J. OPERATOR THEORY
5(1981), 195–211



# THE DUAL OF A SUBNORMAL OPERATOR

JOHN B. CONWAY

Let $S$ be a pure subnormal operator on a Hilbert space $\mathscr{H}$ (that is, $S$ is a subnormal operator with no normal direct summand). If $N$ is the minimal normal extension of $S$ on the Hilbert space $\mathscr{K}$ containing $\mathscr{H}$, then the dual of $S$ is the restriction of $N^*$ to $\mathscr{K} \ominus \mathscr{H}$. This paper studies some of the properties of this dual, as well as those subnormal operators that are unitarily equivalent to their dual.

The reader can consult [3] as well as the material and bibliography of [6] for the properties of subnormal operators.

The theory of subnormal operators is receiving increased attention, especially since it has been proved that they have nontrivial invariant subspaces [5] and, indeed, are reflexive [12]. One of the difficulties with the study of subnormality is the lack of symmetry. The usual symmetry for operator theory (involving the adjoint of the operator) is completely absent. In fact, if both $S$ and $S^*$ are subnormal, then $S$ is a normal operator. It is hoped that the concept of the dual of a subnormal operator will help to fill this void of symmetry.

If $G$ is an open subset of the plane and $f\colon G \to \mathbf{C}$, then the analogue, for $f$, of the adjoint of an operator is given by complex conjugation. That is, the adjoint of the function $f$ is the function $\overline{f}\colon G \to \mathbf{C}$ defined by $\overline{f}(z) = \overline{f(z)}$. If $f$ is analytic, then $\overline{f}$ is analytic iff $f$ is constant on each component of $G$. Thus, for spaces of analytic functions complex conjugation is an inappropriate form of symmetry. The appropriate form of symmetry for analytic functions is to define $f^*(z) = \overline{f(\overline{z})}$ on the open set $G^* = \{z : \overline{z} \in G\}$.

The dual of a subnormal operator is analogous to the function $f^*$ associated with an analytic function $f$. Indeed, there are results that relate the dual of a subnormal operator and the functions $f^*$. (See below.)

In Section 1 some of the elementary properties of the dual are established. Specifically, if $T$ is the dual of the pure subnormal operator $S$, then $S$ is the dual of $T$, $\sigma(T) = \{\lambda : \overline{\lambda} \in \sigma(S)\}$, $S$ is irreducible iff $T$ is irreducible, and if $f$ is an "appropriate" function such that $f(S)$ is defined and a pure subnormal, then the dual of $f(S)$ is $f^*(T)$, where $f^*(z) = \overline{f(\overline{z})}$.

Section 2 studies self-dual subnormal operators; that is, subnormal operators that are unitarily equivalent to their duals. In particular, it is shown that if $S$ is a self-dual, finitely multicyclic, subnormal operator, then $S$ can by written as the direct sum of a trivially self-dual operator and a countable number of irreducible self-dual operators. Thus, in the finitely multicyclic case the study of self-dual subnormal operators is equivalent to the study of those that are irreducible.

Section 3 presents some examples of self-dual subnormal operators. In particular, the self-dual, subnormal, weighted shifts (unilateral or bilateral) are characterized.

In the last section, some open questions are posed.

## 1. SOME BASIC PROPERTIES OF THE DUAL

If $S$ is a subnormal operator on a Hilbert space $\mathscr{H}$, let $N$ be its minimal normal extension on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$. Relative to the decomposition $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^{\perp}$, $N$ can be written as a two-by-two matrix with operator entries,

$$N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}. \tag{1.1}$$

If the decomposition $\mathscr{K} = \mathscr{H}^{\perp} \oplus \mathscr{H}$ is considered, then

$$N^* = \begin{bmatrix} T & X^* \\ 0 & S^* \end{bmatrix}. \tag{1.2}$$

From here it is clear that $T$ is subnormal and $N^*$ is a normal extension of $T$. Olin ([11], Lemma 5.3) has observed that $S$ is pure iff $N^*$ is the minimal normal extension of $T$. It is for this reason that only pure subnormal operators will be considered in this paper.

If $S$ is pure and $N$, the minimal normal extension of $S$, has the representation (1.1), then the operator $T$ in (1.1) is called the *dual of* $S$. Note that since the minimal normal extension of $S$ is unique up to unitary equivalence, the dual of $S$ is unique up to unitary equivalence.

An examination of the matrix representation (1.2) yields the following as an immediate consequence.

1.3. PROPOSITION. *If $S$ is a pure subnormal operator and $T$ is the dual of $S$, then $S$ is the dual of $T$.*

1.4. PROPOSITION. *If $S$ is a pure subnormal operator and $T$ is the dual of $S$, then*

$$\sigma(S) = \{\bar{\lambda} : \lambda \in \sigma(T)\}.$$

*Proof.* If $\lambda \notin \sigma(S)$, then $\lambda \notin \sigma(N)$. Moreover, $(N-\lambda)^{-1}$ leaves $\mathscr{H}$ invariant and $(S-\lambda)^{-1} = (N-\lambda)^{-1}|\mathscr{H}$. If $N$ has the representation (1.1) and, relative to the same decomposition of $\mathscr{K}$,

$$(N-\lambda)^{-1} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

an easy matrix calculation implies that $T^* - \lambda$ is invertible and $C = (T^* - \lambda)^{-1}$. Hence $\lambda \notin \sigma(T^*)$, and so $\bar{\lambda} \notin \sigma(T)$. This shows that $\sigma(S) \supseteq \{\bar{\lambda} : \lambda \in \sigma(T)\}$. The other inclusion follows by using Proposition 1.3 and the preceding argument. ▨

If $\mu$ is a scalar-valued spectral measure for $N$, then there is a measure $\mu_0 \leqslant \mu$ and a bounded open set $G$ such that $\mu_0 \perp (\mu - \mu_0)$ and $P^\infty(\mu)$, the weak star closure of the polynomials in $L^\infty(\mu)$, is given by the formula $P^\infty(\mu) = L^\infty(\mu - \mu_0) \oplus \oplus H^\infty(G, \mu_0)$, where $H^\infty(G, \mu_0)$ is the image of $H^\infty(G)$ in $L^\infty(\mu_0)$ under a natural isometric embedding [13]. ($H^\infty(G)$ is the algebra of bounded analytic functions on $G$.) The set $G^-$ is sometimes called the Sarason hull of $\mu$ and has several additional properties, as does the measure $\mu_0$. (See [13].)

For any Borel subset $\Delta$ of the plane, let $\mu^*(\Delta) = \mu(\Delta^*)$. ($\Delta^* \equiv \{\lambda : \bar{\lambda} \in \Delta\}$.) Then $\mu^*$ is a regular Borel measure. If $\mu$ is a scalar-valued spectral measure for the normal operator $N$, $\mu^*$ is the scalar-valued spectral measure for $N^*$. It follows that $P^\infty(\mu^*) = L^\infty(\mu^* - \mu_0^*) \oplus H^\infty(G^*, \mu_0^*)$. Also, if $\varphi^*(z) = \overline{\varphi(\bar{z})}$, then $\varphi \to \varphi^*$ defines a real-linear isometry of $P^\infty(\mu)$ onto $P^\infty(\mu^*)$ that is a weak-$*$ homeomorphism.

If, as in the situation under consideration in this paper, $N$ is the minimal normal extension of a pure subnormal operator $S$, then $\mu = \mu_0$. So $P^\infty(\mu) = H^\infty(G, \mu)$ and $P^\infty(\mu^*) = H^\infty(G^*, \mu^*)$. Also, if $\varphi \in H^\infty(G)$ and $\varphi$ is not constant on any component of $G$, then $\varphi(S)$ is a pure subnormal operator and $\varphi(N)$ is the minimal normal extension of $\varphi(S)$ ([16], Theorem 6.1 and Corollary 6.4). Similarly, if $T$ is the dual of $S$, $\varphi^*(T)$ is a pure subnormal operator and $\varphi^*(N^*)$ is its minimal normal extension.

