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THE LINEAR SPAN OF PROJECTIONS IN SIMPLE C^*-ALGEBRAS

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Let A be a C^*-algebra and consider the following chain of conditions:

(AF) A is approximately finite-dimensional.
(FS) A_{sa} has a dense set of elements with finite spectrum.
(CP) The convex hull of projections is dense in the unit ball of A_+.
(LP) The linear span of projections is dense in A.
(AP) The algebra generated by projections is dense in A.
($\exists$P) A has non-trivial projections.

Clearly each of the conditions implies every other below it. The question under discussion is whether any lower condition will imply a higher one if A is assumed to be simple. We show that ($\exists$P) implies (AP) (an unpublished result by the authors of [1]), and that ($\exists$P) implies (LP) in a number of interesting cases among which are Cuntz's algebras $\mathcal{O}_n$ ([7]) and Rieffel's irrational rotation C^*-algebras ([11], [12]).

Nevertheless ($\exists$P) does not in general imply (LP), just as (LP) does not imply (CP). It is conceivable that (CP) implies (FS) for any simple C^*-algebra, but it is false that (FS) implies (AF). Finally it should be mentioned that not all simple C^*-algebras satisfy ($\exists$P), see [3] and [4]. For background and terminology we refer to [9].

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POSITIVE RESULTS

Let D be a closed $*$-invariant subspace of a C^*-algebra A. Following [1] we say that A *derives* D if $ad - da \in D$ for every a in A and d in D. We say that D is *unitarily invariant* if $uDu^* = D$ for every unitary u in $\tilde{A}$. The equation

$$\exp(\mathrm{i}th)\, d \exp(-\mathrm{i}th) = d + \mathrm{i}t(hd - dh) + \mathcal{O}(t^2),$$

valid for every real t and every h in A_{sa}, shows that if D is unitarily invariant then A derives D (cf. [1, 5.2]). Since the projections in A is a unitarily invariant set we immediately obtain the following result.

LEMMA 1. *Given a C^*-algebra A let $L(P)$ and $A(P)$, respectively, denote the closed linear span and the C^*-algebra generated by the projections in A. Then A derives both $L(P)$ and $A(P)$.*

PROPOSITION 2. *If A is a simple C^*-algebra with a non-trivial projection then the subalgebra generated by projections is dense in A.*

Proof. Define $A(P)$ as in Lemma 1 and note that $A(P) \neq 0$ and $A(P) \neq \mathbf{C}\,1$ (if $1 \in A$). Since A derives $A(P)$ by Lemma 1, there is by [1, 5.1] a closed ideal I of A, contained in $A(P)$, such that $A(P)/I$ is contained in the center of A/I. If $I = 0$ then $A(P)$ is trivial, since the center of A is trivial; in contradiction with our assumption. Thus $I \neq 0$, whence $I = A$ since A is simple; which implies that $A(P) = A$.

THEOREM 3. *Let A be a simple C^*-algebra with a non-trivial projection and denote by $L(P)$ the closed linear span of projections in A. If $L(P) \neq A$ there is a non-zero, self-adjoint, tracial functional on A vanishing on $L(P)$.*

Proof. By Hahn-Banach's theorem there is a non-zero, self-adjoint functional φ on A vanishing on $L(P)$. Since A derives $L(P)$ this implies that $\varphi(xy) = \varphi(yx)$ for every x in A and y in $L(P)$. But then the same equality holds with y in the C^*-algebra $A(P)$ generated by the projections in A. By assumption A contains a non-trivial projection, whence $A(P) = A$ by Proposition 2. Consequently $\varphi(xy) = \varphi(yx)$ for all x, y in A, as desired.

