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TOPOLOGICAL DIRECT INTEGRALS OF LEFT HILBERT ALGEBRAS. II

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1. INTRODUCTION

This paper is a continuation of [7]. For some continuous field of left Hilbert algebras $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$ (see [7], 1.1) we assume the following condition to be satisfied

(C) For any $x, y \in \Lambda$ the function $(\xi, t) \mapsto (\Delta_\xi^{it} x(\xi), y(\xi))$ is continuous on $\Omega \times \mathbf{R}$.

By [7], 1.3 condition (C) is stronger than condition (L8) in [7], 1.1. We do not know whether (C) follows from the other properties of Λ or not. However condition (C) is satisfied in the case of the central decomposition of a KMS-state which was considered in [7], Section 3. Pursuing the investigations of Section 2 in [7] we mainly intend to prove the following theorem.

1.1. THEOREM. *Let $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$ be a continuous field of* l.H.a.'s *which is defined on the locally compact space Ω such that condition* (C) *is satisfied. Suppose that $\mathfrak{A}_\xi$ contains a unit e_ξ for any $\xi\in\Omega$ and the vector field $\xi \mapsto e_\xi$ is contained in Λ. Let $\mathfrak{A}$ be the direct integral of $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$ with respect to some Radon measure μ on Ω* (see [7], 1.5). *If $\mathscr{E}$ $(E(\xi))$ is the maximal central projection in $\mathscr{L}(\mathfrak{A})$ $(\mathscr{L}(\mathfrak{A}_\xi))$ such that the von Neumann algebra $\mathscr{L}(\mathfrak{A})_{\mathscr{E}}(\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)})$ is of type* III $(\xi\in\Omega)$ *then the following identity holds*

$$\mathscr{E} = \int^{\oplus} E(\xi)\, \mathrm{d}\mu(\xi)$$

(see [5], p. 195).

In order to prove Theorem 1.1 we need some tools which will be developed in Section 2 and Section 3. First, in Section 2 we gather some fundamental properties of crossed products of von Neumann algebras with cyclic an separating vectors. Next, in Section 3 we introduce the covariant continuous field of l. H. a.'s $(\{\mathfrak{B}_\xi\}_{\xi\in\Omega}, \Gamma)$ which is associated with $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$. Now the theory of M. Takesaki in [12], [13] and the results of H. Halpern in [3] allow to reduce our problem to a corresponding problem in the "separable" case. This will be done in Section 4. Finally we achieve the proof by an application of the results of C. Lance in [4].

2. SOME FACTS ABOUT CROSSED PRODUCTS

Let $\mathfrak{A}$ be a l.H.a. with unit e, and let $\mathscr{H}$ be the completion of $\mathfrak{A}$. Furthermore let $\{\sigma_t\}_{t\in\mathbf{R}}$ be the modular automorphism group associated with $\mathfrak{A}$ and let Δ be the modular operator. Next we want to describe the construction how to get a l.H.a. which generates the crossed product $\mathscr{L}(\mathfrak{A}) \otimes_\sigma \mathbf{R}$. For the proofs we refer to [13], [14], and [8].

Let $C_c(\mathbf{R}, \mathscr{H})$ be the vector space of all $\mathscr{H}$-valued continuous functions on $\mathbf{R}$ whose support is compact and let $L^2(\mathbf{R}, \mathscr{H})$ be the Hilbert space whose elements are classes of equivalent square integrable functions from $\mathbf{R}$ into $\mathscr{H}$ with respect to the Lebesgue measure. Since the support of the Lebesgue measure is $\mathbf{R}$ every x in $L^2(\mathbf{R}, \mathscr{H})$ can contain at most one function f in $C_c(\mathbf{R}, \mathscr{H})$. If such a f exists we will identify x with this function. $C_c(\mathbf{R}, \mathscr{H})$ is a module over the ring $C_c(\mathbf{R})$ and we write fx instead of $t \mapsto f(t)x(t)$ if $x \in C_c(\mathbf{R}, \mathscr{H})$, $f \in C_c(\mathbf{R})$. We also write fx instead of $t \mapsto f(t)x$ if $x \in \mathscr{H}$, $f \in C_c(\mathbf{R})$. There is a unique isomorphism U from $\mathscr{H} \otimes L^2(\mathbf{R})$ onto $L^2(\mathbf{R}, \mathscr{H})$ such that

$$U(x \otimes f) = fx \quad \text{holds for } x \in \mathscr{H}, f \in C_c(\mathbf{R}).$$

By this isomorphism we may identify $\mathscr{H} \otimes L^2(\mathbf{R})$ and $L^2(\mathbf{R}, \mathscr{H})$.

Let $\mathscr{B}$ be the linear space of all s^*-continuous functions $t \mapsto Z(t)$ from $\mathbf{R}$ into $\mathscr{L}(\mathfrak{A})$ whose support is compact (i.e. the functions $t \mapsto Z(t)$ and $t \mapsto Z(t)^*$ are strongly continuous). On $\mathscr{B}$ is given a multiplication and an involution as follows

$$(Z_1 \cdot Z_2)(t) := \int \sigma_s(Z_1(s+t))\, Z_2(-s)\, \mathrm{d}s \quad \text{for } Z_1, Z_2 \in \mathscr{B}$$

(2.1)

$$Z^\#(t) := \sigma_t(Z(-t))^* \quad \text{for } Z \in \mathscr{B}.$$

We define a mapping ρ from $\mathscr{B}$ into $C_c(\mathbf{R}, \mathscr{H})$ as follows. For any $Z \in \mathscr{B}$ let $\rho(Z)$ be the function $t \mapsto Z(t)\, e$. ρ is linear and injective. Let $\mathfrak{B}$ be the image of $\mathscr{B}$ with respect to ρ. We may transfer the operations from $\mathscr{B}$ to $\mathfrak{B}$ with respect to ρ. Thus $\mathfrak{B}$ becomes a l.H.a. and $L^2(\mathbf{R}, \mathscr{H})$ is the completion of $\mathfrak{B}$. Furthermore $\mathscr{L}(\mathfrak{B})$ is isomorphic to $\mathscr{L}(\mathfrak{A}) \otimes_\sigma \mathbf{R}$ and we have for any $Z \in \mathscr{B}$

$$\pi(\rho(Z)) = \int \lambda(t)\pi_\sigma(Z(t))\, \mathrm{d}t \tag{2.2}$$

where $\lambda(t)$ is that unitary operator on $L^2(\mathbf{R}, \mathscr{H})$ whose restriction to $C_c(\mathbf{R}, \mathscr{H})$ is given by $(\lambda(t)x)(s) = x(s-t)$ and π_σ is that $*$-isomorphism from $\mathscr{L}(\mathfrak{A})$ into $\mathscr{B}(L^2(\mathbf{R}, \mathscr{H}))$ which is given by $(\pi_\sigma(Z)x)(s) = \sigma_s^{-1}(Z)x(s)$ $(x \in C_c(\mathbf{R}, \mathscr{H}), Z \in \mathscr{L}(\mathfrak{A}))$. The canonical conjugation J^0 associated to $\mathfrak{B}$ is given by

$$(J^0 x)(t) = \Delta^{-it} Jx(-t) \quad \text{for } x \in C_c(\mathbf{R}, \mathscr{H}) \tag{2.3}$$

where J is the canonical conjugation associated to $\mathfrak{A}$.

2.1. LEMMA. *Let $\mathfrak{B}_0$ be the linear subspace of $\mathfrak{B}$ which is generated by all elements of the form fa, where $f \in C_c(\mathbf{R})$ and $a \in \mathfrak{A}$. Then $\mathfrak{B}_0$ is dense in $\mathfrak{B}$ with respect to the $\#$-norm* (see [11], (3.8)).

