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UNITARY EQUIVALENCE OF RESTRICTED SHIFTS

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INTRODUCTION

Suppose D is a finitely connected domain in the plane bounded by $n+1$ analytic Jordan curves. Letting ∂D denote the boundary of D, we write $\partial D = \gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_{n+1}$. Let $H^\infty = H^\infty(D)$ be the Banach space of holomorphic functions bounded on D and suppose $H^2 = H^2(D)$ is the Hilbert space of holomorphic functions f defined on D such that $|f|^2$ has a harmonic majorant.

If $\theta = (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})$ is a point on the n-torus, then by H^2_θ we mean the Hardy class of multiple valued functions, modulus automorphic of index θ. A subspace M of H^2 is called invariant if $fg \in M$ for all $f \in M$ and $g \in H^\infty$. Any such subspace has the form

$$M = \varphi H^2_\theta$$

where φ is a modulus automorphic inner function and θ is the appropriate index. See [10].

In the language of the Abrahamse-Douglas model theory, a "multiplicity one bundle shift on D" is the multiplication operator S^θ on H^2_θ given by

$$S^\theta(f) = z \cdot f.$$

A scalar c_0 model over D is an operator S^θ_φ on the subspace $H^2_\theta \ominus \varphi H^2_\gamma$ given by

$$S^\theta_\varphi(f) = Pzf = \text{"compression of } S^\theta\text{"}.$$

Here, P denotes orthogonal projection onto $H^2_\theta \ominus \varphi H^2_\gamma$, φ is a modulus automorphic inner function, and γ is an index chosen so $\varphi H^2_\gamma \subseteq H^2_\theta$. See [2] and [3].

The main problem in constructing a successful model theory is nonuniqueness. It can occur that S^θ_φ and S^γ_φ are unitarily equivalent, but $\theta \neq \gamma$.

In this paper we prove the following theorem.

THEOREM. *Suppose φ is singular inner and nonatomic. Then S_φ^θ is unitarily equivalent to S_φ^γ if and only if $\theta = \gamma$.*

Thus we describe a situation where the c_0 model is unique.

We greatly prefer to work with single valued functions. This is accomplished by replacing the H_θ^2 spaces by H^2 spaces with weighted inner products. We give the details below.

In terms of single valued functions, an invariant subspace $M \subseteq H^2$ has the form $M = sH^2$ where the single valued function s is a "rigid function" or simply, an "inner function". That is, $s \in H^\infty$, $|s| = 1$, a.e. [ds] on γ_{n+1}, and there are constants c_i, $i = 1, \ldots, n$, such that $|s| = c_i$, a.e. [ds] on γ_i. (We let ds denote arc length measure.) Every such s can be expressed as

$$s = \varphi \cdot G,$$

where φ is a modulus automorphic inner function and G is the modulus automorphic outer function such that

$$|G| = \begin{cases} c_i & \text{on } \gamma_i,\ i = 1, \ldots, n \\ 1 & \text{on } \gamma_{n+1}. \end{cases}$$

Let f and g belong to H^2. Then

$$\langle f, g \rangle = \int_{\partial D} f\bar{g}\, \mathrm{d}s$$

defines an inner product on H^2. Consider an $n + 1$-tuple $\bar{\alpha} = (\alpha(1), \ldots, \alpha(n + 1))$ of positive numbers. Let α be the function on ∂D defined by

$$\alpha(z) = \alpha(i) \quad \text{for } z \in \gamma_i,\ i = 1, \ldots, n + 1.$$

Then α^2ds is the measure

$$\alpha^2 \mathrm{d}s = \alpha(i)^2 \mathrm{d}s \quad \text{on } \gamma_i,\ i = 1, \ldots, n + 1.$$

Thus

$$\langle f, g \rangle_\alpha = \int_{\partial D} f\bar{g}\, \alpha^2\, \mathrm{d}s$$

defines another inner product on H^2.

Let M be the invariant subspace $M = sH^2$. The notion of invariant subspace is independent of inner product. However, the orthogonal complement

$$M_\alpha^\perp = H^2 \underset{\alpha}{\ominus} M$$

depends on the inner product $\langle\ ,\ \rangle_\alpha$.

Let P_α denote projection onto $H^2 \underset{\alpha}{\ominus} M$ with respect to the inner product $\langle\, ,\rangle_\alpha$. We define the "restricted shift" T_α on $M_\alpha^\perp$.