COROLLARY 4. *Let A be a simple C^*-algebra with a non-trivial projection. If A has no tracial states or has a unique tracial state then the linear span of projections is dense in A.*

Proof. Take $L(P)$ as before .If $L(P) \neq A$ there is by Theorem 3 a non-zero self-adjoint tracial functional φ on A vanishing on $L(P)$. Let $\varphi = \varphi_+ - \varphi_-$ be the Jordan decomposition of φ in positive parts (cf. [9, 3.2.5]) and note that both φ_+ and φ_- are finite traces on A. If $\varphi_- = 0$ then $\varphi \geqslant 0$. However, any tracial state on a simple C^*-algebra is faithful, in contradiction with $\varphi(L(P)_+) = 0$. Thus $\varphi_- \neq 0$ and similarly $\varphi_+ \neq 0$. Since $\varphi_+ \perp \varphi_-$ we see that A has at least two linearly independent tracial states whenever $L(P) \neq A$.

COROLLARY 5. *If A is a stable, simple C^*-algebra with a non-trivial pojection then the linear span of projections is dense in A.*

The next result is an observation of Rieffel adapted to a slightly more general situation (see [11], [12]).

LEMMA 6. (Rieffel). *Let A be a unital C^*-algebra containing a pair of unitaries u, v such that $uv = \theta vu$ with $\theta \neq 1$. Then A has a non-trivial projection.*

Proof. We may as well assume that $\mathrm{Sp}(v) = \mathbf{T}$ (the unit circle) since otherwise the C^*-algebra $C^*(v)$ generated by v contains a non-trivial projection. Thus $C^*(v) = C(\mathbf{T})$ and from the equation $uvu^* = \lambda v$ we obtain an automorphism α of $C(\mathbf{T})$ given by

$$\alpha f(t) = f(t+\lambda), \qquad f \in C(\mathbf{T}),$$

writing $\mathbf{T}$ as an additive group $(= \mathbf{R}/2\pi\mathbf{Z})$ with $\lambda = \operatorname{Arg}\theta$.

We may visualize $\mathbf{T}$ as $[0, 2\pi[$, with $0 < \lambda \leqslant \pi$. Take any f in $C(\mathbf{T})$ such that

$$f(\lambda/2) = 1/2, \quad 0 \leqslant f(t) \leqslant 1/2; \quad \text{and} \quad f(t) = 0 \quad \text{if} \quad t \notin [0, \lambda].$$

Now define g in $C(\mathbf{T})$ by

$$g(t) = \begin{cases} 1/2 - (1/4 - f(t)^2)^{1/2} & \text{if} \quad 0 \leqslant t \leqslant \lambda/2 \\ 1/2 + (1/4 - f(t)^2)^{1/2} & \text{if} \quad \lambda/2 \leqslant t \leqslant \lambda \\ 1 - g(t-\lambda) & \text{if} \quad \lambda \leqslant t \leqslant 2\lambda \\ 0 & \text{if} \quad t \notin [0, 2\lambda]. \end{cases}$$

It is straightforward to check that g satisfies the equations

$$(*) \qquad g(t) + g(t+\lambda) = 1 \qquad \text{if } 0 \leqslant t \leqslant \lambda;$$

$$(**) \qquad g(t) - g(t)^2 = f(t)^2 + f(t-\lambda)^2 \quad \text{for all } t.$$

(On the interval $[0, \lambda]$, g is made to satisfy $(**)$ since $f(t-\lambda) = 0$; on $[\lambda, 2\pi]$ we have $f(t) = 0$ and s and $1-s$ are simultaneous solutions to $s - s^2 = f(t-\lambda)^2$; and outside $[0, 2\lambda]$ everything is zero.)