Proof. Let $Z \in \mathscr{B}$ be given and let $0 < r < \infty$ be chosen such that the support K of Z is contained in the open interval $I :=]-r, r[$. Furthermore let $f \in C_c(\mathbf{R})$ be chosen such that $0 \leqslant f \leqslant 1$, $f/_K = 1$ and $f/_{I^c} = 0$. Since Z is s^*-continuous and $\pi(\mathfrak{A})$ is s^*-dense in $\mathscr{L}(\mathfrak{A})$, for any $\varepsilon > 0$ and $t \in K$ there is an open neighbourhood $\mathscr{U}_t$ of t and an element $a_t \in \mathfrak{A}$ such that the following holds for any $s \in \mathscr{U}_t$

$$\|\pi(a_t)e - Z(s)\,e\| + \|\pi(a_t)^* e - Z(s)^* e\| \leqslant \varepsilon/\sqrt{8r}.$$

Since K is compact there is a finite set of neighbourhoods, say $\mathscr{U}_{t_1}, \ldots, U_{t_n}$, which cover K. Let $Z_j := \pi(a_{t_j})$ for $j = 1, \ldots, n$. Let $\{h_1, \ldots, h_n\}$ be a partition of unity on K subordinate to the cover $\{\mathscr{U}_{t_1}, \ldots, \mathscr{U}_{t_n}\}$ and let $Z_\varepsilon \colon t \mapsto f(t) \sum_{j=1}^{n} h_j(t) Z_j$ as well as $x_\varepsilon \colon t \mapsto Z_\varepsilon(t)e$, $x \colon t \mapsto Z(t)e$. Then we obtain

$$\|x_\varepsilon - x\| = \left(\int \|x_\varepsilon(t) - x(t)\|^2 \,\mathrm{d}t\right)^{1/2} \leqslant \left(\int_I \varepsilon^2/8r \,\mathrm{d}t\right)^{1/2} = \varepsilon/2;$$

$$\|x_\varepsilon^\# - x^\#\| = \left(\int \|\sigma_t(Z_\varepsilon(-t))^* e - \sigma_t(Z(-t))^* e\|^2 \,\mathrm{d}t\right)^{1/2} =$$

$$= \left(\int \|\Delta^{it}(Z_\varepsilon(-t)^* - Z(-t)^*)\Delta^{-it} e\|^2 \,\mathrm{d}t\right)^{1/2} =$$

$$= \left(\int \|(Z_\varepsilon(-t)^* - Z(-t)^*)\,e\|^2 \mathrm{d}t\right)^{1/2} \leqslant \left(\int_I \varepsilon^2/8r \,\mathrm{d}t\right)^{1/2} = \varepsilon/2.$$

Hence we have shown that $\|x_\varepsilon - x\|_\# = \|x_\varepsilon - x\| + \|x_\varepsilon^\# - x^\#\| \leqslant \varepsilon$. Our assertion follows from this.

2.2. LEMMA. *Let $\{f_n\}_{n \in \mathbf{N}}$ be a sequence of positive functions in $C_c(\mathbf{R})$ such that $\int f_n(t)\mathrm{d}t = 1$ and $f_n(t) = 0$ if $|t| \geqslant 1/n$ holds for any $n \in \mathbf{N}$. Let $x_n := f_n e$.*

Then the sequence $\{\pi(x_n)\}_{n\in\mathbf{N}}$ *converges strongly to the identical operator* Id *on* $L^2(\mathbf{R},\mathscr{H})$.

Proof. Let $f\in C_c(\mathbf{R})$ and $x:=fe$. By (2.2) we have $\pi(x)=\int f(t)\lambda(t)\,\mathrm{d}t$. Let $g\in C_c(\mathbf{R})$, $a\in\mathscr{H}$ and $y:=ga$. Then we obtain for any $z\in C_c(\mathbf{R},\mathscr{H})$

$$(\pi(x)\,y,z)=\int f(t)(\lambda(t)y,z)\,\mathrm{d}t=$$

$$=\int f(t)\int (y(s-t),z(s))\mathrm{d}s\mathrm{d}t=$$

$$=\int f(t)\int g(s-t)(a,z(s))\mathrm{d}s\mathrm{d}t=$$

$$=\int f*g(s)(a,z(s))\mathrm{d}s=((f*g)a,z).$$

Hence $\pi(x)y=(f*g)\,a$ holds. Thus we obtain

$$\|y-\pi(x_n)y\|=\|ga-(f_n*g)\,a\|=\|g-(f_n*g)\|_2\|a\|.$$

By our assumptions we have $\lim\limits_{n\to\infty}\|y-\pi(x_n)\,y\|=0$. For any $y,z\in C_c(\mathbf{R},\mathscr{H})$ and $n\in\mathbf{N}$ the following is true

$$|(\pi(x_n)\,y,z)|=\left|\int f_n(t)(\lambda(t)\,y,z)\mathrm{d}t\right|\leqslant$$

$$\leqslant\int f_n(t)|(\lambda(t)\,y,z)|\,\mathrm{d}t\leqslant\|y\|\,\|z\|\int f_n(t)\|\lambda(t)\|\mathrm{d}t=$$

$$=\|y\|\,\|z\|\int f_n(t)\,\mathrm{d}t=\|y\|\,\|z\|.$$

This implies that $\|\pi(x_n)\|\leqslant 1$ holds for any $n\in\mathbf{N}$. Since the set $\{fa\,|\,f\in C_c(\mathbf{R}),\ a\in\mathscr{H}\}$ is total in $L^2(\mathbf{R},\mathscr{H})$ (see 2.1) we conclude that the sequence $\{\pi(x_n)\}_{n\in\mathbf{N}}$ converges strongly to Id.

3. THE COVARIANT CONTINUOUS FIELD OF LEFT HILBERT ALGEBRAS

Concerning the general theory of topological direct integrals of Hilbert spaces we use the following notation. Let $(\{\mathscr{H}_\xi\}_{\xi\in\Omega},\Phi)$ be a continuous field of Hilbert spaces which is defined on the locally compact space Ω (see [2], [5]) and let μ be a positive Radon measure on Ω. Let $\mathscr{H}:=\int^{\oplus}\mathscr{H}_{\xi}\mathrm{d}\mu(\xi)$ be the direct integral of $(\{\mathscr{H}_{\xi}\}_{\xi\in\Omega}, \Phi)$ with respect to μ. For any square integrable vector field $\xi\mapsto x(\xi)$ we denote by $\int^{\oplus}x(\xi)\mathrm{d}\mu(\xi)$ or $\tilde{x}$ the corresponding element in $\mathscr{H}$. As in [5], for any strongly measurable and essentially bounded operator field $\xi\to Z(\xi)$ we denote by $\int^{\oplus}Z(\xi)\mathrm{d}\mu(\xi)$ that operator in $\mathscr{B}(\mathscr{H})$ which has the property that $\xi\mapsto Z(\xi)$ is a decomposition of it. For any vector field x we define the support of x to be the set of all $\xi\in\Omega$ such that $x(\xi)$ is not zero. Similarly we define the support of an operator field.

Next we want to prove two Fubini-like theorems for direct integrals. Let Ω_1, Ω_2 be locally compact spaces. Furthermore let $(\{\mathscr{H}_{(\xi,\eta)}\}_{(\xi,\eta)\in\Omega_1\times\Omega_2}, \Phi)$ be a continuous field of Hilbert spaces and let μ_i be a positive Radon measure on Ω_i for $i=1, 2$. For any $\xi\in\Omega_1$ $(\eta\in\Omega_2)$ let

$$\Phi_{\xi}:=\{\eta\mapsto x(\xi,\eta)|x\in\Phi\}\quad(\Phi_{\eta}:=\{\xi\mapsto x(\xi,\eta)|x\in\Phi\}).$$

It is clear that $(\{\mathscr{H}_{(\xi,\eta)}\}_{\eta\in\Omega_2}, \Phi_{\xi})((\{\mathscr{H}_{(\xi,\eta)}\}_{\xi\in\Omega_1}, \Phi_{\eta}))$ is a continuous field of Hilbert spaces. Let

$$\mathscr{H}^{(\xi)}:=\int^{\oplus}\mathscr{H}_{(\xi,\eta)}\mathrm{d}\mu_2(\eta)\quad\left(\mathscr{H}^{(\eta)}:=\int^{\oplus}\mathscr{H}_{(\xi,\eta)}\mathrm{d}\mu_1(\xi)\right).$$

Let Φ_c be the linear space which is generated by all vector fields of the form $(\xi,\eta)\mapsto f(\xi,\eta)x(\xi,\eta)$, where $x\in\Phi, f\in C_c(\Omega_1\times\Omega_2)$.

