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ON THE CONTINUITY OF NON-ANALYTIC FUNCTIONAL CALCULI

ERNST ALBRECHT and MICHAEL NEUMANN

Let $L(X)$ denote the Banach algebra of all continuous linear operators on some complex Banach space X and for $k = 0, 1, \ldots, \infty$ consider the Fréchet algebra $C^k(\Omega)$ of all k-times continuously differentiable complex functions on some open subset Ω of $\mathbf{C}$. In the theory of generalized spectral operators one is in particular interested in operators $T \in L(X)$ which admit a functional calculus (FC for short) on $C^k(\Omega)$ in the sense that there exists an algebraic homomorphism $\Phi: C^k(\Omega) \to L(X)$ with $\Phi(1) = I$ and $\Phi(Z) = T$, where Z denotes the identity function on Ω. It follows from the examples of Badé and Curtis [4, Th. 6.3] and of Dales [7, Th. 1] that these homomorphisms need not be continuous. Therefore, given an operator $T \in L(X)$ with a discontinuous FC on $C^k(\Omega)$, the problem arises to construct a continuous FC on some suitable topological algebra which is algebraically as well as topologically sufficiently near to $C^k(\Omega)$. According to a recent result from [3, Th. 4.6], this problem has a convenient solution if T admits no divisible linear subspace different from $\{0\}$. In this case, every $C^k(\Omega)$-FC for T is necessarily continuous on $C^{2k+1}(\Omega)$ with respect to the $C^{2k+1}(\Omega)$-topology. Conversely, the existence of some continuou $C^\infty(\Omega)$-FC for T implies that $\{0\}$ is the only divisible linear subspace for T. In the present note, we shall attack the above-mentioned problem without any additional assumption on the operator T. Furthermore, we shall investigate the more general situation of non-analytic FC in several variables on a wide class of topological algebras.

For the convenience of the reader, we first recall some definitions. A complex algebra A of complex-valued functions on some nonempty subset Ω of $\mathbf{C}^n$ is called *admissible* if the following three conditions are satisfied:

(a) $1, \pi_1, \ldots, \pi_n \in A$, where $\pi_j: \Omega \to \mathbf{C}$ denotes the coordinate function defined by $\pi_j(z) := z_j$ for all $z = (z_1, \ldots, z_n) \in \Omega$ and $j = 1, \ldots, n$;

(b) A is normal in the sense that for every finite open covering $\{U_1, \ldots, U_m\}$ of $\overline{\Omega}$ there are functions $f_1, \ldots, f_m \in A$ with $0 \leqslant f_j \leqslant 1$ and $\operatorname{supp} f_j \subset U_j$ for $j = 1, \ldots, m$ such that $f_1 + \ldots + f_m = 1$ on Ω;

(c) For every $f \in A$ and every $w \in \mathbf{C}^n \setminus \operatorname{supp} f$ there are functions $f_1, \ldots, f_n \in A$ such that $(w_1 - z_1)f_1(z) + \ldots + (w_n - z_n)f_n(z) = f(z)$ for all $z \in \Omega$.

Here $\operatorname{supp} f$ denotes the closure in $\mathbf{C}^n$ of the set $\{z \in \Omega : f(z) \neq 0\}$. The above definition has been introduced in [1, Def. 2] as the natural extension of the original one in [6, Def. 3.1.2] concerning the case $n = 1$. An admissible algebra is said to be an *admissible* (F)-*algebra* if it is endowed with a complete metrizable vector space topology such that multiplication is separately and hence jointly continuous. Note that for $n = 1$ an admissible (F)-algebra is not assumed to be topologically admissible in the sense of [6, Def. 3.5.1]. Now let A denote an arbitrary admissible algebra on some $\Omega \subset \mathbf{C}^n$ and consider a complex Banach space X. A system $T = (T_1, \ldots, T_n) \in L(X)^n$ is termed *A-scalar* if there exists a homomorphism $\Phi: A \to L(X)$ satisfying $\Phi(1) = I$ and $\Phi(\pi_j) = T_j$ for $j = 1, \ldots, n$. Such a homomorphism is called an A-FC for T. It has been shown in [1, Prop. 7] that for $n = 1$ any A-FC is an A-spectral function in the sense of [6, Def. 3.1.3]. Consequently, the present definition of an A-scalar system coincides with the usual terminology [6, Def. 3.1.18] in the case $n = 1$.

