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REMARKS ON THE SINGULAR EXTENSION IN THE C^*-ALGEBRA OF THE HEISENBERG GROUP

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The representation theory of the Heisenberg group shows that its C^*-algebra $C^*(G)$ is an extension of the form

$$0 \to C_0(\mathbf{R}\setminus\{0\}) \otimes \mathscr{K}(\mathscr{H}) \to C^*(G) \to C_0(\mathbf{R}^2) \to 0$$

where $C_0(X)$, for locally compact X, denotes the continuous functions on X which vanish at infinity and $\mathscr{K}(\mathscr{H})$ denotes the compact operators on the separable complex infinite-dimensional Hilbert space $\mathscr{H}$ (see for instance [11]). In a certain sense this extension of $C_0(\mathbf{R}\setminus\{0\}) \otimes \mathscr{K}(\mathscr{H})$ is localized at the limit point 0 of $\mathbf{R}\setminus\{0\}$, a feature we call singularity in [15] in order to distinguish such extensions from the homogeneous extensions studied in [14].

In this paper we shall exhibit two properties of this extension: a rather strong non-splitting property and what we shall call the existence of a limit distribution.

The non-splitting property we shall prove is that not only the extension is not split, but even the restriction of the extension to a sequence of points in $\mathbf{R}\setminus\{0\}$ which converges to 0, is also not split. In particular, since $C_0(\mathbf{R}^2)$ is generated by three hermitian elements, this implies that there are bounded sequences $(A_n)_{n=1}^{\infty}$, $(B_n)_{n=1}^{\infty}$, $(C_n)_{n=1}^{\infty}$ of compact hermitian operators such that

$$\lim_{n\to\infty} \|[A_n, B_n]\| = \lim_{n\to\infty} \|[A_n, C_n]\| = \lim_{n\to\infty} \|[B_n, C_n]\| = 0$$

for which do not exist sequences $(A'_n)_{n=1}^{\infty}$, $(B'_n)_{n=1}^{\infty}$, $(C'_n)_{n=1}^{\infty}$ of commuting compact hermitian operators

$$[A'_n, B'_n] = [A'_n, C'_n] = [B'_n, C'_n] = 0$$

such that

$$\lim_{n\to\infty} \|A_n - A'_n\| = \lim_{n\to\infty} \|B_n - B'_n\| = \lim_{n\to\infty} \|C_n - C'_n\| = 0.$$

This may be of some interest in connection with the open question concerning approximation of two almost commuting compact hermitian operators. We would like to mention that the rather long ad-hoc argument for the non-splitting result is related in its final part to the non-quasitriangularity of the unilateral shift [9].

We mention that the non-splitting of the extension also follows from one of the results announced in Kasparov's short note [10], but the stronger non-splitting result we obtain seems not to be obtainable in this way.

The property we shall call existence of a limit distribution is a way of expressing as a property of the extension, what is known as the quasiclassical asymptotic of eigenvalues for pseudodifferential operators ([1], [16], [17]). Though, most likely- an easy proof of this property can be based on what is known for pseudodiffe, rential operators, we shall giv e here a proof, using more operator-theory and no pseudodifferential operators. A weaker result in this direction has been noticed in [12]. Let us also mention that asymptotics of eigenvalues naturally arise in the study of ideals of a C^*-algebra associated with singular extensions ([15], [5]), and should therefore play a role in an attempt of characterizing a C^*-algebra such as the C^*-algebra of the Heisenberg group arising from a singular extension.

§ 1

This section is devoted to the proof of the fact that the extension in the C^*-algebra of the Heisenberg group is not split. In fact our result (Theorem 1.1. below) is stronger, namely we prove that the "restriction" of the extension to an arbitrary sequence of points in $\mathbf{R}\setminus\{0\}$ converging to zero is not split.

Let us first recall a few facts concerning the Heisenberg group ([13]).

The Heisenberg group G is the group of matrices

$$[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbf{R}$.

The irreducible representations of G consist of two classes:

1) the infinite-dimensional representations $\pi_\alpha(\alpha \in \mathbf{R}\setminus\{0\})$ on $\mathscr{H} = L^2(\mathbf{R})$, given by

$$(\pi_\alpha([x, y, z])\,\psi)\,(t) = e^{i\alpha(z+ty)}\,\psi(t+x)$$

where $\psi \in \mathscr{H}$;

2) The one-dimensional representations

$$\chi_{a,b}([x, y, z]) = e^{i(ax+by)}\, I_{\mathbf{C}}$$

where $(a, b) \in \mathbf{R}^2$.

Also, it is known that Haar-measure on G coincides with Lebesgue-measure on $\mathbf{R}^3$ via $G \ni [x, y, z] \to (x, y, z) \in \mathbf{R}^3$.

We shall denote also by π_α, $\chi_{a,b}$ the corresponding representations of $L^1(G)$ and of $C^*(G) \supset L^1(G)$.

The surjective $*$-homomorphism

$$q\colon C^*(G) \to C_0(\mathbf{R}^2)$$

is given by

$$(q(m))(a, b) = \chi_{a,b}(m)$$

where $(a, b) \in \mathbf{R}^2$ and $m \in C^*(G)$.

In particular when $F \in L^1(G)$ is of the form

$$F([x, y, z]) = \tilde{f}(x)g(y)h(z)$$

where f, g, $h \in L^1(\mathbf{R})$ and $\tilde{f}(t) = f(-t)$ we have

$$(\pi_\alpha(F)\psi)(t) = \hat{h}(\alpha)\hat{g}(\alpha t)(f * \psi)(t)$$

$$(q(F))(a, b) = \hat{\tilde{f}}(a)\hat{g}(b)\hat{h}(0)$$

where we use the notations

$$\hat{f}(s) = \int e^{ist} f(t)\,dt$$

for the Fourier transform.

With these preparations we can now state our result concerning the non-splitting.

1.1. THEOREM. *Let $(t_j)_{j=1}^{\infty} \subset \mathbf{R}\setminus\{0\}$ be a sequence converging to zero. Then, there do not exist $*$-homomorphisms*

$$\rho_j\colon C_0(\mathbf{R}^2) \to \mathscr{L}(\mathscr{H}) \quad (j \in \mathbf{N})$$

such that

$$\lim_{j\to\infty} \|(\rho_j \circ q)(a) - \pi_{t_j}(a)\| = 0$$

for all $a \in C^*(G)$.

The proof of this theorem constitutes the rest of this section.

We shall assume there exist $*$-homomorphisms $(\rho_j)_{j=1}^{\infty}$ with the above property and develop the consequences of this assumption till we shall reach a contradiction.

In what follows we shall denote π_{t_j} by $\pi^{(j)}$.

We shall choose the elements in $C^*(G)$ to be L^1-functions on G of the form

$$F([x,y,z]) = \widetilde{f}(x)g(y)h(z)$$

where $f, g, h \in L^1(\mathbf{R})$ and h is such that $\hat{h}(t_j) = 1$ for all $j \in \mathbf{N}$. Thus $q(F)$ will be the function $\hat{\widetilde{f}}(a)\hat{g}(b)$.

Let $k \in L^\infty(\mathbf{R})$ and $h \in L^1(\mathbf{R})$; we shall consider the operators

$$(M_k^{(j)}\psi)(t) = k(t_j t)\psi(t)$$

$$T_h\psi = h * \psi$$

where $\psi \in \mathscr{H}$. Then for F as above, we have

$$\pi^{(j)}(F) = M_{\hat{g}}^{(j)} T_f.$$

It is easily seen that

$$\lim_{j\to\infty} \|[M_{\hat{g}}^{(j)}, T_f]\| = 0$$

and

$$[M_{\hat{g}}^{(j)}, T_f] \in \mathscr{K}(\mathscr{H})$$

for $f, g \in L^1(\mathbf{R})$.