Let

$$\Phi_1:=\left\{\xi\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)|x\in\Phi_c\right\}$$

and

$$\Phi_2:=\left\{\eta\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)|x\in\Phi_c\right\}.$$

For convenience we note the following simple lemma which will be used in the sequel. The proof is left to the reader.

3.1. LEMMA. *Let Ω be a topological space, let K be a compact space and let μ be a positive Radon measure on K. Moreover let $f\colon\Omega\times K\mapsto\mathbf{C}$ be a continuous function. Then the function $\Omega\ni\xi\mapsto\int f(\xi,\eta)\,\mathrm{d}\mu(\eta)$ is also continuous.*

3.2. PROPOSITION. *The following assertions are true.*

(a) $(\{\mathscr{H}^{(\xi)}\}_{\xi\in\Omega_1}, \Phi_1)$ *and* $(\{\mathscr{H}^{(\eta)}\}_{\eta\in\Omega_2}, \Phi_2)$ *are continuous fields of Hilbert spaces and, modulo canonical isomorphisms the following identities hold*

$$\int^{\oplus}\mathscr{H}^{(\xi)}\mathrm{d}\mu_1(\xi)=\int^{\oplus}\mathscr{H}_{(\xi,\eta)}\mathrm{d}(\mu_1\times\mu_2)(\xi,\eta)=\int^{\oplus}\mathscr{H}^{(\eta)}\mathrm{d}\mu_2(\eta).$$

(b) *If* $(\xi,\eta)\mapsto x(\xi,\eta)$ *is a continuous vector field whose support is compact then the vector fields*

$$x_1\colon \xi\mapsto\int^{\oplus}x(\xi,\eta)\,\mathrm{d}\mu_2(\eta)\ \textit{and}\ x_2\colon \eta\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)$$

are also continuous, their supports are compact and the following identity holds.

$$\int^{\oplus}x_1(\xi)\,\mathrm{d}\mu_1(\xi)=\int^{\oplus}x_2(\eta)\mathrm{d}\mu_2(\eta).$$

Proof. Clearly Φ_1 is a linear space. Let $x\in\Phi_c$ be given. Then we have for any $\xi\in\Omega_1$

$$\left\|\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right\|^2=\int^{\oplus}\|x(\xi,\eta)\|^2\,\mathrm{d}\mu_2(\eta).$$

Since the function $(\xi,\eta)\mapsto\|x(\xi,\eta)\|^2$ belongs to $C_c(\Omega_1\times\Omega_2)$ we infer from 3.1 that the function $\xi\mapsto\left\|\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right\|$ belongs to $C_c(\Omega_1)$.

We want to show that the set $\{x(\xi)|x\in\Phi_1\}$ is dense in $\mathscr{H}^{(\xi)}$ for any $\xi\in\Omega_1$. Let $\xi\in\Omega_1$ be given. For any $x\in\Phi_c$ and $f\in C_c(\Omega_2)$ the vector field $(\xi,\eta)\mapsto f(\eta)x(\xi,\eta)$ belongs also to Φ_c. Hence any vector field $\eta\mapsto y(\eta)$ which is continuous with respect to Φ_ξ can be uniformly approximated on compact subsets of Ω_2 by vector fields in Φ_ξ (see [2], p. 81, Proposition 6). It follows from this that the set $\{\tilde{y}|\tilde{y}\in\Phi_\xi\}$ is dense in $\mathscr{H}^{(\xi)}$. Thus we have shown that Φ_1 is a fundamental family of vector fields (in the sense of [2], chap. III, §1). Similarly one can show that Φ_2 is a fundamental family of vector fields. Still we have to verify the second part of assertion (a). For any $x\in\Phi_c$ we have

$$\left\|\int^{\oplus}\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right)\mathrm{d}\mu_1(\xi)\right\|^2=$$

$$=\int\left\|\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right\|^2\mathrm{d}\mu_1(\xi)=$$

$$=\int\|x(\xi,\eta)\|^2\mathrm{d}(\mu_1\times\mu_2)(\xi,\eta)=$$

$$=\int\left\|\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)\right\|^2\mathrm{d}\mu_2(\eta)=$$

$$=\left\|\int^{\oplus}\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)\right)\mathrm{d}\mu_2(\eta)\right\|^2.$$

Hence there exists a unique isomorphism $U_1(U_2)$ from

$$\int^{\oplus}\mathscr{H}_{(\xi,\eta)}\mathrm{d}(\mu_1\times\mu_2)(\xi,\eta) \quad \text{onto} \quad \int^{\oplus}\mathscr{H}^{(\xi)}\mathrm{d}\mu_1(\xi)\left(\int^{\oplus}\mathscr{H}^{(\eta)}\mathrm{d}\mu_2(\eta)\right)$$

such that

$$U_1\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}(\mu_1\times\mu_2)(\xi,\eta)\right)=\int^{\oplus}\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right)\mathrm{d}\mu_1(\xi)$$

$$\left(U_2\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}(\mu_1\times\mu_2)(\xi,\eta)\right)=\int^{\oplus}\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)\right)\mathrm{d}\mu_2(\eta)\right)$$

holds for any $x\in\Phi_c$.

(b) By 3.1 the function $\xi\mapsto\left\|\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)\right\|$ belongs to $C_c(\Omega_1)$. Also by 3.1 the function

$$\xi\mapsto\left(\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta),\ \int^{\oplus}y(\xi,\eta)\mathrm{d}\mu_2(\eta)\right)$$

$\left(=\int(x(\xi,\eta),y(\xi,\eta))\,\mathrm{d}\mu_2(\eta)\right)$ belongs to $C_c(\Omega_1)$ if $y\in\Phi_c$. By [2], p. 81 this implies that $\xi\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)$ is continuous. Furthermore it is clear that the support of $\xi\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_2(\eta)$ is compact. Similarly one can see that $\eta\mapsto\int^{\oplus}x(\xi,\eta)\mathrm{d}\mu_1(\xi)$ is continuous and its support is compact. Let U_1, U_2 be defined as in the proof of (a). Then we obtain that $U_2U_1^{-1}$ maps $\int^{\oplus}x_1(\xi)\mathrm{d}\mu_1(\xi)$ onto $\int^{\oplus}x_2(\eta)\mathrm{d}\mu_2(\eta)$ and thus our assertion follows from this.

Now let us consider some special situation. Let $(\{\mathscr{H}_\xi\}_{\xi\in\Omega_1},\Lambda_1)$ be a continuous field of Hilbert spaces. For any $(\xi,\eta)\in\Omega_1\times\Omega_2$ let $\mathscr{H}_{(\xi,\eta)}:=\mathscr{H}_\xi$ and let $\Phi:=\{(\xi,\eta)\mapsto x(\xi)|\ x\in\Lambda\}$. Clearly $(\{\mathscr{H}_{(\xi,\eta)}\}_{(\xi,\eta)\in\Omega_1\times\Omega_2},\Phi)$ is a continuous field of Hilbert spaces.

