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SOME RESULTS ON NORM-IDEAL PERTURBATIONS OF HILBERT SPACE OPERATORS. II

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The present paper is a continuation of the study, we began in [10], of norm-ideal perturbations, based on the invariant $k_{\Phi}(\tau)$.

The results we obtain here, when applied to $\mathscr{C}_n^-$ perturbations of commuting n-tuples of hermitian operators, yield for $n \geqslant 2$ the invariance up to unitary equivalence of the absolutely continuous part and for $n \geqslant 3$ an existence theorem for wave-operators. Let us mention that our result on wave-operators improves the result of J. Voigt [11]. Namely, the ideal $\mathscr{C}_n^-$ is strictly bigger than the union of the ideals $\mathscr{C}_p$ with $p < n$ and the wave-operators we consider are more general. Moreover, we show that for $n \geqslant 3$ all the wave-operators are equal.

Our approach to these questions is based on the following considerations. For a commuting n-tuple of hermitian operators the singular part can be characterized as the greatest invariant subspace, such that the restriction of the n-tuple admits a quasicentral approximate unit relative to the ideal $\mathscr{C}_n^-$. This leads us to consider a generalization in this sense of the decomposition into singular and absolutely continuous part to the framework of representations of C^*-algebras and arbitrary norm-ideals $\mathfrak{S}_{\Phi}^{(0)}$. Under certain conditions the non-vanishing of k_{Φ} on the analogue of the absolutely continuous part implies that sequences of elements of $\mathfrak{S}_{\Phi}^{(0)}$ which commute asymptotically with the algebra considered, must converge strongly to zero, which is the source for our results on the existence of wave-operators.

The present paper has three sections.

In § 1 we give the generalization of the decomposition into singular and absolutely continuous part and prove three theorems, which represent our results on the invariance of the generalized absolutely continuous part and the existence of wave-operators.

In § 2 we apply the results of § 1 to commuting n-tuples of hermitian operators. Besides that, we show how the technique of taking Cesaro-means used in § 1 can also be used to give a new proof for the vanishing of k_n for n-tuples ($n \geqslant 2$) of commuting hermitian operators.

In § 3 we give an example showing that the generalized absolutely continuous part is not invariant in the absence of additional assumptions, like those in § 1. The algebra considered here is the C^*-algebra of a free group and the norm-ideal is the Macaev ideal ($\mathscr{C}_\infty^-$ in our notations).

The present paper is a revised and augmented version of INCREST-preprint No. 37/1979 which wore the same title number III. The difference in numbers is due to the fact that our two first preprints on this subject have been published as a single paper [10].

§ 1.

Let H be a separable infinite-dimensional complex Hilbert space. By $L(H)$, $K(H)$, $P(H)$, $R(H)$, $R_1^+(H)$ (or simply L, K, P, R, R_1^+ when H is fixed) we shall respectively denote the bounded operators on H, the compact operators on H, the finite rank orthogonal projections on H, the finite rank operators on H, and the finite rank positive contractions on H. For n-tuples of operators and norm-ideals $\mathfrak{S}_\Phi^{(0)}$ we shall use the same notations as in [10]. Also the definition and basic properties of the invariant k_Φ are given in [10].

LEMMA 1.1. *Let* $\tau = (T_1, \ldots, T_n) \in (L(H))^n$. *The following assertions are equivalent:*

(i) $k_\Phi(\tau) = 0$;

(ii) *there is a sequence* $(A_m)_1^\infty \subset R$, $A_m \geqslant 0$ *and* $A \in L(H)$ *such that*

$$\text{w-}\lim_{m\to\infty} A_m = A$$

$$\operatorname{Ker} A = 0$$

$$\lim_{m\to\infty} \|[A_m, \tau]\|_\Phi = 0.$$

Proof. Clearly (i) $\Rightarrow$ (ii) so we shall concentrate on (ii) $\Rightarrow$ (i). Since the sequence $(A_m)_1^\infty$ is weakly convergent there is $C > 0$ such that $0 \leqslant A_m \leqslant CI$ for all $m \in \mathbf{N}$. Clearly $[A, \tau] = 0$. Consider the spectral projection P_ε of A corresponding to $[\varepsilon, C]$ where $\varepsilon > 0$. Then it will be sufficient to prove that $k_\Phi(\tau | P_\varepsilon H) = 0$ for all $\varepsilon > 0$, since $k_\Phi(\tau) = \lim_{\varepsilon \downarrow 0} k_\Phi(\tau | P_\varepsilon H)$ by Proposition 1.4 in [10]. Then replacing A_m by $P_\varepsilon A_m | P_\varepsilon H$ it is easily seen that it will be sufficient to prove (ii) $\Rightarrow$ (i) only in the case when the assumption $\operatorname{Ker} A = 0$ is replaced by the stronger assumption $A \geqslant \varepsilon I$ for some $\varepsilon > 0$.

Also, since $\text{w-}\lim_{m\to\infty} A_m = A$ there is a subsequence $A_{m_1}, A_{m_2}, \ldots$ such that for

$$B_k = \frac{1}{k}(A_{m_1} + \ldots + A_{m_k})$$

we have $\operatorname{s-lim}_{k\to\infty} B_k = A$. Also, clearly $\lim_{k\to\infty} |[B_k, \tau]|_\Phi = 0$. Thus replacing the sequence $(A_m)_1^\infty$ by the sequence $(B_k)_1^\infty$ we may assume $\operatorname{s-lim}_{m\to\infty} A_m = A$.

Let now φ be a C^∞-function on $\mathbf{R}$, with compact support such that $0 \leqslant \varphi \leqslant 1$ and $\varphi(0) = 0$, $\varphi(t) = 1$ for $\varepsilon \leqslant t \leqslant C$. Then, $\operatorname{s-lim}_{m\to\infty} \varphi(A_m) = I$. Indeed, let P be a polynomial such that $|P(t) - \varphi(t)| < \delta$ for all $t \in [0, C]$ and let $\xi \in H$, $\|\xi\| = 1$. Then we have

$$\limsup_{m\to\infty} \|\varphi(A_m)\xi - \xi\| \leqslant$$

$$\leqslant \limsup_{m\to\infty} (\|\varphi(A_m)\,\xi - P(A_m)\,\xi\| +$$

$$+ \|P(A_m)\xi - P(A)\xi\| + \|P(A)\xi - \varphi(A)\xi\|) \leqslant 2\delta.$$

Since $\delta > 0$ is arbitrary this shows that $\operatorname{s-lim}_{m\to\infty} \varphi(A_m) = I$.

We have $0 \leqslant \varphi(A_m) \leqslant I$, $\varphi(A_m) \in R$ and $\operatorname{s-lim}_{m\to\infty} \varphi(A_m) = I$, so by Proposition 1.1 in [10], we shall have $k_\Phi(\tau) = 0$ if we can prove that $\lim_{m\to\infty} |[\varphi(A_m), \tau]|_\Phi = 0$.

This will be achieved by the Fourier-transform method from [7]. Remark first that

$$|[\exp(\mathrm{i}t\, A_m), \tau]|_\Phi \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \frac{|t|^n}{n!} n |[A_m, \tau]|_\Phi \, \|A_m\|^{n-1} =$$

$$= |[A_m, \tau]|_\Phi \, |t| \exp(|t| \, \|A_m\|)$$

so that

$$\sup_{|t|\leqslant 1} |[\exp(\mathrm{i}t\, A_m), \tau]|_\Phi \leqslant |[A_m, \tau]|_\Phi \, \mathrm{e}^C.$$

Then taking $n \in \mathbf{N}$ such that $|t| \leqslant n \leqslant |t| + 1$ we obtain

$$|[\exp(\mathrm{i}t\, A_m), \tau]|_\Phi \leqslant n \left|\left[\exp\left(\mathrm{i}\frac{t}{n} A_m\right), \tau\right]\right|_\Phi \leqslant$$

$$\leqslant n\, \mathrm{e}^C |[A_m, \tau]|_\Phi \leqslant (|t| + 1)\, \mathrm{e}^C \, |[A_m, \tau]|_\Phi .$$

Thus denoting by $\hat{\varphi}$ the Fourier-transform of φ we have

$$|[\varphi(A_m), \tau]|_\Phi = \left|\left[\int_{\mathbf{R}} \hat{\varphi}(t) \exp(2\pi\, \mathrm{i}t\, A_m)\, \mathrm{d}t, \tau\right]\right|_\Phi \leqslant$$

$$\leqslant \mathrm{e}^C \, |[A_m, \tau]|_\Phi \int_{\mathbf{R}} |\hat{\varphi}(t)| \, (2\pi|t| + 1)\, \mathrm{d}t$$

and since the last integral is finite we have

$$\lim_{m\to\infty} |[\varphi(A_m), \tau]|_\Phi = 0.$$

Q.E.D.

Consider now a separable C^*-algebra $\mathscr{A}$ with unit and $\mathscr{B} \subset \mathscr{A}$, $\bar{\mathscr{B}} = \mathscr{A}$, $\mathscr{B} \ni 1$, a dense $*$-subalgebra which has an at most countable basis as a vector space over $\mathbf{C}$.

Let $\rho: \mathscr{A} \to L(H)$ be a non-degenerate $*$-representation. We shall denote by $\mathscr{P}_\Phi(\rho)$ (or simply $\mathscr{P}_\Phi$ when ρ is fixed) the set of all selfadjoint projections $P \in (\rho(\mathscr{A}))'$ such that $\rho(\mathscr{B})|PH$ is Φ-well-behaved (see 2.1 in [10]), i.e. such that for all $T_1, \ldots, T_m \in \rho(\mathscr{B})$ we have $k_\Phi(T_1|PH, \ldots, T_m|PH) = 0$.