We shall choose

$$\hat{f}_0(s) = 2\mathrm{i}(s+\mathrm{i})^{-1}$$

so that

$$f_0(t) = \begin{cases} 0 & \text{for } t < 0 \\ 2\mathrm{e}^{-t} & \text{for } t \geqslant 0. \end{cases}$$

Since $1 - \hat{f}_0(s) = (s-\mathrm{i})(s+\mathrm{i})^{-1}$ which admits an analytic extension to the upper half-plane, it is known, using the Paley-Wiener theorem, that $I - T_{f_0}$ is a unitary operator, the restriction of which to $L^2((a,\infty))$ $(a \in \mathbf{R})$ is a unilateral shift of multiplicity one (see [19], [20]). We shall write T for T_{f_0} and S for $I - T$.

We shall also consider functions $g_k \in L^1(\mathbf{R})$ $(k = 1,2,3,4)$ such that $0 \leqslant \hat{g}_k \leqslant 1$ and $\operatorname{supp}\hat{g}_1 \subset [-3/2, 3/2]$, $\hat{g}_1(t) = 1$ for $-1 \leqslant t \leqslant 1$, $\operatorname{supp}\hat{g}_3 \subset [1/2, 5/2]$, $\hat{g}_3(t) = 1$ for $1 \leqslant t \leqslant 2$ and $\hat{g}_2(t) = \hat{g}_1(t/2)$, $\hat{g}_4(t) = \hat{g}_3(-t)$. We shall denote $M_{\hat{g}_k}^{(j)}$ by $M_{k,j}$.

Further we shall consider the projections $P_{k,j} = M_{\psi_k}^{(j)}$ $(k = 1,2,3,4)$ and $P_j = M_{\psi_+}^{(j)}$ where $\psi_1, \psi_2, \psi_3, \psi_4, \psi_+$ are the characteristic functions of $[-1,1]$, $[-2,2]$, $[1,2]$, $[-2,-1]$ and $[-1,\infty)$.

Assuming the existence of the ρ_j's consider the normal operators $R_j = \rho_j(h)$ where $h \in C_0(\mathbf{R}^2)$ is the function $h(s, t) = \hat{\tilde{f}}_0(s)\hat{g}_1(t)$ with f_0, g_1 the functions chosen before. We have:

$$\lim_{j\to\infty} \|R_j - M_{1,j}T\| = 0.$$

Let $0 < \delta < 1$ and consider $Q_{\delta,j}$ the spectral projection of $R_j^* R_j$ corresponding to $[\delta, \infty)$. Then we shall have $[Q_{\delta,j}, \rho_j(q(a))] = 0$ for all $a \in C^*(G)$ and this will give

$$\lim_{j\to\infty} \|[Q_{\delta,j}, \pi^{(j)}(a)]\| = 0$$

and in particular

$$\lim_{j\to\infty} \|[Q_{\delta,j}, M_{\hat{g}}^{(j)}T_f]\| = 0$$

for all $g, f \in L^1(\mathbf{R})$.

Remark also that, since $M_{1,j}T$ is compact, the essential norm of R_j tends to zero for $j \to \infty$, and hence for fixed δ with the exception of a finite number of j's, the projections $Q_{\delta,j}$ will have finite rank.

1.2. LEMMA. *Let $g, f \in L^1(\mathbf{R})$. We have:*

$$\lim_{j\to\infty} \|[M_{\hat{g}}^{(j)}, Q_{\delta,j}]\| = 0$$

$$\lim_{j\to\infty} \|[T_f, Q_{\delta,j}]\| = 0.$$

Proof. We have $R_j^* R_j = \rho_j(|h|^2)$ where $h \in C_0(\mathbf{R}^2)$, $h(s, t) = \hat{\tilde{f}}_0(s)\hat{g}_1(t)$. Consider $\varphi, \gamma \in L^1(\mathbf{R})$ such that $\hat{\tilde{\varphi}} \geqslant 0$, $\hat{\gamma} \geqslant 0$ and $\hat{\tilde{\varphi}}(s)\hat{\gamma}(t) = 1$ on

$$\{(s, t) \in \mathbf{R}^2 \mid |h(s, t)|^2 \geqslant \delta\}.$$

Then for $\Phi(s, t) = \hat{\tilde{\varphi}}(s)\,\hat{\gamma}(t)$ we have

$$\lim_{j\to\infty} \|M_{\hat{\gamma}}^{(j)}T_\varphi - \rho_j(\Phi)\| = 0$$

and

$$\rho_j(\Phi)Q_{\delta,j} = Q_{\delta,j}\rho_j(\Phi) = Q_{\delta,j}.$$

It follows that

$$\lim_{j\to\infty} \|M_{\hat{\gamma}}^{(j)}T_\varphi Q_{\delta,j} - Q_{\delta,j}\| = \lim_{j\to\infty} \|Q_{\delta,j}\, M_{\hat{\gamma}}^{(j)}T_\varphi - Q_{\delta,j}\| = 0.$$

Then:

$$\limsup_{j\to\infty} \|[M_{\hat{g}}^{(j)}, Q_{\delta,j}]\| =$$

$$= \limsup_{j\to\infty} \|M_{\hat{g}}^{(j)} M_{\hat{\gamma}}^{(j)} T_\varphi Q_{\delta,j} - Q_{\delta,j} M_{\hat{\gamma}}^{(j)} T_\varphi M_{\hat{g}}^{(j)}\| \leqslant$$

$$\leqslant \limsup_{j\to\infty} (\|[M_{\hat{g}\hat{\gamma}}^{(j)} T_\varphi, Q_{\delta,j}]\| + \|[M_{\hat{\gamma}}^{(j)} T_\varphi, M_{\hat{\gamma}}^{(j)}]\|) = 0.$$

Similarly

$$\limsup_{j\to\infty} \|[T_f, Q_{\delta,j}]\| =$$

$$= \limsup_{j\to\infty} \|T_f M_{\hat{\gamma}}^{(j)} T_\varphi Q_{\delta,j} - Q_{\delta,j} M_{\hat{\gamma}}^{(j)} T_\varphi T_f\| \leqslant$$

$$\leqslant \limsup_{j\to\infty} (\|[M_{\hat{\gamma}}^{(j)} T_{\varphi * f}, Q_{\delta,j}]\| + \|[T_f, M_{\hat{\gamma}}^{(j)} T_\varphi]\|) = 0.$$

Q.E.D.

1.3. Lemma. *We have*

$$\lim_{j\to\infty} \|(I - P_{2,j}) Q_{\delta,j}\| = 0$$

$$\lim_{j\to\infty} \|P_{3,j} Q_{\delta,j} P_{4,j}\| = 0.$$

Proof. We have

$$\|(I - P_{2,j}) Q_{\delta,j}\| \leqslant \delta^{-1/2} \|(I - P_{2,j}) R_j\|$$

and

$$\limsup_{j\to\infty} \|(I - P_{2,j}) R_j\| = \limsup_{j\to\infty} \|(I - P_{2,j}) M_{1,j} T\| = 0$$

since $P_{2,j} M_{1,j} = M_{1,j}$. This proves the first assertion.

For the second assertion, using Lemma 1.2 we have

$$\limsup_{j\to\infty} \|P_{3,j} Q_{\delta,j} P_{4,j}\| = \limsup_{j\to\infty} \|P_{3,j} M_{3,j} Q_{\delta,j} M_{4,j} P_{4,j}\| \leqslant$$

$$\leqslant \limsup_{j\to\infty} \|M_{3,j} Q_{\delta,j} M_{4,j}\| = \limsup_{j\to\infty} \|Q_{\delta,j} M_{3,j} M_{4,j}\| = 0.$$

Q.E.D.

Consider now $\varepsilon > 0$ and $E_{\varepsilon,j}$ the spectral projection of $P_{1,j} T^* T P_{1,j}$ corresponding to $[\varepsilon, \infty)$.