Set $p = g + fu + u^*f$. Since $C^*(v)(= C(\mathbf{T}))$ is contained in A we have $p \in A_{\mathrm{sa}}$. Moreover, since $\alpha = \mathrm{Ad}u$ we see from $(*)$ and $(**)$ that

$$p^2 = g^2 + f^2 + \alpha^{-1}(f^2) + (gf + f\alpha(g))u + u^*(fg + \alpha(g)f) +$$
$$+ f\alpha(f)u^2 + u^{*2}\alpha(f)f = g + (g + \alpha(g))fu + u^*f(g + \alpha(g)) = p.$$

THEOREM 7. *Let A be a simple, unital C^*-algebra and assume that there is an ergodic, compact, abelian group of automorphisms of A. Then the linear span of projections is dense in A.*

Proof. Denote by G the ergodic, compact, abelian group in $\mathrm{Aut}(A)$. By [8, 2.3 & 2.11] there is a family $\{u(\gamma) \mid \gamma \in \hat{G}\}$ of unitary (eigen) operators in A (relative to the action of G) such that A is the closed linear span of the $u(\gamma)$'s. Moreover, by [8, 4.5 & 6.1] the function $\chi: \hat{G} \times \hat{G} \to \mathbf{T}$ given by

$$\chi(\gamma, \delta) = u(\gamma)\, u(\delta)\, u(\gamma)^* u(\delta)^*$$

is a symplectic bicharacter (and is a complete invariant for the conjugacy class of A relative to G). Since A is simple, χ is injective in the sense that if $\gamma \neq 0$ there is a δ such that $\chi(\gamma, \delta) \neq 1$ (cf. [8, 5.8]). Applying Lemma 6 to $u(\gamma)$ and $u(\delta)$ we see that A contains non-trivial projections. Finally, A has a unique tracial state by [8, 6.3], so that the linear span of projections lies dense in A by Corollary 4.

REMARK 8. In any unital AF-algebra the unitary group is connected. This fact is used in [8, 6.5] to show that the simple (nuclear) C^*-algebras described in Theorem 6 are AF-algebras if and only if G is totally disconnected.

REMARK 8. In the class $\mathscr{S}$ of separable, simple, unital C^*-algebras with unique tracial state there is an obvious invariant; viz. the range of the trace on the set of projections, which gives a countable subset E of $[0, 1]$. Take A in $\mathscr{S}$ and suppose that u, v are unitaries in A with $uv = \theta uv$. Inspection of the proof of Lemma 6 shows that $(2\pi)^{-1} \operatorname{Arg}\theta \in E(A)$. Since $u^n v = \theta^n v u^n$ it follows that actually

$$(*) \qquad (\mathbf{Z}(2\pi)^{-1} \operatorname{Arg}\theta + \mathbf{Z}) \cap [0, 1] \subset E(A).$$

The recent results of Pimsner and Voiculescu [10] show that if u and v are generators for A then equality holds in $(*)$.

Assume now that A is a C^*-algebra in $\mathscr{S}$ as in Theorem 7 and denote by G the ergodic, compact, abelian group of automorphisms. Then A is (linearly) generated by a family $\{u(\gamma) \mid \gamma \in \hat{G}\}$ of unitaries such that

$$\chi(\gamma, \delta) = u(\gamma)\, u(\delta)\, u(\gamma)^* u(\delta)^* \in \mathbf{T}$$

for all γ, δ in $\hat{G}$. It follows from the above that

$$(**) \qquad \bigoplus_{\gamma, \delta} (\mathbf{Z} \operatorname{Arg} \chi(\gamma, \delta)) \cap [0, 1] \subset E(A).$$

Maybe an elaborated version of the construction in [10] could give an embedding of A as a subalgebra of an AF-algebra, in such a way that equality could be established in $(**)$.

NEGATIVE RESULTS

PROPOSITION 9. (Blackadar) *There is a separable, simple, unital C^*-algebra with a non-trivial set of projections whose linear span is not dense in the algebra.*

Proof. Put $\Delta = [0, 1]$ and let A_Δ be the separable, simple, unital projectionless C^*-algebra constructed in [4, 4.4]. If $M = A_\Delta \otimes M_2$ there is exactly one equivalence class of non-trivial projections in M (see [4, 4.11]). Since A_Δ has two extremal tracial states, if follows that M has the same property. But $\tau(p) = 1/2$ for any tracial state τ and any non-trivial projection p of M, and thus the linear span of projections cannot be dense in M.