We now prove that any FC on an admissible (F)-algebra A is continuous on a fairly large part of A. This fact is an easy consequence of the automatic continuity theory for generalized local linear operators which has been established in [2]; see also [3] for some related results.

PROPOSITION. *Let A denote an admissible* (F)-*algebra on some $\Omega \subset \mathbf{C}^n$. Then for every homomorphism $\Phi: A \to L(X)$ with $\Phi(1) = I$ there exists a finite set Λ in $\mathbf{C}^n$ such that Φ is continuous on the closed subalgebra*

$$\mathscr{E}_A(F) := \{f \in A : \operatorname{supp} f \subset F\}$$

of A for every closed $F \subset \mathbf{C}^n$ with $F \cap \Lambda = \emptyset$. In the particular case $A = C^k(\Omega)$ for some open $\Omega \subset \mathbf{C}^n$ and some $k = 0, 1, \ldots, \infty$, one may assume in addition that $\Lambda \subset \Omega$.

Proof. Let $\mathscr{F}(\mathbf{C}^n)$ denote the family of all closed subsets of $\mathbf{C}^n$ and let $\mathscr{S}(Y)$ stand for the family of all closed linear subspaces of some topological vector space Y. From the normality assumption on A we obtain

$$\mathscr{E}_A(F) = \{f \in A : fg = 0 \text{ for all } g \in A \text{ with } f \cap \operatorname{supp} g = \emptyset\}$$

for every $F \in \mathscr{F}(\mathbf{C}^n)$. Therefore $\mathscr{E}_A(F)$ is in fact a closed subalgebra of A. Moreover, it is easily seen that the mapping $\mathscr{E}_A : \mathscr{F}(\mathbf{C}^n) \to \mathscr{S}(A)$ is a 2-spectral precapacity in the sense of [2, Def. 1.1]. Next define

$$\mathscr{E}_{L(X)}(F) := \{S \in L(X) : \Phi(g)\, S = 0 \text{ for all } g \in A \text{ with } F \cap \operatorname{supp} g = \emptyset\}$$

for all $F \in \mathscr{F}(\mathbf{C}^n)$. According to [1, Th. 4], the induced mapping $\mathscr{E}_{L(X)} : \mathscr{F}(\mathbf{C}^n) \to \mathscr{S}(L(X))$ is a countably $\cap$-stable precapacity [2, Def. 1.1]. Since Φ is obviously local in the sense that

$$\Phi \mathscr{E}_A(F) \subset \mathscr{E}_{L(X)}(F) \quad \text{for all } F \in \mathscr{F}(\mathbf{C}^n),$$

the general automatic continuity theory [2, Th. 3.7] supplies us with some finite subset Λ of $\mathbf{C}^n$ such that for every $F \in \mathscr{F}(\mathbf{C}^n)$ with $F \cap \Lambda = \emptyset$ the restriction of Φ to $\mathscr{E}_A(F)$ is closed and hence continuous by the closed graph theorem. In the special case $A = C^k(\Omega)$, it is shown in [3, Lemma 3.7] that Φ vanishes on $\mathbf{C}^n \setminus K$ for some compact subset K of Ω. By means of an $f \in A$ with $\operatorname{supp} f \cap (\Lambda \setminus \Omega) = \emptyset$ and $f = 1$ on some neighbourhood of K, one easily verifies that in this case Λ can be replaced by $\Omega \cap \Lambda$. The assertion follows.