We shall denote by $E_\Phi^0(\rho)$ (or simply E_Φ^0 when ρ is fixed) the projection

$$E_\Phi^0(\rho) = \bigvee_{P \in \mathscr{P}_\Phi} P.$$

We shall also write $E_\Phi(\rho) = I - E_\Phi^0(\rho)$ and in case ρ is fixed we shall write E_Φ for $E_\Phi(\rho)$.

THEOREM 1.2. *With the above notations we have* $E_\Phi^0(\rho) \in \mathscr{P}_\Phi(\rho)$ *and* $E_\Phi^0(\rho)$ *is a central projection in* $(\rho(\mathscr{A}))'$. *If* $(A_m)_1^\infty \subset \mathfrak{S}_\Phi^{(0)}$ *is such that*

$$\sup_m \|A_m\| < \infty$$

and

$$\lim_{m\to\infty} |[A_m, \rho(b)]|_\Phi = 0$$

for all $b \in \mathscr{B}$ *then*

$$\operatorname*{s-lim}_{m\to\infty} A_m E_\Phi = 0.$$

Proof. We shall first prove that if $P \in \mathscr{P}_\Phi$ and E is the central support (see [13]) of P in $(\rho(\mathscr{A}))'$ then $E \in \mathscr{P}_\Phi$. Since H is separable there are selfadjoint projections $P_j \in (\rho(\mathscr{A}))'$, $P_j \leqslant P$ and partial isometries $V_j \in (\rho(\mathscr{A}))'$ such that $V_j^* V_j = P_j$ and $\sum_{j=1}^{\infty} V_j V_j^* = E$ (this implies $k \neq j \Rightarrow (V_j V_j^*)(V_k V_k^*) = 0$). For an n-tuple τ of elements of $\rho(\mathscr{B})$ we clearly have

$$k_\Phi(\tau|P_j H) = k_\Phi(\tau|V_j V_j^* H)$$

and in view of Proposition 1.4 in [10], we have

$$k_\Phi(\tau|P_j H) \leqslant k_\Phi(\tau|PH) = 0$$

so that $k_\Phi(\tau|V_jV_j^*H) = 0$. Again by Proposition 1.4 in [10] we have

$$k_\Phi(\tau|EH) \leqslant \lim_{n\to\infty} \sum_{j=1}^{n} k_\Phi(\tau|V_jV_j^*H) = 0.$$

Thus $E \in \mathscr{P}_\Phi$.

Denoting by $\mathscr{Z}$ the center of $(\rho(\mathscr{A}))'$ we have

$$E_\Phi^0 = \bigvee_{E\in\mathscr{P}_\Phi\cap\mathscr{Z}} E.$$

Since $P' \in (\rho(A))'$, $P' \leqslant P$, $P \in \mathscr{P}_\Phi \Rightarrow P' \in \mathscr{P}_\Phi$, it follows that we can find $(E_j)_1^\infty \subset \mathscr{P}_\Phi \cap \mathscr{Z}$ such that $i \neq j \Rightarrow E_iE_j = 0$ and $\sum_{j=1}^{\infty} E_j = E_\Phi^0$.

That $E_\Phi^0 \in \mathscr{P}_\Phi \cap \mathscr{Z}$ follows now from Proposition 1.4 in [10].

For the remaining assertion of the theorem remark that if $(A'_m)_1^\infty \subset R$ is such that $\lim_{m\to\infty} |A_m - A'_m|_\Phi = 0$ we may replace $(A_m)_1^\infty$ by $(A'_m)_1^\infty$ and thus we may assume $(A_m)_1^\infty \subset R$. Also since $\rho(\mathscr{B})$ is selfadjoint, the assumptions on the A_m's are also satisfied by their hermitian and antihermitian parts, so that we may assume $A_m = A_m^*$. Consider $B_m = E_\Phi A_m^2 E_\Phi$. Clearly $0 \leqslant B_m \leqslant CI$ for some $C>0$ and it will be sufficient to prove that $\text{w-}\lim_{m\to\infty} B_m = 0$. Assume the contrary, and then replacing the B_m's by some subsequence we may assume

$$\text{w-}\lim_{m\to\infty} B_m = X \neq 0.$$

Consider the projection P onto $H \ominus \operatorname{Ker} X$. We have $P \in (\rho(\mathscr{A}))'$, $P \leqslant E_\Phi$ and $\text{w-}\lim_{m\to\infty} PB_mP|PH = X|PH$. It is also clear that

$$\lim_{m\to\infty} |[(PB_mP|PH),\ (\rho(b)|PH)]|_\Phi = 0$$

for all $b \in \mathscr{B}$. Using Lemma 1.1 we infer that $k_\Phi((\rho(b_1)|PH), \ldots, (\rho(b_n)|PH)) = 0$ for every $(b_1, \ldots, b_n) \in \mathscr{B}$. But this means that $P \in \mathscr{P}_\Phi$ and hence $P \leqslant E_\Phi^0$ so that necessarily $P = 0$ and hence $X = 0$.

Q.E.D.

THEOREM 1.3. *Let ρ_1, ρ_2 be non-degenerate $*$-representations of $\mathscr{A}$ in H and assume there are unitaries $U_n \in I + \mathfrak{S}_\Phi^{(0)}$ such that*

$$\lim_{n\to\infty} |U_n\rho_1(b)U_n^* - \rho_2(b)|_\Phi = 0$$

for all $b \in \mathscr{B}$. Then the following strong limits exist

$$V_1 = \text{s-}\lim_{n\to\infty} U_nE_\Phi(\rho_1)$$

$$V_2 = \text{s-}\lim_{n\to\infty} U_n^*E_\Phi(\rho_2),$$

and $V_1 = V_2^*$, $V_1^*V_1 = E_\Phi(\rho_1)$, $V_1V_1^* = E_\Phi(\rho_2)$. *In particular* $\rho_1|E_\Phi(\rho_1)$ *and* $\rho_2|E_\Phi(\rho_2)$ *are unitarily equivalent.*

Proof. Let $\xi \in E_\Phi(\rho_1)H$ and assume that $(U_n\xi)_{n=1}^\infty$ is not a Cauchy sequence; we shall prove that this leads to a contradiction.

Indeed, if $(U_n\xi)_{n=1}^\infty$ is not a Cauchy sequence then there is $\varepsilon > 0$ and there are $m_1 < m_2 < \ldots$ such that $\| U_{m_k}\xi - U_{m_{k+1}}\xi\| \geqslant \varepsilon$ for all $k \in \mathbf{N}$. For $A_k = U_{m_{k+1}}^* U_{m_k} - I$ we have $A_k \in \mathfrak{S}_\Phi^{(0)}$, $\|A_k\| \leqslant 2$ and

$$\limsup_{k\to\infty} \| A_k\xi \| \geqslant \varepsilon.$$

On the other hand for $b \in \mathscr{B}$ we have

$$\limsup_{k\to\infty} |[A_k, \rho_1(b)]|_\Phi =$$

$$= \limsup_{k\to\infty} |[U_{m_{k+1}}^* U_{m_k}, \rho_1(b)]|_\Phi =$$

$$= \limsup_{k\to\infty} |U_{m_{k+1}}^* U_{m_k}\rho_1(b)U_{m_k}^* U_{m_{k+1}} - \rho_1(b)|_\Phi \leqslant$$

$$\leqslant \limsup_{k\to\infty} (|U_{m_k}\rho_1(b)U_{m_k}^* - \rho_2(b)|_\Phi +$$

$$+ |U_{m_{k+1}}^*\rho_2(b)U_{m_{k+1}} - \rho_1(b)|_\Phi) = 0.$$

Hence using Theorem 1.2 we infer that $\underset{k\to\infty}{\text{s-lim}}\, A_kE_\Phi(\rho_1) = 0$ and hence $\lim_{k\to\infty} \| A_k\xi \| = 0$, a contradiction.

Thus, the strong limit

$$V_1 = \underset{n\to\infty}{\text{s-lim}}\, U_nE_\Phi(\rho_1)$$

exists.

Reversing the roles of ρ_1 and ρ_2, and replacing U_n by U_n^* we also get that

$$V_2 = \underset{n\to\infty}{\text{s-lim}}\, U_n^*E_\Phi(\rho_2)$$

exists.

By the same symmetry it is easily seen that for the remaining assertions it will be sufficient to prove that

$$X = \underset{n\to\infty}{\text{s-lim}}\, E_\Phi^0(\rho_2)U_nE_\Phi(\rho_1) = 0.$$

Assume $X \neq 0$, then $\rho_2(b)X = X\rho_1(b)$ for all $b \in \mathscr{B}$ and for W the partial isometry in the polar decomposition of X we have that W interwines $\rho_1|W^*WH$

and $\rho_2|WW^*H$. But $\rho_1(\mathscr{B})|W^*WH$ is not Φ-well-behaved since $W^*W \leqslant E_\Phi(\rho_1)$ whereas $\rho_2(\mathscr{B})|WW^*H$ is Φ-well-behaved since $WW^* \leqslant E^0_\Phi(\rho_2)$. Thus we must have $X = 0$.

Q.E.D.

Before giving the next theorem, we shall point out two facts that will be used in its proof.

Let $(X_{p,q})_{p,q \in \mathbf{N}} \subset \mathfrak{S}^{(0)}_\Phi$ be such that $\operatorname{s-lim}_{q\to\infty} X_{p,q} = \operatorname{s-lim}_{q\to\infty} X^*_{p,q} = 0$ for all $p \in \mathbf{N}$. Then there are $k_1 < k_2 < k_3 < \ldots$ such that

$$\lim_{j\to\infty} \frac{1}{j} \left| |X_{p,k_1} + \ldots + X_{p,k_j}|_\Phi - |X_{p,k_1} \oplus \ldots \oplus X_{p,k_j}|_\Phi \right| = 0$$

for all $p \in \mathbf{N}$.