1.4. LEMMA. *We have*

$$\limsup_{j\to\infty} \|(I - Q_{\delta,j})E_{\varepsilon,j}\| \leqslant 5\delta^{1/4}\varepsilon^{-1/2}.$$

Proof. Consider $\eta_j \in (I - Q_{\delta,j})(\mathscr{H})$, $\|\eta_j\| = 1$. We shall prove that

$$\limsup_{j\to\infty} \|E_{\varepsilon,j}\eta_j\| \leqslant 5\delta^{1/4}\varepsilon^{-1/2}$$

which will give the desired conclusion.

Since $\eta_j \in (I - Q_{\delta,j})(\mathscr{H})$, we have:

$$\delta^{1/2} \geqslant \limsup_{j\to\infty} \|R_j\eta_j\| = \limsup_{j\to\infty} \|TM_{1,j}\eta_j\|.$$

Let $P'_{3,j} = M^{(j)}_{\psi'_3}$, $P'_{4,j} = M^{(j)}_{\psi'_4}$ be the projections given by the characteristic functions ψ'_3, ψ'_4 of $(1, \infty)$ and $(-\infty, -1)$. Let further $\alpha_j^{(1)} = P'_{3,j}M_{1,j}\eta_j$, $\alpha_j^{(2)} = P_{1,j}M_{1,j}\eta_j$, $\alpha_j^{(3)} = P'_{4,j}M_{1,j}\eta_j$ and remark that

$$\alpha_j^{(1)} + \alpha_j^{(2)} + \alpha_j^{(3)} = M_{1,j}\eta_j$$

and

$$\alpha_j^{(2)} = P_{1,j}\eta_j.$$

We have $P'_{4,j}TM_{1,j}(I - P'_{4,j}) = 0$ so that

$$\begin{aligned}\delta^{1/2} &\geqslant \limsup_{j\to\infty} \|TM_{1,j}\eta_j\| \geqslant \\ &\geqslant \limsup_{j\to\infty} \|P'_{4,j}TM_{1,j}\eta_j\| = \\ &= \limsup_{j\to\infty} \|P'_{4,j}T\alpha_j^{(3)}\| = \\ &= \limsup_{j\to\infty} \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|\alpha_j^{(3)} - S\alpha_j^{(3)}\|^2 &= \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|^2 + \|(I - P'_{4,j})S\alpha_j^{(3)}\|^2 = \\ &= \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|^2 + \|\alpha_j^{(3)}\|^2 - \|P'_{4,j}S\alpha_j^{(3)}\|^2 \leqslant \\ &\leqslant \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|^2 + \|\alpha_j^{(3)}\|^2 - \\ &- (\max\{0, \|\alpha_j^{(3)}\| - \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|\})^2 \leqslant \\ &\leqslant \max\{2\|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|^2,\ 2\|\alpha_j^{(3)}\|\,\|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|\} \leqslant \\ &\leqslant 2\max\{\|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|^2, \|\alpha_j^{(3)} - P'_{4,j}S\alpha_j^{(3)}\|\}.\end{aligned}$$

Since $\delta < 1$ this gives

$$\limsup_{j\to\infty} \|\alpha_j^{(3)} - S\alpha_j^{(3)}\| \leqslant 2\delta^{1/4}.$$

Remark that T being normal we have

$$\|TM_{1,j}\eta_j\| = \|T^*M_{1,j}\eta_j\|$$

and dealing in the same way with T^* as we did with T, using now the projection $P'_{3,j}$, we obtain

$$\limsup_{j\to\infty} \|\alpha_j^{(1)} - S^*\alpha_j^{(1)}\| \leqslant 2\delta^{1/4}$$

or equivalently

$$\limsup_{j\to\infty} \|\alpha_j^{(1)} - S\alpha_j^{(1)}\| \leqslant 2\delta^{1/4}.$$

Thus we have

$$\limsup_{j\to\infty} \|T(\alpha_j^{(1)} + \alpha_j^{(3)})\| =$$

$$= \limsup_{j\to\infty} \|(I - S)(\alpha_j^{(1)} + \alpha_j^{(3)})\| \leqslant 4\delta^{1/4}$$

and hence

$$\limsup_{j\to\infty} \|T\alpha_j^{(2)}\| \leqslant \delta^{1/2} + 4\delta^{1/4} \leqslant 5\delta^{1/4}.$$

This finally gives:

$$\limsup_{j\to\infty} \|E_{\varepsilon,j}\eta_j\|^2 \leqslant$$

$$\leqslant \varepsilon^{-1} \limsup_{j\to\infty} \|TP_{1,j}\eta_j\|^2 =$$

$$= \varepsilon^{-1} \limsup_{j\to\infty} \|T\alpha_j^{(2)}\|^2 \leqslant 25\varepsilon^{-1}\delta^{1/2}.$$

Q.E.D.

Using the preceding three lemmas it is easily seen that we can find a sequence $\{\delta_j\}_1^\infty$ sufficiently slowly decreasing to zero, so that for $Q_j = Q_{\delta_j,j}$, $E_j = E_{\varepsilon_j,j}$ where $\varepsilon_j = \delta_j^{1/4}$ we have rank$Q_j < \infty$ (except possibly a finite number of indices, which will be omitted) and

$$\lim_{j\to\infty} \|[T, Q_j]\| = 0$$

$$\lim_{j\to\infty} \|(I - P_{2,j})Q_j\| = 0$$

$$\lim_{j\to\infty} \|P_{3,j}Q_jP_{4,j}\| = 0$$

$$\lim_{j\to\infty} \|(I - Q_j)E_j\| = 0.$$

1.5. LEMMA. *There are finite-rank projections* $\widetilde{Q}_j$ *such that*

$$\lim_{j\to\infty} \|Q_j - \widetilde{Q}_j\| = 0$$

$$\widetilde{Q}_j \leqslant P_{2,j},\ E_j \leqslant \widetilde{Q}_j$$

$$\lim_{j\to\infty} \|[\widetilde{Q}_j, T]\| = 0,\ \lim_{j\to\infty} \|P_{3,j}\widetilde{Q}_j P_{4,j}\| = 0.$$

Proof. Consider $\widetilde{Q}'_j$ the projection onto the range of $P_{2,j}\,Q_j$. Since $\lim_{j\to\infty} \|(I - P_{2,j})Q_j\| = 0$ it is easily seen that

$$P_{2,j} \geqslant \widetilde{Q}'_j \quad \text{and} \quad \lim_{j\to\infty} \|Q_j - \widetilde{Q}'_j\| = 0.$$

Now we have $\lim_{j\to\infty} \|(I - \widetilde{Q}'_j)E_j\| = 0$ so that for E'_j the projection onto the range of $\widetilde{Q}'_j E_j$ we shall have $\lim_{j\to\infty} \|E'_j - E_j\| = 0$. Taking then $\widetilde{Q}_j = \widetilde{Q}'_j - E'_j + E_j$ it is easily seen that $\widetilde{Q}_j$ are projections such that $E_j \leqslant \widetilde{Q}_j \leqslant P_{2,j}$ and $\lim_{j\to\infty} \|Q_j - \widetilde{Q}_j\| = 0$. The other assertions follow from $\lim_{j\to\infty} \|Q_j - \widetilde{Q}_j\| = 0$. Q.E.D.