For $f \in M_\alpha^\perp$

$$T_\alpha f = P_\alpha zf.$$

We claim that for the correct choice of α, T_α is unitarily equivalent to S_φ^0, where $s = \varphi \cdot G$.

Let $\bar{\alpha} = (\alpha(1), \ldots, \alpha(n), 1)$, where the $\alpha(i)$ are chosen so that if F is the modulus automorphic outer function that satisfies $|F| = \alpha$ on ∂D, then the index of F is 0. It is well known that it is possible to do this; see [1], Prop. 1.15.

Consider the map $A: H^2 \to H_0^2$, given by $Af = F \cdot f$. Then we have the equations

$$\int_{\partial D} |f|^2 \alpha^2 \mathrm{d}s = \int_{\partial D} |fF|^2 \mathrm{d}s = \int_{\partial D} |Af|^2 \mathrm{d}s$$

and A is an isometry of H^2 (with inner product $\langle\, ,\rangle_\alpha$) onto H_0^2. Furthermore,

$$A(H^2 \underset{\alpha}{\ominus} sH^2) = H_0^2 \ominus \varphi H_\gamma^2 .$$

It is easy to check that

$$AT_\alpha A^* = S_\varphi^0.$$

We now ask the following question. When are two restricted shifts unitarily equivalent ?

We deal with this problem in two stages. First we prove a result on similarity. Let s_1 and s_2 be two inner functions and let $\langle\, ,\rangle_{\alpha_1}$ and $\langle\, ,\rangle_{\alpha_2}$ be two inner products. Theorem 1.1 says that if the corresponding restricted shifts are similar then $s_1 = s_2$.

We then suppose that s is singular inner and nonatomic. Our main result is Theorem 2.1.

THEOREM 2.1. *Let s be singular inner and nonatomic. Let $M_1 = H^2 \underset{\alpha_1}{\ominus} sH^2$, $M_2 = H^2 \underset{\alpha_2}{\ominus} sH^2$, and let T_1 and T_2 be the corresponding restricted shifts. Then there exists an isometry $U: M_1 \to M_2$ of M_1 onto M_2 such that $UT_1 = T_2U$ if and only if there is an $H^\infty(D)$ function $F(z)$ such that:*

(i) *F is an outer function;*
(ii) *$|F| = \alpha_1/\alpha_2$ on ∂D;*
(iii) *$Uf = F \cdot f$.*

The proof of the theorem relies on the isometry $V: L^2(\mathrm{d}\sigma) \to H^2 \ominus sH^2$ constructed by the author in [5] which generalizes the isometry of Ahern and Clark, [4].

§ 1.

We begin the proof on similarity. Let s be an inner function and let P be orthogonal projection onto $H^2 \ominus sH^2$ with respect to some fixed inner product.

LEMMA 1.1. *If $f \in H^\infty$, then $P(fg) = P(f \cdot Pg)$ for all $g \in H^2$.*

Proof. Let J be orthogonal projection onto sH^2. Then, for $g \in H^2$, since $g = Jg + Pg$ and $f \cdot Jg \in sH^2$, it follows from the linearity of P that $P(fg) = P(f \cdot Pg)$.

Now let t be another inner function. Set $M_1 = H^2 \underset{\alpha_1}{\ominus} sH^2$, $M_2 = H^2 \underset{\alpha_2}{\ominus} tH^2$, and $P_i =$ orthogonal projection onto M_i with respect to $\langle\ ,\ \rangle_{\alpha_i}$, $i = 1,2$. Let $T_1 = P_1 z$ and $T_2 = P_2 z$ be the corresponding restricted shifts on M_1 and M_2, respectively. Suppose T_1 and T_2 are similar. That is, suppose $W: M_1 \to M_2$ is a bounded operator which is 1:1 and onto, and $WT_1 = T_2 W$.

LEMMA 1.2. *Let $R(z)$ be a rational function with poles off $\bar{D}$, the closure of D. Then for $g \in H^2$,*

$$W(P_1 Rg) = P_2(R \cdot WP_1 g).$$

Proof. It is enough to prove this for $R = 1/Q$, where Q is a polynomial. We write

$$WP_1 g = WP_1 \left(\frac{Q}{Q} \cdot g \right) = WP_1(Q \cdot P_1(g/Q)),$$

which follows from Lemma 1.1. Thus

$$WP_1 g = P_2(Q \cdot WP_1(g/Q)),$$

since W commutes with $P_1 z$ and $P_2 z$. It follows that

$$WP_1 g = Q \cdot WP_1(g/Q) + tm,$$

where $m \in H^2$. Therefore, since W maps into M_2, we have

$$P_2 \left(\frac{1}{Q} \cdot WP_1 g \right) = WP_1(g/Q).$$

This completes the proof.