Indeed, since $\operatorname{s-lim}_{q\to\infty} X_{p,q} = \operatorname{s-lim}_{q\to\infty} X^*_{p,q} = 0$ it is easily seen that we can find $k_1 < k_2 < k_3 < \ldots$ and finite rank orthogonal projections Q_j such that

$$m \neq n \Rightarrow Q_m Q_n = 0$$

$$0 \leqslant p \leqslant j \Rightarrow |Q_j X_{p,k_j} Q_j - X_{p,k_j}|_\Phi \leqslant \frac{1}{j}.$$

Then we have

$$\limsup_{j\to\infty} \frac{1}{j} \left| |X_{p,k_1} + \ldots + X_{p,k_j}|_\Phi - |X_{p,k_1} \oplus \ldots \oplus X_{p,k_j}|_\Phi \right| \leqslant$$

$$\leqslant \limsup_{j\to\infty} \frac{1}{j} (|X_{p,k_1} - Q_1 X_{p,k_1} Q_1|_\Phi + \ldots + |X_{p,k_j} - Q_j X_{p,k_j} Q_j|_\Phi) +$$

$$+ \limsup_{j\to\infty} \frac{1}{j} |(X_{p,k_1} - Q_1 X_{p,k_1} Q_1) \oplus \ldots \oplus (X_{p,k_j} - Q_j X_{p,k_j} Q_j)|_\Phi = 0.$$

The second fact we shall use is that if $\mathfrak{S}^{(0)}_\Phi \neq \mathscr{C}_1$ then for every $X \in \mathfrak{S}^{(0)}_\Phi$ we have

$$\lim_{j\to\infty} \frac{1}{j} |\underbrace{X \oplus \ldots \oplus X}_{j\text{-times}}|_\Phi = 0.$$

When X is a rank-one orthogonal projection this is a result of Kuroda (see [8], ch. X, § 2, the proof of Theorem 2.3). For general $X \in \mathfrak{S}^{(0)}_\Phi$ this follows immediately now from the fact that rank-one orthogonal projections are total in $\mathfrak{S}^{(0)}_\Phi$.

THEOREM 1.4. *Assume $\mathfrak{S}^{(0)}_\Phi \neq \mathscr{C}_1$ and let ρ_1, ρ_2 be non-degenerate $*$-representations of $\mathscr{A}$ on H, so that $E^0_\Phi(\rho_1) = 0$, $E^0_\Phi(\rho_2) = 0$ and $\rho_1(b) - \rho_2(b) \in \mathfrak{S}^{(0)}_\Phi$ for all*

$b \in \mathscr{B}$. Assume moreover there is a sequence of unitaries $u_n \in \mathscr{B} \cap Z(\mathscr{A})$ ($Z(\mathscr{A})$ the center of $\mathscr{A}$) such that

$$\text{w-}\lim_{n\to\infty} \rho_1(u_n) = \text{w-}\lim_{n\to\infty} \rho_2(u_n) = 0$$

and $\rho_2(u_n^*)\rho_1(u_n)$ is weakly convergent for $n \to \infty$. Then for

$$W = \text{w-}\lim_{n\to\infty} \rho_2(u_n^*)\rho_1(u_n)$$

we have $\operatorname{Ker} W = 0$, $\operatorname{Ker} W^* = 0$ *and $\rho_2(b)W = W\rho_1(b)$ for all $b \in \mathscr{B}$. In particular ρ_1 and ρ_2 are unitarily equivalent.*

Proof. That $\rho_2(b)W = W\rho_1(b)$ is easily seen, as follows. We have

$$W\rho_1(b) - \rho_2(b)W = \text{w-}\lim_{n\to\infty} \rho_2(u_n^*)(\rho_1(b) - \rho_2(b))\rho_1(u_n)$$

and since $\rho_1(b) - \rho_2(b)$ is compact and $\text{w-}\lim\limits_{n\to\infty} \rho_1(u_n) = 0$ we have

$$\text{s-}\lim_{n\to\infty} (\rho_1(b) - \rho_2(b))\, \rho_1(u_n) = 0$$

so that

$$\text{w-}\lim_{n\to\infty} \rho_2(u_n^*)\, (\rho_1(b) - \rho_2(b))\, \rho_1(u_n) = 0.$$

Thus if we can prove that $\operatorname{Ker} W = \operatorname{Ker} W^* = 0$ it will be immediate that ρ_1 and ρ_2 are unitarily equivalent. Also clearly, by symmetry, it will be sufficient to prove that $\operatorname{Ker} W = 0$.

Let $P \in (\rho(\mathscr{A}))'$ denote the projection onto $\operatorname{Ker} W$ and denote by W_n the unitary $\rho_2(u_n^*)\rho_1(u_n)$. Assume $P \neq 0$ and we shall show that this leads to a contradiction.

Thus we assume there is $\xi \in PH$, $\|\xi\| = 1$. Since $\text{w-}\lim\limits_{n\to\infty} W_n P = 0$, replacing the u_n's by some subsequence we may assume that $m \neq n \Rightarrow \|W_n\xi - W_m\xi\| > 1$.

Put $A_k = W_{n_{k+1}}^* W_{n_k} - I$ for a sequence $n_1 < n_2 < \ldots$ which we shall define recurrently below. Clearly P has infinite rank, since otherwise $P \leqslant E_{\Phi}^0(\rho_1)$, so that there is an orthogonal basis $(\xi_k)_{k=1}^{\infty}$ for PH. Let also $(b_n)_{n=1}^{\infty}$ be a basis for the vector space $\mathscr{B}$. We take $n_1 = 1$. Suppose $n_1 < \ldots < n_k$ have been chosen. Then we can find $n_{k+1} > n_k$ such that

$$\|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_{k+1}})\, \rho_2(u_{n_k}^*)\, \rho_1(u_{n_k})\, \xi_i\| < 1/k,$$

$$\|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_k})\, \rho_2(u_{n_{k+1}}^*)\, \rho_1(u_{n_{k+1}})\, \xi_i\| < 1/k$$

for $1 \leqslant i, j \leqslant k + 1$. This is indeed possible, since $\rho_1(b_j) - \rho_2(b_j)$ being compact and $\rho_2(u_m) \xrightarrow{w} 0$, $\rho_2(u_m^*)\rho_1(u_m)P \xrightarrow{w} 0$ we have

$$\lim_{m\to\infty} \|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_m)\, \rho_2(u_{n_k}^*)\, \rho_1(u_{n_k})\xi_i\| = 0$$

$$\lim_{m\to\infty} \|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_k})\, \rho_2(u_m^*)\, \rho_1(u_m)\, \xi_i\| = 0.$$

For the above choice of the sequence $n_1 < n_2 < \dots$ we shall prove that

$$\underset{k\to\infty}{\text{s-lim}}\, [PA_kP, \rho_1(b)] = 0$$

$$\underset{k\to\infty}{\text{s-lim}}\, [PA_k^*P, \rho_1(b)] = 0$$

for all $b \in \mathscr{B}$.

Since $[PA_kP, \rho_1(b)] = P[A_k, \rho_1(b)]P$, $[PA_k^*P, \rho_1(b)] = P[A_k^*, \rho_1(b)]P$, it is easily seen that it will be sufficient to prove that for all $i, j \in \mathbf{N}$ we have

$$\lim_{k\to\infty} \|[A_k, \rho_1(b_j)]\,\xi_i\| = 0$$

$$\lim_{k\to\infty} \|[A_k^*, \rho_1(b_j)]\,\xi_i\| = 0.$$

We have

$$[A_k, \rho_1(b_j)] = [W_{n_{k+1}}^* W_{n_k}, \rho_1(b_j)] =$$

$$= [\rho_1(u_{n_{k+1}}^*)\, \rho_2(u_{n_{k+1}})\, \rho_2(u_{n_k}^*)\, \rho_1(u_{n_k}),\ \rho_1(b_j)] =$$

$$= \rho_1(u_{n_{k+1}}^*)\, \rho_2(u_{n_{k+1}} u_{n_k}^*)\, (\rho_1(b_j) - \rho_2(b_j))\, \rho_1(u_{n_k}) -$$

$$- \rho_1(u_{n_{k+1}}^*)\, (\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_{k+1}})\, \rho_2(u_{n_k}^*)\, \rho_1(u_{n_k}).$$

Thus

$$\limsup_{k\to\infty} \|[A_k, \rho_1(b_j)]\,\xi_i\| \leqslant$$

$$\leqslant \limsup_{k\to\infty} (\|(\rho_1(b_j) - \rho_2(b_j))\, \rho_1(u_{n_k})\, \xi_i\| +$$

$$+ \|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_{k+1}})\, \rho_2(u_{n_k}^*)\, \rho_1(u_{n_k})\, \xi_i\|) = 0$$

because of $\underset{k\to\infty}{\text{w-lim}}\, \rho_1(u_{n_k}) = 0$, the compactness of $\rho_1(b_j) - \rho_2(b_j)$ and the choice of the n_k's.