Let $0 < \alpha < 1$ and consider Q'_j the spectral projection of $\widetilde{Q}_j P_j \widetilde{Q}_j$ corresponding to $[1 - \alpha, \infty)$. Clearly $Q'_j \geqslant E_j$. Moreover, we have

$$\|(I - P_j)Q'_j\| = \|Q'_j(I - P_j)Q'_j\|^{1/2} =$$

$$= \|Q'_j - Q'_j(\widetilde{Q}_j P_j \widetilde{Q}_j)\|^{1/2} \leqslant \alpha^{1/2}.$$

As already mentioned $S \mid P_j(\mathscr{H})$ is a unilateral shift of multiplicity one. Consider $\xi_j \in \mathrm{Ker}((S \mid P_j(\mathscr{H}))^*)$, $\|\xi_j\| = 1$. In fact $\xi_j(t) = \beta_j \xi(t + t_j)$ for some $\beta_j \in \mathbf{C}$, $|\beta_j| = 1$, where ξ is a vector in $\mathrm{Ker}((S \mid L^2(0, \infty))^*)$, $\|\xi\| = 1$. Since $P_{1,j}$ is the projection of $L^2(\mathbf{R})$ onto $L^2(-t_j^{-1}, t_j^{-1})$, it is easily seen that

$$\lim_{j\to\infty} \|(I - P_{1,j})\,\xi_j\| = 0.$$

But even more, we have:

$$\lim_{j\to\infty} \|(I - E_j)\xi_j\| = 0.$$

Indeed, otherwise there would be $\eta_j \in (P_{1,j} - E_j)(\mathscr{H})$, $\|\eta_j\| = 1$ such that

$$\limsup_{j\to\infty} | < \xi_j,\ \eta_j > | > 0.$$

Since

$$\lim_{j\to\infty} \| S\eta_j - \eta_j\|^2 = \lim_{j\to\infty} \|T\eta_j\|^2 \leqslant \lim_{j\to\infty} \varepsilon_j = 0$$

we would have

$$\limsup_{j\to\infty} | \langle \xi_j, \eta_j \rangle | =$$

$$= \limsup_{j\to\infty} | \langle \xi_j, S\eta_j \rangle | =$$

$$= \limsup_{j\to\infty} | \langle S^*\xi_j, P_{1,j}\, \eta_j \rangle | = 0$$

a contradiction, which proves our assertion.

Thus we also have

$$\lim_{j\to\infty} \|(I - Q_j')\, \xi_j\| = 0$$

$$\lim_{j\to\infty} \|Q_j'\xi_j\| = 1$$

and hence

$$\limsup_{j\to\infty} \|(Q_j' S Q_j')^* Q_j'\xi_j\| =$$

$$= \limsup_{j\to\infty} \|Q_j' S^*\xi_j\| \leqslant \alpha^{1/2} + \limsup_{j\to\infty} \|Q_j' P_j S^* \xi_j\| = \alpha^{1/2}.$$

Since Q_j' has finite rank it follows that we can find $\zeta_j \in Q_j'(\mathscr{H})$, $\|\zeta_j\| = 1$ such that

$$\limsup_{j\to\infty} \|Q_j' S Q_j' \zeta_j\| \leqslant \alpha^{1/2}.$$

Taking into account that $Q_j'\zeta_j = \zeta_j$ this gives

$$\limsup_{j\to\infty} | \langle S\, \zeta_j, \zeta_j \rangle | \leqslant \alpha^{1/2}$$

and hence

$$2(1 - \alpha^{1/2}) \leqslant \liminf_{j\to\infty} \|(S - I)\zeta_j\|^2 =$$

$$= \liminf_{j\to\infty} \|(S - I)\,(I - (P_{1,j} - E_j))\zeta_j\|^2 \leqslant$$

$$\leqslant \|S - I\|^2 \liminf_{j\to\infty} \| \,(I - (P_{1,j} - E_j))\zeta_j\, \|^2 =$$

$$= 4(1 - \limsup_{j\to\infty} \|(P_{1,j} - E_j)\zeta_j\|^2)$$

so that

$$\limsup_{j\to\infty} \|(P_{1,j} - E_j)\zeta_j\|^2 \leqslant \frac{1 + \alpha^{1/2}}{2}.$$

Now, using the fact that T is normal, we have

$$\lim_{j\to\infty} \|(P_{1,j} - E_j)T\| = 0$$

and hence

$$\limsup_{j\to\infty} \|(P_{1,j} - E_j)S\zeta_j\|^2 \leqslant \frac{1 + \alpha^{1/2}}{2}.$$

Remark also the following estimate:

$$\|(I - P_j)S\zeta_j\| \leqslant \|(I - P_j)SQ'_j\| =$$
$$= \|(I - P_j)S(I - P_j)Q'_j\| \leqslant \|(I - P_j)Q'_j\| \leqslant \alpha^{1/2}$$

Summing up the preceding discussion, we have proved:

$$(*) \qquad \begin{cases} \|(I - P_j)Q_j'\| \leqslant \alpha^{1/2} \\ \limsup_{j\to\infty} \|Q'_j S\zeta_j\| \leqslant \alpha^{1/2} \\ \limsup_{j\to\infty} \|(P_{1,j} - E_j)S\zeta_j\|^2 \leqslant \dfrac{1 + \alpha^{1/2}}{2} \\ \|(I - P_j)S\zeta_j\| \leqslant \alpha^{1/2} \end{cases}$$

where $\zeta_j \in Q'_j(\mathscr{H})$ and $\|\zeta_j\| = 1$.

The next thing we want to prove, is that

$$\lim_{j\to\infty} \|(\widetilde{Q}_j - Q'_j)P_{3,j}\| = 0.$$

To this end, remark that there is M_j, $\|M_j\| \leqslant (1 - \alpha)^{-1}$ such that

$$M_j(\widetilde{Q}_j - \widetilde{Q}_j P_j \widetilde{Q}_j) = \widetilde{Q}_j - Q'_j.$$

This gives

$$\limsup_{j\to\infty} \|(\widetilde{Q}_j - Q'_j)\, P_{3,j}\| \leqslant$$
$$\leqslant (1 - \alpha)^{-1} \limsup_{j\to\infty} \|(\widetilde{Q}_j - \widetilde{Q}_j P_j \widetilde{Q}_j)P_{3,j}\| =$$
$$= (1 - \alpha)^{-1} \limsup_{j\to\infty} \|\widetilde{Q}_j(I - P_j)\, \widetilde{Q}_j P_{3,j}\| =$$
$$= (1 - \alpha)^{-1} \limsup_{j\to\infty} \|\widetilde{Q}_j P_{4,j} \widetilde{Q}_j P_{3,j}\| = 0$$

which is the desired result. Since $(\tilde{Q}_j - Q_j')E_j = 0$, we also have

$$\lim_{j\to\infty} \|(\tilde{Q}_j - Q_j')(P_{3,j} + E_j)\| = 0.$$

Using this fact and the relations (*) we have:

$$\limsup_{j\to\infty} \|\tilde{Q}_j S\zeta_j\| \leqslant$$

$$\leqslant \limsup_{j\to\infty} (\|Q_j' S\zeta_j\| + \|(\tilde{Q}_j - Q_j')(E_j + P_{3,j})S\zeta_j\| +$$

$$+ \|(\tilde{Q}_j - Q_j')(P_{1,j} - E_j)S\zeta_j\| + \|(\tilde{Q}_j - Q_j')(I - P_j)S\zeta_j\|) \leqslant$$

$$\leqslant \alpha^{1/2} + \left(\frac{1 + \alpha^{1/2}}{2}\right)^{1/2} + \alpha^{1/2}.$$

This further gives

$$\liminf_{j\to\infty} \|(I - \tilde{Q}_j)S\zeta_j\| \geqslant 1 - \left(2\alpha^{1/2} + \left(\frac{1 + \alpha^{1/2}}{2}\right)^{1/2}\right).$$

Since $\tilde{Q}_j\zeta_j = \zeta_j$, $\|\zeta_j\| = 1$ we also have

$$\liminf_{j\to\infty} \|(I - \tilde{Q}_j)S\tilde{Q}_j\| \geqslant 1 - \left(2\alpha^{1/2} + \left(\frac{1 + \alpha^{1/2}}{2}\right)^{1/2}\right).$$

Because $0 < \alpha < 1$ is arbitrary, for α small enough we obtain

$$\liminf_{j\to\infty} \|(I - \tilde{Q}_j)S\tilde{Q}_j\| \geqslant 1/4.$$

This last relation contradicts

$$\lim_{j\to\infty} \|[S, \tilde{Q}_j]\| = \lim_{j\to\infty} \|[T, \tilde{Q}_j]\| = 0$$

and this contradiction concludes the proof of Theorem 1.1.