The singularity set Λ occurring in the preceding proposition need not be empty. In order to construct homomorphisms which are everywhere defined and continuous on reasonable admissible algebras, we need an additional assumption on A. For each compact $K \subset \mathbf{C}^n$ let $H(K)$ denote the algebra of all germs of complex functions which are holomorphic in some neighbourhood of K. For each function g holomorphic in some neighbourhood of K let $[g]$ denote the corresponding germ in $H(K)$. We equip $H(K)$ with the usual inductive limit topology. Now consider an admissible (F)-algebra A on some $\Omega \subset \mathbf{C}^n$. We shall say that the *holomorphic functions operate on A* if for every $f \in A$ with compact support we have that $fg \in A$ for all g holomorphic in some neighbourhood of $\operatorname{supp} f$ and, moreover, that the corresponding (well-defined) linear mapping from $H(\operatorname{supp} f)$ into A given by $[g] \mapsto fg$ is continuous. This condition will enable us to apply the holomorphic FC in several variables.

THEOREM. *Let A denote an admissible* (F)*-algebra on some $\Omega \subset \mathbf{C}^n$ and assume that the holomorphic functions operate on A. Then for every A-scalar system $T = (T_1, \ldots, T_n) \in L(X)^n$ there exists an admissible topological algebra A_0 on some $\Omega_0 \subset \mathbf{C}^n$ such that the following four conditions are fulfilled:*

(i) *$\Omega \subset \Omega_0$ and $f|\Omega \in A$ for all $f \in A_0$;*

(ii) *A_0 is the countable inductive limit of* (F)*-algebras;*

(iii) *The restriction mapping from A_0 into A is continuous;*

(iv) *T admits a continuous A_0-*FC.

In the particular case $A = C^k(\Omega)$ for some open $\Omega \subset \mathbf{C}^n$ and some $k = 0, 1, \ldots, \infty$, one may choose $\Omega_0 = \Omega$ and hence $A_0 \subset A$ as a continuously embedded subalgebra.

Proof. (1) The proof is long and will be divided into several steps. We first proceed to the construction of A_0. Let $\Phi: A \to L(X)$ denote some FC for T and fix a corresponding finite subset $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ of $\mathbf{C}^n$ according to the above proposition. We define $\Omega_0 := \Omega \cup U$, where U is some bounded open neighbourhood of Λ in $\mathbf{C}^n$. Clearly, we may assume $\Omega_0 = \Omega$ whenever Ω is a neighbourhood of Λ. Now let A_0 consist of all functions $f: \Omega_0 \to \mathbf{C}$ with the following three properties:

(a) $f|\Omega \in A$;
(b) f is bounded on $\Omega_0 \setminus \Omega$;
(c) f is holomorphic on some open neighbourhood U_f of Λ.

One easily verifies that the algebra A_0 is normal on Ω_0 and contains $1, \pi_1, \ldots, \pi_n$. Let us demonstrate that for all $f \in A_0$ and all $w \in \mathbf{C}^n \setminus \operatorname{supp} f$ there are functions $f_1, \ldots, f_n$ in A_0 such that

$$(w_1 - z_1)f_1(z) + \ldots + (w_n - z_n)f_n(z) = f(z) \quad \text{for all } z \in \Omega_0.$$

From the corresponding property of A we first obtain $g_1, \ldots, g_n \in A$ such that $(w_1 - z_1)g_1(z) + \ldots + (w_n - z_n)g_n(z) = f(z)$ for all $z \in \Omega$. For $j = 1, \ldots, n$ we may extend g_j to Ω_0 by taking

$$g_j(z) := f(z)(\overline{w}_j - \overline{z}_j)\left(\sum_{l=1}^{n} |w_l - z_l|^2\right)^{-1} \qquad \text{for all } z \in \Omega_0 \setminus \Omega.$$

It is obvious that all these functions are bounded on $\Omega_0 \setminus \Omega$ and fulfill

$$(w_1 - z_1)g_1(z) + \ldots + (w_n - z_n)g_n(z) = f(z) \qquad \text{for all } z \in \Omega_0.$$