Similarly we have

$$[A_k^*, \rho_1(b_j)] =$$

$$= \rho_1(u_{n_k}^*)\, \rho_2(u_{n_k} u_{n_{k+1}}^*)\, (\rho_1(b_j) - \rho_2(b_j))\, \rho_1(u_{n_{k+1}}) -$$

$$- \rho_1(u_{n_k}^*)\, (\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_k})\, \rho_2(u_{n_{k+1}}^*)\, \rho_1(u_{n_{k+1}}),$$

so that

$$\limsup_{k\to\infty} \|[A_k^*, \rho_1(b_j)]\,\xi_i\| \leqslant$$

$$\leqslant \limsup_{k\to\infty} (\|(\rho_1(b_j) - \rho_2(b_j))\, \rho_1(u_{n_{k+1}})\, \xi_i\| +$$

$$+ \|(\rho_1(b_j) - \rho_2(b_j))\, \rho_2(u_{n_k})\, \rho_2(u_{n_{k+1}}^*)\, \rho_1(u_{n_{k+1}})\, \xi_i\|) = 0$$

again because of $\text{w-}\lim_{k\to\infty} \rho_1(u_{n_{k+1}}) = 0$, the compactness of $\rho_1(b_j) - \rho_2(b_j)$ and the choice of the n_k's.

Remark now that

$$A_k^* A_k = A_k A_k^* = -A_k - A_k^*$$

since $A_k + I$ is unitary. Thus, we shall have

$$\text{s-}\lim_{k\to\infty} [PA_k^* A_k P, \rho_1(b)] =$$
$$= \text{s-}\lim_{k\to\infty} [\rho_1(b), PA_k P + PA_k^* P] = 0,$$

for all $b \in \mathscr{B}$.

Since $\mathscr{B}$ is selfadjoint we clearly have also:

$$\text{s-}\lim_{k\to\infty} ([PA_k^* A_k P, \rho_1(b)])^* = 0,$$

for all $b \in \mathscr{B}$. Remark further that $u_n \in \mathscr{B}$ implies $W_n \in I + \mathfrak{S}_\Phi^{(0)}$ and hence $A_k \in \mathfrak{S}_\Phi^{(0)}$. The $*$-strong convergence of $[PA_k^* A_k P, \rho_1(b)]$ to 0, easily gives that we can find $k_1 < k_2 < \ldots$ such that

$$\lim_{j\to\infty} \Big| \Big| \frac{1}{j} ([PA_{k_1}^* A_{k_1} P, \rho_1(b)] + \ldots + [PA_{k_j}^* A_{k_j} P, \rho_1(b)]) \Big|_\Phi -$$
$$- \frac{1}{j} |([PA_{k_1}^* A_{k_1} P, \rho_1(b)]) \oplus \ldots \oplus ([PA_{k_j}^* A_{k_j} P, \rho_1(b)])|_\Phi \Big| = 0$$

for all $b \in \mathscr{B}$.

Now, since

$$[PA_k^* A_k P, \rho_1(b)] = P[A_k^* A_k, \rho_1(b)]P =$$
$$= P[\rho_1(b), A_k + A_k^*]P = P(V_k(\rho_1(b) - \rho_2(b))V_k' + V_k''(\rho_1(b) -$$
$$- \rho_2(b))V_k''')P + P(\widetilde{V}_k(\rho_1(b) - \rho_2(b))\widetilde{V}_k' + \widetilde{V}_k''(\rho_1(b) - \rho_2(b))\widetilde{V}_k''')P$$

where $V_k, V_k', V_k'', V_k''', \widetilde{V}_k, \widetilde{V}_k', \widetilde{V}_k'', \widetilde{V}_k'''$ are unitaries, it follows that

$$|([PA_{k_1}^* A_{k_1} P, \rho_1(b)]) \oplus \ldots \oplus ([PA_{k_j}^* A_{k_j} P, \rho_1(b)])|_\Phi \leqslant$$
$$\leqslant 4 | \underbrace{(\rho_1(b) - \rho_2(b)) \oplus \ldots \oplus (\rho_1(b) - \rho_2(b))}_{j\text{-times}} |_\Phi .$$

Since $\mathfrak{S}_\Phi^{(0)} \neq \mathscr{C}_1$ by the remark preceding the theorem, we have

$$\lim_{j\to\infty} \frac{1}{j} | \underbrace{(\rho_1(b) - \rho_2(b)) \oplus \ldots \oplus (\rho_1(b) - \rho_2(b))}_{j\text{-times}} |_\Phi = 0$$

so that for $B_j = \frac{1}{j}(PA_{k_1}^* A_{k_1} P + \ldots + PA_{k_j}^* A_{k_j} P)$ we have $\lim_{j\to\infty} |[B_j, \rho_1(b)]|_\Phi = 0$ for all $b \in \mathscr{B}$. Since $E_\Phi^0(\rho_1) = 0$ and $\|B_j\| \leqslant 4$ using Theorem 1.2, we obtain

$$\text{s-}\lim_{j\to\infty} B_j = 0.$$

To end the proof, recall now that $\|W_n\xi - W_m\xi\| > 1$ for $m \neq n$, so that $\|A_k\xi\| = \|A_k P\xi\| > 1$. This gives $\langle PA_k^* A_k P\xi, \xi\rangle > 1$ so that also $\langle B_j\xi, \xi\rangle > 1$ which is a contradiction.

Q.E.D.

REMARK. The assumption that $\rho_2(u_n^*)\rho_1(u_n)$ is weakly convergent for $n \to \infty$, in the statement of Theorem 1, is not necessary for deriving the unitary equivalence of ρ_1 and ρ_2 since we may always replace $(u_n)_1^\infty$ by some subsequence so that $\rho_2(u_n^*)\,\rho_1(u_n)$ becomes weakly convergent.

For a more restricted class of norm-ideals a much stronger existence result for wave-operators is given in the next theorem.

THEOREM 1.5. *Assume the norm-ideal $\mathfrak{S}_\Phi^{(0)}$ is such that*

$$\lim_{n\to\infty} n^{-1/2}\,\Phi(\underbrace{1, \ldots, 1}_{n\text{-times}}, 0, 0 \ldots) = 0$$

(such norm-ideals are for instance $\mathscr{C}_p$ and $\mathscr{C}_p^-$ for $p > 2$). Let ρ_1, ρ_2 be non-degenerate $*$-representations of $\mathscr{A}$ on H, such that $\rho_1(b) - \rho_2(b) \in \mathfrak{S}_\Phi^{(0)}$ for all $b \in \mathscr{B}$. Assume there are unitaries $u_n \in \mathscr{B} \cap Z(\mathscr{A})$ ($Z(\mathscr{A})$ the center of $\mathscr{A}$) such that

$$\text{w-}\lim_{n\to\infty} \rho_1(u_n) E_\Phi(\rho_1) = 0.$$

Then the following strong limit exists:

$$W = \text{s-}\lim_{n\to\infty} \rho_2(u_n^*)\,\rho_1(u_n)\,E_\Phi(\rho_1).$$

Proof. Put $W_n = \rho_2(u_n^*)\rho_1(u_n)$ and assume that the $W_n E_\Phi(\rho_1)$ are not strongly convergent. We shall prove that this leads to a contradiction. Since the $W_n E_\Phi(\rho_1)$ are not strongly convergent we can find $\xi \in E_\Phi(\rho_1)H$, $\|\xi\| = 1$, $\varepsilon > 0$ and $n_1 < n_2 < \ldots$ such that $\|W_{n_{j+1}}\xi - W_{n_j}\xi\| > \varepsilon$ for all $j \in \mathrm{N}$. Let $A_k = W_{n_{k+1}}^* W_{n_k} - I$. Clearly $\|A_k\xi\| > \varepsilon$ and $A_k \in \mathfrak{S}_\Phi^{(0)}$. For $b \in \mathscr{B}$ we have:

$$\begin{aligned} &E_\Phi(\rho_1)\,[A_k, \rho_1(b)]\,E_\Phi(\rho_1) = \\ &= E_\Phi(\rho_1)\,\rho_1(u_{n_{k+1}}^*)\,\rho_2(u_{n_{k+1}})\,\rho_2(u_{n_k}^*)\,(\rho_1(b) - \rho_2(b))\rho_1(u_{n_k})\,E_\Phi(\rho_1) - \\ &- E_\Phi(\rho_1)\,\rho_1(u_{n_{k+1}}^*)\,(\rho_1(b) - \rho_2(b))\,\rho_2(u_{n_{k+1}})\,\rho_2(u_{n_k}^*)\,\rho_1(u_{n_k})\,E_\Phi(\rho_1) = \\ &= E_\Phi(\rho_1)\,V_k\sigma(b)\,V_k' E_\Phi(\rho_1) + E_\Phi(\rho_1)\,V_k''\sigma(b)\,V_k''' E_\Phi(\rho_1) \end{aligned}$$

where $\sigma(b) = \rho_1(b) - \rho_2(b)$, $V_k = \rho_1(u^*_{n_{k+1}})\,\rho_2(u_{n_{k+1}})\,\rho_2(u^*_{n_k})$, $V'_k = \rho_1(u_{n_k})$, $V''_k = -\rho_1(u^*_{n_{k+1}})$, $V'''_k = \rho_2(u_{n_{k+1}})\,\rho_2(u^*_{n_k})\,\rho_1(u_{n_k})$. We have

$$\operatorname*{w-lim}_{k\to\infty} V'_k E_\Phi(\rho_1) = \operatorname*{w-lim}_{k\to\infty} (V''_k)^* E_\Phi(\rho_1) = 0$$

so that, $\sigma(b)$ being compact, we have

$$\operatorname*{s-lim}_{k\to\infty} E_\Phi(\rho_1)\, V_k \sigma(b)\, V'_k E_\Phi(\rho_1) = 0$$

$$\operatorname*{s-lim}_{k\to\infty} (E_\Phi(\rho_1)\, V''_k \sigma(b)\, V'''_k E_\Phi(\rho_1))^* = 0.$$

Similarly we have:

$$E_\Phi(\rho_1)\,[A^*_k, \rho_1(b)]\,E_\Phi(\rho_1) =$$

$$= E_\Phi(\rho_1)\,\rho_1(u^*_{n_k})\,\rho_2(u_{n_k})\,\rho_2(u^*_{n_{k+1}})\,\sigma(b)\,\rho_1(u_{n_{k+1}})\,E_\Phi(\rho_1) -$$

$$- E_\Phi(\rho_1)\,\rho_1(u^*_{n_k})\,\sigma(b)\,\rho_2(u_{n_k})\,\rho_2(u^*_{n_{k+1}})\,\rho_1(u_{n_{k+1}})\,E_\Phi(\rho_1) =$$

$$= E_\Phi(\rho_1)\,\widetilde{V}_k\sigma(b)\,\widetilde{V}'_k E_\Phi(\rho_1) + E_\Phi(\rho_1)\,\widetilde{V}''_k\sigma(b)\,\widetilde{V}'''_k E_\Phi(\rho_1)$$

where $\widetilde{V}_k = \rho_1(u^*_{n_k})\,\rho_2(u_{n_k})\,\rho_2(u^*_{n_{k+1}})$, $\widetilde{V}'_k = \rho_1(u_{n_{k+1}})$, $\widetilde{V}''_k = -\rho_1(u^*_{n_k})$, $\widetilde{V}'''_k = \rho_2(u_{n_k})\,\rho_2(u^*_{n_{k+1}})\,\rho_1(u_{n_{k+1}})$.