LEMMA 1.3. *Let $f \in H^\infty$. Then for $g \in H^2$*

$$WP_1(fg) = P_2(f \cdot WP_1 g).$$

Proof. We may use Runge's theorem to choose a sequence of rational functions, $\{R_n\}$, with poles off $\bar{D}$, such that $R_n \to f$ in H^2 and $\lim_{n\to\infty} \|R_n\|_\infty \leqslant \|f\|_\infty$. It follows that $R_n g \to fg$ in H^2 and $R_n \cdot WP_1 g \to fWP_1 g$ in H^2. Thus

$$WP_1(fg) = \lim_{n\to\infty} WP_1(R_n g) =$$
$$= \lim_{n\to\infty} P_2(R_n WP_1 g) =$$
$$= P_2(fWP_1 g),$$

as claimed.

THEOREM 1.1. *Suppose M_1, M_2, s, t and W are as above. Then $s = t$.*

Proof. If $f \in H^\infty$ and $g \in H^2$, then $WP_1(fg) = P_2(f \cdot WP_1 g)$. Set $g = 1$ and $f = s$. Then

$$0 = WP_1(s) = P_2(s \cdot WP1).$$

Thus t divides $s \cdot WP1$. We claim $WP1$ can have no common inner factor with t. To prove this, observe that

$$\{WP_1(f): f \in H^\infty\} = \{P_2(f \cdot WP1): f \in H^\infty\}.$$

The first set is dense in M_2. If $WP1$ did have a common inner factor with t, then the closure of $\{P_2(f \cdot WP1): f \in H^\infty\}$ would be a proper subspace of M_2, which is absurd. Thus t divides s. Similarly s divides t. This proves the theorem.

§ 2.

We are now ready to begin the proof of Theorem 2.1. Using the notation of the last section, we assume that $U: M_1 \to M_2$ is an isometry of M_1 onto M_2 such that $UT_1 = T_2 U$. By Theorem 1.1, $s = t$; that is, $M_1 = H^2 \underset{\alpha_1}{\ominus} sH^2$ and $M_2 = H^2 \underset{\alpha_2}{\ominus} sH^2$.

We assume that s is singular and nonatomic. This means

$$s(z) = \exp\left(- \int_{\partial D} P(z, \xi) d\bar{\sigma}(\xi) \right),$$

where $\bar{\sigma}$ is a positive measure on ∂D which is nonatomic and singular with respect to arc length. The kernel $P(z, \xi)$ plays the role that $\frac{\xi + z}{\xi - z}$ plays on the unit disk. A discussion can be found in [8].

Let $\gamma: [0, L] \to \partial D$ be a parameterization of ∂D by arc length. Let σ be the measure on $[0, L]$ induced by $\bar{\sigma}$. For $0 \leqslant \lambda \leqslant L$, define

$$s_\lambda(z) = \exp\left(-\int_0^\lambda P(z, \gamma(\tau))\,\mathrm{d}\sigma(\tau)\right).$$

Then s_λ is an inner function which divides s. In particular, there are constants $c_\lambda(i)$ such that

$$|s_\lambda| = c_\lambda(i), \quad \text{a.e. } [\mathrm{d}s] \text{ on } \gamma_i, \; i = 1, \ldots, n.$$

Let $\beta(\lambda)$ be the function on $[0, L]$ defined by

$$\beta(\lambda) = c_i(\lambda) \qquad \text{if } \gamma(\lambda) \in \gamma_i.$$

Then

$$\mathrm{d}\sigma_1(\lambda) = \beta(\lambda)^2 \mathrm{d}\sigma(\lambda)$$

defines a measure on $[0, L]$.