Next consider the finite sets $\Lambda_1 := \Lambda \cap \operatorname{supp} f$ and $\Lambda_2 := \Lambda \setminus \operatorname{supp} f$. Then it is easy to find a bounded open neighbourhood V_1 of Λ_1 in $\mathbf{C}^n$ with $w \notin V_1$ and $\Lambda_2 \cap V_1 = \emptyset$ such that for suitable holomorphic functions $u_1, \ldots, u_n : V_1 \to \mathbf{C}$ we have

$$(w_1 - z_1)u_1(z) + \ldots + (w_n - z_n)u_n(z) = 1 \qquad \text{for all } z \in V_1.$$

Let V_2 denote an open neighbourhood of Λ_2 with the property $(V_1 \cup \operatorname{supp} f) \cap V_2 = \emptyset$. According to the normality of A_0 on Ω_0, we may choose $\varphi_1, \varphi_2 \in A_0$ such that $\operatorname{supp} \varphi_i \subset V_i$ and $\operatorname{supp}(1 - \varphi_i) \cap \Lambda_i = \emptyset$ for $i = 1,2$. Since the holomorphic functions operate on A, we conclude that $\varphi_1 u_1, \ldots, \varphi_1 u_n \in A_0$. Because of $\varphi_1 + \varphi_2 = 1$ on some neighbourhood of Λ, it follows that

$$f_j := (1 - \varphi_1 - \varphi_2)g_j + \varphi_1 u_j f \in A_0 \qquad \text{for } j = 1, \ldots, n.$$

These functions have the desired properties since

$$(w_1 - z_1)f_1(z) + \ldots + (w_n - z_n)f_n(z) =$$

$$= (1 - \varphi_1 - \varphi_2)(z) \sum_{j=1}^{n} (w_j - z_j)g_j(z) + f(z) \sum_{j=1}^{n} (w_j - z_j)u_j(z)\varphi_1(z) =$$

$$= f(z) - \varphi_2(z)f(z) = f(z)$$

holds for arbitrary $z \in \Omega_0$. Therefore, A_0 turns out to be an admissible algebra.

(2) In order to define a natural topology on A_0, we fix a sequence $(U_m)_{m \in \mathbf{N}}$ of open neighbourhoods U_m of Λ such that $U_{m+1} \subset \overline{U}_{m+1} \subset U_m \subset \overline{U}_m \subset U$ for all $m \in \mathbf{N}$ and $U_m \downarrow \Lambda$ as $m \to \infty$. For each $m \in \mathbf{N}$ we consider the algebra A_m consisting

of all functions $f \in A_0$ which are holomorphic on U_m and continuous on $\overline{U}_m$. Then A_m is an (F)-algebra with respect to the (F)-norm $|\cdot|_m$ on A_m given by

$$|f|_m := |(f|\Omega)| + \sup\{|f(z)|: z \in \overline{U}_m \cup (\Omega_0 \setminus \Omega)\} \quad \text{for all } f \in A_m,$$

where $|\cdot|$ denotes an (F)-norm defining the topology of A. In view of $A_m \uparrow A_0$ as $m \to \infty$, we may endow A_0 with the inductive limit topology with respect to the sequence of (F)-algebras A_m. Note that A_0 carries the locally convex inductive limit topology if A and hence each of the algebras A_m is locally convex. Moreover, A_0 is the countable inductive limit of Banach algebras if A is supposed to be a Banach algebra. Obviously, the restriction mapping from A_0 into A is continuous. It remains to construct a continuous A_0-FC for T. For this end, we need some auxiliary observations which will be carried out in the following four paragraphs.

(3) For each $F \in \mathscr{F}(\mathbf{C}^n)$ we consider the closed linear subspace

$$\mathscr{E}_X(F) := \{x \in X: \Phi(g)x = 0 \quad \text{for all } g \in A \text{ with } F \cap \operatorname{supp} g = \emptyset\}$$

of X. By [1, Th. 4] we know that

$$T_j \mathscr{E}_X(F) \subset \mathscr{E}_X(F) \quad \text{for } j = 1, \ldots, n \text{ and } \sigma(T, \mathscr{E}_X(F)) \subset F,$$

where $\sigma(T, \mathscr{E}_X(F))$ denotes the joint spectrum of the system $T = (T_1, \ldots, T_n)$ with respect to the Banach space $\mathscr{E}_X(F)$ in the sense of Taylor [8]. Occasionally we shall make use of the fact that

$$\Phi(f) = 0 \quad \text{on } \mathscr{E}_X(F) \quad \text{for all } f \in A \text{ with } F \cap \operatorname{supp} f = \emptyset,$$

which is an immediate consequence of the inclusion $\sigma(T, \mathscr{E}_X(F)) \subset F$ and of [1, Th. 6].