Since

$$\operatorname*{w-lim}_{k\to\infty} \widetilde{V}'_k E_\Phi(\rho_1) = \operatorname*{w-lim}_{k\to\infty} (\widetilde{V}''_k)^* E_\Phi(\rho_1) = 0$$

and $\sigma(b)$ is compact, we have

$$\operatorname*{s-lim}_{k\to\infty} E_\Phi(\rho_1)\,\widetilde{V}_k\sigma(b)\,\widetilde{V}'_k E_\Phi(\rho_1) = 0$$

$$\operatorname*{s-lim}_{k\to\infty} (E_\Phi(\rho_1)\,\widetilde{V}''_k\sigma(b)\,\widetilde{V}'''_k E_\Phi(\rho_1))^* = 0.$$

Let $(b_i)_{i=1}^{\infty} \subset \mathscr{B}$ be a basis of $\mathscr{B}$ and put

$$X_{k,i} = E_\Phi(\rho_1)\, V_k \sigma(b_i)\, V_k' E_\Phi(\rho_1) + E_\Phi(\rho_1)\, \tilde{V}_k \sigma(b_i)\, \tilde{V}_k' E_\Phi(\rho_1)$$

$$Y_{k,i} = E_\Phi(\rho_1)\, V_k'' \sigma(b_i)\, V_k''' E_\Phi(\rho_1) + E_\Phi(\rho_1)\, \tilde{V}_k'' \sigma(b_i)\, \tilde{V}_k''' E_\Phi(\rho_1)$$

so that $X_{k,i}$, $Y_{k,i} \in \mathfrak{S}_\Phi^{(0)}$ and

$$\operatorname*{s-lim}_{k\to\infty} X_{k,i} = \operatorname*{s-lim}_{k\to\infty} Y_{k,i}^* = 0$$

$$E_\Phi(\rho_1)\,[A_k + A_k^*, \rho_1(b_i)]\, E_\Phi(\rho_1) = X_{k,i} + Y_{k,i}.$$

For $\alpha > 0$ let $\sigma_\alpha(b_i)$ be a finite rank operator such that $|\sigma(b_i) - \sigma_\alpha(b_i)|_\Phi < \alpha/4$. By $X_{k,i}^{(\alpha)}$, $Y_{k,i}^{(\alpha)}$ we shall denote the operators given by the same formulae as $X_{k,i}$, $Y_{k,i}$ with $\sigma(b_i)$ replaced by $\sigma_\alpha(b_i)$.

Remark now that we can find $k_1 < k_2 < \ldots$ and finite rank orthogonal projections $(Q_j)_1^\infty$ such that

$$m \neq n \Rightarrow Q_m Q_n = 0$$

$$1 \leqslant i \leqslant j \Rightarrow \begin{cases} |X_{k_j,i} - X_{k_j,i} Q_j|_\Phi < 1/j \\ |Q_j Y_{k_j,i} - Y_{k_j,i}|_\Phi < 1/j. \end{cases}$$

Then

$$\limsup_{j\to\infty} \frac{1}{j}\left(\left|\sum_{s=1}^{j} X_{k_s,i}\right|_\Phi + \left|\sum_{s=1}^{j} Y_{k_s,i}\right|_\Phi\right) =$$

$$= \limsup_{j\to\infty} \frac{1}{j}\left(\left|\sum_{s=1}^{j} X_{k_s,i} Q_s\right|_\Phi + \left|\sum_{s=1}^{j} Q_s Y_{k_s,i}\right|_\Phi\right) \leqslant$$

$$\leqslant \alpha + \limsup_{j\to\infty} \frac{1}{j}\left(\left|\sum_{s=1}^{j} X_{k_s,i}^{(\alpha)} Q_s\right|_\Phi + \left|\sum_{s=1}^{j} Q_s Y_{k_s,i}^{(\alpha)}\right|_\Phi\right).$$

Consider left and right polar decompositions $X_{k_s,i}^{(\alpha)} Q_s = L_{s,i} R_{s,i}$, $Q_s Y_{k_s,i}^{(\alpha)} = R_{s,i}' L_{s,i}'$ where

$$R_{s,i} = (Q_s X_{k_s,i}^{(\alpha)*} X_{k_s,i}^{(\alpha)} Q_s)^{1/2}, \quad R_{s,i}' = (Q_s Y_{k_s,i}^{(\alpha)} Y_{k_s,i}^{(\alpha)*} Q_s)^{1/2}.$$

Consider further

$$M_{j,i} = j^{-1/2} \sum_{s=1}^{j} L_{s,i}, \quad M_{j,i}' = j^{-1/2} \sum_{s=1}^{j} L_{s,i}'.$$

It is easily seen that

$$\|M_{j,i}\| \leqslant 1, \quad \|M_{j,i}'\| \leqslant 1.$$

We have:

$$\limsup_{j\to\infty}\frac{1}{j}\left(\left|\sum_{s=1}^{j}X_{k_s,i}^{(\alpha)}Q_s\right|_{\Phi}+\left|\sum_{s=1}^{j}Q_sY_{k_s,i}^{(\alpha)}\right|_{\Phi}\right)=$$

$$=\limsup_{j\to\infty}j^{-1/2}\left(\left|M_{j,i}\sum_{s=1}^{j}R_{s,i}\right|_{\Phi}+\left|\left(\sum_{s=1}^{j}R'_{s,i}\right)M'_{j,i}\right|_{\Phi}\right)\leqslant$$

$$\leqslant\limsup_{j\to\infty}j^{-1/2}\left(\left|\sum_{s=1}^{j}R_{s,i}\right|_{\Phi}+\left|\sum_{s=1}^{j}R'_{s,i}\right|_{\Phi}\right)=\limsup_{j\to\infty}j^{-1/2}\left(\left|\bigoplus_{s=1}^{j}R_{s,i}\right|_{\Phi}+\right.$$

$$\left.+\left|\bigoplus_{s=1}^{j}R'_{s,i}\right|_{\Phi}\right)=\limsup_{j\to\infty}j^{-1/2}\left(\left|\bigoplus_{s=1}^{j}X_{k_s,i}^{(\alpha)}Q_s\right|_{\Phi}+\left|\bigoplus_{s=1}^{j}Q_sY_{k_s,i}^{(\alpha)}\right|_{\Phi}\right)\leqslant$$

$$\leqslant\limsup_{j\to\infty}j^{-1/2}\left(\left|\bigoplus_{s=1}^{j}X_{k_s,i}^{(\alpha)}\right|_{\Phi}+\left|\bigoplus_{s=1}^{j}Y_{k_s,i}^{(\alpha)}\right|_{\Phi}\right)\leqslant$$

$$\leqslant\limsup_{j\to\infty}j^{-1/2}\left(\left|\bigoplus_{s=1}^{j}E_{\Phi}(\rho_1)V_{k_s}\sigma_\alpha(b_i)V'_{k_s}E_{\Phi}(\rho_1)\right|_{\Phi}+\right.$$

$$+\left|\bigoplus_{s=1}^{j}E_{\Phi}(\rho_1)V''_{k_s}\sigma_\alpha(b_i)V'''_{k_s}E_{\Phi}(\rho_1)\right|_{\Phi}+\left|\bigoplus_{s=1}^{j}E_{\Phi}(\rho_1)\widetilde{V}_{k_s}\sigma_\alpha(b_i)\widetilde{V}'_{k_s}E_{\Phi}(\rho_1)\right|_{\Phi}+$$

$$\left.+\left|\bigoplus_{s=1}^{j}E_{\Phi}(\rho_1)\widetilde{V}''_{k_s}\sigma_\alpha(b_i)\widetilde{V}'''_{k_s}E_{\Phi}(\rho_1)\right|_{\Phi}\right)\leqslant$$

$$\leqslant\limsup_{j\to\infty}4j^{-1/2}\,|\underbrace{\sigma_\alpha(b_i)\oplus\ldots\oplus\sigma_\alpha(b_i)}_{j\text{-times}}|_{\Phi}.$$

Now if X is a finite rank operator we have $\lim_{j\to\infty}j^{-1/2}|\underbrace{X\oplus\ldots\oplus X}_{j\text{-times}}|_{\Phi}=0.$
Indeed if X is a rank-one orthogonal projection, this is just our assumption on Φ and the general case follows from the fact that every finite rank operator is a linear combination of such projections. This gives

$$\limsup_{j\to\infty}\frac{1}{j}\left(\left|\sum_{s=1}^{j}X_{k_s,i}\right|_{\Phi}+\left|\sum_{s=1}^{j}Y_{k_s,i}\right|_{\Phi}\right)\leqslant\alpha$$

and since $\alpha>0$ is arbitrary we have

$$\lim_{\to\infty}\frac{1}{j}\left(\left|\sum_{s=1}^{j}X_{k_s,i}\right|_{\Phi}+\left|\sum_{s=1}^{j}Y_{k_s,i}\right|_{\Phi}\right)=0.$$

Since $A_k + I$ is unitary, we have $A_k^* A_k = -A_k - A_k^*$ and hence, by what has been just proved, we have

$$\lim_{j\to\infty} \frac{1}{j} \left| E_\Phi(\rho_1) \left[\sum_{s=1}^{j} A_{k_s}^* A_{k_s},\ \rho_1(b_i) \right] E_\Phi(\rho_1) \right|_\Phi = 0.$$

Thus for $B_j = \frac{1}{j} \sum_{s=1}^{j} E_\Phi(\rho_1)\, A_{k_s}^* A_{k_s} E_\Phi(\rho_1)$ we have $\|B_j\| \leqslant 4$, $B_j \geqslant 0$, $B_j \in \mathfrak{S}_\Phi^{(0)}$ and

$$\lim_{j\to\infty} \|[B_j,\ \rho_1(b)]\|_\Phi = 0$$

for all $b \in \mathscr{B}$.