1.5. PROPOSITION. *With the notation of the preceding paragraphs, if $T$ is the dual of the pure subnormal operator $S$ and $\varphi \in H^\infty(G)$, then $\varphi^*(T)$ is the dual of $\varphi(S)$.*

*Proof.* Note that if $p$ is any polynomial, then $p(T^*) = p^*(T)^*$. Using the matrix representation (1.1), for any polynomial $p$

$$p(N) = \begin{bmatrix} p(S) & X_p \\ 0 & p(T^*) \end{bmatrix} = \begin{bmatrix} p(S) & X_p \\ 0 & p^*(T)^* \end{bmatrix},$$

where $X_p$ is some operator from $\mathscr{H}^\perp$ into $\mathscr{H}$. Let $\{p_i\}$ be a net of polynomials that converges to $\varphi$ in the weak-$*$ topology of $L^\infty(\mu)$. It follows by elementary arguments that $p_i(N) \to \varphi(N)$ and $p_i(S) \to \varphi(S)$ weakly. But $p_i^* \to \varphi^*$ weak $*$ in $L^\infty(\mu^*)$, so $p_i^*(T)^* \to \varphi^*(T)^*$ weakly. It follows that

$$\varphi(N) = \begin{bmatrix} \varphi(S) & X_\varphi \\ 0 & \varphi^*(T)^* \end{bmatrix}$$

for some $X_\varphi : \mathscr{H}^\perp \to \mathscr{H}$. ▨

Let $S$ be the unilateral shift of multiplicity one, represented as multiplication by $z$ on $H^2$. The minimal normal extension $N$ of $S$ is the operator defined by multiplication by $z$ on $L^2(\lambda) = L^2$, where $\lambda$ is normalized Lebesgue measure on the unit circle $\partial\mathbf{D}$. The dual of $S$, $T$, is the operator defined on $K^2 \equiv L^2 \ominus H^2$ by $Tf = \bar{z}f$. For $f$ in $L^2$ define $f^\#$ in $L^2$ by

$$f^\#(z) = f(\bar{z}).$$

Note that $f \to f^\#$ is a unitary map on $L^2$. Define $W: L^2 \to L^2$ by $Wf = \bar{z}f^\#$. $W$ is a unitary transformation and $W^2 = 1$. Hence $W$ is a *symmetry* (a hermitian unitary operator). Moreover $WNWf = WN(\bar{z}f^\#) = W(f^\#) = \bar{z}(f^\#)^\# = \bar{z}f = N^*f$. Also, for $n \geqslant 0$, $Wz^n = \bar{z}^{(n+1)}$; hence $WH^2 = K^2$. It follows that if $U = W| H^2$, then $U: H^2 \to K^2$ is an isomorphism and $USU^{-1} = T$. So $S$ is unitarily equivalent to its dual. If $\varphi \in H^\infty (\equiv H^\infty(\mathbf{D}))$, it follows that $U\varphi(S)\,U^{-1} = \varphi(T)$. Combining this fact with the preceding proposition yields the following.

1.6. Corollary. *If $\varphi \in H^\infty$ and $T_\varphi$ is the corresponding analytic Toeplitz operator, then the dual of $T_\varphi$ is unitarily equivalent to $T_{\varphi^*}$.*

For any operator $A$, let $\mathscr{W}^*(A)$ denote the von Neumann algebra generated by $A$ and let $\mathscr{W}^*(A)'$ be its commutant. The next result appears as Theorem 8 in [3].

1.6. Proposition. *Let $S$ be a subnormal operator on $\mathscr{H}$ with minimal normal extension $N$ on $\mathscr{K}$, and let $P$ be the projection of $\mathscr{K}$ onto $\mathscr{H}$. If $\mathscr{A} = \mathscr{W}^*(N)' \cap \{P\}'$ and $\rho: \mathscr{A} \to \mathscr{W}^*(S)'$ is defined by $\rho(A) = A|\mathscr{H}$, then $\rho$ is a $*$-isomorphism.*

Maintaining the notation introduced in the preceding proposition, if $T$ is the dual of $S$, then $\mathscr{W}^*(T)'$ is isomorphic to $\mathscr{W}^*(N^*)' \cap \{1 - P\}' = \mathscr{A}$. This proves the following result.

1.7. Proposition. *If $S$ is a pure subnormal operator with dual $T$, then $\mathscr{W}^*(S)'$ and $\mathscr{W}^*(T)'$ are $*$-isomorphic von Neumann algebras.*

1.8. Corollary. *If $S$ is a pure subnormal operator with dual $T$, then $S$ is irreducible iff $T$ is irreducible.*

*Proof.* The reducing subspaces for any operator $A$ are precisely the ranges of projections in $W^*(A)'$. To say that $A$ is irreducible is to say that $W^*(A)' = \mathbf{C}$. By the preceding proposition, $W^*(S)' = \mathbf{C}$ iff $W^*(T)' = \mathbf{C}$. ▨

Let $\lambda$ be planar Lebesgue measure restricted to the open unit disk, let $N=$ multiplication by $z$ on $L^2(\lambda)$, and let $S$ be the restriction of $N$ to $L^2_a(\mathbf{D})$, the space of analytic functions in $L^2(\lambda)$. The operator $S$ is also called the *Bergman shift* because of its action on the normalization of the basis $\{1, z, z^2, \dots\}$ for $L^2_a(\mathbf{D})$. It is not difficult to show that $S^*S - SS^*$ is a trace class operator, though this is a consequence of a more general theorem [2].

Now, if the matrix representation (1.1) for $N$ is substituted in the equation $N^*N - NN^* = 0$, it is seen that $XX^* = S^*S - SS^*$. Thus, for the Bergman shift $S$, $X$ is compact. If $\sigma_e$ denotes the essential spectrum, then $\mathbf{D}^- = \sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*)$. But $\sigma_e(S) = \partial\mathbf{D}$, so $\sigma_e(T) = \mathbf{D}^-$. Also $S$ is irreducible, so, by Corollary 1.8, $T$ is irreducible. Therefore the dual of the Bergman shift is an irreducible subnormal operator with essential spectrum equal to that of its minimal normal extension.

It is worth remarking that the dual of multiplication by $z$ on $L^2_a(G)$, for any bounded open subset $G$ of $\mathbf{C}$, can be represented as multiplication by $\bar{z}$ on the Sobolev space $W^{1,2}_0(G)$.

We end this section by remarking that the dual of a cyclic subnormal operator is seldom cyclic. For example, if $S$ is a subnormal unilateral weighted shift, then there is a strongly continuous unitary representation $\omega \to U_\omega$ of the circle group such that $U_\omega S U_\omega^* = \omega S$ for all $\omega$. (If $\{e_0, e_1, \dots\}$ is the shifted orthonormal basis, let $U_\omega e_n = \omega^n e_n$.) Gellar [7] has shown that this property characterizes the subnormal unilateral weighted shifts among all cyclic subnormal operators.