3.3. Proposition. *Let $(\xi,\eta)\mapsto Z(\xi,\eta)$ be an operator field which is continuous with respect to Φ and whose support is compact. Then $\eta\mapsto\int Z(\xi,\eta)\mathrm{d}\mu_2(\eta)$ is continuous with respect to Λ (the integral which occurs is to be understood in the sense of vector valued integration) and $\eta\mapsto\int^{\oplus}Z(\xi,\eta)\mathrm{d}\mu_1(\xi)$ is strongly continuous. Moreover the following holds*

$$\int^{\oplus}\int Z(\xi,\eta)\mathrm{d}\mu_2(\eta)\mathrm{d}\mu_1(\xi)=\int\int^{\oplus}Z(\xi,\eta)\mathrm{d}\mu_1(\xi)\mathrm{d}\mu_2(\eta).$$

Proof. Since Z is bounded the function $\eta \mapsto Z(\xi, \eta)$ is strongly continuous for any $\xi \in \Omega_1$. Let $x \in \Lambda$, $\xi_0 \in \Omega_1$ and let $\varepsilon > 0$ be given. Since Z is continuous and the support of Z is compact we can find some $y_1, \ldots, y_n \in \Lambda$ and $f_1, \ldots, f_n \in C_c(\Omega_2)$ such that the following holds

$$h(\xi_0) < \varepsilon \quad \text{where } h(\xi) := \int \left\| Z(\xi, \eta)\, x(\xi) - \sum_{i=1}^{n} f_i(\eta) y_i(\xi) \right\| \mathrm{d}\mu_2(\eta).$$

By 3.1, h is a continuous function. Let $y := \sum_{i=1}^{n} \int f_i(\eta) \mathrm{d}\mu_2(\eta) y_i$. By a well known inequality in vector valued integration theory we obtain

$$\left\| \int Z(\xi, \eta) \mathrm{d}\mu_2(\eta) x(\xi) - y(\xi) \right\| \leqslant h(\xi) \qquad \text{for any } \xi \in \Omega_1.$$

Since $h(\xi_0) < \varepsilon$ holds and h is continuous there is an open neighbourhood $\mathscr{U}$ of ξ_0 such that

$$\left\| \int Z(\xi, \eta) \mathrm{d}\mu_2(\eta) x(\xi) - y(\xi) \right\| \leqslant \varepsilon \quad \text{holds} \quad \text{for any} \quad \xi \in \mathscr{U}.$$

Since Z is bounded we infer from this that $\xi \mapsto \int Z(\xi, \eta) \mathrm{d}\mu_2(\eta)$ is continuous in ξ_0 (see [2], p. 84, Proposition 9).

From 3.2(b) we obtain immediately that $\eta \mapsto \int^{\oplus} Z(\xi, \eta) \mathrm{d}\mu_1(\xi)$ is strongly continuous.

Finally for any continuous vector fields $\xi \mapsto x(\xi)$, $\xi \mapsto y(\xi)$ with compact support we have

$$\left(\int^{\oplus} \int Z(\xi, \eta) \mathrm{d}\mu_2(\eta) \mathrm{d}\mu_1(\xi) \tilde{x}, \tilde{y} \right) =$$

$$= \iint (Z(\xi, \eta) x(\xi), y(\xi)) \mathrm{d}\mu_1(\xi) \mathrm{d}\mu_2(\eta) =$$

$$= \left(\iint^{\oplus} Z(\xi, \eta) \mathrm{d}\mu_1(\xi) \mathrm{d}\mu_2(\eta) \tilde{x}, \tilde{y} \right).$$

Thus our last assertion follows from this.

Now let $(\{\mathfrak{A}_\xi\}_{\xi \in \Omega}, \Lambda)$ be a continuous field of left Hilbert algebras which satisfies the condition (C) in 1.1.

Furthermore we assume that $\mathfrak{A}_\xi$ contains a unit e_ξ for any $\xi \in \Omega$ and the vector field $\xi \mapsto e_\xi$ is contained in Λ. By [7], 1.1 (L6) this implies that Ω is compact. As in [7], for any $\xi \in \Omega$ let $\mathscr{H}_\xi$ be the completion of $\mathfrak{A}_\xi$, let $\{\sigma_t^\xi\}_{t \in \mathbf{R}}$ be the modular automorphism group associated with $\mathfrak{A}_\xi$, and let Δ_ξ be the corresponding modular operator.

Let μ be a positive Radon measure on Ω, let $\mathscr{H}:=\int^{\oplus}\mathscr{H}_\xi\,d\mu(\xi)$ be the direct integral of $(\{\mathscr{H}_\xi\}_{\xi\in\Omega}, \Lambda)$ with respect to μ and let $\mathfrak{A}=\int^{\oplus}\mathfrak{A}_\xi\,d\mu(\xi)$ be the direct integral of $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$ with respect to μ (see [7], 1.6). For any $n\in\mathbf{N}$ we set $\mathscr{H}_{(\xi, t_1,\dots,t_n)}:=\mathscr{H}_\xi$ if $\xi\in\Omega$ and $(t_1,\dots,t_n)\in\mathbf{R}^n$. Moreover we set $\Lambda_n:=\{(\xi, t_1,\dots,t_n)\mapsto x(\xi)\,|\,x\in\Lambda\}$. Clearly $(\{\mathscr{H}_{(\xi, t_1,\dots,t_n)}\}_{(\xi, t_1,\dots,t_n)\in\Omega\times\mathbf{R}^n}, \Lambda_n)$ is a continuous field of Hilbert spaces.

The following lemma is an immediate consequence of the validity of condition (C) (see also [2], p. 81 and [5], 1.4).

3.4. LEMMA. *The operator field $(\xi, t)\mapsto\Delta_\xi^{it}$ is continuous with respect to Λ_1.*

Next we want to define the covariant continuous field of l. H. a.'s associated with $(\{\mathfrak{A}_\xi\}_{\xi\in\Omega}, \Lambda)$. Let $\mathfrak{B}_0$ ($\mathfrak{B}_\xi$) be the l.H.a. which is associated with $\mathfrak{A}$ ($\mathfrak{A}_\xi$) as in Section 2 and let $\mathscr{L}$ ($\mathscr{L}_\xi$) be the completion of $\mathfrak{B}$ ($\mathfrak{B}_\xi$, $\xi\in\Omega$). For any $\xi\in\Omega$ let J_ξ^0 be the canonical conjugation associated with $\mathfrak{B}_\xi$. Let $\mathscr{R}$ be the smallest linear subspace of $\prod_{(\xi,t)\in\Omega\times\mathbf{R}}\mathscr{B}(\mathscr{H}_{(\xi,t)})$ which satisfies the following conditions.

(3.1) For any $x\in\Lambda$ and $f\in C_c(\mathbf{R})$ the operator field $(\xi, t)\mapsto f(t)\pi_\xi(x(\xi))$ belongs to $\mathscr{R}$.

(3.2) If the operator fields $(\xi, t)\mapsto Z_1(\xi, t)$ and $(\xi, t)\mapsto Z_2(\xi, t)$ belong to $\mathscr{R}$ then the same is true for the operator fields

$$(\xi, t)\mapsto\int\sigma_t^\xi(Z_1(\xi, s+t))Z_2(\xi, -s)\,ds$$

and

$$(\xi, t)\mapsto\sigma_t^\xi(Z_1(\xi, -t))^* \qquad \text{(see (2.1))}.$$

By the results in Section 2 $\mathscr{R}$ is well defined. Finally let

$$\Gamma:=\left\{\xi\mapsto\int^{\oplus}Z(\xi, t)e_\xi dt\,|\,Z\in\mathscr{R}\right\}.$$

3.5. THEOREM. (a) *$(\{\mathfrak{B}_\xi\}_{\xi\in\Omega}, \Gamma)$ is a continuous field of* l. H. a.'s.

(b) *There is a sequence $\{x_n\}_{n\in\mathbf{N}}$ in Γ such that for any $\xi\in\Omega$ the set $\{Zx_n(\xi)\,|\,Z\in\mathscr{L}(\mathfrak{B}_\xi)', n\in\mathbf{N}\}$ is total in $\mathscr{L}_\xi$.*

(c) *For $\mathfrak{B}:=\int^{\oplus}\mathfrak{B}_\xi d\mu(\xi)$ we have $\mathfrak{B}\subseteq\mathfrak{B}_0$ and $\mathfrak{B}''=\mathfrak{B}_0''$* (i.e. *$\mathfrak{B}$ and $\mathfrak{B}_0$ are equivalent in the sense of* [11], *Definition* 5.1).