Now, $\langle E_\Phi(\rho_1) A_k^* A_k E_\Phi(\rho_1)\xi, \xi\rangle \geqslant \varepsilon^2$ and hence also $\langle B_j\xi, \xi\rangle \geqslant \varepsilon^2$. But by virtue of Theorem 1.2, we must have

$$\text{s-}\lim_{j\to\infty} B_j = \text{s-}\lim_{j\to\infty} B_j E_\Phi(\rho_1) = 0.$$

This contradiction concludes the proof.

Q.E.D.

REMARK. The wave-operator W in the preceding theorem does not depend on the particular sequence $(u_n)_{n\geqslant 0}$. Indeed, mixing two different such sequences, one gets a new sequence of unitaries for which the theorem applies.

COROLLARY 1.6. *Assume $\mathfrak{S}_\Phi^{(0)}$ is such that*

$$\lim_{n\to\infty} n^{-1/2}\, \Phi\, (\underbrace{1, \ldots, 1}_{n\text{-times}}, 0, 0, \ldots) = 0.$$

Let ρ be a non-degenerate $*$-representation of $\mathscr{A}$ on H and let $T \in L(H)$ be such that $[T, \rho(\mathscr{B})] \subset \mathfrak{S}_\Phi^{(0)}$. Assume there are unitaries $u_n \in \mathscr{B} \cap Z(\mathscr{A})$ such that

$$\text{w-}\lim_{n\to\infty} \rho(u_n)\, E_\Phi(\rho) = 0.$$

Then the following strong limit exists:

$$\text{s-}\lim_{n\to\infty} \rho(u_n^*)\, T\, \rho(u_n) E_\Phi(\rho).$$

Proof. First, note that it will be sufficient to prove the corollary for hermitian operators T. Indeed, if $[T, \rho(\mathscr{B})] \subset \mathfrak{S}_\Phi^{(0)}$ then also $[T + T^*, \rho(\mathscr{B})] \subset \mathfrak{S}_\Phi^{(0)}$ and $[T - T^*, \rho(\mathscr{B})] \subset \mathfrak{S}_\Phi^{(0)}$.

Next, remark that the corollary for T unitary implies the corollary for T hermitian.

Indeed, if T is hermitian, there is no loss of generality in assuming $\|T\| < 1$. Using as in Lemma 1.1 the Fourier-transform method of [7], we have that $[T, \rho(\mathscr{B})] \subset \mathfrak{S}_{\Phi}^{(0)}$ implies that $[(I - T^2)^{1/2}, \rho(\mathscr{B})] \subset \mathfrak{S}_{\Phi}^{(0)}$. Then $T_1 = T + \mathrm{i}(I - T^2)^{1/2}$ is unitary and $[T_1, \rho(\mathscr{B})] \subset \mathfrak{S}_{\Phi}^{(0)}$. The corollary applied to the unitaries T_1, T_1^* will imply the corollary for the hermitian operator T.

Thus it will be sufficient to consider only the case when T is unitary. Defining in this case $\rho_2(a) = T^*\rho(a)T$ for $a \in \mathscr{A}$ and $\rho_1 = \rho$, the representations ρ_1 and ρ_2 satisfy the assumptions of the preceding theorem. It follows that

$$\operatorname*{s-lim}_{n\to\infty} \rho(u_n^*)T\rho(u_n)E_{\Phi}(\rho) =$$

$$= T \operatorname*{s-lim}_{n\to\infty} \rho_2(u_n^*)\,\rho_1(u_n)\,E_{\Phi}(\rho_1)$$

exists.

Q.E.D.

From the preceding corollary a two-spaces version of Theorem 1.5 is easily obtained.

COROLLARY 1.7. *Assume* $\mathfrak{S}_{\Phi}^{(0)}$ *is such that*

$$\lim_{n\to\infty} n^{-1/2}\ \Phi(\underbrace{1, \ldots, 1}_{n\text{-times}}, 0, 0, \ldots) = 0.$$

Let ρ_1, ρ_2 *be non-degenerate* $*$*-representations of* $\mathscr{A}$ *on Hilbert spaces* H_1 *and respectively* H_2 *and let* $J \in L(H_1, H_2)$ *be such that*

$$\rho_2(b)J - J\rho_1(b) \in \mathfrak{S}_{\Phi}^{(0)}$$

for all $b \in \mathscr{B}$. *Assume there are unitaries* $u_n \in \mathscr{B} \cap \mathrm{Z}(\mathscr{A})$ *such that*

$$\operatorname*{w-lim}_{n\to\infty} \rho_k(u_n)\,E_{\Phi}(\rho_k) = 0 \qquad (k = 1, 2).$$

Then the following strong limit exists:

$$W = \operatorname*{s-lim}_{n\to\infty} \rho_2(u_n^*)\,J\rho_1(u_n)E_{\Phi}(\rho_1).$$

Proof. Consider $H = H_1 \oplus H_2$, $\rho = \rho_1 \oplus \rho_2$ and

$$T = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}.$$

The present corollary follows immediately from the preceding corollary applied to ρ and T.

Q.E.D.

§ 2.

This section deals with applications of § 1 to n-tuples of commuting hermitian operators.

First we show how Theorem 1.4 can be used to prove the invariance under $\mathscr{C}_n^-$-perturbations of the absolutely continuous part for $n \geqslant 2$, thus improving the result we had obtained in [10] where this was obtained only for the case when the multiplicity-function of the absolutely continuous part is integrable. We also formulate explicitely the consequences of Theorem 1.5 for commuting n-tuples of hermitian operators.

Next we show how the technique of taking Cesaro-means, used in § 1, can be also used to prove that the invariant k_Φ for certain norm-ideals like the $\mathscr{C}_p$-classes with $p > 1$, takes only the values 0 and ∞. This seems to indicate that in general the results about k_p with $p > 1$ are not sharp. We also show that this gives a new proof for the vanishing of k_n for commuting n-tuples ($n \geqslant 2$) of hermitian operators.

We begin with the application of Theorem 1.4 to the invariance of absolutely continuous parts.

Let $\delta = (D_1, \ldots, D_n)$ be such an n-tuple, $K \subset \mathbf{R}^n$ its spectrum and $H = H_a \oplus \oplus H_s$ the decomposition of H into the absolutely continuous and singular subspace for δ. Consider $\mathscr{B} \subset C(K)$, $\mathscr{B} \ni 1$, the $*$-subalgebra generated by the restriction to K of the coordinate functions on $\mathbf{R}^n$ and of some sequence $(\varphi_j)_1^\infty$ of C^∞-functions on $\mathbf{R}^n$. Let further ρ be the representation of $C(K)$ which takes the coordinate functions into $D_1, \ldots, D_n$. Then we assert that $E^0_{\Psi_n}(\rho)$, $E_{\Psi_n}(\rho)$, where $\mathfrak{S}^{(0)}_{\Psi_n} = \mathscr{C}_n^-$, are just the projections onto H_s and H_a.

We have $E^0_{\Psi_n}(\rho)H \subset H_s$ since $k_n^-(\delta | E^0_{\Psi_n}(\rho)H) = 0$, which by Corollary 4.6 in [10] implies that $\delta | E^0_{\Psi_n}(\rho)H$ has spectral measure singular with respect to Lebesgue-measure. In order to prove also the reverse inclusion, $H_s \subset E^0_{\Psi_n}(\rho)H$, we must show that for $g_1, \ldots, g_m \in \mathscr{B}$ we have $k_n^-((\rho(g_1)|H_s), \ldots, (\rho(g_m)|H_s)) = 0$. Now, by Proposition 4.1 in [10] we have $k_n^-(\delta|H_s) = 0$, so that there are $X_j \in R_1^+(H_s)$ such that $X_j \uparrow I$ and

$$\lim_{j\to\infty} |[X_j, (\delta|H_s)]|_n^- = 0.$$

We shall prove that

$$\lim_{j\to\infty} |[X_j, (\rho(g_k)|H_s)]|_n^- = 0$$

for $k = 1, \ldots, m$. This will be shown by the Fourier-transform method of [7]. Let $h_1, \ldots, h_m$ be C^∞-functions on $\mathbf{R}^n$ with compact support, such that $h_k|K = g_k$. We have

$$\rho(g_k) = \int_{\mathbf{R}^n} \exp(2\pi i(\xi_1 D_1 + \ldots + \xi_n D_n))\hat{h}_k(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n$$

where $\hat{h}_k$ is the Fourier-transform of h_k. We have

$$|[(\exp(2\pi i(\xi_1 D_1 + \ldots + \xi_n D_n))|H_s), X_j]|_n^- \leqslant$$
$$\leqslant C(1 + |\xi_1| + \ldots + |\xi_n|)\, |[(\delta|H_s), X_j]|_n^-$$

for some constant $C > 0$.