If $N$ is the minimal normal extension of $S$, then it is easy to see that there is a unitary representation $\omega \to W_\omega$ of the circle group on $\mathcal{K}$ such that $W_\omega N W_\omega^* = \omega N$ and $W_\omega | \mathcal{H} = U_\omega$. If $V_\omega = W_\omega | \mathcal{K} \ominus \mathcal{H}$, then $V_\omega T V_\omega^* = \omega T$. By Gellar's Theorem, if $T$ is cyclic, $T$ must be a unilateral weighted shift. If $\|S\| = 1$, then $\|T\| = 1$ and, because $T$ is a weighted shift, $\sigma_e(T) = \partial\mathbf{D}$. Since $X$ in (1.1) is compact when $S$ is a weighted shift, $\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*) = \partial\mathbf{D}$. Hence $\sigma(N)$ differs from $\partial\mathbf{D}$ by at most a countable number of points in $\mathbf{D}$ that accumulate on the unit circle. By the circular symmetry of $N$, the only possible point in $\sigma(N)\setminus\sigma_e(N)$ is the origin. Hence $\sigma(N) = \partial\mathbf{D}$ or $\sigma(N) = \partial\mathbf{D} \cup \{0\}$. By various representation theorems for subnormal weighted shifts (see [14], for example), this means that $S$ is the shift with weight sequence $\{1, 1, \dots\}$ or $\{\alpha, 1, 1, \dots\}$ for some $\alpha$, $0 < \alpha < 1$. Since these weighted shifts have cyclic duals, we have shown that these are the only weighted unilateral shifts with cyclic duals.

## 2. SELF-DUAL SUBNORMAL OPERATORS

A subnormal operator is said to be *self-dual* if it is unitarily equivalent to its dual. As we saw in the preceding section, the unilateral shift of multiplicity one is self-dual. If $S_1, S_2, \dots$ are self-dual, then it is easy to see that $S_1 \oplus S_2 \oplus \dots$

is self-dual. Hence, any pure isometry is self-dual. A quasinormal operator [9] is subnormal, and it follows from the structure theory of such operators [4] that every pure quasinormal operator is self-dual. Other examples of self-dual subnormal operators will be given later.

The first result is immediate from Proposition 1.4.

2.1. PROPOSITION. *If $S$ is a self-dual subnormal operator, $\sigma(S)$ is symmetric with respect to the real axis.*

Let $S$ be a pure subnormal operator with minimal normal extension $N$ given by (1.1). If $U:\mathscr{H} \to \mathscr{K} \ominus \mathscr{H}$ is an isomorphism such that $USU^{-1} = T$, then, because $N^*$ is the minimal normal extension of $T$, there is a unitary operator $W:\mathscr{K} \to \mathscr{K}$ such that $WNW^{-1} = N^*$ and $W|\mathscr{H} = U$ ([3] or [9], Solution 155). Because $W$ and $U$ are isometries, $W(\mathscr{K} \ominus \mathscr{H}) = \mathscr{H}$. Define $V = W|\mathscr{K} \ominus \mathscr{H}$; so $V:\mathscr{K} \ominus \mathscr{H} \to \mathscr{H}$ is an isomorphism.

Relative to the decomposition $\mathscr{K} = \mathscr{H} \oplus \mathscr{H}^{\perp}$,

$$W = \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}. \tag{2.2}$$

Substituting this matrix and the matrix (1.1) into the equation $WNW^{-1} = N^*$, we obtain the equations

$$VTV^{-1} = S,$$

$$UXV^{-1} = X^*.$$

2.3. LEMMA. *If $N$ is normal and $W$ is a unitary operator such that $WNW^{-1} = N^*$, then there is a unitary operator $W_1$ in $\mathscr{W}^*(W^2)$ and a symmetry $W_2$ such that $W = W_1W_2 = W_2W_1$ and $W_2NW_2 = N^*$.*

*Proof.* Because $WNW^{-1} = N^*$, $WN^*W^{-1} = N$. Hence $N = WN^*W^{-1} = W(WNW^{-1})W^{-1} = W^2N(W^2)^{-1}$. So $W^2 \in \{N\}' = \mathscr{W}^*(N)'$ (Fuglede's Theorem). Let $W_1$ be a unitary in $\mathscr{W}^*(W^2)$ such that $W_1^2 = W^2$. So $W_1$ is a function of $W^2$ and must be in $\{N\}'$. Since $W^2 \in \mathscr{W}^*(W)$, $W_1 \in \mathscr{W}^*(W)$. Hence $W_1W = WW_1$.

Let $W_2 = WW_1^*$. Then $W_2$ is unitary and $W_2^2 = W^2W_1^{*2} = 1$. Hence $W_2$ is a symmetry.

Clearly $W_1W_2 = W_2W_1 = W$. Finally, $N^* = WNW^{-1} = W_2W_1NW_1^{-1}W_2^{-1} = W_2NW_2^{-1}$ since $W_1 \in \{N\}'$. ▨

2.4. PROPOSITION. *If $S$ is a pure subnormal operator on $\mathscr{H}$ with minimal normal extension $N$ on $\mathscr{K}$, the following statements are logically equivalent:*

(a) *$S$ is self-dual.*

(b) *There is a unitary operator $W$ on $\mathscr{K}$ such that $WNW^{-1} = N^*$ and $W\mathscr{H} = \mathscr{H}^{\perp}$.*

(c) *There is a symmetry $W$ on $\mathscr{K}$ such that $WNW = N^*$ and $W\mathscr{H} = \mathscr{H}^\perp$.*

*Proof.* That (a) and (b) are equivalent was shown prior to the statement of Lemma 2.3. Clearly (c) implies (b), so it remains to prove that (b) implies (c).

Let $W$ be as in the statement of (b) and let $W_1$ and $W_2$ be as in Lemma 2.3. If $U = W|\mathscr{H}$ and $V = W|\mathscr{H}^\perp$, then using (2.2) we see that

$$W^2 = \begin{bmatrix} VU & 0 \\ 0 & UV \end{bmatrix}.$$

Thus $\mathscr{H}$ reduces $W^2$. Since $W_1$ is a function of $W^2$, $\mathscr{H}$ also reduces $W_1$. Hence $W_1\mathscr{H} = \mathscr{H}$ and $W_1\mathscr{H}^\perp = \mathscr{H}^\perp$. Therefore $W_2\mathscr{H} = WW_1^{-1}\mathscr{H} = W\mathscr{H} = \mathscr{H}^\perp$. So $W_2$ is the symmetry required to establish part (c). ▨

Using the preceding proposition, the next result can be proved in the same manner as Corollary 1.6. Indeed, it generalizes that corollary.

2.5. PROPOSITION. *If $S$ is a self-dual subnormal operator with minimal normal extension $N$, $\mu$ is a scalar-valued spectral measure for $N$, and $P^\infty(\mu) = H^\infty(G, \mu)$, then for any $\varphi$ in $H^\infty(G)$, the dual of $\varphi(S)$ is unitarily equivalent to $\varphi^*(S)$.*

Note that because $N$ and $N^*$ are unitarily equivalent, $G = G^*$. So $\varphi^* \in H^\infty(G)$ whenever $\varphi \in H^\infty(G)$.

2.6. COROLLARY. *If $S$ is a self-dual subnormal operator and $\varphi \in H^\infty(G)$ such that $\varphi(\bar{z}) = \overline{\varphi(z)}$, then $\varphi(S)$ is a self-dual subnormal operator.*

If $S$ is any pure subnormal operator with dual $T$, then $S \oplus T$ is a self-dual subnormal operator. Indeed, the dual of $S \oplus T$ is $T \oplus S$ and its minimal normal extension is $N \oplus N^*$. Any self-dual subnormal operator that is unitarily equivalent to a subnormal operator of the form $S \oplus T$, where $T$ is the dual of $S$, will be called a *trivially self-dual* subnormal operator. The existence of trivially self-dual operators would seem to be a hindrance to the study of duality, since every subnormal operator appears as part of a trivially self-dual operator. However, in the case of a large class of subnormal operators trivially self-dual operators present no difficulty at all.