PROPOSITION 11. (Blackadar) *There is a separable, simple, unital C^*-algebra M such that the linear span of projections is dense in M, but the convex span of projections is not dense in the unit ball of M_+.*

Proof. Let A be the separable, simple, unital, projectionless C^*-algebra constructed in Section 3 of [4], and put $M = A \otimes M_2$. By [4, 4.9] there is exactly one equivalence class of non-trivial projections in M. Since A has a unique tracial state ([4, 4.2]), so does M, and it follows from Corollary 4 that the linear span of projections is dense in M. However, the tracial state τ on M takes arbitrarily small values on the unit sphere of M_+ (consider the restriction of τ to the commutative C^*-subalgebra generated by a positive element with infinite spectrum), so there is an x in M_+ with $\|x\| = 1$ and $\tau(x) \leqslant \frac{1}{4}$. If now $\sum \lambda_n p_n$ is a convex combination of projections with λ_0 corresponding to $p_0 = 0$ then

$$\|\sum \lambda_n p_n - x\| \geqslant \sum \lambda_n \tau(p_n) - \tau(x) \geqslant \frac{1}{2}(1 - \lambda_0) - \frac{1}{4};$$

$$\|\sum \lambda_n p_n - x\| \geqslant 1 - \|\sum \lambda_n p_n\| \geqslant 1 - (1 - \lambda_0) = \lambda_0;$$

from which we conclude that $\|\sum \lambda_n p_n - x\| \geqslant \frac{1}{6}$.

REMARK 12. In the two previous (counter) examples we have used the fact that when A is projectionless and unital then $A \otimes M_2$ has a single equivalence class of non-trivial projections. As pointed out by Blackadar this is the result of his choice of A (rather his choice of order unit in the dimension group for the AF-algebra underlying the construction of A). Indeed, for any n there is a projectioness A as above such that $A \otimes M_2$ has n distinct equivalence classes of projections.

REMARK 13. It is appropriate to mention here the condition:

(HP) For each hereditary C^*-subalgebra B of A there is an approximate unit in B consisting of projections.

Note that we do not assume in (HP) that the projections form an increasing net. Condition (HP) arose in conversation with Blackadar and Elliott, and we arrived at the following characterization.

PROPOSITION 14. *In any C^*-algebra we have the implications*

$$\text{(FS)} \Rightarrow \text{(HP)} \Rightarrow \text{(CP)}.$$

Proof. (FS) $\Rightarrow$ (HP) (see the proof of [5, 3.1]). Let B be a hereditary C^*-subalgebra of A and take $x_1, \ldots, x_n$ in B and $\varepsilon > 0$. We must find a projection p in B such that $\|(1-p)x_k\| \leqslant \varepsilon$ for all k. Since

$$\|(1-p)x_k\|^2 = \|(1-p)x_k x_k^*(1-p)\| \leqslant \|(1-p)(\textstyle\sum x_k x_k^*)^{\frac{1}{2}}\|^2,$$

it suffices to consider a single element x in B_+, and by normalization we may take $\|x\| = 1$.

Choose $\delta > 0$ such that $6\delta < \varepsilon - \varepsilon^2$. Then choose n so large that $1 - \delta^{2/n} \leqslant \delta$. By assumption there is a y in A_{sa} with finite spectrum such that $\|x - y\| \leqslant \delta$. By spectral theory we may assume that $0 \leqslant y \leqslant 1$ and since $t \to t^{1/n}$ is continuous we may further assume that $\|x^{1/n} - y^{1/n}\| \leqslant \delta$ (cf. [9, 1.1.9]). Let q denote the spectral projection of y corresponding to the interval $[\delta, 1]$, so that $\|(1-q)y\| \leqslant \delta$. By our choice of n, $\|(1-y^{2/n})q\| \leqslant \delta$ whence

$$\|(1-q)x\| \leqslant 2\delta \text{ and } \|x^{1/n}qx^{1/n} - q\| \leqslant 3\delta.$$