Hence

$$|[X_j, (\rho(g_k)|H_s)]|_n^- \leqslant$$
$$\leqslant C|[(\delta|H_s), X_j]|_n^- \int_{\mathbf{R}^n} (1 + |\xi_1| + \ldots + |\xi_n|)\, |\hat{h}_k(\xi_1, \ldots, \xi_n)|\, d\xi_1 \ldots d\xi_n$$

and the last integral being finite we have

$$\lim_{j\to\infty} |[X_j, (\rho(g_k)|H_s)]|_n^- = 0$$

thus establishing our assertion.

LEMMA 2.1. *Let $\delta, \delta' \in (L(H))^n$ be n-tuples, with $n \geqslant 2$, of commuting hermitian operators. Assume the spectral measures of δ and δ' are absolutely continuous with respect to Lebesgue measure and assume $\delta - \delta' \in (\mathscr{C}_n^-)^n$. Then δ and δ' are unitarily equivalent.*

Proof. Let $K \subset \mathbf{R}^n$ be a compact set containing the spectra of δ and δ' (which are in fact equal) and consider $\mathscr{B} \subset C(K)$, $\mathscr{B} \ni 1$ the $*$-subalgebra generated by the coordinate-functions and by $f(x_1, \ldots, x_n) = \exp(ix_1)$, restricted to K. Denote by ρ_1, ρ_2 the representations of $C(K)$ which take the coordinate functions into δ and δ' respectively.

It is easily seen that $\rho_1(b) - \rho_2(b) \in \mathscr{C}_n^-$ for all $b \in \mathscr{B}$. Also, since the spectral measures of δ, δ' are absolutely continuous with respect to Lebesgue measure it is easily seen that

$$\text{w-}\lim_{m\to\infty} \rho_1(f^m) = \text{w-}\lim_{m\to\infty} \rho_2(f^m) = 0.$$

Moreover, by the discussion preceding the statement of the present lemma, we have $E^{(0)}_{\Psi_n}(\rho_1) = E^{(0)}_{\Psi_n}(\rho_2) = 0$. Since $n \geqslant 2$ we may then use Theorem 1.4 and the Remark following Theorem 1.4 with $\mathscr{A} = C(K)$, $\Phi = \Psi_n$, $u_n = f^n$ to obtain that ρ_1, ρ_2 are unitarily equivalent.

Q.E.D.

THEOREM 2.2. *Let $\delta, \delta' \in (L(H))^n$ be n-tuples, with $n \geqslant 2$, of commuting hermitian operators, such that $\delta - \delta' \in (\mathscr{C}_n^-)^n$. Then the absolutely continuous parts of δ and δ' are unitarily equivalent.*

Proof. Consider a third n-tuple $\delta'' \subset (L(H))^n$ of commuting hermitian operators, the spectral measure of which is absolutely continuous with respect to Lebesgue measure, the spectrum of which contains the spectra of δ, δ' and which has a cyclic vector. Clearly the spectrum and the essential spectrum of δ'' are equal. Let now $\delta = \delta_a \oplus \delta_s$, $\delta' = \delta'_a \oplus \delta'_s$ be the decompositions of δ and δ' into absolutely continuous and singular part. Let $\mathscr{A}$ denote the C^*-algebra generated by $\delta_a \oplus \delta''$ which is isomorphic to $C(K)$ where K is the spectrum of δ''. Let further $\mathscr{B} \subset \mathscr{A}$ be the $*$-algebra generated by I and $\delta_a \oplus \delta''$. Since the spectrum and essential spectrum of $\delta_a \oplus \delta''$ are equal, it follows that the canonical map $p\colon L \to L/K$ is injective on $\mathscr{A}$. Thus $p(\mathscr{A})$ is isomorphic to $C(K)$ and since the spectrum of δ_s is contained in K there is a non-degenerate $*$-representation $\rho\colon p(\mathscr{A}) \to L(H)$ such that $\rho(p(\delta_a \oplus \delta'')) = \delta_s$. Since the spectral measure of δ_s is singular it follows by Proposition 4.1 of [10] that $\rho(p(\mathscr{B}))$ is Ψ_n-well-behaved. Applying Theorem 2.4 of [10] to ρ, $\mathscr{A}$, $\mathscr{B}$ which we have just defined, we obtain that there is a n-tuple $\tau \in (L(H \oplus H))^n$ which is unitarily equivalent to $\delta_a \oplus \delta''$ and such that $(\delta_s \oplus \delta_a \oplus \delta'') - \tau = (\delta \oplus \delta'') - \tau \in \mathscr{C}_n^-$. By the same kind of argument there is τ', unitarily equivalent to $\delta'_a \oplus \delta''$ and such that $(\delta' \oplus \delta'') - \tau' \in \mathscr{C}_n^-$. Since $\delta - \delta' \in \mathscr{C}_n^-$ it follows that $\tau - \tau' \in \mathscr{C}_n^-$ and hence by Lemma 2.1 τ and τ' are unitarily equivalent. This implies that $\delta_a \oplus \delta''$ and $\delta'_a \oplus \delta''$ are unitarily equivalent. Thus for m, m', m'' the multiplicity functions of δ_a, δ'_a, δ'' we have $m + m'' = m' + m''$ almost everywhere with respect to Lebesgue measure. Now since δ'' has a cyclic vector, it follows that m'' is an L^∞-function, so that the preceding equality implies $m = m'$ almost everywhere.

Q.E.D.

We pass now to the application of Theorem 1.5 to commuting n-tuples of hermitian operators. This runs much in the same way as the application of Theorem 1.4, so that we shall only state the result and leave the details to the reader.

THEOREM 2.3. *Let $\delta, \delta' \in (L(H))^n$ be n-tuples of commuting hermitian operators, with $n \geqslant 3$ and such that $\delta - \delta' \in \mathscr{C}_n^-$. Let $(f_m)_{m=1}^\infty$ be C^∞-functions in some neighborhood of the union of the spectra of δ and δ', such that $|f_m| = 1$ and assume $(f_m)_1^\infty$ is weakly convergent to zero in $L^2(K)$ (K is the spectrum of δ and the L^2-space is with respect to the restriction of Lebesgue-measure to K).*

Then the following strong-limit exists,

$$W = \operatorname*{s\text{-}lim}_{m \to \infty} (f_m(\delta'))^* f_m(\delta) E_a$$

where E_a is the projection onto the absolutely continuous subspace of δ. Moreover, W does not depend on the particular sequence $(f_m)_1^\infty$.

In the same way there are corresponding results for the two corollaries of Theorem 1.5 which the reader will have no difficulty to state.

We pass now to the application of the technique of Cesaro-means.

We shall say that $\mathfrak{S}_\Phi^{(0)}$ has *property* (Σ) if the following condition is satisfied: *For every sequence* $(X_n)_1^\infty \subset R(H)$ *such that* $\sup\limits_n |X_n|_\Phi < \infty$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} |X_1 \oplus \ldots \oplus X_n|_\Phi = 0.$$

It is easily seen that the ideals $\mathscr{C}_p$ with $p > 1$ have property (Σ), and that the ideals $\mathscr{C}_p^-$ do not have property (Σ).

PROPOSITION 2.4. *Let* $\tau = (T_1, \ldots, T_n) \in (L(H))^n$ *and assume* $\mathfrak{S}_\Phi^{(0)}$ *has property* (Σ). *Then* $k_\Phi(\tau)$ *is either* 0 *or* ∞.

Proof. It will be sufficient to prove that $k_\Phi(\tau) < \infty$ implies $k_\Phi(\tau) = 0$.

Thus assume $k_\Phi(\tau) < \infty$. Then in view of the definition of k_Φ it is easily seen that we can find a sequence $(A_m)_1^\infty \subset R_1^+$, $A_m \uparrow I$ such that the subspaces

$$H_m = \sum_j [A_m, T_j]H + \sum_j [A_m, T_j]^* H$$

are pairwise orthogonal and $\sup\limits_m |[A_m, \tau]|_\Phi < \infty$. Defining $B_m = \dfrac{1}{m}(A_1 + \ldots + A_m)$ we have $B_m \in R_1^+$, $B_m \uparrow I$ and

$$\lim_{m\to\infty} |[B_m, \tau]|_\Phi = \lim_{m\to\infty} \frac{1}{m} \left| \bigoplus_{j=1}^\infty [A_m, \tau] \right|_\Phi = 0$$

because of property (Σ).

Q.E.D.

Proposition 2.4 can be used to give *a new proof for the fact that* $k_n(\tau) = 0$ *for an n-tuple* τ *of commuting hermitian operators, when* $n \geqslant 2$. Indeed, it is sufficient to prove this only for the case when τ has a cyclic vector. Using Proposition 2.4 we have to prove only that $k_n(\tau) < \infty$ which can be seen easily as follows. Let ξ be the cyclic vector of τ and assume also $\|\tau\| < 1$, which is no loss of generality. Then there are $(2m)^n$ disjoint Borel sets Σ_j in $\mathbf{R}^n$, of diameter $\leqslant n/m$, which cover the spectrum of τ. Let E_j be the spectral projection of τ for the Borel set Σ_j and consider the orthogonal projection P_m onto

$$\mathbf{C}E_1\xi + \ldots + \mathbf{C}E_{(2m)^n}\xi.$$

Then $|[P_m, \tau]|_n^n \leqslant (2m)^n \cdot \left(\dfrac{2n}{m}\right)^n = (4n)^n$ and $P_m \xrightarrow{w} I$, which shows that $k_n(\tau) < \infty$.