§2

In this section we shall be concerned with the existence of a limit distribution for the extension in the C^*-algebra of the Heisenberg group. As pointed out in the introduction this corresponds to what is known as the quasiclassical asymptotic of eigenvalues and can be most likely easily obtained from the results known for pseudodifferential operators. Our aim here is to avoid pseudodifferential operators and to try to emphasize the operator-theoretical side of this fact, connected to singular extensions.

We begin by considering a more general situation than $C^*(G)$.

Let B denote the C^*-algebra of bounded $\mathscr{K}(\mathscr{H})$-valued functions $k\colon (0,\infty) \to \mathscr{K}(\mathscr{H})$ such that $\lim_{t\to\infty} \|k(t)\| = 0$. Let further $B_0 \subset B$ denote the ideal consisting of those functions $k\colon (0, \infty) \to \mathscr{K}(\mathscr{H})$ in B, such that $\lim_{t\to 0} \|k(t)\| = 0$. The canonical map $B \to B/B_0$ will be denoted by p.

Consider further Y a locally compact metrizable space, $C_0(Y)$ the C^*-algebra of continuous functions vanishing at infinity, μ a measure on Y and $\rho\colon C_0(Y) \to B/B_0$ a $*$-homomorphism.

We shall also denote by $\mathscr{V}_\varepsilon$ for $\varepsilon > 0$, the set of continuous increasing functions $\varphi\colon [0, \infty) \to [0, \infty)$ such that φ is equal 0 on $[0, \varepsilon]$. Also $\mathscr{V}$ will denote $\bigcup_{\varepsilon>0} \mathscr{V}_\varepsilon$.

For Y and μ as above we shall consider subsets $\Gamma \subset (C_0(Y))_+$ (the positive part of $C_0(Y)$) satisfying the following conditions:

ASSUMPTION (Γ)

(i) *for every* $g \in \Gamma$ *and* $\psi \in \mathscr{V}$ *we have* $\psi \circ g \in \Gamma$.

(ii) *for every* $\varepsilon > 0, f \in (C_0(Y))_+$ *and* W *neighborhood of* supp$f$ *there are* $n \in \mathbf{N}$ *and* $g_1, \ldots, g_n \in \Gamma$ *such that* supp$g_j \subset W$ $(j = 1, \ldots, n)$ *and*

$$\|f - (g_1 + \ldots + g_n)\| < \varepsilon.$$

(iii) *for every* $\varepsilon > 0$ *and* $f \in (C_0(Y))_+$ *with compact support there are* $n \in \mathbf{N}$ *and* $g_1, \ldots, g_n \in \Gamma$ *such that*

$$i \neq j \Rightarrow \operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$$

$$g_1 + \ldots + g_n \leqslant f$$

$$\int (f - (g_1 + \ldots + g_n))\, \mathrm{d}\mu < \varepsilon.$$

With these preparations we can state the following theorem.

2.1. THEOREM. *Let* $\Gamma \subset (C_0(Y))_+$ *satisfy Assumption* (Γ). *Then the following two conditions are equivalent*:

(i) *for every* $g \in \Gamma$ *there is* $h \in B_+$ *such that* $p(h) = \rho(g)$ *and*

$$\lim_{t\downarrow 0} t \operatorname{Tr}((\psi(h))(t)) = \int (\psi \circ g)\, \mathrm{d}\mu$$

for all $\psi \in \mathscr{V}$;

(ii) *for every* $f \in (C_0(Y))_+$ *and* $h \in B_+$ *such that* $p(h) = \rho(f)$ *we have*

$$\lim_{t\downarrow 0} t \operatorname{Tr}(\psi(h))(t)) = \int (\psi \circ f)\, \mathrm{d}\mu$$

for all $\psi \in \mathscr{V}$.

We shall precede the proof of the theorem by two lemmas.

2.2. LEMMA. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})$, $K_1 \geqslant 0$, $K_2 \geqslant 0$ and $0 < \delta < \varepsilon$. Suppose $\psi \in \mathcal{V}_\varepsilon$ and $K_1 + \delta I \geqslant K_2$. Then we have*

$$\mathrm{Tr}\psi(K_1 + \delta I) \geqslant \mathrm{Tr}\psi(K_2).$$

Proof. For $r > \delta$ it follows from the minimax principle (see [8], ch. II, § 1) that for the spectral projections we have:

$$\mathrm{Tr}E((r, \infty), K_1 + \delta I) \geqslant \mathrm{Tr}E((r, \infty), K_2).$$

This gives

$$\mathrm{Tr}\psi(K_1 + \delta I) = \int_\varepsilon^\infty \mathrm{Tr}E((r, \infty), K_1 + \delta I)\mathrm{d}\psi(r) \geqslant$$

$$\geqslant \int_\varepsilon^\infty \mathrm{Tr}E((r, \infty), K_2)\,\mathrm{d}\psi(r) = \mathrm{Tr}\,\psi(K_2).$$

Q.E.D.

We shall use several times the following consequence of the preceding lemma:

Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})$ be finite rank, $K_1 \geqslant 0$, $K_2 \geqslant 0$ and suppose

$$K_1 + \theta I \geqslant K_2$$

where $\theta > 0$. Then we have

$$\mathrm{Tr}K_1 \geqslant \mathrm{Tr}(K_2 - \theta I)_+.$$

The second lemma we shall need is:

2.3. LEMMA. *Given $M > 0$ and $n \in \mathbf{N}$ there is a continuous function $\omega\colon [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$, such that the following holds: whenever $K, K_1, \ldots, K_n \in \mathcal{K}(\mathcal{H})$ are finite rank and positive, and $0 < \theta \leqslant 1$ so that*

$$K + \theta I \geqslant \sum_{i=1}^n K_i$$

$$i \neq j \Rightarrow \|K_i K_j\| \leqslant \theta$$

$$\|K\| + 1 \leqslant M$$

then we have

$$\mathrm{Tr}K \geqslant \sum_{i=1}^n \mathrm{Tr}(K_i - \omega(\theta)I)_+.$$

Proof. Let $\varepsilon > 0$ and consider P_i the spectral projections of K_i corresponding to $[\varepsilon, \infty)$. Then we have

$$\|P_i P_j\| \leqslant \varepsilon^{-2} \|K_i K_j\| \leqslant \theta\varepsilon^{-2}.$$

Consider further

$$Q_i = \bigvee_{k=1}^{i} P_k - \bigvee_{k=1}^{i-1} P_k.$$

Then for fixed $n \in \mathbf{N}$, there is a continuous functions ω_1: $[0, \infty) \to [0, \infty)$ with $\omega_1(0) = 0$, such that

$$\|Q_i - P_i\| \leqslant \omega_1(\theta\varepsilon^{-2}).$$

This is a folklore-type fact which can be obtained by iteration from the fact that given selfadjoint projections R_1, R_2 with $\|R_1 R_2\| \leqslant \delta \leqslant 10^{-1}$ we have $\|R_1 \vee R_2 - (R_1 + R_2)\| \leqslant 3\delta$.