Suppose $\bar{\alpha}$ is an $n+1$-tuple as in the introduction. We will be interested in the cases where $\bar{\alpha} = \bar{\alpha}_1$ and $\bar{\alpha} = \bar{\alpha}_2$. There is an isometry, $V: L^2(\mathrm{d}\sigma_1) \to H^2 \ominus_\alpha sH^2$, which is natural with respect to the restricted shift. We will have two V's; $V_i: L^2(\mathrm{d}\sigma_1) \mapsto H^2 \ominus_{\alpha_i} sH^2$, $i = 1,2$. The idea behind the proof of the theorem is to study the operators $V_i^* T_i V_i$, $i = 1, 2$, and the isometry $V_2^* U V_1$ on $L^2(\mathrm{d}\sigma_1)$. We will need a detailed description of each V_i and $V_i^* T_i V_i$.

By $|s_\lambda|^2\alpha^2\mathrm{d}s$, we mean the measure on ∂D given by $|s_\lambda(z)|^2\alpha^2\mathrm{d}s$ for $z \in \partial D$. Let $\langle f, g\rangle_{|s_\lambda|\alpha}$ be the inner product defined by

$$\langle f, g\rangle_{|s_\lambda|\alpha} = \int_{\partial D} f\bar{g}|s_\lambda|^2\alpha^2\,\mathrm{d}s.$$

Now let $\zeta \in D$. By $B(z, \zeta, |s_\lambda|\alpha)$, we mean the unique H^2 function of z with the reproducing property

$$\langle f, B(\cdot, \zeta, |s_\lambda|\alpha)\rangle_{|s_\lambda|\alpha} = f(\zeta)$$

for all $f \in H^2$. For a discussion see [5], [7], or [9].

We can now describe the operator $V: L^2(\mathrm{d}\sigma_1) \to H^2 \ominus_\alpha sH^2$. For $c \in L^2(\mathrm{d}\sigma_1)$,

$$(Vc)(z) = \int_0^L \sqrt{2}\,c(\lambda)s_\lambda(z)B(z, \gamma(\lambda), |s_\lambda|\alpha)\alpha(\gamma(\lambda))\mathrm{d}\sigma_1(\lambda).$$

The operator V is an isometry onto $H^2 \ominus_\alpha sH^2$. The operator V^*PzV takes the form

$$V^*PzV = M + R,$$

where $(Mc)(\tau) = \gamma(\tau)c(\tau)$ is a multiplication operator and

$$(\dagger)\qquad (Rc)(\tau) = \int_0^\tau c(\lambda)u(\lambda, \tau)\mathrm{d}\sigma(\lambda)$$

is a compact integral operator. It will be important to observe that $u(\lambda, \tau) \in \in L^2(\mathrm{d}\sigma \times \mathrm{d}\sigma)$ and that with the notation

$$u_\lambda(\tau) = u(\lambda, \tau),$$

we have $u_\lambda \in L^2(\mathrm{d}\sigma(\tau))$ a.e. $[\mathrm{d}\sigma(\lambda)]$ and

$$Vu_\lambda = P(\sqrt{2}\, s_\lambda \cdot (z - \gamma(\lambda)) \cdot B(z, \gamma(\lambda), |s_\lambda|\alpha)\alpha(\gamma(\lambda))).$$

For the proofs of these theorems see [5] or [6].

Let $W = V_2^* U V_1$, where the $V_i : L^2(\mathrm{d}\sigma_1) \to H^2 \underset{\alpha_i}{\ominus} sH^2$ are defined as above.

LEMMA 2.1. *If* $c \in L^2(\mathrm{d}\sigma_1)$ *then* $(Wc)(\lambda) = \mathrm{e}^{ik(\lambda)} \cdot c(\lambda)$ *where* $k(\lambda)$ *is independent of* c.

Before we prove Lemma 2.1 we need the following result.

LEMMA 2.2. *A subspace* M *contained in* $H^2 \ominus sH^2$ *is invariant under the restricted shift* $T = Pz$ *if and only if*

$$M = (H^2 \ominus sH^2) \ominus (H^2 \ominus tH^2)$$

where t *is an inner function which divides* s.

This fact is quite familiar and the proof will be omitted.