§ 3.

In this section we shall study k_∞^- for the regular representation of the free group on n-generators F_n. This will provide an example where the generalized absolutely continuous part is not invariant up to unitary equivalence under perturbations.

The n generators of the free group F_n ($n \geqslant 2$) will be denoted by $g_1, \ldots, g_n$. The C^*-algebra of F_n will be denoted by $\mathscr{A}_n = C^*(F_n)$ and we shall consider $\mathscr{B}_n \subset \mathscr{A}_n$ the $*$-subalgebra generated by the unitaries $u_1, \ldots, u_n$ which correspond to the generators $g_1, \ldots, g_n$. The norm-ideal $\mathscr{C}_\infty^-$ is defined as $\mathfrak{S}_{\Psi_\infty}^{(0)}$ where

$$\Psi_\infty((\xi_j)_{j\in\mathbf{N}}) = \sum_{j=1}^{\infty} \xi_j^* j^{-1}$$

with ξ_j^* denoting the decreasing rearrangement of $(|\xi_j|)_{j\in\mathbf{N}}$. This ideal coincides with the well-known Macaev-ideal (see for instance [5], § 15, where this ideal is denoted $\mathfrak{S}_\omega^{(0)}$).

We begin with a result that will not be used in the sequel, but which seems to be independently of interest.

Proposition 3.1. *Let $\tau = (T_1, \ldots, T_n) \in (L(H))^n$. Then we have*

$$k_\infty^-(\tau) \leqslant 2\|\tau\| \ln(2n+1).$$

Proof. In view of Proposition 1.4 of [10] there is no loss of generality if we assume that there is a finite set $\{\xi_1, \ldots, \xi_m\} \subset H$ which is cyclic for (τ, τ^*). We define recurrently finite-dimensional subspaces H_k by $H_0 = \mathbf{C}\xi_1 + \ldots + \mathbf{C}\xi_m$ and $H_{k+1} = H_k + \sum_{j=1}^{n} T_j H_k + \sum_{j=1}^{n} T_j^* H_k$. It is easily seen that $\dim H_k \leqslant m(2n+1)^k$ and that the matrix of τ with respect to the decomposition $H = H_0 \oplus (H_1 \ominus H_0) \oplus \ldots \ldots \oplus (H_{k+1} \ominus H_k) \oplus \ldots$ is tri-diagonal. Define

$$A_k = \frac{1}{k}(P_0 + \ldots + P_{k-1})$$

where P_k is the orthogonal projection onto H_k. Then $\|[A_k, \tau]\| \leqslant (2/k)\|\tau\|$ and the ank of $A_k, T_j]$ is $\leqslant 2m(2n+1)^k$. This gives:

$$|[A_k, \tau]|_\infty^- \leqslant \frac{2}{k}\|\tau\|\Psi_\infty(\underbrace{1, \ldots\ldots, 1}_{2m(2n+1)^k\text{-times}}, 0, 0, \ldots)$$

and hence

$$\limsup_{k\to\infty} |[A_k, \tau]|_\infty^- \leqslant 2\|\tau\| \ln(2n+1).$$

Since $A_k \uparrow I$ this concludes the proof.

Q.E.D.

We return now to the left regular representation of F_n on $l^2(F_n)$. Let $U_1, \ldots, U_n$ be the unitaries in $L(l^2(F_n))$ corresponding to $g_1, \ldots, g_n$, i.e.

$$U_j(\xi_g)_{g\in F_n} = (\eta_g)_{g\in F_n}$$

where $\eta_g = \xi_{g_j^{-1}g}$ for $(\xi_g)_{g\in F_n} \in l^2(F_n)$. As in § 3 of [10] we shall also consider the operators $T_1, \ldots, T_n$ acting on $l_\infty^-(F_n)$ defined by

$$T_j(\xi_g)_{g\in F_n} = (\eta_g)_{g\in F_n}$$

where $\eta_g = \xi_{g_j g}$ for $(\xi_g)_{g\in F_n} \in l_\infty^-(F_n)$.

LEMMA 3.2. *For $n \geqslant 2$ we have*

$$k_\infty^-(U_1, \ldots, U_n) > 0.$$

Proof. In view of Proposition 3.1 of [10] it will be sufficient to prove that

$$\inf\{\max_{1\leqslant j\leqslant n} |T_j\eta - \eta|_\infty^- \mid \eta \in l_\infty^-(F_n), \eta_e = 1\} > 0$$

where $e \in F_n$ is the neutral element of F_n.

To this end we shall construct elements $G_1, \ldots, G_n$ in the dual of $l_\infty^-(F_n)$ such that

$$\sum_{j=1}^{n} ((G_j \circ T_j^{-1}) - G_j) = \delta_e$$

where δ_e is the functional on $l_\infty^-(F_n)$ corresponding to evaluation at e.

Let F_n^+ be the semi-group in F_n generated by $\{e, g_1, \ldots, g_n\}$ and denote for $g = g_{i_1}^{k_1} g_{i_2}^{k_2} \ldots g_{i_m}^{k_m} \in F_n^+$ its length by $l(g) = k_1 + \ldots + k_m$ (of course $l(e) = 0$). We define G_j as a function on F_n by

$$G_j(g) = \begin{cases} 0 & \text{if } g \notin g_j F_n^+, \\ n^{-l(g)} & \text{if } g \in g_j F_n^+. \end{cases}$$

It is easily checked that

$$\sum_{j=1}^{n} (G_j(g_j g) - G_j(g)) = \begin{cases} 0 & \text{if } g \neq e, \\ 1 & \text{if } g = e. \end{cases}$$

To see that G_j defines a functional on $l_\infty^-(F_n)$ remark that the non-zero values of G_j in decreasing order are

$$n^{-1}, \underbrace{n^{-2}, \ldots, n^{-2}}_{n\text{-times}}, \underbrace{n^{-3}, \ldots, n^{-3}}_{n^2\text{-times}}, \ldots$$

and hence for $(\xi_g)_{g\in F_n} \in l_\infty^-(F_n)$ we have

$$\sum_{g\in F_n} |G_j(g)\xi_g| \leqslant n^{-1}\xi_1^* + n^{-2}(\xi_2^* + \ldots + \xi_{n+1}^*) +$$

$$+ n^{-3}(\xi_{n+2}^* + \ldots + \xi_{n^2+n+1}^*) + \ldots \leqslant |\xi|_\infty^-$$

where $(\xi_k^*)_{k\geqslant 1}$ denotes the decreasing rearrangement of $(|\xi_g|)_{g\in F_n}$.

Thus for $\eta \in l_\infty^-(F_n)$ with $\eta_e = 1$ we have

$$1 = \left| \sum_{g \in F_n} \left(\sum_{j=1}^{n} (G_j(g_j g) - G_j(g)) \right) \eta_g \right| = \left| \sum_{g \in F_n} \sum_{j=1}^{n} G_j(g) \cdot (\eta_{g_j^{-1} g} - \eta_g) \right| \leqslant$$

$$\leqslant \sum_{j=1}^{n} |T_j^{-1}\eta - \eta|_\infty^- \leqslant n \max_{1 \leqslant j \leqslant n} |T_j^{-1}\eta - \eta|_\infty^- =$$

$$= n \max_{1 \leqslant j \leqslant n} |T_j \eta - \eta|_\infty^- .$$

Q.E.D.

Consider now $\mathscr{A}_n = C^*(F_n)$ and $\mathscr{B}_n \subset \mathscr{A}_n$ the $*$-subalgebra generated by $u_1, \ldots, u_n$. Let ρ_r be the left regular representation of $C^*(F_n)$ on $l^2(F_n)$.

LEMMA 3.3. *For $n \geqslant 2$ we have*

$$E_{\Psi_\infty}^{(0)}(\rho_r) = 0.$$

Proof. Since $k_\infty^-(\rho_r(u_1), \ldots, \rho_r(u_n)) > 0$ by the preceding lemma, it follows that $E_{\Psi_\infty}(\rho_r) \neq 0$ and since ρ_r is a factor-representation and $E_{\Psi_\infty}(\rho_r)$ is in the center of $(\rho_r(\mathscr{A}_n))''$ we infer $E_{\Psi_\infty}(\rho_r) = I$.

Q.E.D.

With these preparations we can now give the example announced at the beginning of this section.

Consider a faithful representation ρ_0 of $\mathscr{A}_n$ which is a direct sum of finite-dimensional representations (it is folklore that such a representation exists). Consider further unitaries $U_1', \ldots, U_n' \in L(l^2(F_n))$ such that each of them is diagonal (i.e. for each U_j' there is a basis of eigenvectors) and $U_j - U_j' \in \mathscr{C}_\infty^-$ for $1 \leqslant j \leqslant n$. Let ρ' denote the representation of $\mathscr{A}_n$ on $l^2(F_n)$ such that $\rho'(u_j) = U_j'$.

Then, for $\rho_1 = \rho_0 \oplus \rho_r$ and $\rho_2 = \rho_0 \oplus \rho'$ we have $\rho_1(b) - \rho_2(b) \in \mathscr{C}_\infty^-$ for $b \in \mathscr{B}_n$.

Moreover $E_{\Psi_\infty}(\rho_1)$ is the projection onto $l^2(F_n)$ by Lemma 3.3 and $E_{\Psi_\infty}(\rho_2) \leqslant$ $\leqslant E_{\Psi_\infty}(\rho_1)$. On the other hand it is clear that ρ_r and ρ' are disjoint representations of $\mathscr{A}_n$ and hence the generalized absolutely continuous parts of ρ_1 and ρ_2 are disjoint.

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