Consider now the polar decomposition $Q_i P_i = V_i(P_i Q_i P_i)^{1/2}$. We have:

$$P_i(1 - \omega_1(\theta\varepsilon^{-2})) \leqslant P_i Q_i P_i \leqslant P_i(1 + \omega_1(\theta\varepsilon^{-2}))$$

and hence for $\omega_1(\theta\varepsilon^{-2})$ small enough we have:

$$\|P_i - V_i\| = \|P_i - Q_i P_i((I - P_i) + P_i Q_i P_i)^{-1/2}\| \leqslant$$

$$\leqslant \omega_1(\theta\varepsilon^{-2}) \|((I - P_i) + P_i Q_i P_i)^{-1/2}\| +$$

$$+ \|((I - P_i) + P_i Q_i P_i)^{-1/2} - I\| \leqslant$$

$$\leqslant \omega_1(\theta\varepsilon^{-2}) (1 - \omega_1(\theta\varepsilon^{-2}))^{-1/2} +$$

$$+ \max\{1 - (1 + \omega_1(\theta\varepsilon^{-2}))^{-1/2}, (1 - \omega_1(\theta\varepsilon^{-2})^{-1/2} - 1\}.$$

Thus there is a continuous function ω_2: $[0, \infty) \to [0, \infty)$, $\omega_2(0) = 0$ such that

$$\|P_i - V_i\| \leqslant \omega_2(\theta\varepsilon^{-2}).$$

Returning now to the K_i's for $i \neq j$, we have

$$\left\| Q_i \left(\sum_{k=1}^{n} K_k P_k \right) Q_j \right\| \leqslant M \sum_{k=1}^{n} \|Q_i P_k\| \, \|P_k Q_j\| \leqslant$$

$$\leqslant M \sum_{k=1}^{n} (\|Q_i Q_k\| + \omega_1(\theta\varepsilon^{-2}))(\|Q_j Q_k\| + \omega_1(\theta\varepsilon^{-2})) \leqslant$$

$$\leqslant Mn\omega_1(\theta\varepsilon^{-2})(1 + \omega_1(\theta\varepsilon^{-2})).$$

Also for $i \neq j$ we have:

$$\left\| Q_i \left(\sum_{k=1}^{n} K_k P_k \right) Q_i - V_i K_i P_i V_i^* \right\| \leqslant$$

$$\leqslant nM\omega_1(\theta\varepsilon^{-2}) + \|Q_i(K_iP_i - V_iK_iP_iV_i^*)\, Q_i\| \leqslant$$

$$\leqslant nM\omega_1(\theta\varepsilon^{-2}) + 2M\omega_2(\theta\varepsilon^{-2}).$$

Hence, there is a continuous function $\omega_3\colon [0, \infty) \to [0, \infty)$, $\omega_3(0) = 0$, depending only on M and n, such that

$$\left\| \sum_{k=1}^{n} K_i P_i - \sum_{i=1}^{n} V_i K_i P_i V_i^* \right\| \leqslant \omega_3(\theta\varepsilon^{-2}).$$

Thus we have

$$K + (\omega_3(\theta\varepsilon^{-2}) + \theta) I \geqslant \sum_{i=1}^{n} V_i K_i P_i V_i^*.$$

Taking $\varepsilon = \theta^{1/3}$ and $\omega_4(\theta) = \theta + \omega_3(\theta^{1/3})$ we have

$$K_i P_i \geqslant (K_i - \theta^{1/3} I)_+$$

and hence

$$K + \omega_4(\theta) I \geqslant \sum_{i=1}^{n} V_i (K_i - \theta^{1/3} I)_+ V_i^*.$$

Using Lemma 2.2 we have:

$$\mathrm{Tr} K \geqslant \mathrm{Tr}\left(\left(\sum_{i=1}^{n} V_i (K_i - \theta^{1/3} I)_+ V_i^* - \omega_4(\theta) I \right)_+ \right) =$$

$$= \mathrm{Tr}\left(\sum_{i=1}^{n} V_i (K_i - (\theta^{1/3} + \omega_4(\theta)) I)_+ V_i^* \right) =$$

$$= \sum_{i=1}^{n} \mathrm{Tr}((K_i - (\theta^{1/3} + \omega_4(\theta)) I)_+).$$

Thus we may take $\omega(\theta) = \theta^{1/3} + \omega_4(\theta)$.

Q.E.D.

Proof of Theorem 2.1. It is obvious that (ii) $\Rightarrow$ (i), so that it will be sufficient to prove that (i) $\Rightarrow$ (ii).

Consider $f \in (C_0(Y))_+$, $h \in B_+$ such that $p(h) = \rho(f)$ and $\psi \in \mathscr{V}$. Remarking that $\psi \circ f$ has compact support it follows using Assumption (Γ) (ii) that for given $\delta > 0$ we can find $g_1, \ldots, g_m \in \Gamma$ such that

$$\|\psi \circ f - (g_1 + \ldots + g_m)\| < \delta$$

$$\int |\psi \circ f - (g_1 + \ldots + g_m)| \, d\mu < \delta.$$

It is easily seen that we can find $\chi_j \in \mathscr{V}$ such that

$$\|\psi \circ f - (\chi_1 \circ g_1 + \ldots + \chi_m \circ g_m)\| < \delta$$

$$\int |\psi \circ f - (\chi_1 \circ g_1 + \ldots + \chi_m \circ g_m)| \, \mathrm{d}\mu < \delta.$$

Consider now $y_1, \ldots, y_m \in B_+$ such that $p(y_j) = g_j$ and

$$\lim_{t \downarrow 0} t \cdot \mathrm{Tr}((\varphi(y_j))(t)) = \int (\varphi \circ g_j) \, \mathrm{d}\mu$$

for all $\varphi \in \mathscr{V}$.

Now for $\theta > \delta$ there is $t(\theta)$ such that for $0 < t \leqslant t(\theta)$ we have

$$\sum_{j=1}^{m} (\chi_j(y_j))(t) + \theta I \geqslant (\psi(h))(t).$$

Using Lemma 2.2, we have for $0 < t \leqslant t(\theta)$

$$\mathrm{Tr}\left(\sum_{j=1}^{m} (\chi_j(y_j))(t)\right) \geqslant \mathrm{Tr}(((\varphi_\theta \circ \psi)(h))(t))$$

where

$$\varphi_\theta(x) = \max\{x - \theta, 0\}.$$

Hence we have

$$\limsup_{t \downarrow 0} \; t \cdot \mathrm{Tr}((\varphi_\theta \circ \psi)(h)(t)) \leqslant$$

$$\leqslant \lim_{t \downarrow 0} t \cdot \mathrm{Tr}\left(\sum_{j=1}^{m} (\chi_j(y_j))(t)\right) \leqslant$$

$$\leqslant \int (\psi \circ f) \, \mathrm{d}\mu + \delta.$$

Since $\delta > 0$ is arbitrary, we have proved the following fact:

for $f \in (C_0(Y))_+$, $\psi \in \mathscr{V}$ *and* $h \in B_+$ *such that* $p(h) = \rho(f)$ *we have for all* $\theta > 0$

$$\limsup_{t \downarrow 0} t \cdot \mathrm{Tr}((\varphi_\theta \circ \psi)(h)(t)) \leqslant$$

$$\leqslant \int (\psi \circ f) \, \mathrm{d}\mu.$$

Now, given $\psi \in \mathscr{V}_\varepsilon$, consider for $0 < \theta < \varepsilon$ the function

$$\psi_\theta(t) = \begin{cases} 0 & \text{for } 0 \leqslant t < \varepsilon - \theta \\ t - (\varepsilon - \theta) & \text{for } \varepsilon - \theta \leqslant t < \varepsilon \\ \psi(t) + \theta & \text{for } \varepsilon \leqslant t. \end{cases}$$

We have $\varphi_\theta \circ \psi_\theta = \psi$, so that the preceding result shows that:

$$\limsup_{t \downarrow 0} t \cdot \mathrm{Tr}((\psi(h))(t)) \leqslant$$

$$\leqslant \int (\psi_\theta \circ f) \, \mathrm{d}\mu.$$

Since $\theta \in (0, \varepsilon)$ is arbitrary it follows from

$$\lim_{\theta \downarrow 0} \int (\psi_\theta \circ f) \, \mathrm{d}\mu = \int (\psi \circ f) \, \mathrm{d}\mu$$

that

$$\limsup_{t \downarrow 0} t \cdot \mathrm{Tr}((\psi(h))(t)) \leqslant$$

$$\leqslant \int (\psi \circ f) \, \mathrm{d}\mu.$$

Thus, the remaining part of the proof will be to show that

$$\liminf_{t \downarrow 0} t \cdot \mathrm{Tr}((\psi(h))(t)) \geqslant \int (\psi \circ f) \, \mathrm{d}\mu.$$