(4) We now claim that the homomorphism $\Phi_F: A \to L(\mathscr{E}_X(F))$ given by

$$\Phi_F(f) := \Phi(f)|\mathscr{E}_X(F) \in L(\mathscr{E}_X(F)) \quad \text{for all } f \in A$$

is continuous whenever $F \in \mathscr{F}(\mathbf{C}^n)$ satisfies $F \cap \Lambda = \emptyset$. To prove this assertion, we use the normality of A to obtain some $\varphi \in A$ with $\operatorname{supp} \varphi \cap \Lambda = \emptyset$ and $\operatorname{supp}(1 - \varphi) \cap F = \emptyset$. Since multiplication by φ defines a continuous mapping from A into $\mathscr{E}_A(\operatorname{supp} \varphi)$ and since Φ is continuous on $\mathscr{E}_A(\operatorname{supp} \varphi)$, the mapping $\Theta_F: A \to L(\mathscr{E}_X(F))$ given by

$$\Theta_F(f) := \Phi(\varphi f)|\mathscr{E}_X(F) \in L(\mathscr{E}_X(F)) \quad \text{for all } f \in A$$

is certainly continuous on A. Since Φ_F vanishes on $\mathbf{C}^n \setminus F$, we conclude that $\Phi_F = \Theta_F$ which proves the continuity of Φ_F on A.

(5) Given a compact subset K of $\mathbf{C}^n$, let B_K denote the closed commutative subalgebra of $L(\mathscr{E}_X(K))$ generated by the operators $\Phi(f)|\mathscr{E}_X(K)$ for all $f \in A$. It follows

from [1, Th. 6] that the Taylor spectrum $\sigma(T, \mathscr{E}_X(K))$ coincides with the usual joint spectrum of $T_1|\mathscr{E}_X(K), \ldots, T_n|\mathscr{E}_X(K)$ with respect to the Banach algebra B_K. We may therefore consider the continuous analytic FC

$$\Psi_K : H(\sigma(T, \mathscr{E}_X(K))) \to B_K$$

for the operators $T_1, \ldots, T_n$ acting on $\mathscr{E}_X(K)$ in the sense of Bourbaki [5]. Now let $H(P)$ denote the usual (F)-algebra of all holomorphic functions on some open polydisc P in $\mathbf{C}^n$. In view of (3), for every $g \in H(P)$ the germ $[g]$ is well-defined in $H(\sigma(T, \mathscr{E}_X(K)))$ whenever $K \subset P$ is compact. Note that

$$\Psi_K([g]) = \Psi_L([g])|\mathscr{E}_X(K) \quad \text{for all compact } K, L \subset P \text{ with } K \subset L.$$

This follows immediately from the continuity of the analytic FC and from the fact that the complex polynomials in n variables are dense in $H(P)$.

(6) Again, let P denote an open polydisc in $\mathbf{C}^n$. For every compact subset K of $P \cap (\mathbf{C}^n \setminus \Lambda)$ we assert that

$$\Psi_K([g]) = \Phi_K(f) \quad \text{for all } f \in A \text{ and } g \in H(P) \text{ with } f|\Omega \cap P = g|\Omega \cap P.$$