An operator $A$ is *essentially normal* iff $A^*A - AA^*$ is compact. That is, $A$ is essentially normal if its image in the Calkin algebra under the natural map is normal. According to [2], any finitely multicyclic hyponormal is essentially normal. In fact for such a hyponormal operator $A$, $A^*A - AA^*$ is in the trace class.

The next result can be found in [1], and a generalization can be found in [8].

2.7. PROPOSITION. *If $A$ is an essentially normal operator on $\mathscr{H}$ and $Z_0, Z_1, \ldots$ are the minimal central projections in $\mathscr{W}^*(A)$, then:*

(a) *$Z_i \perp Z_j$ for $i \neq j$ and $Z_0 + Z_1 + \ldots = 1$.*

(b) *$A|Z_0\mathscr{H}$ is normal.*

(c) *If* $n \geqslant 1$, $\mathcal{W}^*(A|Z_n\mathcal{H})$ *is a type* I *factor and* $A|Z_n\mathcal{H}$ *is unitarily equivalent to the direct sum of an irreducible essentially normal operator with itself a finite number of times.*

For any integer $k$ and any Hilbert space $\mathcal{L}$, let $\mathcal{L}^{(k)}$ be the direct sum of $\mathcal{L}$ with itself $k$ times. If $B \in \mathcal{B}(\mathcal{L})$, $B^{(k)}$ is the operator defined on $\mathcal{L}^{(k)}$ by $B^{(k)}(f_1,\dots,f_k) = (Bf_1,\dots,Bf_k)$. If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{L})$, $\mathcal{A}^{(k)} = \{B^{(k)}: B \in \mathcal{A}\}$.

If $A$ and $Z_n$ are as in the preceding proposition, put $A_n = A|Z_n\mathcal{H}$. Part (c) states that for each $n \geqslant 1$, there is a Hilbert space $\mathcal{L}_n$, an irreducible operator $B_n$ on $\mathcal{L}_n$, and an integer $k_n$ such that $A_n$ is unitarily equivalent to $B_n^{(k_n)}$ on $\mathcal{L}_n^{(k_n)}$. Thus, $\mathcal{W}^*(A_n)$ is spatially isomorphic to $\mathcal{B}(\mathcal{L}_n)^{(k_n)}$.

The next result is an immediate consequence of the preceding proposition.

2.8. Corollary. *If $S$ is an essentially normal operator, then* $S = S_0 \oplus S_1 \oplus \oplus S_2 \oplus \dots$, *where $S_0$ is normal and $S_1, S_2, \dots$ are irreducible.*

As stated, some of the operators $S_n (n \geqslant 1)$ in Corollary 2.8 could be operators on one-dimensional spaces, and hence normal. If it is assumed that no $S_n$ $(n \geqslant 1)$ acts on a one-dimensional space (all such summands can be grouped together with $S_0$), then the decomposition is unique up to unitary equivalence and the ordering of the summands. This can be proved by first proving a uniqueness statement for Proposition 2.7 and using some standard results in the theory of von Neumann algebras [15]. This uniqueness will be used in the next result.

2.9. Theorem. *If $S$ is a self-dual, essentially normal, subnormal operator, then* $S = S_0 \oplus S_1 \oplus S_2 \oplus \dots$, *where $S_0$ is a trivially self-dual subnormal operator and $S_1, S_2, \dots$ are irreducible, self-dual, subnormal operators.*

*Proof.* Let $N =$ the minimal normal extension of $S$ and suppose it has the representation (1.1).

Because $S$ is self-dual, it must be pure. By the preceding corollary, $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$, where each $\mathcal{H}_n$ is an infinite dimensional reducing subspace for $S$ and $S_n \equiv S|\mathcal{H}_n$ is irreducible. Note that each $\mathcal{H}_n$ is a minimal reducing subspace for $S$.

Let $\mathcal{K}_n$ be the smallest reducing subspace for $N$ that contains $\mathcal{H}_n$; so $N_n = N|\mathcal{K}_n$ is the minimal normal extension of $S_n$. Also, $\mathcal{K}_n$ equals the closed linear span of $\{N^{*k}f : f \in \mathcal{H}_n, k \geqslant 0\}$. If $n \neq m$, $f_n \in \mathcal{H}_n$, and $f_m \in \mathcal{H}_m$, then for $k$ and $j \geqslant 0$

$$\langle N^{*k}f_n, N^{*j}f_m\rangle = \langle N^j f_n, N^k f_m\rangle =$$

$$= \langle S^j f_n, S^k f_m\rangle = 0,$$

since $\mathcal{H}_n \perp \mathcal{H}_m$. Thus $\mathcal{K}_n \perp \mathcal{K}_m$ for $n \neq m$. It follows from the fact that $N$ is the minimal normal extension of $S$ that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \dots$ .

Because $\mathcal{K}_n \subseteq \mathcal{K}_m^\perp$ for $n \neq m$, $\mathcal{K}_n \perp \mathcal{H}_m$ for $m \neq n$. Hence $\mathcal{M}_n \equiv \mathcal{K}_n \ominus \mathcal{H}_n \subseteq \mathcal{H}^\perp$. It is easy to see that $\mathcal{M}_n$ reduces $T$ and $T|\mathcal{M}_n$ is the dual of $S_n$. Because $S_n$ is irreducible, so is $T|\mathcal{M}_n$ (Corollary 1.8). Therefore $\mathcal{M}_n$ is a minimal reducing subspace for $T$. Moreover

$$\mathcal{H}^\perp = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \dots .$$

By Proposition 2.4, there is a symmetry $W$ on $\mathcal{K}$ such that $WNW = N^*$ and $W\mathcal{H} = \mathcal{H}^\perp$. Let $\mathcal{L}_n = W\mathcal{H}_n$. It follows that $\mathcal{L}_1, \mathcal{L}_2, \dots$ are pairwise orthogonal, minimal, reducing subspaces for $T$, and $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots = \mathcal{H}^\perp$. Also $W\mathcal{L}_n = \mathcal{H}_n$ since $W$ is a symmetry.

Therefore,

$$T = (T|\mathcal{M}_1) \oplus (T|\mathcal{M}_2) \oplus \dots$$

and

$$T = (T|\mathcal{L}_1) \oplus (T|\mathcal{L}_2) \oplus \dots$$

are two decompositions of $T$ into the direct sum of irreducible operators. Because of the uniqueness part of Corollary 2.8 (see the discussion following that corollary), there is a permutation $\tau: \mathbf{N} \to \mathbf{N}$ such that for each $n$ in $\mathbf{N}$,

$$\text{dual}(S_n) = T|\mathcal{M}_n \cong T|\mathcal{L}_{\tau(n)}. \tag{2.10}$$

On the other hand, $W\mathcal{H}_n = \mathcal{L}_n$, and so the isomorphism between $\mathcal{H}_n$ and $\mathcal{L}_n$ defined by restricting $W$ to $\mathcal{H}_n$ shows that $S_n \cong T|\mathcal{L}_n$. Combining this with (2.10) gives

$$S_n \cong T|\mathcal{L}_n \cong T|\mathcal{M}_{\tau^{-1}(n)} = \text{dual}(S_{\tau^{-1}(n)}). \tag{2.11}$$

But $S_{\tau^{-1}(n)} \cong T|\mathcal{L}_{\tau^{-1}(n)}$, and unitarily equivalent operators have unitarily equivalent duals. Thus

$$S_n \cong \text{dual}(T|\mathcal{L}_{\tau^{-1}(n)}) \cong \text{dual}(T|\mathcal{M}_{\tau^{-2}(n)}) = S_{\tau^{-2}(n)}.$$

That is,

$$S_n \cong S_{\tau^2(n)} \cong S_{\tau^{-2}(n)}, \quad n \geqslant 1. \tag{2.12}$$

Decompose $\mathbf{N}$ into the orbits of $\tau$. That is, decompose $\mathbf{N}$ into pairwise disjoint sets $\{\mathbf{N}_k: k \geqslant 1\}$ such that for each $k$, $\mathbf{N}_k = \{\tau^j(n): j \in \mathbf{Z}\}$ for some (any) $n$ in $\mathbf{N}_k$. Define

$$\mathcal{R}_k = \oplus \{\mathcal{H}_n: n \in \mathbf{N}_k\}.$$

Note that the spaces $\mathcal{R}_k$ reduce $S$, are pairwise orthogonal, and $\mathcal{H} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \dots$.