Using Assumption (Γ)(iii) it is easily seen that, given $\delta > 0$, we can find $g_1, \ldots, g_m \in \Gamma$, $\chi_1, \ldots, \chi_m \in \mathscr{V}$ such that

$$i \neq j \Rightarrow \mathrm{supp}(\chi_i \circ g_i) \cap \mathrm{supp}(\chi_j \circ g_j) = \emptyset$$

$$\chi_1 \circ g_1 + \ldots + \chi_m \circ g_m \leqslant \psi \circ f$$

$$\int (\psi \circ f - (\chi_1 \circ g_1 + \ldots + \chi_m \circ g_m)) \, \mathrm{d}\mu < \delta.$$

Consider again $y_1, \ldots, y_m \in B_+$ such that $p(y_j) = g_j$ and

$$\lim_{t \downarrow 0} t \cdot \mathrm{Tr}((\varphi(y_j))(t)) = \int (\varphi \circ g_j) \, \mathrm{d}\mu$$

for all $\varphi \in \mathscr{V}$. Thus, we have

$$\lim_{t \downarrow 0} \left\| \left((\psi(h))(t) - \sum_{i=1}^{m} (\chi_j(y_j))(t) \right)_+ \right\| = 0$$

$$i \neq j \Rightarrow \lim_{t \downarrow 0} \|(\chi_j(y_j))(t)\,(\chi_i(y_i))(t)\| = 0.$$

Using Lemma 2.3, it follows that for all $\theta > 0$ we have:

$$\liminf_{t \downarrow 0} t \cdot \mathrm{Tr}(\psi(h)(t)) \geqslant$$

$$\geqslant \lim_{t \downarrow 0} \sum_{i=1}^{m} t \cdot \mathrm{Tr}((\varphi_\theta \circ \chi_i)(y_i)(t)) =$$

$$= \sum_{i=1}^{m} \int (\varphi_\theta \circ \chi_i \circ g_i)\, \mathrm{d}\mu.$$

Now, since $\theta > 0$ is arbitrary we have

$$\liminf_{t \downarrow 0} t \cdot \mathrm{Tr}(\psi(h)(t)) \geqslant$$

$$\geqslant \lim_{\theta \downarrow 0} \sum_{i=1}^{m} \int (\varphi_\theta \circ \chi_i \circ g_i)\, \mathrm{d}\mu =$$

$$= \sum_{i=1}^{m} \int (\chi_i \circ g_i)\, \mathrm{d}\mu \geqslant \int (\psi \circ f)\, \mathrm{d}\mu - \delta.$$

Since $\delta > 0$ is arbitrary we finally get

$$\liminf_{t \downarrow 0} t \cdot \mathrm{Tr}(\psi(h)(t)) \geqslant \int (\psi \circ f)\, \mathrm{d}\mu.$$

Q.E.D.

If the homomorphism ρ is injective, then it determines a singular extension of

$$C_0((0, \infty)) \otimes \mathcal{K}(\mathcal{H}) \simeq B_0 \qquad \text{by } C_0(Y),$$

and we shall say that the extension determined by ρ *has limit distribution* μ *if condition* (ii) *in Theorem* 2.1 *is satisfied.* If ρ_1, ρ_2 determine equivalent extensions, i.e. if there is a $*$-strongly continuous unitary-valued function $U: (0, \infty) \to \mathcal{L}(\mathcal{H})$ such that $Up^{-1}(\rho_1(f))\, U^* = p^{-1}(\rho_2(f))$ for all $f \in C_0(Y)$, then it is easily seen that the extension determined by ρ_1 has limit distribution μ if and only if ρ_2 has limit distribution μ.

The C^*-algebra of the Heisenberg group is given by an extension of $C_0(\mathbf{R}\setminus\{0\}) \otimes \otimes \mathcal{K}(\mathcal{H})$ by $C_0(\mathbf{R}^2)$. Since $\mathbf{R}\setminus\{0\}$ has two components $(-\infty, 0)$, $(0, +\infty)$ for which the situations are quite similar, we shall only consider $(0, \infty)$ and the extension of $C_0((0, \infty)) \otimes \mathcal{K}(\mathcal{H})$ by $C_0(\mathbf{R}^2)$ which is obtained from the extension of $C_0(\mathbf{R}\setminus\{0\}) \otimes \otimes \mathcal{K}(\mathcal{H})$ by $C_0(\mathbf{R}^2)$.

The extension of $C_0((0, \infty)) \otimes \mathcal{K}(\mathcal{H})$ by $C_0(\mathbf{R}^2)$ can be described as follows. Consider the $*$-homomorphism $\pi_+ : C^*(G) \to B$ given by $(\pi_+(a))(t) = \pi_t(a)$ for

$t \in (0, \infty)$. The homomorphism $\rho_+ : C_0(\mathbf{R}^2) \to B/B_0$ is then defined by $\rho_+(q(a)) = p(\pi_+(a))$ where $a \in C^*(G)$.

2.4. THEOREM. *The singular extension of $C_0((0, \infty)) \otimes \mathcal{K}(\mathcal{H})$ by $C_0(\mathbf{R}^2)$ determined by ρ_+ has limit distribution $\pi\lambda$ where λ denotes Lebesgue-measure on $\mathbf{R}^2$.*

The proof of Theorem 2.4 will reduce essentially after an application of Theorem 2.1 to computations of eigenvalues of certain integral operators, so we shall precede the proof with a lemma about the eigenvalues of these integral operator. This lemma is clearly not new and a proof has been given only for the sake of some completeness.

2.5. LEMMA. *The integral operator*

$$(Tf)(x) = \int_{\mathbf{R}} K(x, y) f(y) \, \mathrm{d}y$$

with kernel $K(x, y) = \exp(-\omega_1(x^2 + y^2) - \omega_2(x - y)^2 - dx - \bar{d}y)$ *where* $\omega_1 > 0$, $\omega_2 > 0$, $d \in \mathbf{C}$, *is a positive compact operator on* $L^2(\mathbf{R})$. *The eigenvalues of T are*:

$$\tau_n = A \sqrt{\pi} \, \beta^n \quad (n = 0, 1, 2, \ldots)$$

where

$$A = \sqrt{2} \, \mathrm{e}^{(\mathrm{Re}d)^2/2\omega_1} (\sqrt{\omega_1} + \sqrt{\omega_1 + 2\omega_2})^{-1}$$

$$\beta = 2\omega_2(\sqrt{\omega_1} + \sqrt{\omega_1 + 2\omega_2})^{-2}.$$

Proof. It is immediate that T is selfadjoint and compact. The positivity assertion will be a consequence of the assertion concerning the eigenvalues.

Since

$$K(x, y) = \exp(-\mathrm{i}x \, \mathrm{Im}d) \exp(-\omega_1(x^2 + y^2) - \omega_2(x - y)^2 - \mathrm{Re}d(x + y)) \cdot \exp(\mathrm{i}y \, \mathrm{Im}d)$$

it follows that the eigenvalues of T do not depend on $\mathrm{Im}d$ and thus it will be sufficient to consider only the case when d is real.

