mu(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2).$$

As proved in ([6, Theorem 1, p. 96] for the case $\mu = 2$, and [7, Theorem 4.4, p. 13] for the general $\mu \geqslant 1$), the biquadratic form $F(\mathbf{x}; \mathbf{y}) \geqslant 0$ for all real x_i, y_j, but F cannot be written as sum of squares of real bilinear forms.

Note that the fact $F \geqslant 0$ above is equivalent to $\Phi(S) \geqslant 0$ for every rank-1 positive semidefinite 3×3 *real* matrix S. Given a general rank-1 positive semidefinite *complex* matrix $[\bar{\alpha}_i\alpha_j]_{i,j=1}^3$, let $\lambda_1, \lambda_2, \lambda_3$ be complex numbers of modulus 1 with $\alpha_j = \lambda_j|\alpha_j|$. Then by the definition of Φ,

$$\Phi \begin{bmatrix} \bar{\alpha}_1\alpha_1 & \bar{\alpha}_1\alpha_2 & \bar{\alpha}_1\alpha_3 \\ \bar{\alpha}_2\alpha_1 & \bar{\alpha}_2\alpha_2 & \alpha_2\alpha_3 \\ \bar{\alpha}_3\alpha_1 & \bar{\alpha}_3\alpha_2 & \bar{\alpha}_3\alpha_3 \end{bmatrix} = \begin{bmatrix} \bar{\lambda}_1 & 0 & 0 \\ 0 & \bar{\lambda}_2 & 0 \\ 0 & 0 & \bar{\lambda}_3 \end{bmatrix} \cdot \Phi \begin{bmatrix} |\alpha_1|^2 & |\alpha_1\alpha_2| & |\alpha_1\alpha_3| \\ |\alpha_2\alpha_1| & |\alpha_2|^2 & |\alpha_2\alpha_3| \\ |\alpha_3\alpha_1| & |\alpha_3\alpha_2| & |\alpha_3|^2 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

turns out to be positive semidefinite too. Hence by linearity, $\Phi(A) \geqslant 0$ for every positive semidefinite complex matrix A.

Now suppose that Φ is decomposable; then there exist 3×3 complex matrices W_k such that $\Phi(A) = \sum W_k^* A W_k$ for every matrix $A = A^{tr}$. Thus, the real biquadratic form F admits the expression

$$F(\mathbf{x}; \mathbf{y}) = \sum_k [y_1\ \ y_2\ \ y_3]\, W_k^* \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix} W_k \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

$$= \sum_k [y_1\ \ y_2\ \ y_3]\, W_k^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} [x_1\ \ x_2\ \ x_3]\, W_k \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} =$$

$$= \sum_k \overline{L_k(\mathbf{x}, \mathbf{y})}\, L_k(\mathbf{x}, \mathbf{y}),$$

where each $L_k(\mathbf{x}, \mathbf{y}) = [x_1 \;\; x_2 \;\; x_3]\, W_k \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ is a complex-coefficient bilinear form in real indeterminates **x**, **y**. Write $L_k = L_k^{(1)} + \sqrt{-1}\, L_k^{(2)}$ where $L_k^{(1)}$, $L_k^{(2)}$ are real bilinear forms. Then

$$F = \sum (L_k^{(1)} - \sqrt{-1}\, L_k^{(2)})\,(L_k^{(1)} + \sqrt{-1}\, L_k^{(2)}) = \sum (L_k^{(1)2} + L_k^{(2)2})$$

is sum of squares of real bilinear forms; that is impossible as mentioned before. Therefore Φ cannot be decomposable. ▨

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REFERENCES

1. Ando, T., Concavity of certain maps among positive definite matrices and its application to Hadamard products, *Linear Algebra and Appl.*, **26** (1979), 203–241.
2. Bram, J., Subnormal operators, *Duke Math. J.*, **22** (1955), 75–94.
3. Choi, M. D., Positive linear maps on C^*-algebras, *Canad. J. Math.*, **24** (1972), 520–529.
4. Choi, M. D., A Schwarz inequality for positive linear maps on C^*-algebras, *Illinois J. Math.*, **18** (1974), 565–574.
5. Choi, M. D., Completely positive linear maps on complex matrices, *Linear Algebra and Appl.*, **10** (1975), 285–290.
6. Choi, M. D., Positive semidefinite biquadratic forms, *Linear Algebra and Appl.*, **12** (1975), 95–100.
7. Choi, M. D.; Lam, T. Y., Extremal positive semidefinite forms, *Math. Ann.*, **231** (1977), 1–18.
8. Davis, Ch., A Schwarz inequality for convex operator functions, *Proc. Amer. Math. Soc.*, **8** (1957), 42–44.
9. Halmos, P. R., *A Hilbert space problem book,* Van Nostrand, Princeton, 1967.
10. Kadison, R. V., A generalized Schwarz inequality and algebraic invariants for C^*-algebras, *Ann. of Math.*, **56** (1952), 494–503.
11. Lieb, E. H.; Ruskai, M. B., Some operator inequalities of the Schwarz type, *Advances in Math.*, **12** (1974), 269–273.
12. Russo, B.; Dye, H. A., A note on unitary operators in C^*-algebras, *Duke Math. J.*, **33** (1966), 413–416.
13. Smul'jan, Ju. L., An operator Hellinger integral (Russian), *Mat. Sb.*, **91** (1959), 381–430.
14. Stinespring, W. F., Positive functions on C^*-algebras, *Proc. Amer. Math. Soc.*, **6** (1955), 211-216.
15. Størmer, E., Positive linear maps of operator algebras, *Acta Math.*, **110** (1963), 233–278.

16. STØRMER, E., Positive linear maps of C^*-algebras, *Lecture Notes in Physics*, Vol. 29, pp. 85–106, Springer-Verlag, 1974.
17. STØRMER, E., Decomposition of positive projections on C^*-algebras, to appear.
18. WORONOWICZ, S. L., Positive maps of low dimensional matrix algebras, *Rep. Mathematical Phys.*, **10** (1976), 165–183.
19. WORONOWICZ, S. L., Nonextendible positive maps, *Comm. Math. Phys.*, **51** (1976), 243–282.

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Added in proofs. E. Kirchberg has recently shown by a non-constructive proof that there exists a counter-example to Woronowicz's Conjecture. Hence, Conjecture 3.4 is also false by virtue of Proposition 3.7.