Now no set $\mathbf{N}_k$ is infinite. Indeed, if $\mathbf{N}_k$ were infinite, then for any $n$ in $\mathbf{N}_k$, $n, \tau^2(n), \tau^4(n), \ldots$ are all distinct integers. By (2.12), $S_n \cong S_{\tau^2(n)} \cong S_{\tau^4(n)} \cong \ldots$. Thus

$$(S|\mathscr{R}_k)^*(S|\mathscr{R}_k) - (S|\mathscr{R}_k)(S|\mathscr{R}_k)^* = (S^*S - SS^*)|\mathscr{R}_k$$

cannot be compact, a contradiction. Hence $\mathbf{N}_k$ is finite for each $k$.

Suppose that $\mathbf{N}_k$ has an even number of elements; say $\mathbf{N}_k = \{n, \tau(n), \ldots, \tau^{2p+1}(n)\}$, $p \geqslant 0$, $\tau^i(n) \neq \tau^j(n)$ for $0 \leqslant i < j \leqslant 2p+1$. Define operators by

$$A = S_n \oplus S_{\tau^2(n)} \oplus \ldots \oplus S_{\tau^{2p}(n)},$$

$$B = S_{\tau(n)} \oplus S_{\tau^3(n)} \oplus \ldots \oplus S_{\tau^{2p+1}(n)}.$$

But (2.11) implies that the dual of $S_{\tau^j(n)}$ is unitarily equivalent to $S_{\tau^{j+1}(n)}$. Hence dual $(A) \cong B$. Since $S|\mathscr{R}_k \cong A \oplus B$, $S|\mathscr{R}_k$ is a trivially self-dual subnormal operator.

If $\mathbf{N}_k$ has an odd number of elements, let $\mathbf{N}_k = \{n, \tau(n), \ldots, \tau^{2p}(n)\}$, $p \geqslant 0$, $\tau^i(n) \neq \tau^j(n)$ for $0 \leqslant i < j \leqslant 2p$. Relation (2.12) implies that $S_n \cong S_{\tau^2(n)} \cong \ldots \cong S_{\tau^{2p}(n)}$. But $\tau(\mathbf{N}_k) = \{\tau(n), \ldots, \tau^{2p+1}(n)\} = \mathbf{N}_k$. Hence $\tau^{2p+1}(n) = n$. Thus $S_{\tau^{2p}(n)} \cong S_{\tau(n)} \cong \ldots \cong S_{\tau^{2p-1}(n)}$. So $S_n \cong S_m$ for all $n$ and $m$ in $\mathbf{N}_k$ and, hence,

$$S|\mathscr{R}_k \cong S_n \oplus S_n \oplus \ldots \oplus S_n \quad (2p+1 \text{ times}). \tag{2.13}$$

By (2.11), dual$(S_n) = T|\mathscr{M}_n \cong S_{\tau(n)} \cong S_n$; so $S_n$ is self-dual.

Since the direct sum of any number of trivially self-dual subnormal operators is a trivially self-dual operator, the theorem is proved. ▨

This section concludes with a characterization of the symmetries $W$ such that $WNW = N^*$, when $N$ is a cyclic normal operator. If $N$ is cyclic, then $N$ is unitarily equivalent to multiplication by $z$ on $L^2(\mu)$ for some compactly supported measure $\mu$ on the plane. If $\mu^*(\Delta) = \mu(\bar{\Delta})$ for every Borel subset $\Delta$ of $\mathbf{C}$, then $\mu^*$ is a regular Borel measure and, because $N \cong N^*$, $\mu$ and $\mu^*$ are mutually absolutely continuous. Thus $\mu$ and $\mu + \mu^*$ are mutually absolutely continuous. Moreover, $(\mu + \mu^*)(\Delta) = (\mu + \mu^*)(\bar{\Delta})$ for any Borel set $\Delta$. Thus we may assume that $N =$ multiplication by $z$ on $L^2(\mu)$ and $\mu(\bar{\Delta}) = \mu(\Delta)$ for every Borel set $\Delta$.

2.14. Proposition. *Let $N$ be multiplication by $z$ on $L^2(\mu)$ where $\mu(\bar{\Delta}) = \mu(\Delta)$ for every Borel subset $\Delta$ of the plane. Define $V\colon L^2(\mu) \to L^2(\mu)$ by $(Vf)(z) = f^*(z) \equiv f(\bar{z})$. If $W\colon L^2(\mu) \to L^2(\mu)$ is a symmetry such that $WNW = N^*$, then there is a function $w$ in $L^\infty(\mu)$ such that $|w| = 1$ a.e. $[\mu]$, $w(z) = w^*(z)\ (\equiv \overline{w(\bar{z})})$, and $Wf = V(wf)$. Conversely, if $w$ has these properties and $Wf = V(wf)$ for all $f$ in $L^2(\mu)$, then $W$ is a symmetry such that $WNW = N^*$.*

*Proof.* It is easy to check that $V$ is a symmetry and $VNV = N^*$. Hence $(VW)N(VW)^* = V(WNW)V = VN^*V = N$. So $VW \in \{N\}'$. Because $N$ is cyclic, there is a function $w$ in $L^\infty(\mu)$ such that $VW = M_w$ (multiplication by $w$). Because $VW$ is unitary, $|w| = 1$ a.e. $[\mu]$. Thus, $W = VM_w$.

Now $W$ is a symmetry, so $W = (VM_w)^* = M_{\bar{w}}V = VM_w$. Hence, $M_w = VM_{\bar{w}}V$. Applying this operator equation to the function 1 gives that $w = (VM_{\bar{w}}V)(1) = VM_{\bar{w}}(1) = V(\bar{w}) = w^*$. The converse is an easy computation. ▨

If, in the preceding proposition, it is not assumed that $\mu(\bar{\Delta}) = \mu(\Delta)$ for each Borel set $\Delta$, then $W$ can be characterized; but the formula involves $\mathrm{d}\mu/\mathrm{d}\mu^*$.

## 3. SOME EXAMPLES OF THE DUAL OF A SUBNORMAL OPERATOR

It was pointed out in the preceding section that every pure quasinormal operator is self-dual, and if $S$ is self-dual and $\varphi$ is an "appropriate" analytic function such that $\varphi = \varphi^*$, then $\varphi(S)$ is self-dual. In this section a few more examples are presented. The first of these is a weighted shift.