To prove this identity, we choose some $\varphi \in A$ with $\operatorname{supp}\varphi \subset P \cap (\mathbf{C}^n \setminus \Lambda)$ and $\operatorname{supp}(1 - \varphi) \cap K = \emptyset$. Since the holomorphic functions operate on A, it makes sense to define

$$\Xi(h) := \Phi_K(\varphi h) \quad \text{for all } h \in H(P).$$

Furthermore, from (4) it is clear that the mapping $\Xi : H(P) \to L(\mathscr{E}_X(K))$ is continuous. Since Φ_K vanishes on $\mathbf{C}^n \setminus K$, we have $\Xi(h) = \Phi_K(k)$ whenever $h \in H(P)$ and $k \in A$ are identical on some open neighbourhood of K. In particular, we obtain $\Xi(h) = h(T)|\mathscr{E}_X(K)$ for all complex polynomials h in n variables. Since these polynomials are dense in $H(P)$, the continuity of Ξ and of the analytic FC implies that $\Xi(h) = \Psi_K([h])$ holds for all $h \in H(P)$. We conclude that

$$\Psi_K([g]) = \Xi(g) = \Phi_K(\varphi g) = \Phi_K(f)$$

whenever $f \in A$ and $g \in H(P)$ fulfill $f|\Omega \cap P = g|\Omega \cap P$.

(7) We now turn to the construction of a continuous A_0-FC Φ_0 for T. Let f denote an arbitrary function in A_0 and choose pairwise disjoint open polydiscs $P_1(f), \ldots, P_r(f)$ centered at $\lambda_1, \ldots, \lambda_r$ respectively such that f is holomorphic on $P_1(f) \cup \ldots \cup P_r(f)$. Again by normality there are $\varphi_1, \ldots, \varphi_r \in A$ such that $\operatorname{supp}\varphi_j \subset P_j(f)$ and $\lambda_j \notin \operatorname{supp}(1 - \varphi_j)$ for $j = 1, \ldots, r$. Because of $\Phi(\varphi_j)X \subset \subset \mathscr{E}_X(\operatorname{supp}\varphi_j)$ for $j = 1, \ldots, r$, it makes sense to consider

$$\Phi_0(f) := \sum_{j=1}^{r} \Psi_{\operatorname{supp}\varphi_j}([f])\,\Phi(\varphi_j) + \Phi\left(\left(1 - \sum_{j=1}^{r} \varphi_j\right) f|\Omega\right).$$

Let us check that this definition does not depend on the particular choice of the functions $\varphi_1, \dots, \varphi_r$. Let $\chi_1, \dots, \chi_r \in A$ denote another system of functions such that $\operatorname{supp} \chi_j \subset P_j(f)$ and $\lambda_j \notin \operatorname{supp}(1 - \chi_j)$ for $j = 1, \dots, r$. For arbitrary $j = 1, \dots, r$ consider the compact subsets $K_j := \operatorname{supp}(\varphi_j - \chi_j)$ and $L_j := \operatorname{supp}\varphi_j \cup \operatorname{supp} \chi_j$ of the polydisc $P_j(f)$. Then $K_j \cap \Lambda = \emptyset$, $K_j \subset L_j$ and $\Phi(\varphi_j - \chi_j)X \subset \mathscr{E}_X(K_j)$. Hence from (5), (6) and the multiplicativity of Φ we conclude that

$$\sum_{j=1}^{r} \Psi_{\operatorname{supp} \varphi_j}([f])\, \Phi(\varphi_j) + \Phi\left(\left(1 - \sum_{j=1}^{r} \varphi_j\right) f | \Omega \right) -$$

$$- \sum_{j=1}^{r} \Psi_{\operatorname{supp} \chi_j}([f])\, \Phi(\chi_j) - \Phi\left(\left(1 - \sum_{j=1}^{r} \chi_j\right) f | \Omega \right) =$$

$$= \sum_{j=1}^{r} \Psi_{L_j}([f])\, \Phi(\varphi_j - \chi_j) + \sum_{j=1}^{r} \Phi((\chi_j - \varphi_j) f | \Omega) =$$

$$= \sum_{j=1}^{r} \Psi_{K_j}([f])\, \Phi(\varphi_j - \chi_j) + \sum_{j=1}^{r} \Phi((\chi_j - \varphi_j) f | \Omega) =$$

$$= \sum_{j=1}^{r} \Phi(f | \Omega)\, \Phi(\varphi_j - \chi_j) + \sum_{j=1}^{r} \Phi((\chi_j - \varphi_j) f | \Omega) = 0.$$

Consequently $\Phi_0(f) \in L(X)$ is well-defined.