Proof. (a) We have to show that Γ satisfies the conditions (L1) up to (L8) in [7], 1.1. Obviously (L1) is fulfilled, i.e. Γ is a linear space. From (3.1) and (3.2) we infer that any $Z\in\mathscr{R}$ is bounded and Z has a compact support. Next we want to show that for any $Z\in\mathscr{R}$ the following is true

(i) $(\xi, t)\mapsto Z(\xi, t)$ and $(\xi, t)\mapsto Z(\xi, t)^*$ are continuous with respect to Λ_1.

Clearly any operator field Z of the form $(\xi, t) \mapsto f(t)\, \pi_\xi(x(\xi))$ satisfies the condition (i). Now let Z_1 and Z_2 be operator fields in $\mathscr{R}$ such that (i) is satisfied. Then the operator fields

$$(\xi, s, t) \mapsto Z_1(\xi, s+t) \quad \text{and} \quad (\xi, s, t) \mapsto Z_2(\xi, -s)$$

are continuous with respect to Λ_2. Hence by 3.4 the operator field

$$(\xi, s, t) \mapsto \sigma_s^\xi(Z_1(\xi, s+t))Z_2(\xi, -s)$$

is also continuous with respect to Λ_2 and its support is compact. By 3.3 we obtain from this (we set $\Omega_1 := \Omega \times \mathbf{R}$ and $\Omega_2 := \mathbf{R}$) that the operator field

$$(\xi, t) \mapsto \int \sigma_s^\xi(Z_1(\xi, s+t))Z_2(\xi, -s)\mathrm{d}s$$

is continuous with respect to Λ_1. Since

$$\left(\int \sigma_s^\xi(Z_1(\xi, s+t))Z_2(\xi, -s)\mathrm{d}s\right)^* =$$

$$= \int Z_2(\xi, -s)^* \sigma_s^\xi(Z_1(\xi, s+t)^*)\mathrm{d}s$$

holds one can see in the same manner that the operator field

$$(\xi, t) \mapsto \left(\int \sigma_s^\xi(Z_1(\xi, s+t))\, Z_2(\xi, -s)\mathrm{d}s\right)^*$$

is continuous with respect to Λ_1. Thus we have shown that this operator field satisfies (i). From 3.4 we infer that the operator fields

$$(\xi, t) \mapsto \sigma_t^\xi(Z_1(\xi, -t)) \text{ and } (\xi, t) \mapsto \sigma_t^\xi(Z_1(\xi, -t))^*$$

are continuous with respect to Λ_1. Hence $(\xi, t) \mapsto \sigma_t^\xi(Z_1(\xi, -t))$ satisfies also the condition (i). By the definition of $\mathscr{R}$ we conclude now that any $Z \in \mathscr{R}$ satisfies (i). Finally from 3.2 (b) we obtain that for any $x \in \Gamma$ the function $\xi \mapsto \|x(\xi)\|$ belongs to $C_c(\Omega)$, i.e. (L2) is satisfied.

The validity of (L3) and (L4) is an immediate consequence of the definition of Γ. (L5) follows from the fact that (L5) is satisfied for Λ. (L6) is an immediate consequence of 2.1.

In order to verify (L7) it is sufficient to show that for any $x \in \Gamma$ which has the form $\xi \mapsto \int^{\oplus} f(t)y(\xi)\,\mathrm{d}t$, where $y \in \Lambda$ and $f \in C_c(\mathbf{R})$, the operator field $\xi \mapsto \pi_\xi(x(\xi))$ is bounded (see (3.1) and (3.2)). Hence let $x \in \Lambda$ and $f \in C_c(\mathbf{R})$ be given. By (2.2) we have for any $\xi \in \Omega$

$$\pi_\xi\left(\int^{\oplus} f(t)x(\xi)\mathrm{d}t\right) = \int f(t)\lambda(t)\pi_{\sigma_\xi}(\pi_\xi(x(\xi)))\mathrm{d}t =$$

$$= \int f(t)\lambda(t)\mathrm{d}t\, \pi_{\sigma_\xi}(\pi_\xi(x(\xi))).$$

Hence we obtain for any $a, b \in \mathscr{H}_\xi$

$$\left|\left(\pi_\xi\left(\int^{\oplus} f(t)x(\xi)\mathrm{d}t\right)a,\ b\right)\right| =$$

$$= \left|\int f(t)(\lambda(t)\ \pi_{\sigma_\xi}(\pi_\xi(x(\xi)))\ a, b)\ \mathrm{d}t\right| \leqslant$$

$$\leqslant \|\pi_{\sigma_\xi}(\pi_\xi(x(\xi)))\|\ \|a\|\ \|b\| \int |f(t)|\ \mathrm{d}t =$$

$$= \|\pi_\xi(x(\xi))\|\ \|a\|\ \|b\| \int |f(t)|\ \mathrm{d}t.$$

From this we infer

$$\left\|\pi_\xi\left(\int^{\oplus} f(t)x(\xi)\mathrm{d}t\right)\right\| \leqslant \|\pi_\xi(x(\xi))\| \int |f(t)|\ \mathrm{d}t.$$

Since Λ satisfies (L7) the operator field $\xi \mapsto \pi_\xi(x(\xi))$ is bounded. Hence the operator field $\xi \mapsto \pi_\xi\left(\int^{\oplus} f(t)x(\xi)\mathrm{d}t\right)$ is also bounded.

Finally let us prove that (L8) holds. By 3.4 the operator field $(\xi, t) \mapsto \Delta_\xi^{it} J_\xi$ is continuous with respect to Λ_1. Hence for any vector field x: $(\xi, t) \mapsto f(t)y(\xi)$, where $y \in \Lambda$, $f \in C_c(\mathbf{R})$, the vector field $(\xi, t) \mapsto \Delta_\xi^{-it} J_\xi x(\xi, -t)$ is also continuous. By (2.3) and (3.2) this implies that $\xi \mapsto J_\xi^0\left(\int^{\oplus} x(\xi, t)\mathrm{d}t\right)$ is continuous. From [7], 1.2 (a) we conclude that $\xi \mapsto J_\xi^0$ is continuous (see also the remark (2) to [7], 1.1).

(b) Let the sequence $\{f_n\}_{n \in \mathbf{N}}$ in $C_c(\mathbf{R})$ be chosen as in 2.2. For any $n \in \mathbf{N}$ let $x_n\colon \xi \mapsto \int^{\oplus} f_n(t)e_\xi \mathrm{d}t$. By 2.2 the sequence $\{\pi_\xi(x_n(\xi))\}_{n \in \mathbf{N}}$ converges strongly to the identical operator on $\mathscr{L}_\xi$. Therefore the set $\{\pi_\xi(x_n(\xi))y \mid y \in \mathfrak{B}'_\xi,\ n \in \mathbf{N}\} \subseteq \subseteq \{Zx_n(\xi) \mid Z \in \mathscr{L}(\mathfrak{B}_\xi)',\ n \in \mathbf{N}\}$ is total in $\mathscr{L}_\xi$ for any $\xi \in \Omega$.

(c) By 3.2 (b) we have for any $x \in \Lambda$, $f \in C_c(\mathbf{R})$

$$\int^{\oplus} f(t)\tilde{x}\mathrm{d}t = \int^{\oplus}\int^{\oplus} f(t)x(\xi)\mathrm{d}t\ \mathrm{d}\mu(\xi).$$

Thus our assertion follows immediately from the definition of $\mathfrak{B}_0$ and $\mathfrak{B}$ as well as from 2.1 and [11], Lemma 5.2.