Let $S$ be an irreducible subnormal, weighted shift on $\mathscr{H}$. So there is an orthonormal basis $\{e_0, e_1, \ldots\}$ for $\mathscr{H}$ such that $Se_n = \alpha_n e_{n+1}$, for some positive scalars $\alpha_0, \alpha_1, \ldots$ Suppose that $1 = \|S\|$ $(= \sup\{\alpha_n : n \geqslant 0\})$. Because $S$ is subnormal, there is a unique probability measure $\nu$ on $[0, 1]$ such that for $n \geqslant 1$

$$(3.1) \qquad (\alpha_0\,\alpha_1 \ldots \alpha_{n-1})^2 = \int r^{2n}\mathrm{d}\nu(r).$$

(See [10], pp. 895—896.) Define a measure $\mu$ on $\mathbf{C}$ by

$$(3.2) \qquad \int f\mathrm{d}\mu = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 f(r\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\nu(r)\,\mathrm{d}\theta$$

for every continuous function $f$ on $\mathbf{C}$ with compact support. It is not difficult to see that $S$ is unitarily equivalent to multiplication by $z$ on $H^2(\mu)$, the completion of the polynomials in $L^2(\mu)$. In fact, the appropriate unitary is the one that sends the basic vector $e_n$ into

$$\|z^n\|^{-1}z^n = \left(\int r^{2n}\ \mathrm{d}\nu(r)\right)^{-\frac{1}{2}} z^n.$$

Conversely, if $\nu$ is a probability measure on $[0, 1]$ and $\alpha_0, \alpha_1, \ldots$ are defined to be the positive numbers satisfying (3.1), then $Se_n = \alpha_n e_{n+1}$ is an irreducible, subnormal, weighted shift.

Using the above notation, let $\nu = \beta\delta_0 + \alpha\delta_1$, where $\alpha, \beta > 0$, $\alpha + \beta = 1$, and $\delta_0$ and $\delta_1$ are the unit point masses at 0 and 1. The "rotated" measure $\mu$ is $\beta\delta_0 + \alpha\lambda$, where $\lambda$ is normalized arc length on the unit circle $\partial\mathbf{D}$. Let $e_0 = 1$ and let $\chi$ be the characteristic function of $\partial\mathbf{D}$. Clearly $\|e_0\| = 1$. For $n \in \mathbf{Z}\setminus\{0\}$, let $e_n = \alpha^{-\frac{1}{2}} z^n \chi$. It is easy to see that $\{e_n : n \in \mathbf{Z}\}$ is an orthonormal sequence. If $\tilde{e}_0 = (\alpha\beta)^{-\frac{1}{2}}(-\alpha\chi_0 + \beta\chi)$, where $\chi_0$ is the characteristic function of $\{0\}$, then $\{\tilde{e}_0\} \cup \{e_n : n \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mu)$.

An easy computation shows that $Ne_n = e_{n+1}$ for $n \geqslant 1$ and $n \leqslant -2$; $Ne_{-1} = \beta^{\frac{1}{2}} \tilde{e}_0 + \alpha^{\frac{1}{2}} e_0$; $Ne_0 = \alpha^{\frac{1}{2}} e_1$; $N\tilde{e}_0 = \beta^{\frac{1}{2}} e_1$. Similarly, $N^* e_n = e_{n-1}$ for $n \geqslant 2$ or $n \leqslant -1$; $N^* e_1 = \beta^{\frac{1}{2}} \tilde{e}_0 + \alpha^{\frac{1}{2}} e_0$; $N^* e_0 = \alpha^{\frac{1}{2}} e_{-1}$; $N^* \tilde{e}_0 = \beta^{\frac{1}{2}} e_{-1}$.

If $S = N | H^2(\mu)$, then $S$ is the subnormal weighted shift with weight sequence $\{\alpha^{\frac{1}{2}}, 1, 1, \ldots\}$. The dual of $S$ is $T = N^* | H^2(\mu)^\perp$. Since $\{\tilde{e}_0, e_{-1}, e_{-2}, \ldots\}$ is an orthonormal basis for $H^2(\mu)^\perp$, it follows that $T$ is a subnormal weighted shift with weight sequence $\{\beta^{\frac{1}{2}}, 1, 1, \ldots\}$. Hence $S$ is a self-dual iff $\alpha = \beta = \frac{1}{2}$.

3.3. PROPOSITION. *Let $S$ be a subnormal weighted shift with weight sequence $\{\alpha_0, \alpha_1, \ldots\}$, $\|S\| = 1$, and let $\nu$ be the unique probability measure on $[0, 1]$ satisfying (3.1). The subnormal operator $S$ is self-dual iff $\nu = \delta_1$ or $\nu = \frac{1}{2}(\delta_0 + \delta_1)$.*

*Proof.* Let $N$ be the minimal normal extension of $S$ and let $\mu$ be the rotated measure defined by (3.2). Then $N =$ multiplication by $z$ on $L^2(\mu)$ and $\sigma(N) =$ the support of $\mu$. Suppose $N$ satisfies (1.1). Because $S$ is cyclic, $X$ is compact. So $\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*)$. Now $\sigma_e(S) = \partial\mathbf{D}$ ([14], pp. 66–72). So if $S \cong T$, $\sigma_e(N) = \partial\mathbf{D}$. But $\sigma(N)$ differs from $\sigma_e(N)$ by at most a countable number of points that must accumulate on $\sigma_e(N)$. It is clear from (3.2) that $\sigma(N) =$ support of $\mu$ is the union of circles and, possibly, the origin. Hence if $S \cong T$, $\sigma(N) = \partial\mathbf{D}$ or $\sigma(N) = \partial\mathbf{D} \cup \{0\}$. Thus, either $\nu = \delta_1$ or $\nu = \beta\delta_0 + \alpha\delta_1$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. If the latter is the case, then the argument preceding this proposition implies that $\alpha = \beta = \frac{1}{2}$.

The converse was established before the statement of the proposition. ▨

Now suppose that $S$ is an irreducible, bilateral, subnormal, weighted shift with weight sequence $\{\alpha_n : n \in \mathbf{Z}\}$, and assume that $\|S\| = 1$. Once more, there is a unique probability measure $\nu$ on $[0, 1]$ such that $\int r^n \, d\nu(r) < \infty$ for all $n \leqslant -1$, (3.1) is satisfied for $n \geqslant 1$, and for $n \geqslant 1$

(3.4) $$(\alpha_{-1}\alpha_{-2} \ldots \alpha_{-n})^{-2} = \int r^{-2n} d\nu(r).$$

Notice that these conditions imply that

(3.5) $$\nu(\{0\}) = 0.$$

Once again define $\mu$ by (3.2) and let $R^2(\mu)$ be the closed linear span of $\{z^n : n \in \mathbf{Z}\}$ in $L^2(\mu)$. If $e_n$ is mapped into $\|z^n\|^{-1}z^n$, then this extends to a unitary of $\mathscr{H}$ onto $R^2(\mu)$ that shows that $S$ is unitarily equivalent to multiplication by $z$ on $R^2(\mu)$.

Let $0 < r < 1$ and put $\nu = \alpha\delta_1 + \beta\delta_r$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. So $\mu = \alpha\lambda_1 + \beta\lambda_r$, where $\lambda_r$ is normalized arclength on the circle $|z| = r$. For $n$ in $\mathbf{Z}$, let $\mathscr{M}_n$ be the closed linear span of $\{z^n|z|^k : k \in \mathbf{Z}\}$. It is easy to see that $\mathscr{M}_n$ is two-dimensional and

$$\mathscr{M}_n = \{az^n + bz^n|z|^2 : a, b \in \mathbf{C}\}.$$

Moreover, $\mathscr{M}_n \perp \mathscr{M}_m$ for $n \neq m$. Put

$$e_n = (\beta r^{2n} + \alpha)^{\frac{1}{2}} z^n.$$

If

$$f_n = r^{-n}(1 - r^2)^{-1}(\alpha\beta)^{-\frac{1}{2}} (\beta r^{2n} + \alpha)^{-\frac{1}{2}} [z^n|z|^2 - (\beta r^{2n+2} + \alpha)(\beta r^{2n} + \alpha)^{-1}z^n],$$

then $\|e_n\| = \|f_n\| = 1$, $f_n \perp e_n$, and $M_n$ is the linear span of $e_n$ and $f_n$.