Let $\lambda > 0$, $\mu \in \mathbf{R}$ and let further $P(y)$ be a polynomial. We have

$$\int_{\mathbf{R}} K(x, y) P(y) \exp(-\lambda y^2 - \mu y) \, \mathrm{d}y =$$

$$= \frac{1}{\sqrt{\omega_1 + \omega_2 + \lambda}} \exp\left(-x^2(\omega_1 + \omega_2) - xd + \left(\frac{\omega_2 x - (d + \mu)/2}{\sqrt{\omega_1 + \omega_2 + \lambda}}\right)^2\right) \times$$

$$\times \int_{\mathbf{R}} \exp(-y^2) \, P\left(\frac{y}{\sqrt{\omega_1 + \omega_2 + \lambda}} + \frac{\omega_2 x - (d + \mu)/2}{\omega_1 + \omega_2 + \lambda}\right) \mathrm{d}y.$$

Taking $\lambda = (\omega_1^2 + 2\omega_1\omega_2)^{1/2}$, $\mu = d((\omega_1 + 2\omega_2)/\omega_1)^{1/2}$ we obtain:

$$\int_{\mathbf{R}} K(x, y)\, P(y) \exp(-\lambda y^2 - \mu y)\, \mathrm{d}y =$$

$$= A \exp(-\lambda x^2 - \mu x) \int_{\mathbf{R}} \exp(-y^2)\, P(\alpha y + \beta x + \gamma)\, \mathrm{d}y$$

where

$$A = \sqrt{2} \exp(d^2/2\omega_1)\, (\sqrt{\omega_1} + \sqrt{\omega_1 + 2\omega_2})^{-1}$$

$$\alpha = 2/(\sqrt{\omega_1} + \sqrt{\omega_1 + 2\omega_2})$$

$$\beta = 2\omega_2(\sqrt{\omega_1} + \sqrt{\omega_1 + 2\omega_2})^{-2}$$

$$\gamma = -(d/2)\cdot(\omega_1 + \sqrt{\omega_1^2 + 2\omega_1\omega_2})^{-1}.$$

Using the formulae

$$P(\alpha y + \beta x + \gamma) = \sum_{k \geqslant 0} (k!)^{-1} P^{(k)}(\beta x + \gamma)\, (\alpha y)^k$$

$$\int_{\mathbf{R}} \exp(-y^2)\, y^{2k}\, \mathrm{d}y = \sqrt{\pi}\, (2k)!\, 2^{-k}(k!)^{-1}$$

$$\int_{\mathbf{R}} \exp(-y^2)\, y^{2k+1}\, \mathrm{d}y = 0$$

we obtain

$$\int_{\mathbf{R}} \exp(-y^2)\, P(\alpha y + \beta x + \gamma)\, \mathrm{d}y = Q(x)$$

where Q is the polynomial

$$Q(x) = \sum_{k \geqslant 0} 2^{-k}(k!)^{-1}\, \alpha^{2k} P^{(2k)}(\beta x + \gamma).$$

Consider now $L\colon \mathbf{C}[X] \to \mathbf{C}[X]$ the linear map which transforms the polynomial P into Q. Then L does not increase the degree of a polynomial. The restriction of L to the space of polynomials of degree $\leqslant n$ can be expressed as an upper triangular matrix with respect to the basis $1, X, X^2, \ldots, X^n$. The diagonal elements o this matrix are $\sqrt{\pi}, \sqrt{\pi}\,\beta, \ldots, \sqrt{\pi}\,\beta^n$ and since $0 < \beta < 1$ it follows that for every $n \geqslant 0$, there is a polynomial P_n of degree n such that

$$LP_n = \sqrt{\pi}\, \beta^n P_n.$$

Returning to the integral operator T, this gives:

$$\int_{\mathbf{R}} K(x, y)\, P_n(y) \exp(-\lambda y^2 - \mu y)\, \mathrm{d}y =$$

$$= \sqrt{\pi}\, \beta^n\, A P_n(x) \exp(-\lambda x^2 - \mu x).$$

Thus $P_n(x)\exp(-\lambda x^2 - \mu x)$ is an eigenvector of T corresponding to the eigenvalue $\sqrt{\pi}\,\beta^n A$.

Now since T is selfadjoint it will be sufficient to see that these eigenvectors span $L^2(\mathbb{R})$. But since every function $P(x)\exp(-\lambda x^2 - \mu x)$ with P a polynomial is a linear combination of eigenvectors this reduces to the known fact that the function $P(x)\exp(-\lambda x^2 - \mu x)$ are dense in $L^2(\mathbb{R})$.

Q.E.D.

Proof of Theorem 2.4. Consider the set $\Gamma \subset (C_0(\mathbb{R}^2))_+$ defined as follows

$$\Gamma = \{\psi \circ f_{a,b,r} \mid \psi \in \mathscr{V}, \quad (a, b, r) \in \mathbb{R}\times\mathbb{R}\times(0, \infty)\}$$

where $f_{a,b,r}(x, y) = \exp(-r((x-a)^2 + (y-b)^2))$. It is easy to check that Γ satisfies Assumption (Γ). Hence, in view of Theorem 2.1, it will be sufficient to find for every $f_{a,b,2r}$ an element $h \in B_+$ such that $\rho_+(f_{a,b,2r}) = p(h)$ and

$$\lim_{t\downarrow 0} t\cdot \mathrm{Tr}(\varphi(h)(t)) = \int_{\mathbb{R}^2} (\varphi \circ f_{a,b,2r})\, d\lambda.$$

Consider $F \in L^1(G) \subset C^*(G)$ given by

$$F([x, y, z]) = \tilde{f}(x)\, g(y)\, k(z)$$

where $\hat{k}(t)$ is equal 1 in some neighborhood of 0, and

$$\tilde{f}(x) = (2\sqrt{\pi r})^{-1} e^{-x^2/4r - iax}$$

$$\hat{g}(y) = e^{-r(y-b)^2}.$$

Then $q(F) = f_{a,b,r}$. On the other hand $\pi_t(F)$ is given by the kernel

$$K_t(s_1, s_2) = \hat{k}(t)\, \hat{g}(ts_1)\, f(s_1 - s_2)$$

and hence computing, we find that $\pi_t(F)(\pi_t(F))^*$ is given by the kernel

$$G_t(s_1, s_2) = \frac{|\hat{k}(t)|^2}{2\sqrt{2\pi r}}\, e^{\,2rb^2} e^{-\omega_1(s_1^2 + s_2^2) - \omega_2(s_1 - s_2)^2 - ds_1 - d\cdot}$$

where $\omega_1 = rt^2$, $\omega_2 = \dfrac{1}{8r}$, $d = -2rtb - ia$.

Now, we have

$$\rho_+(f_{a,b,2r}) = p(\pi_+(F)(\pi_+(F))^*).$$

Thus we may take $h = \pi_+(F)(\pi_+(F))^*$. Using Lemma 2.5 we find that the eigenvalues $\tau_{n,t}$ of $h(t)$ are

$$\tau_{n,t} = A_t \beta_t^n, \quad (n = 0,1,2,\dots),$$

where

$$A_t = \frac{|\hat{k}(t)|^2}{2\sqrt{r}} \left(\sqrt{r}\, t^2 + \sqrt{r t^2 + \frac{1}{4r}} \right)^{-1}$$

$$\beta_t = \frac{1}{4r} \left(\sqrt{r}\, t^2 + \sqrt{r t^2 + \frac{1}{4r}} \right)^{-2}.$$

It is easy to see that

$$\lim_{t \downarrow 0} A_t = 1, \quad 0 < \beta_t < 1$$

for t sufficiently close to 0, and

$$\lim_{t \downarrow 0} \frac{|\ln \beta_t|}{t} = 4r.$$

Thus, putting $\psi(e^{-t}) = \varphi(t)$, we have

$$\mathrm{Tr}\psi(h(t)) = \sum_{n=0}^{\infty} \varphi(-\ln A_t + n\,|\ln \beta_t|)$$

and hence

$$\lim_{t \downarrow 0} t\, \mathrm{Tr}\psi(h(t)) =$$

$$= \lim_{t \downarrow 0} \frac{t}{\ln|\beta_t|} |\ln \beta_t| \sum_{n=0}^{\infty} \varphi(-\ln A_t + n|\ln \beta_t|) =$$

$$= \frac{1}{4r} \int_0^{\infty} \varphi(t)\, dt = \frac{1}{4r} \int_0^{\infty} \psi(e^{-t})\, dt.$$

On the other hand, we have

$$\int_{\mathbf{R}^2} (\psi \circ f_{a,b,2r})\, d\lambda = \int_0^{\infty} \psi(e^{-4rs^2})\, 2\pi s\, ds = \frac{\pi}{4r} \int_0^{\infty} \psi(e^{-t})\, dt,$$

so that

$$\lim_{t \downarrow 0} t \cdot \mathrm{Tr}\psi(h(t)) = \pi \int_{\mathbf{R}^2} (\psi \circ f_{a,b,2r})\, d\lambda.$$

Q.E.D.

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