3.7 Definition. We call $(\{\mathfrak{B}_\xi\}_{\xi \in \Omega}, \Gamma)$ *the covariant continuous field* of l.H.a.'s associated with $(\{\mathfrak{A}_\xi\}_{\xi \in \Omega}, \Lambda)$.

By 3.5 (b) the set Γ satisfies the condition (L9) in [7]. Thus the results of [7], Section 2 are available for the field $(\{\mathfrak{B}_\xi\}_{\xi \in \Omega}, \Gamma)$.

Let us recall that the dual action $\{\hat{\sigma}_t\}_{t\in\mathbf{R}}$ of $\{\sigma_t\}_{t\in\mathbf{R}}$ on $\mathscr{L}(\mathfrak{B})$ is given by

$$\hat{\sigma}_t(Z) = v(t)Zv(t)^* \qquad \text{for } Z\in\mathscr{L}(\mathfrak{B}),\ t\in\mathbf{R}$$

where $v(t)$ is determined by

$$v(t)x(s) = \mathrm{e}^{-\mathrm{i}st}x(s) \qquad \text{if } x\in C_c(\mathbf{R},\mathscr{H}),\ s\in\mathbf{R}.$$

Similarly let $\{\hat{\sigma}_t^\xi\}_{t\in\mathbf{R}}$ be the dual action of $\{\sigma_t^\xi\}_{t\in\mathbf{R}}$ on $\mathscr{L}(\mathfrak{B}_\xi)$, and let $v^\xi(t)$ be the unitary which is determined by $v^\xi(t)x(s) = \mathrm{e}^{-\mathrm{i}st}x(s)$ for $x\in C_c(\mathbf{R},\mathscr{H}_\xi)$, $s\in\mathbf{R}$.

3.8. PROPOSITION. *$\xi\mapsto v^\xi(t)$ is a continuous operator field for any $t\in\mathbf{R}$ and we have $v(t)=\int^\oplus v^\xi(t)\mathrm{d}\mu(\xi)$.*

Proof. The operator field $(\xi,t)\mapsto \mathrm{e}^{-\mathrm{i}st}\mathrm{Id}_\xi$ is continuous with respect to Λ_1 (Id_ξ = identical operator on $L^2(\mathbf{R},\mathscr{H}_\xi)$). Thus our assertion follows from 3.2(b).

3.9. PROPOSITION. *There is a positive semifinite normal and faithful trace τ (τ_ξ) on $\mathscr{L}(\mathfrak{B})^+$ ($\mathscr{L}(\mathfrak{B}_\xi)^+$) which is relatively invariant,* i.e. $\tau\circ\hat{\sigma}_t=\mathrm{e}^{-t}\tau$ ($\tau_\xi\circ\sigma_t^\xi=\mathrm{e}^{-t}\tau_\xi$) *holds* for *any* $t\in\mathbf{R}$, *such that for any* $Z=\int^\oplus Z(\xi)\,\mathrm{d}\mu(\xi)\in\mathscr{L}(\mathfrak{B})^+$ *the function* $\xi\mapsto\tau_\xi(Z(\xi))$ *is measurable and the following identity holds*

$$\tau(Z)=\int\tau_\xi(Z(\xi))\mathrm{d}\mu(\xi).$$

Proof. For any $n\in\mathbf{N}$ let

$$f_n(t) = (2\pi)^{-1}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}st+s/2}\chi_{[-n,n]}\,\mathrm{d}s,\quad t\in\mathbf{R}$$

and

$$v_n:=f_ne\ ;\ v_n^\xi:=f_ne_\xi \quad \text{if } \xi\in\Omega.$$

Moreover for any $n\in\mathbf{N}$ we define a positive functional τ_n on $\mathscr{L}(\mathfrak{B})$ as follows

$$\tau_n(Z) = (Zv_n,v_n) \quad \text{if } Z\in\mathscr{L}(\mathfrak{B}).$$

In the same manner we define a positive functional τ_n^ξ on $\mathscr{L}(\mathfrak{B}_\xi)$ for any $n\in\mathbf{N}$. By [14], 3.5 of part 2, the sequence $\{\tau_n\}_{n\in\mathbf{N}}$ ($\{\tau_n^\xi\}_{n\in\mathbf{N}}$) is monotonely increasing and

$$\tau:=\sup_{n\in\mathbf{N}}\tau_n \quad \text{on } \mathscr{L}(\mathfrak{B})^+ \quad (\tau^\xi:=\sup_{n\in\mathbf{N}}\tau_n^\xi \quad \text{on } \mathscr{L}(\mathfrak{B}_\xi)^+)$$

is a positive semifinite normal and faithful trace on $\mathscr{L}(\mathfrak{B})^+$ ($\mathscr{L}(\mathfrak{B}_\xi)^+$) which is relatively invariant. Since $\xi\mapsto v_n^\xi$ is continuous and $v_n=\int^\oplus v_n^\xi\mathrm{d}\mu(\xi)$ holds (see 3.2(b)) for any positive $Z = \int^{\oplus} Z(\xi) d\mu(\xi)$ the function $\xi \mapsto \tau_n^{\xi}(Z(\xi))$ is measurable and $\tau_n(Z) = \int \tau_n^{\xi}(Z(\xi)) d\mu(\xi)$ holds. Hence the function $\xi \mapsto \tau^{\xi}(Z(\xi))$ is measurable and from Lebesgue's monotone convergence theorem we obtain $\tau(Z) = \int \tau^{\xi}(Z(\xi)) d\mu(\xi)$.

4. PROOF OF THEOREM 1.1

We keep the notation which has been introduced in the last part of Section 3. Let $\mathscr{E}$ be the maximal central projection in $\mathscr{L}(\mathfrak{A})$ such that $\mathscr{L}(\mathfrak{A})_{\mathscr{E}}$ is of type III. Let $\mathscr{F} := \mathscr{E} \otimes 1$. Then $\mathscr{L}(\mathfrak{B})_{\mathscr{F}}$ is isomorphic to the crossed product $\mathscr{L}(\mathfrak{A})_{\mathscr{F}} \otimes_{\sigma/\mathscr{L}(\mathfrak{A})_{\mathscr{F}}} \mathbf{R}$ (see [14], 4.1 of part 2). For convenience we want to denote the restriction of the dual action $\hat{\sigma}$ to $\mathscr{L}(\mathfrak{B})_{\mathscr{F}}$ by $\hat{\sigma}$ again. By the proof of [3], Proposition 14 there is a countably generated von Neumann subalgebra $\mathscr{N}_0$ of $\mathscr{L}(\mathfrak{B})_{\mathscr{F}}$ such that the following conditions are satisfied

(4.1) $\mathscr{N}_0$ is invariant under the dual action $\{\hat{\sigma}_t\}_{t \in \mathbf{R}}$.

(4.2) $\mathscr{N}_0$ is properly infinite.

(4.3) The restriction of τ to $\mathscr{N}_0^+$ is semifinite.

(4.4) The center of $\mathscr{N}_0$ is contained in the center of $\mathscr{L}(\mathfrak{B})_{\mathscr{F}}$.

(4.5) The crossed product $\mathscr{N}_0 \otimes_{\hat{\sigma}} \mathbf{R}$ which may be considered to be canonically embedded into $\mathscr{L}(\mathfrak{B})_{\mathscr{F}} \otimes_{\hat{\sigma}} \mathbf{R}$ is of type III.

Let $\mathscr{Z}$ be the algebra of all diagonalisable operators in $\mathscr{L}(\mathfrak{B})$ and let $\mathscr{N}$ be the von Neumann algebra generated by $\mathscr{N}_0$ and $\mathscr{F}\mathscr{Z}$. It is evident that $\mathscr{N}$ also satisfies the conditions (4.1), (4.2), (4.3), (4.4). By the same argument which was used in [3], Proposition 14 to verify (4.5) for $\mathscr{N}_0$ one can see that $\mathscr{N}$ satisfies (4.5) too.

Next let us look how the situation reduces to the components.