Proof of Lemma 2.1. It is enough to show that W commutes with multiplication by the characteristic function of an interval. That is, for $0 \leqslant a < b \leqslant L$ and $c \in L^2$,

$$W(\chi_{a,b} \cdot c) = \chi_{a,b} \cdot (Wc).$$

Let $s_{a,b}(z) = \exp\left(-\int_a^b P(z, \gamma(\lambda))\mathrm{d}\sigma(\lambda)\right)$. Set $M_{a,b} = H^2 \underset{\alpha_1}{\ominus} s_{a,b}H^2$. By Lemma 2.2, $M_1 \ominus M_{a,b}$ is invariant under $T_1 = P_1 z$. Now

$$M_1 = M_{a,b} \oplus (M_1 \ominus M_{a,b})$$

and hence

$$M_2 = U(M_1) = U(M_{a,b}) \oplus U(M_1 \ominus M_{a,b}) =$$

$$= U(M_{a,b}) \oplus (M_2 \ominus U(M_{a,b}))$$

since U is an isometry. Furthermore, $M_2 \ominus U(M_{a,b})$ is invariant under $T_2 = P_2 z$. By Lemma 2.2, $U(M_{a,b}) = H^2 \underset{\alpha_2}{\ominus} tH^2$, where t divides s. We claim that $t = s_{a,b}$.

For this, let $P_{a,b}$ be projection onto $M_{a,b}$ and $Q_{a,b}$ be projection onto $U(M_{a,b})$. Since U is an isometry, $UP_{a,b} = Q_{a,b}U$. Thus

$$UP_{a,b}T_1 = Q_{a,b}UT_1 = Q_{a,b}T_2U.$$

This says that the operators $P_{a,b}T_1$ and $Q_{a,b}T_2$ on $M_{a,b}$ and $U(M_{a,b})$ respectively, are unitarily equivalent. But $P_{a,b}T_1$ is the restricted shift on $H^2 \underset{\alpha_1}{\ominus} s_{a,b}H^2$ and $Q_{a,b}T_2$ is the restricted shift on $H^2 \underset{\alpha_2}{\ominus} tH^2$. By Theorem 1.1, $t = s_{a,b}$.

We now complete the proof that $W\chi_{a,b} = \chi_{a,b}W$.

Let $W(\chi_{a,b}L^2) = N$ and let Q denote projection onto N. If $P =$ "multiplication by $\chi_{a,b}$" denotes projection onto $\chi_{a,b}L^2(d\sigma)$, then $WP = QW$. Thus all we need to show is that $N = \chi_{a,b}L^2(d\sigma_1)$.

By the properties of the isometry V_1,

$$V_1(\chi_{a,b}L^2) = (H^2 \underset{\alpha_1}{\ominus} s_{0,b}H^2) \underset{\alpha_1}{\ominus} (H^2 \underset{\alpha_1}{\ominus} s_{0,a}H^2).$$

Since $U(M_{a,b}) = H^2 \underset{\alpha_2}{\ominus} s_{a,b}H^2$, we have

$$U(V_1\chi_{a,b}L^2) = (H^2 \underset{\alpha_2}{\ominus} s_{0,b}H^2) \underset{\alpha_2}{\ominus} (H^2 \underset{\alpha_2}{\ominus} s_{0,a}H^2) =$$

$$= V_2(\chi_{a,b}L^2).$$

Thus

$$(V_2^* UV_1)(\chi_{a,b}L^2) = \chi_{a,b}L^2,$$

as desired.

It now follows that W takes the form $(Wc)(\lambda) = e^{ik(\lambda)}c(\lambda)$.

LEMMA 2.3. *There exists a holomorphic function $F(z)$ such that $(Uf)(z) = F(z)f(z)$.*

Proof. Recall that $V_i^* T_i V_i = M + R_i$ where M is a multiplication operator and R_i is an integral operator given by equation (†) with kernel u_i satisfying

$$V_i u_i(\lambda, \cdot) = P_i(\sqrt{2}\, s_\lambda \cdot (z - \gamma(\lambda))B(z, \gamma(\lambda), |s_\lambda|\alpha_i)\alpha_i(\gamma(\lambda))) \tag{1}$$

a.e. $[d\sigma(\lambda)]$, where $u_i(\lambda, \cdot) \in L^2(d\sigma(\tau))$, a.e. $[d\sigma(\lambda)]$, $i = 1,2$. Thus

$$W(M + R_1) = (M + R_2)W$$

and

$$WR_1 = R_2W,$$

since $MW = WM$. Therefore, if $c \in L^2(d\sigma_1)$,

$$\int_0^L [u_1(\lambda, \tau)e^{ik(\tau)} - u_2(\lambda, \tau)e^{ik(\lambda)}]\, c(\lambda)d\sigma(\lambda) = 0$$

a.e. $[d\sigma(\tau)]$. Let $g \in L^\infty(d\sigma_1)$. Then

$$\int_0^L\int_0^L [u_1(\lambda, \tau)e^{ik(\tau)} - u_2(\lambda, \tau)e^{ik(\lambda)}]\, c(\lambda)g(\tau)d\sigma(\tau)d\sigma(\lambda) = 0.$$