The last inequality shows that with $z = x^{1/n}qx^{1/n}$ we have $\|z - z^2\| \leqslant 6\delta$, so that the spectrum of z is contained in $[0, \varepsilon] \cup [1-\varepsilon, 1]$ by our choice of δ. Consequently the spectral projection p of z corresponding to the interval $\left[\frac{1}{2}, 1\right]$ belongs to B (since $z \in B$) and $\|z - p\| \leqslant \varepsilon$. Finally

$$\|(1-p)x\| \leqslant \varepsilon + \|(1-z)x\| \leqslant \varepsilon + 3\delta + \|(1-q)x\| < \varepsilon + 5\delta < 2\varepsilon.$$

(HP) $\Rightarrow$ (CP) Given x in A_+ with $\|x\| = 1$ fix n and for $1 \leqslant k \leqslant n$ let p_k denote the spectral projection of x (computed in A'') corresponding to the interval $[k/n, 1]$. Then

$$\frac{1}{n}\sum p_k \leqslant x \leqslant \frac{1}{n} + \frac{1}{n}\sum p_k.$$

For each k the set $B_k = p_k A'' p_k \cap A$ is the hereditary C^*-subalgebra of A corresponding to the open projection p_k (cf. [9, 3.11.10]). By assumption there is a net $\{p_{k\lambda}\}$ of projections in B_k converging strongly (in A'') to p_k. Consequently the net $\left\{\frac{1}{n}\sum p_{k\lambda}\right\}$ is strongly convergent to $\frac{1}{n}\sum p_k$. With $n \to \infty$ this shows that the convex hull of projections in A contains x as a strong limit point. Since $x \in A$ if follows from Hahn-Banach's theorem that the convex set contains x as a limit point in norm, as desired.

REMARK 15. Note the condition (HP) in fact implies that any x in the unit ball of A_+ can be approximated in norm *from below* with convex combinations of projections; a fact which is not immediate from condition (FS).

Finally let us remark that even condition (FS) does not imply (AF). An obvious counterexample arises by taking A to be a factor of type III and noting that any unital AF-algebra has a tracial state. The undesirable fact that A is non-separable is circumvented by our last example.

PROPOSITION 16. (Elliott) *There is a separable, simple, unital C^*-algebra A such that A_{sa} contains a dense set of elements with finite spectrum, but A is not approximately finite-dimensional.*

Proof. Let M be a factor of type III on a separable Hilbert space and let A_0 be a separable C^*-subalgebra of M containing a proper isometry u (i.e. $u^*u = 1$ but $uu^* \neq 1$). Assume that we have defined an increasing chain of separable C^*-subalgebras A_k of M, $0 \leqslant k \leqslant 2n$, such that A_{2k} is simple if $k \geqslant 1$ and such that the closure of the set of self-adjoint elements in A_{2k+1} with finite spectrum contains $(A_{2k})_{sa}$ for $k \geqslant 0$. Then choose a countable family E of spectral projections from a dense set in $(A_{2n})_{sa}$ such that in the (separable) C^*-algebra A_{2n+1} generated by A_{2n} and E the self-adjoint elements in A_{2n+1} with finite spectrum lie dense in $(A_{2n})_{sa}$. Now use [2, 2.2] (applied to A_{2n+1} and M) to find a separable simple C^*-algebra A_{2n+2} with $A_{2n+1} \subset A_{2n+2} \subset M$. Working by induction we can thus define a sequence (A_n), and we let A be the closure of $\cup A_n$. Then A is simple because it is the inductive limit of the simple C^*-algebras A_{2n}. Moreover, the elements with finite spectrum in A_{sa} lie dense in $(A_{2n})_{sa}$ for all n, hence lie dense in A_{sa}. Finally A is not AF because it contains a proper isometry (cf. [7]).

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