4.1 LEMMA. *For any* $Z = \int^{\oplus} Z(\xi) d\mu(\xi) \in \mathscr{L}(\mathfrak{A})$ *we have*

$$Z \otimes 1 = \int^{\oplus} Z(\xi) \otimes 1 d\mu(\xi).$$

Proof. Let $x, y \in \Lambda$ and $f, g \in C_c(\mathbf{R})$. Then the following identities hold

$$(Z(\xi) \otimes 1 fx(\xi), gy(\xi)) = (f, g)(Z(\xi)x(\xi), y(\xi)),$$

$$\|Z(\xi) \otimes 1 fx(\xi)\| = \|f\|_2 \|Z(\xi)x(\xi)\| \quad (\xi \in \Omega)$$

and

$$(Z \otimes 1 f\tilde{x}, g\tilde{y}) = (f, g)(Z\tilde{x}, \tilde{y}).$$

Hence the functions $\xi \mapsto (Z(\xi) \otimes 1\ fx(\xi),\ gy(\xi))$ and $\xi \mapsto \|Z(\xi) \otimes 1\ fx(\xi)\|$ are continuous on some compact subset $K \subseteq \Omega$ if the functions $\xi \mapsto (Z(\xi)x(\xi), y(\xi))$ and $\xi \mapsto \|Z(\xi)x(\xi)\|$ are continuous on K and we have

$$(Z \otimes 1 f\tilde{x}, g\tilde{y}) = \int (Z(\xi) \otimes 1\ fx(\xi),\ gy(\xi))\mathrm{d}\mu(\xi).$$

Since $\xi \mapsto Z(\xi)$ is strongly measurable and the set $\{fx(\xi) \mid f \in C_c(\mathbf{R}),\ x \in \Lambda\}$ is total in $\mathscr{L}_\xi$ for any $\xi \in \Omega$ it follows that $\xi \mapsto Z(\xi) \otimes 1$ is strongly measurable (see [2], p. 81 and [7], 1.2(a)). Furthermore, since $\{f\tilde{x} \mid f \in C_c(\mathbf{R}),\ x \in \Lambda\}$ is total in $\mathscr{L}$ the operator field $\xi \mapsto Z(\xi) \otimes 1$ is a decomposition of $Z \otimes 1$. Thus our assertion follows from this.

Let $\mathscr{E} = \int^{\oplus} E(\xi)\mathrm{d}\mu(\xi)$ and $\mathscr{F} = \int^{\oplus} F(\xi)\mathrm{d}\mu(\xi)$. From 4.1 we infer that $F(\xi) = E(\xi) \otimes 1$ holds a.e.. Let $Z_1, Z_2, \ldots$ be a generating sequence for $\mathscr{N}_0$ and let

$$Z_n = \int^{\oplus} Z_n(\xi)\,\mathrm{d}\mu(\xi) \qquad \text{for any } n \in \mathbf{N}.$$

For any $\xi \in \Omega$ let $\mathscr{N}_\xi$ be the weakly closed involutive algebra generated by the sequence $\{Z_n(\xi)\}_{n\in\mathbf{N}}$. $\mathscr{N}_\xi$ is a von Neumann algebra on the Hilbert space $F(\xi)\mathscr{L}_\xi$ a.e.. One can show that the family $\{\mathscr{N}_\xi\}_{\xi\in\Omega}$ is essentially independent of the selection of the generating sequence $\{Z_n\}_{n\in\mathbf{N}}$, i.e. if

$$\check{Z}_1 = \int^{\oplus} \check{Z}_1(\xi)\mathrm{d}\mu(\xi),\ \check{Z}_2 = \int^{\oplus} \check{Z}_2(\xi)\mathrm{d}\mu(\xi),\ \ldots$$

is another generating sequence for $\mathscr{N}_0$ and $\check{\mathscr{N}}_\xi$ is the weakly closed involutive algebra generated by $\{\check{Z}_n(\xi)\}_{n\in\mathbf{N}}$ then $\mathscr{N}_\xi = \check{\mathscr{N}}_\xi$ holds a.e..

4.2. LEMMA. *$\mathscr{N}_\xi$ is invariant under the dual action $\{\hat{\sigma}_t^\xi\}_{t\in\mathbf{R}}$* a.e..

Proof. Let $\mathscr{A}$ ($\mathscr{A}_\xi$) be the involutive algebra generated by $\{Z_n\}_{n\in\mathbf{N}}$ ($\{Z_n(\xi)\}_{n\in\mathbf{N}}$; $\xi \in \Omega$). By [1], p. 31, Corollary, for any $n \in \mathbf{N}$ there is a sequence $\{Z_n^{(i)}\}_{i\in\mathbf{N}}$ in $\mathscr{A}$ which converges strongly to $\hat{\sigma}_t(Z_n)$. For any $i, n \in \mathbf{N}$ let $Z_n^{(i)} = \int^{\oplus} Z_n^{(i)}(\xi)\mathrm{d}\mu(\xi)$ such that $Z_n^{(i)}(\xi) \in \mathscr{A}_\xi$ for every $\xi \in \Omega$. From 3.8 it follows that

$$\hat{\sigma}_t(Z_n) = \int^{\oplus} \hat{\sigma}_t^\xi(Z_n(\xi))\mathrm{d}\mu(\xi).$$

From [7], 2.2 we infer that some subsequence of $\{Z_n^{(i)}(\xi)\}_{i\in\mathbf{N}}$ converges strongly to $\hat{\sigma}_t^\xi(Z_n(\xi))$ a.e.. In particular $\hat{\sigma}_t^\xi(Z_n(\xi))$ belongs to $\mathscr{N}_\xi$ for any $n \in \mathbf{N}$ a.e.. Hence we conclude that $\hat{\sigma}_t^\xi(\mathscr{N}_\xi) \subseteq \mathscr{N}_\xi$ holds a.e..

4.3. LEMMA. *The trace τ_ξ is semifinite on $\mathscr{N}_\xi^+$* a.e..

Proof. Since τ is semifinite on $\mathscr{N}_0^+$ there is a monotonely increasing sequence $\{P_n\}_{n\in\mathbf{N}}$ of projections in $\mathscr{N}_0$ such that $\sup_{n\in\mathbf{N}} P_n = \mathrm{Id}$ and $\tau(P_n)<\infty$ holds for any $n\in\mathbf{N}$. Let $P_n=\int^{\oplus} P_n(\xi)\,\mathrm{d}\mu(\xi)$ for any $n\in\mathbf{N}$. By the same argument we used in the proof of 4.2 we can see that $P_n(\xi)\in\mathscr{N}_\xi$ holds a.e.. Hence by 3.9 there is a subset $M\subseteq\Omega$ such that $\Omega\setminus M$ has measure zero and for any $\xi\in M$ the following hold:

$P_n(\xi)$ is a projection in the von Neumann algebra $\mathscr{N}_\xi\subseteq\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\subseteq\mathscr{L}(\mathfrak{B}_\xi)$ satisfying $\tau_\xi(P_n(\xi))<\infty$;

the sequence $\{P_n(\xi)\}_{n\in\mathbf{N}}$ is monotonely increasing;

$\sup_{n\in\mathbf{N}} P_n(\xi)=\mathrm{Id}_\xi$ holds.

Let $Y\in\mathscr{N}_\xi^+$ for some $\xi\in M$. Then the sequence $\{Y^{1/2}P_n(\xi)Y^{1/2}\}_{n\in\mathbf{N}}$ is monotonely increasing and we have $Y=\sup_{n\in\mathbf{N}} Y^{1/2}P_n(\xi)Y^{1/2}$ as well as $\tau_\xi(Y^{1/2}P_n(\xi)Y^{1/2})<\infty$. This shows that τ_ξ is semifinite on $\mathscr{N}_\xi^+$ for $\xi\in M$.