Since linear combinations of functions $c(\lambda)g(\tau)$ are dense in $L^2(d\sigma \times d\sigma)$, we have shown that

$$e^{ik(\tau)}u_1(\lambda, \tau) = e^{ik(\lambda)}u_2(\lambda, \tau), \quad \text{a.e. } [d\sigma \times d\sigma].$$

Now for all but a σ measure zero set of values λ we have the $L^2(d\sigma(\tau))$ functions

$$(u_1)_\lambda(\tau) = u_1(\lambda, \tau)$$

and

$$(u_2)_\lambda(\tau) = u_2(\lambda, \tau).$$

Thus

$$e^{-ik(\lambda)}W(u_1)_\lambda = (u_2)_\lambda, \quad \text{a.e. } [d\sigma(\lambda)].$$

Applying V_2 to both sides and using the fact that $V_2W = UV_1$, we get

(2) $$e^{-ik(\lambda)}UV_1(u_1)_\lambda = V_2(u_2)_\lambda, \quad \text{a.e. } [d\sigma(\lambda)].$$

Now define the functions

$$f_\lambda(z) = e^{-ik(\lambda)}(z - \gamma(\lambda))B(z, \gamma(\lambda), |s_\lambda|\alpha_1)\alpha_1(\gamma(\lambda))$$

and

$$g_\lambda(z) = (z - \gamma(\lambda))B(z, \gamma(\lambda), |s_\lambda|\alpha_2)\alpha_2(\gamma(\lambda)).$$

Observe that f_λ and g_λ are regular on a neighborhood of $\bar{D}$. See [5] for details. Thus equations (1) and (2) yield

(*) $$UP_1(s_\lambda f_\lambda) = P_2(s_\lambda g_\lambda), \qquad \text{a.e. } [d\sigma(\lambda)].$$

Let $\lambda' < \lambda$, where λ and λ' are in the set where (*) holds. Choose a polynomial m, so $m/f_{\lambda'}$ is regular on a neighborhood of $\bar{D}$. Then

$$UP_1(ms_\lambda f_\lambda) = P_2(mUP_1s_\lambda f_\lambda) = P_2(ms_\lambda g_\lambda).$$

On the other hand

$$UP_1(ms_\lambda f_\lambda) = UP_1\left[\frac{m}{f_{\lambda'}} s_{\lambda',\lambda} f_\lambda s_{\lambda'} f_{\lambda'}\right] =$$

$$= P_2\left[\frac{m}{f_{\lambda'}} s_{\lambda',\lambda} f_\lambda UP_1(s_{\lambda'} f_{\lambda'})\right] = P_2\left[\frac{m}{f_{\lambda'}} s_{\lambda',\lambda} f_{\lambda'} \cdot (s_{\lambda'} g_{\lambda'})\right].$$

Thus

$$ms_\lambda g_\lambda - \frac{m}{f_{\lambda'}} s_\lambda f_\lambda g_{\lambda'} = st,$$

for some $t \in H^2$. This yields

$$mg_\lambda f_{\lambda'} - mfg_{\lambda'} = (s/s_\lambda)tf_{\lambda'}.$$

But the left side is holomorphic on a neighborhood of $\bar{D}$ and can have no singular inner factor. This means $t = 0$. Thus

$$f_{\lambda'} g_\lambda = f_\lambda g_{\lambda'},$$

where λ and λ' belong to a set of full measure with respect to $\mathrm{d}\sigma$. It follows that the quotient

$$g_\lambda / f_\lambda = \mathrm{e}^{ik(\lambda)} \cdot \frac{B(z, \gamma(\lambda), |s_\lambda|\alpha_2)\alpha_2(\gamma(\lambda))}{B(z, \gamma(\lambda), |s_\lambda|\alpha_1)\alpha_1(\gamma(\lambda))}$$

is independent of λ, as λ varies over the aforementioned set. Let $F(z)$ be the function such that $F(z)f_\lambda(z) = g_\lambda(z)$.