It is easy to see that

$$Ne_n = [(\beta r^{2n+2} + \alpha)/(\beta r^{2n} + \alpha)]^{\frac{1}{2}} e_{n+1}. \tag{3.6}$$

If $\mathscr{H} = R^2(\mu)$, then $\{e_n : n \in \mathbf{Z}\}$ is an orthonormal basis for $\mathscr{H}$ and $S = N|\mathscr{H}$ is the bilateral shift under consideration.

It is not difficult to see that $N\mathscr{M}_n \subseteq \mathscr{M}_{n+1}$ (in fact, $N\mathscr{M}_n = \mathscr{M}_{n+1}$). Hence $N^*\mathscr{M}_n \subseteq \mathscr{M}_{n-1}$. Since $N^*\mathscr{H}^\perp \subseteq \mathscr{H}^\perp$, it must be that for every integer $n$ there is a scalar $\gamma_n$ such that $N^*f_n = \gamma_n f_{n-1}$. Since $N^*f_n = \bar{z} f_n$, $\gamma_n = \langle \bar{z} f_n, f_{n-1} \rangle$. A calculation reveals that

$$N^*f_n = r[(\beta r^{2n-2} + \alpha)/(\beta r^{2n} + \alpha)]^{\frac{1}{2}} f_{n-1}. \tag{3.7}$$

Put $h_n = f_{-n}$. Thus $N^*h_n = N^*f_{-n} = \gamma_{-n} f_{-n-1} = \gamma_{-n} h_{n+1}$. So $N^*|\mathscr{H}^\perp$ is a bilateral shift. $S$ is self-dual iff there is an integer $k$ such that

$$\left[\frac{\beta r^{2n+2k+2} + \alpha}{\beta r^{2n+2k} + \alpha}\right]^{\frac{1}{2}} = \gamma_{-n} = r\left[\frac{\beta r^{-2n-2} + \alpha}{\beta r^{-2n} + \alpha}\right]^{\frac{1}{2}} \tag{3.8}$$

for all $n$ ([14], p. 53). Elementary algebraic manipulation demonstrates that (3.8) holds iff $\beta^2 r^{2k} = \alpha^2$. However $1 = \alpha + \beta = \beta r^k + \beta = \beta(r^k + 1)$. Hence

$$\beta = (r^k + 1)^{-1}, \quad \alpha = r^k(r^k + 1)^{-1}.$$

So for each integer $k$ there is a normal operator $N_k$ and a bilateral shift $S_k = N_k|\mathscr{H}$. Substituting the above values of $\alpha$ and $\beta$ in equation (3.6) gives

$$S_k e_n = [(1 + r^{2n-k+2})/(1 + r^{2n-k})]^{\frac{1}{2}} e_{n+1}.$$

Thus

$$S_{2k}e_n = [(1 + r^{2(n-k)+2})/(1 + r^{2(n-k)})]^{\frac{1}{2}} e_{n+1},$$

$$S_{2k+1}e_n = [(1 + r^{2(n-k)+1})/(1 + r^{2(n-k)-1})]^{\frac{1}{2}} e_{n+1}.$$

By translating these weights, it can be seen that $S_{2k} \cong S_0$ and $S_{2k+1} \cong S_1$ for all $k$.

However, $S_1$ and $S_0$ are not unitarily equivalent. In fact, if they were it would follow that there is an integer $q$ such that

$$(1 + r^{2n+2})/(1 + r^{2n}) = (1 + r^{2(n+q)+1})/(1 + r^{2(n+q)-1}).$$

Again, algebraic manipulations show that this equation implies that $r^{2q-1} = 1$, which is impossible since $q$ is an integer.

3.9. Proposition. *If $S$ is an irreducible, bilateral, subnormal, weighted shift with $\|S\| = 1$ and $v$ is the unique probability measure on* $[0,1]$ *associated with $S$ by* (3.1) *and* (3.4), *then the following statements are logically equivalent:*

(a) *$S$ is self-dual.*

(b) *There is a number $r$, $0 < r < 1$, such that $S$ is unitarily equivalent to the shift corresponding to the measure $v = \frac{1}{2}(\delta_r + \delta_1)$ or $v = (1 + r)^{-1} (r\delta_r + \delta_1)$.*

(c) *There is a number $r$, $0 < r < 1$, such that the weight sequence of $S$ is a translate of either*

$$\left\{\left[\frac{1 + r^{2n+2}}{1 + r^{2n}}\right]^{\frac{1}{2}}\right\}_{n=-\infty}^{\infty} \quad or \quad \left\{\left[\frac{1 + r^{2n+1}}{1 + r^{2n-1}}\right]^{\frac{1}{2}}\right\}_{n=-\infty}^{\infty}.$$

*Proof.* The fact that (b) and (c) are equivalent is a result of a direct computation. Since it was shown prior to the statement of this proposition that (b) implies (a), it remains to show that (a) implies (b).

Assume $S$ is self-dual, let $N$ be its minimal normal extension, and let $N$ have the matrix representation (1.1). A computation proves that $XX^* = S^*S - SS^*$ is trace class. As in the proof of Proposition 3.3, this implies $\sigma_e(S) = \sigma_e(N)$. Using the circular symmetry of $\sigma(S)$ and $\sigma(N)$ and (3.5), $\sigma(N) = \sigma_e(N) = \sigma_e(S)$. Since $\sigma_e(S)$ is the boundary of an annulus [14] and $\|S\| = 1$, $v = \beta\delta_r + \alpha\delta_1$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. ▨

## 4. CONCLUDING REMARKS AND OPEN QUESTIONS

It was proved in Proposition 1.7 that if $S$ and $T$ are dual subnormal operators, then $W^*(S)'$ and $W^*(T)'$ are $*$-isomorphic.

4.1. Question. *If $S$ and $T$ are dual subnormal operators, are $\mathscr{W}^*(S)$ and $\mathscr{W}^*(T)$ $*$-isomorphic?*

If $S$ is essentially normal and its minimal normal extension is given by (1.1), then $XX^* = S^*S - SS^*$, and hence is compact. Because $X^*X = T^*T - TT^*$, it follows that $T$ is essentially normal.

Apply Proposition 2.7 to the operator $A = S$, and let $Z_1, Z_2, \ldots$ be the central projections in $\mathscr{W}^*(S)$ obtained there. It follows that if $\mathscr{X}_k = Z_k\mathscr{H}$, then

$$\mathscr{W}^*(S) = \mathscr{W}^*(S \mid \mathscr{X}_1) \oplus \mathscr{W}^*(S \mid \mathscr{X}_2) \oplus \ldots,$$

$$\mathscr{W}^*(S)' = \mathscr{W}^*(S \mid \mathscr{X}_1)' \oplus \mathscr{W}^*(S \mid \mathscr{X}_2)' \oplus \ldots .$$

Because $\mathscr{W}^*(S)'$ and $\mathscr{W}^*(T)'$ are $*$-isomorphic, there are central projections $Y_1, Y_2, \ldots$ in $\mathscr{W}^*(T)$ such that if $\mathscr{Y}_k = Y_k(\mathscr{H}^\perp)$,

$$\mathscr{W}^*(T)' = \mathscr{W}^*(T \mid \mathscr{Y}_1)' \oplus \mathscr{W}^*(T \mid \mathscr{Y}_2)' \oplus \ldots,$$

and for each $k$,

$$\mathscr{W}^*(T \mid \mathscr{Y}_k)' \approx \mathscr{W}^*(S \mid \mathscr{X}_k)'. \tag{4.2}$$

(The symbol $\approx$ stands for "$*$-isomorphic", while $\cong$ represents "spatially isomorphic".)