(8) Next we show that the mapping $\Phi_0 \colon A_0 \to L(X)$ is in fact an FC for T. Linearity is obvious, and from (6) one easily deduces that $\Phi_0(1) = I$ and $\Phi_0(\pi_j) = T_j$ for $j = 1, \dots, n$. To prove that Φ_0 preserves multiplication, let $f, g \in A_0$ be arbitrarily given. Then there are pairwise disjoint open polydiscs $P_1, \dots, P_r$ centered at $\lambda_1, \dots, \lambda_r$ respectively such that f and g are both holomorphic on $P_1 \cup \dots \cup P_r$. We choose $\varphi_1, \dots, \varphi_r \in A$ such that $\operatorname{supp}\varphi_j \subset P_j$ as well as $\lambda_j \notin \operatorname{supp}(1 - \varphi_j)$ for $j = 1, \dots, r$ and define $\varphi := \varphi_1 + \dots + \varphi_r$. Clearly $\varphi_j \varphi_m = 0$ if $j \neq m$ which implies $\varphi_j(1 - \varphi) = \varphi_j(1 - \varphi_j)$ for $j = 1, \dots, r$ and

$$(1 - \varphi)^2 + 2 \sum_{j=1}^{r} \varphi_j(1 - \varphi_j) = 1 - \sum_{j=1}^{r} \varphi_j^2.$$

For $j = 1, \dots, r$ we consider the compact subsets $K_j := \operatorname{supp} \varphi_j(1 - \varphi_j)$ and $L_j := \operatorname{supp} \varphi_j$ of the polydisc P_j and note that $K_j \cap \Lambda = \emptyset$, $K_j \subset L_j$ and $\Phi(\varphi_j(1 - \varphi_j))X \subset \mathscr{E}_X(K_j)$. We also observe that Ψ_K maps into the commutative algebra B_K for all compact $K \subset \mathbf{C}^n$. Combining all these facts with (5) and (6) we obtain

$$\Phi_0(f)\Phi_0(g) =$$

$$= \left(\sum_{j=1}^{r} \Psi_{L_j}([f])\, \Phi(\varphi_j) + \Phi((1-\varphi)\, f|\Omega) \right) \left(\sum_{m=1}^{r} \Psi_{L_m}([g])\, \Phi(\varphi_m) + \Phi((1-\varphi)\, g|\Omega) \right) =$$

$$= \sum_{j=1}^{r} \sum_{m=1}^{r} \Psi_{L_j}([f])\, \Phi(\varphi_j)\, \Psi_{L_m}([g]) \Phi(\varphi_m) + \Phi((1-\varphi)^2 fg|\Omega) +$$

$$+ \sum_{j=1}^{r} \Psi_{L_j}([f])\, \Phi(\varphi_j(1-\varphi)\, g|\Omega) + \sum_{j=1}^{r} \Phi((1-\varphi) f|\Omega) \Psi_{L_j}([g])\, \Phi(\varphi_j) =$$

$$= \sum_{j=1}^{r} \Psi_{L_j}([fg])\, \Phi(\varphi_j^2) + \Phi((1-\varphi)^2 fg|\Omega) +$$

$$+ \sum_{j=1}^{r} \Psi_{K_j}([f])\, \Phi(\varphi_j(1-\varphi_j) g|\Omega) + \sum_{j=1}^{r} \Psi_{K_j}([g])\, \Phi(\varphi_j(1-\varphi_j) f|\Omega) =$$

$$= \sum_{j=1}^{r} \Psi_{L_j}([fg]) \Phi(\varphi_j^2) + \Phi((1-\varphi)^2 fg|\Omega) + 2 \sum_{j=1}^{r} \Phi(\varphi_j(1-\varphi_j) fg|\Omega) =$$

$$= \sum_{j=1}^{r} \Psi_{L_j}([fg])\, \Phi(\varphi_j^2) + \Phi\left(\left(1 - \sum_{j=1}^{r} \varphi_j^2 \right) fg|\Omega \right).$$

In view of $\operatorname{supp} \varphi_j^2 = \operatorname{supp} \varphi_j \subset P_j$ and $\lambda_j \notin \operatorname{supp}(1-\varphi_j^2) \subset \operatorname{supp}(1-\varphi_j)$ for $= 1, \ldots, r$, the last expression is $= \Phi_0(fg)$. Hence Φ_0 respects multiplication.