Let Ω_0 be the set of all $\xi\in\Omega$ such that the von Neumann algebra $\mathscr{N}_\xi\subseteq\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\subseteq\mathscr{L}(\mathfrak{B}_\xi)$ is invariant under the action $\{\hat{\sigma}_t^\xi\}_{t\in\mathbf{R}}$ and τ_ξ is semifinite on $\mathscr{N}_\xi^+$. By 4.2 and 4.3 the set $\Omega\setminus\Omega_0$ has measure zero. For any $\xi\in\Omega_0$ let α_t^ξ be the restriction of $\hat{\sigma}_t^\xi$ to $\mathscr{N}_\xi$. The crossed product $\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R}$ is canonically embedded into $\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\otimes_{\hat{\sigma}^\xi}\mathbf{R}$ as a von Neumann subalgebra. From [13], 5.19 it can be seen that the dual weight $\hat{\tau}_\xi$ of τ_ξ is semifinite on $(\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R})^+$. From [13], 5.15 it follows that $\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R}$ is invariant under the modular automorphism group associated with $\hat{\tau}_\xi$. Hence we infer from the theorem in [12], Section 3 that there is a σ-weakly continuous faithful projection of norm one from $\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\otimes_{\hat{\sigma}^\xi}\mathbf{R}$ onto $\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R}$. If $\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R}$ is of type III then by [9], 2.6.5, $\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\otimes_{\hat{\sigma}^\xi}\mathbf{R}$ is also of type III and by [13], 4.5, $\mathscr{L}(\mathfrak{B}_\xi)_{F(\xi)}\otimes_{\hat{\sigma}^\xi}\mathbf{R}$ is isomorphic to $\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}$, in particular $\mathscr{L}(\mathfrak{A}_\xi)_{E(\xi)}$ is of type III. Therefore Theorem 1.1 would be proved if we can show that $\mathscr{N}_\xi\otimes_{\alpha^\xi}\mathbf{R}$ is of type III a.e. on Ω_0.

Let the sequence $\{f_n\}_{n\in\mathbf{N}}$ in $C_c(\mathbf{R})$ be chosen as in 2.2. By 2.2 the sequence $\{f_ne\}_{n\in\mathbf{N}}\left(e=\int^{\oplus} e_\xi\mathrm{d}\mu(\xi)\right)$ is cyclic for $\mathscr{L}(\mathfrak{B})$. Hence the sequence $\{J^\circ(f_ne)\}_{n\in\mathbf{N}}$ is separating for $\mathscr{L}(\mathfrak{B})$. Similarly the sequence $\{J_\xi^0(f_ne_\xi)\}_{n\in\mathbf{N}}$ is separating for $\mathscr{L}(\mathfrak{B}_\xi)$. For any $n\in N$ we denote by y_n the vector field $\xi\mapsto J_\xi^0(f_ne_\xi)$.

Let μ_0 be the restriction of the measure μ to Ω_0. Henceforth we work exclusively with the reduced measure space (Ω_0,μ_0). Let Σ be the set of all vector fields which are rational linear combinations of vector fields of the form $\xi\mapsto F(\xi)Z(\xi)y_n(\xi)$, $n\in\mathbf{N}$, where $\xi\mapsto Z(\xi)$ is an operator field which is a finite product of operator fields of the form $\xi\mapsto Z_m(\xi)$ or $\xi\mapsto Z_m(\xi)^*$. Let $\mathscr{H}^0$ ($\mathscr{H}_\xi^0$) be the Hilbert space which is generated by the set $\{\mathscr{F}Z\tilde{y}_n\mid Z\in\mathscr{N},n\in\mathbf{N}\}$ ($\{x(\xi)\mid x\in\Sigma\}$). Chosing some enumeration for the set Σ this set becomes a fundamental sequence of vector fields in the sense of [1], p. 142, Definition 1. Moreover we obtain

$$\mathscr{H}^0 = \int^{\oplus} \mathscr{H}^0_\xi \mathrm{d}\mu_0(\xi)$$

in the sense of [1], p. 147, Definition 3, i.e. $\mathscr{H}^0$ is the direct integral of the Hilbert spaces $\mathscr{H}^0_\xi$.

By the definition of the Hilbert space $\mathscr{H}^0$ ($\mathscr{H}^0_\xi$) the mapping

$$\Phi: \mathscr{N} \ni Z \mapsto Z/_{\mathscr{H}^0} \quad (\Phi_\xi: \mathscr{N}_\xi \ni Z \mapsto Z/\mathscr{H}^0_\xi)$$

is an isomorphism from $\mathscr{N}$ ($\mathscr{N}_\xi$) onto some von Neumann algebra $\mathscr{M}$($\mathscr{M}_\xi$) on $\mathscr{H}^0$ ($\mathscr{H}^0_\xi$; $\xi \in \Omega_0$).

The operator field $\xi \mapsto \Phi_\xi(Z_n(\xi))$ is measurable for any $n \in \mathbf{N}$ (in the sense of [1], p. 156, Definition 1) and the sequence $\{\Phi_\xi(Z_n(\xi))\}_{n\in\mathbf{N}}$ generates $\mathscr{M}_\xi$ for any $\xi \in \Omega_0$. Hence $\xi \mapsto \mathscr{M}_\xi$ is a measurable field of von Neumann algebras in the sense of [1], p. 173, Definition 1, and we have

$$\Phi(Z_n) = \int^{\oplus} \Phi_\xi(Z_n(\xi))\, \mathrm{d}\mu_0(\xi), \qquad n \in \mathbf{N}$$

(in the sense of [1], p. 159, Definition 2). Since $\mathscr{M}$ is generated by the sequence $\{\Phi(Z_n)\}_{n\in\mathbf{N}}$ and by the diagonalisable operators on $\mathscr{H}^0$ we obtain

$$\mathscr{M} = \int^{\oplus} \mathscr{M}_\xi \mathrm{d}\mu_0(\xi)$$

(in the sense of [1], p. 174, Definition 2).

For any $t \in \mathbf{R}$ let $\beta_t := \Phi \circ \alpha_t \circ \Phi^{-1}$ ($\beta^\xi_t := \Phi_\xi \circ \alpha^\xi_t \circ \Phi^{-1}_\xi$). It follows from [13] 3.4 that $\mathscr{N} \otimes_\alpha \mathbf{R}$ ($\mathscr{N}_\xi \otimes_{\alpha^\xi} \mathbf{R}$) is isomorphic to $\mathscr{M} \otimes_\beta \mathbf{R}$ ($\mathscr{M}_\xi \otimes_{\beta^\xi} \mathbf{R}$). Hence $\mathscr{M} \otimes_\beta \mathbf{R}$ is of type III and we are done if we can show that $\mathscr{M}_\xi \otimes_{\alpha^\xi} \mathbf{R}$ is of type III a.e. on Ω_0.

By the definition of the dual action $\{\hat{\sigma}^\xi_t\}_{t\in\mathbf{R}}$ it follows that for any $Z = \int^{\oplus} Z(\xi)\mathrm{d}\mu_0(\xi) \in \mathscr{M}$ and for any measurable vector field $\xi \mapsto x(\xi)$, $\xi \mapsto y(\xi)$ the function

$$(\xi, t) \mapsto (\beta_t(Z(\xi))x(\xi), y(\xi))$$

is measurable on $\Omega_0 \times \mathbf{R}$. Using this one can show that the following is true (see [10], 1.4)

$$\mathscr{M} \otimes_\beta \mathbf{R} = \int^{\oplus} \mathscr{M}_\xi \otimes_{\beta^\xi} \mathbf{R}\, \mathrm{d}\mu_0(\xi).$$

Since $\mathscr{M} \otimes_\beta \mathbf{R}$ is of type III it follows from [4], 4.2 that $\mathscr{M}_\xi \otimes_{\beta^\xi} \mathbf{R}$ is of type III a.e.. Thus our proof is complete.

Concluding remark. The situation considered in Theorem 1.1 applies to the central decomposition of a KMS-state on a C^*-algebra with unit which we investigated in [7], Section 3.

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