Now Proposition 2.7 also implies that there is an infinte dimensional Hilbert space $\mathscr{L}$ such that for each $k$, there is an integer $n_k$ and an irreducible subnormal operator $S_k$ on $\mathscr{L}$ such that $S \mid \mathscr{X}_k \cong S_k^{(n_k)}$ ($\equiv$ the direct sum of $S_k$ with itself $n_k$ times). Thus $\mathscr{W}^*(S \mid \mathscr{X}_k)'$ is $*$-isomorphic to the $n_k \times n_k$ matrices, and

$$\mathscr{W}^*(S \mid \mathscr{X}_k) \cong \mathscr{B}(\mathscr{L})^{(n_k)}.$$

By (4.2), $\mathscr{W}^*(T \mid \mathscr{Y}_k)'$ is $*$-isomorphic to the $n_k \times n_k$ matrices. Because $T \mid \mathscr{Y}_k$ is pure, it must be that $\mathscr{W}^*(T \mid \mathscr{Y}_k)'$ is spatially isomorphic to the $n_k \times n_k$ matrices acting on $\mathscr{L}^{(n_k)}$. Therefore

$$\mathscr{W}^*(T \mid \mathscr{Y}_k) \cong \mathscr{B}(\mathscr{L})^{(n_k)}.$$

Hence Question 4.1 can be answered affirmatively when $S$ is essentially normal.

4.3. QUESTION. *Can the self-dual subnormal operators be characterized?*

This question seems intractable at this point, but there are several specializations of it. For example, Theorem 2.9 seems to hold out some hope for characterizing the essentially normal, self-dual, subnormal operators. Since cyclic subnormal operators are essentially normal, a good place to start such an investigaton is by answering the following.

4.4. QUESTION. *Can the self-dual, cyclic, irreducible, subnormal operators be characterized?*

It is known that if $S$ is a cyclic subnormal operator, then there is a compactly supported measure $\mu$ on the plane such that $S$ is unitarily equivalent to multiplication by $z$ on $H^2(\mu)$, the closure of the polynomials in $L^2(\mu)$ [3]. Let $W\colon L^2(\mu) \to L^2(\mu)$ be a symmetry such that $WNW = N^*$ and $WH^2(\mu) = L^2(\mu) \ominus H^2(\mu)$. Let $\nu(\Delta) = \mu(\overline{\Delta})$ for every Borel subset $\Delta$ of the plane. If $\mu = \nu$, then the form of $W$ is given by Proposition 2.14.

However, although $\mu$ and $\nu$ are mutually absolutely continuous, it cannot be assumed that $\mu$ and $\nu$ are equal since $\mu$ was chosen and fixed when $S$ was represented on $H^2(\mu)$. Nevertheless, a formula for $W$ can be obtained that is similar to the one obtained in Proposition 2.14, but where the formula contains the Radon-Nikodym derivative $d\mu/d\nu$.

Let $K$ be a compact subset of the plane. To say that $K$ is thin is, for many authors, a precise statement. In this paper the word "thin" will be used as a non-technical descriptive term. For us, the statement that $K$ is thin could mean that $K$ has no interior, $K$ has planar Lebesgue measure zero, or $R(K) = C(K)$ (where $R(K)$ is the uniform closure in $C(K)$ of the rational functions with poles off $K$).

After a small amount of investigation, the reader will realize that it seems likely that if $S$ is a cyclic, irreducible, self-dual, subnormal operator with minimal normal extension $N$, then $\sigma(N)$ should be thin. Indeed, it would seem that $\sigma(N)$ is thin if it is only assumed that $S$ and its dual, $T$, are both cyclic. Let $S$ be multiplication by $z$ on $H^2(\mu)$. If $T$ is also cyclic, then there is a function $f$ in $L^2(\mu)$ such that $\{p + f\bar{q}: p$ and $q$ are polynomials$\}$ is dense in $L^2(\mu)$. This suggests an affirmative answer to the following.

4.5. QUESTION. *If S is a cyclic subnormal operator with cyclic dual, is the spectrum of the minimal normal extension thin?*

Proposition 2.4 implies that if $S$ is self-dual and $N$ is its minimal normal extension, then $N \cong N^*$. The next question is concerned with a converse to this result.

Let $\mu$ be a scalar-valued spectral measure for $N$, and let $G$ be the bounded open set, and $\mu_0$ the measure such that $\mu_0 \perp (\mu - \mu_0)$ and $P^\infty(\mu) = L^\infty(\mu - \mu_0) \oplus H^\infty(G, \mu_0)$. (See the material following Proposition 1.4.) If $\mu \neq \mu_0$, then $N$ cannot be the minimal normal extension of any pure subnormal operator. Call a normal operator $N$ *completely nonreductive* if $\mu = \mu_0$; that is, if $P^\infty(\mu)$ is isomorphic to the algebra of bounded analytic functions on some bounded open set.

4.6. QUESTION. *If N is completely nonreductive and $N \cong N^*$, is N the minimal normal extension of a self-dual subnormal operator?*

Since self-dual subnormal operators are pure, an affirmative answer to Question 4.6 would shed light on the as yet open question of which completely nonreductive normal operators are the minimal normal extensions of a pure subnormal operators. Until this latter question is answered, it seems unlikely that Question 4.6 will be answered. However, it is known that every completely nonreductive cyclic normal operator is the minimal normal extension of a pure subnormal operator ([6], p. 50). Question 4.6 is open and interesting for cyclic normal operators.

*This research was partially supported by a grant from the National Science Foundation of the United States.*

## REFERENCES


1. BEHNCKE, H., Structure of certain nonnormal operators, *J. Math. Mech.*, **18**(1968), 103–107.
2. BERGER, C. A.; SHAW, B. I., Selfcommutators of multicyclic hyponormal operators are trace class, *Bull. Amer. Math. Soc.*, **79**(1973), 1193–1199.
3. BRAM, J., Subnormal operators, *Duke Math. J.*, **22**(1955), 75–94.
4. BROWN, A., On a class of operators, *Proc. Amer. Math. Soc.*, **4**(1953), 723–728.
5. BROWN, S. W., Some invariant subspaces for subnormal operators, *Integral Equations and Operator Theory*, **1**(1978), 310–333.
6. CONWAY, J. B.; OLIN, R. F., *A functional calculus for subnormal operators*. II, Memoirs Amer. Math. Soc., vol. 184, 1977.
7. GELLAR, R., Circularly symmetric normal and subnormal operators, *J. d'Analyse Math.*, **32** (1977), 93–117.
8. GILFEATHER, F., On the Suzuki structure theory for nonselfadjoint operators on Hilbert space, *Acta Sci. Math. (Szeged)*, **32**(1971), 239–249.
9. HALMOS, P. R., *A Hilbert space problem book*, D. Van Nostrand Co., Princeton, 1967.
10. HALMOS, P. R., Ten problems in Hilbert space, *Bull. Amer. Math. Soc.*, **76**(1970), 887–933.
11. OLIN, R. F., Functional relationships between a subnormal operator and its minimal normal extension, *Pacific J. Math.*, **63**(1976), 221–229.
12. OLIN, R. F.; THOMSON, J. E., Algebras of subnormal operators, *J. Functional Anal.*, to appear.
13. SARASON, D., Weak-star density of polynomials, *J. fur Reine Angew. Math.*, **252**(1972), 1–15.
14. SHIELDS, A. L., Weighted shift operators and analytic function theory, *Math. Surveys*, **13** (1974), 49–128.
15. TOPPING, D. M., *Lectures on von Neumann algebras*, D. Van Nostrand Co., London, 1971.



*JOHN B. CONWAY*
*Department of Mathematics,*
*Indiana University,*
*Bloomington, IN 47405,*
*U.S.A.*


Received November 13, 1979.