We now compute $(Uf)(z)$ for $f \in M_1$. Set $f = V_1 c$ for $c \in L^2(\mathrm{d}\sigma_1)$. Then $Uf = UV_1 c = V_2 W c$. Thus

$$(V_2 W c)(z) = \int_0^L \sqrt{2}\, \mathrm{e}^{ik(\lambda)} c(\lambda) s_\lambda(z) B(z, \gamma(\lambda), |s_\lambda|\alpha_2)\, \alpha_2 \mathrm{d}\sigma_1 =$$

$$= \int_0^L \sqrt{2}\, \mathrm{e}^{ik(\lambda)} c(\lambda) s_\lambda(z) F(z)\, \mathrm{e}^{-ik(\lambda)} B(z, \gamma(\lambda), |s_\lambda|\, \alpha_1)\, \alpha_1 \mathrm{d}\sigma_1 =$$

$$= F(z)(V_1 c)(z).$$

We have shown that U is multiplication by F.

It only takes a little more work to show that F is outer and that $|F|$ has the right boundary values.

If $F(\zeta) = 0$ for some $\zeta \in D$, it follows that for each $f \in M_2$, $f(\zeta) = 0$. In particular, $P_2(B(\cdot, \zeta, \alpha_2)) = 0$. But

$$P_2B(z, \zeta, \alpha_2) = B(z, \zeta, \alpha_2) - \overline{s(\zeta)}\, s(z)B(z, \zeta, |s|\alpha_2)$$

and we would have

$$B(z, \zeta, \alpha_2) = \overline{s(\zeta)} s(z)B(z, \zeta, |s|\alpha_2).$$

This contradicts the fact that $B(\cdot, \zeta, \alpha_2)$ can have no singular inner part. So $F \neq 0$ on D.

Next we observe that $f \in M_1$ implies

$$f(\zeta) = \langle f, P_1B(\cdot, \zeta, \alpha_1)\rangle_{\alpha_1} =$$

$$= \langle Uf, UP_1B(\cdot, \zeta, \alpha_1)\rangle_{\alpha_2}.$$

So

$$F(\zeta)f(\zeta) = \langle Ff, \overline{F(\zeta)}FP_1B(\cdot, \zeta, \alpha_1)\rangle_{\alpha_2}.$$

Since Ff ranges over all of M_2 we have proven that

$$P_2B(\cdot, \zeta, \alpha_2) = \overline{F(\zeta)}FP_1B(\cdot, \zeta, \alpha_1)$$

and hence

$$B(z, \zeta, \alpha_2) - \overline{s(\zeta)}s(z)B(z, \zeta, |s|\alpha_2) =$$

$$= \overline{F(\zeta)}F(z)[B(z, \zeta, \alpha_1) - \overline{s(\zeta)}s(z)B(z, \zeta, |s|\alpha_1)].$$

Thus

$$B(z, \zeta, \alpha_2) - \overline{F(\zeta)}F(z)B(z, \zeta, \alpha_1) =$$

$$= \overline{s(\zeta)}s(z)[B(z, \zeta, |s|\alpha_2) - \overline{F(\zeta)}F(z)B(z, \zeta, |s|\alpha_1)].$$

Now, $F = g_\lambda/f_\lambda$ is meromorphic on a neighborhood of $\bar{D}$. The functions $B(\cdot, \zeta, \alpha_i)$, $i = 1, 2$, are both regular on $\bar{D}$. If we multiply both sides of the last equation by f_λ the left hand side will be regular on $\bar{D}$ and the right hand side will have a singular factor. Thus both sides vanish. In other words

$$B(z, \zeta, \alpha_2) = \overline{F(\zeta)}F(z)B(z, \zeta, \alpha_1).$$

Let $\zeta = z$. We get

$$\|B(\cdot, \zeta, \alpha_2)\|_{\alpha_2}^2 = |F(\zeta)|^2 \|B(\cdot, \zeta, \alpha_1)\|_{\alpha_1}^2$$

and

$$|F(\zeta)|^2 = \frac{\|B(\cdot, \zeta, \alpha_2)\|_{\alpha_2}^2}{\|B(\cdot, \zeta, \alpha_1)\|_{\alpha_1}^2}.$$

As $\zeta \to \zeta^* \in \partial D$ along a normal line to ∂D at ζ^*, the right hand side of the last equation tends to α_1^2/α_2^2. See [5] or [7]. Since $F \neq 0$ and $|F|$ has no nontangential limit equal to 0 at any point on ∂D, it follows that F is outer and $|F| = \alpha_1/\alpha_2$ on ∂D. This completes the proof of Theorem 2.1.

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