(9) Let us finally prove that Φ_0 is continuous. For arbitrary $m \in \mathbf{N}$ we have to show that Φ_0 is continuous on A_m with respect to $|\cdot|_m$. We fix pairwise disjoint open polydiscs $P_1, \ldots, P_r$ centered at $\lambda_1, \ldots, \lambda_r$ respectively such that $P_1 \cup \ldots \cup P_r \subset U_m$ and consider $\varphi_1, \ldots, \varphi_r \in A$ such that $\operatorname{supp} \varphi_j \subset P_j$ as well as $\lambda_j \notin \operatorname{supp}(1-\varphi_j)$ for $j = 1, \ldots, r$. Then $F := \operatorname{supp}(1 - \varphi_1 - \ldots - \varphi_r)$ fulfills $F \cap \Lambda = \emptyset$. From (4) we conclude that Φ_F is continuous on A which obviously implies the continuity of the mapping

$$f \mapsto \Phi(f|\Omega)\, \Phi\left(1 - \sum_{j=1}^{r} \varphi_j\right) = \Phi\left(\left(1 - \sum_{j=1}^{r} \varphi_j\right) f|\Omega\right) \quad \text{on } A_m.$$

On the other hand, the mapping given by

$$f \mapsto \sum_{j=1}^{r} \Psi_{\operatorname{supp} \varphi_j}([f])\, \Phi(\varphi_j)$$

is certainly continuous on A_m. Hence Φ_0 is continuous on A_m which completes the proof of the theorem.

We finally remark that, in the preceding theorem, one may arrive at $A_0 \subset A$ as a continuously embedded subalgebra whenever Ω is a neighbourhood of the Taylor joint spectrum $\sigma(T, X)$. This follows immediately from the above proof and from [1, Th. 6]. Also, a slight modification of the above construction will sometimes be more appropriate. For example, in the case $A = C^k(\Omega)$ for some open subset Ω of $\mathbf{R}^n$ and some $k = 0, 1, \ldots, \infty$, one may use real-analytic functions in order to construct a continuous FC on some continuously embedded subalgebra of $C^k(\Omega)$.

REFERENCES

1. ALBRECHT, E.; FRUNZĂ, ŞT., Non-analytic functional calculi in several variables, *Manuscripta Math.*, **18**(1976), 327–336.
2. ALBRECHT, E.; NEUMANN, M., Automatic continuity of generalized local linear operators, *Manuscripta Math.*, **32**(1980), 263–294.
3. ALBRECHT, E.; NEUMANN, M., Stetigkeitsaussagen für Homomorphismen zwishcen topologischen Algebren, preprint.
4. BADÉ, W. G.; CURTIS, P. C., The structure of module derivations of Banach algebras of differentiable functions, *J. Functional Analysis*, **28**(1978), 226–247.
5. BOURBAKI, N., *Théories spectrales. Éléments de mathématique, Fasc. 32*, Hermann et Cie, Paris, 1967.
6. COLOJOARĂ, I.; FOIAŞ, C., *Theory of generalized spectral operators*, Gordon and Breach, New York-London-Paris, 1968.
7. DALES, H. G., Discontinuous homomorphisms from topological algebras, *Amer. J. Math.*, **101**(1979), 635–646.
8. TAYLOR, J. L., A joint spectrum for several commuting operators, *J. Functional Analysis*, **6**(1970), 172–191.

ERNST ALBRECHT
Fachbereich Mathematik der
Universität des Saarlandes,
D-6600 Saarbrücken,
West Germany.

MICHAEL NEUMANN
Fachbereich Mathematik der
Universität Essen-GHS,
D-4300 Essen,
West Germany.

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